Discrete Channels

**Def:** A discrete channel is denoted by \((\mathcal{X}, P_{Y|X}, \mathcal{Y})\) where \(\mathcal{X}\) is a finite input set, \(\mathcal{Y}\) is a finite output set and \(P_{Y|X}\) is a conditional probability distribution, d.h.

\[
\forall x \in \mathcal{X} \quad \forall y \in \mathcal{Y} : P_{Y|X}(y|x) = 1
\]

\[
\forall x \in \mathcal{X} : \sum_{y} P_{Y|X}(y|x) = 1
\]

\[P_{Y|X}(y|x) = \text{the probability that the channel outputs } y \text{ when given } x \text{ as input}\]
Noisy-Channel Coding

Def: A \((M,n)\)-code for the channel \((\mathcal{X}, P_{Y|X}, \mathcal{Y})\) consists of

1. message set: \([M] = \{1, 2, \ldots, M\}\)
2. encoding function: \(e : [M] \rightarrow \mathcal{X}^n\)
   codebook: \(\{e(1), e(2), \ldots, e(M)\}\)
3. deterministic decoding function assigning a guess to each possible received vector \(d : \mathcal{Y}^n \rightarrow [M]\)

The rate of a \((M,n)\)-code denotes the transmitted bits per channel use \(R := \frac{\log M}{n}\)
Rate and Error

The rate of a \((M,n)\)-code denotes the transmitted bits per channel use:

\[
R := \frac{\log M}{n}
\]

The probability of error when sending \(w \in [M]\):

\[
\lambda_w := \Pr[\tilde{W} = d(Y^n) \neq w \mid X^n = e(w)]
\]

Maximal probability of error:

\[
\lambda^{(n)} := \max_{w \in [M]} \lambda_w
\]

Average probability of error:

\[
p_{e}^{(n)} := \frac{1}{M} \sum_{w=1}^{M} \lambda_w
\]
Encoding with Feedback

The rate of a $(M,n)$-code denotes the transmitted bits per channel use:

$$R := \frac{\log M}{n}$$

probability of error when sending $w \in [M]$

$$\lambda_w := \Pr[\tilde{W} = d(Y^n) \neq w \mid X^n = e(w)]$$

maximal probability of error:

$$\lambda^{(n)} := \max_{w \in [M]} \lambda_w$$

encoding with feedback:

$X_{i+1}$ can depend on $w, X_1, \ldots, X_i$ and $Y_1, \ldots, Y_i$