Solutions to Exercise Set 3

Solution 3.1 Working out $H|x\rangle \otimes H|y\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle + (-1)^x|1\rangle\big) \otimes \frac{1}{\sqrt{2}}\big(|0\rangle + (-1)^y|1\rangle\big)$ and applying *CNOT* yields

$$\begin{aligned}
CNOT(H|x\rangle \otimes H|y\rangle) &= \frac{1}{2} CNOT\big(|0\rangle|0\rangle + (-1)^x|1\rangle|0\rangle + (-1)^y|0\rangle|1\rangle + (-1)^{x+y}|1\rangle|1\rangle\big) \\
&= \frac{1}{2}\big(|0\rangle|0\rangle + (-1)^x|1\rangle|1\rangle + (-1)^y|0\rangle|1\rangle + (-1)^{x+y}|1\rangle|0\rangle\big) \\
&= \frac{1}{2}\Big(\big(|0\rangle + (-1)^{x+y}|1\rangle\big) \otimes |0\rangle + \big((-1)^y|0\rangle + (-1)^x|1\rangle\big) \otimes |1\rangle\Big) \\
&= \frac{1}{\sqrt{2}}\big(H|x \oplus y\rangle \otimes |0\rangle + (-1)^y H|x \oplus y\rangle \otimes |1\rangle\big) \\
&= H|x \oplus y\rangle \otimes H|y\rangle\,.
\end{aligned}$$

In words, on the Hadamard basis *CNOT* acts as a control-NOT but with the *second* qubit as the control qubit.

An alternative (and more insightful) solution is to recall from Exercise 1 that $HXH = Z$ (and of course $H\mathbb{I}H = \mathbb{I}$), and thus $H_2 CNOT_{12} H_2 = H_2 C_1(X_2) H_2 = C_1(Z_2)$, where we use the convention to write H_2 for $\mathbb{I} \otimes H$, etc. Furthermore, $C(Z)$ is symmetric in that $C_1(Z_2) = C_2(Z_1)$; indeed, $C_1(Z_2)|x\rangle|y\rangle = |x\rangle \otimes Z^x|y\rangle = (-1)^{xy}|x\rangle|y\rangle = Z^y|x\rangle \otimes |y\rangle = C_2(Z_1)|x\rangle|y\rangle$. Thus,

$$\begin{aligned}
CNOT_{12} H_1 H_2 &= H_2 H_2 CNOT_{12} H_2 H_1 = H_2 C_1(Z_2) H_1 \\
&= H_2 C_2(Z_1) H_1 = H_2 H_1 CNOT_{21} H_1 H_1 = H_1 H_2 CNOT_{21}\,.
\end{aligned}$$

Solution 3.2 Straightforward calculations show that

$$\begin{aligned}
V^\dagger V &= (1+i)(\mathbb{I} - iX)(1-i)(\mathbb{I} + iX)/4 = 2(\mathbb{I} + X^2)/4 = 2(\mathbb{I} + \mathbb{I})/4 = \mathbb{I} \qquad \text{and} \\
V^2 &= (1-i)(\mathbb{I} + iX)(1-i)(\mathbb{I} + iX)/4 = -2i(\mathbb{I} + 2iX - X^2)/4 = X\,.
\end{aligned}$$

Solution 3.3 For $x, y \in \{0,1\}$ and $|\varphi\rangle \in \mathcal{H}$, the vector $|x\rangle|y\rangle|\varphi\rangle$ indeed gets mapped to

$$\begin{aligned}
|x\rangle|y\rangle|\varphi\rangle &\mapsto |x\rangle|y\rangle V^y|\varphi\rangle \mapsto |x\rangle|x \oplus y\rangle V^y|\varphi\rangle \mapsto |x\rangle|x \oplus y\rangle (V^\dagger)^{x\oplus y} V^y|\varphi\rangle \\
&\mapsto |x\rangle|y\rangle (V^\dagger)^{x\oplus y} V^y|\varphi\rangle \mapsto |x\rangle|y\rangle V^x (V^\dagger)^{x\oplus y} V^y|\varphi\rangle = |x\rangle|y\rangle U^{xy}|\varphi\rangle\,,
\end{aligned}$$

where the final equality is verified by checking the different cases for x and y, and using that $V^\dagger V = \mathbb{I} = VV^\dagger$ and $V^2 = U$.

Solution 3.4 For $n = 1$ and $x \in \{0,1\}$, we have

$$H|x\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle + (-1)^x|1\rangle\big) = \frac{1}{\sqrt{2}} \sum_{y \in \{0,1\}} (-1)^{x\cdot y}|y\rangle$$

by definition. For the general case with $x = (x_1, \ldots, x_n) \in \{0,1\}^n$, we then have

$$\begin{aligned}
H^{\otimes n}|x\rangle &= \bigotimes_{i=1}^n H|x_i\rangle = \bigotimes_{i=1}^n \left(\frac{1}{\sqrt{2}} \sum_{y_i \in \{0,1\}} (-1)^{x_i \cdot y_i}|y_i\rangle\right) \\
&= \frac{1}{2^{n/2}} \sum_{y_1,\ldots,y_n} (-1)^{x_1\cdot y_1} \cdots (-1)^{x_n \cdot y_n}|y_1\rangle \cdots |y_n\rangle = \frac{1}{2^{n/2}} \sum_{y \in \{0,1\}^n} (-1)^{x\cdot y}|y\rangle\,,
\end{aligned}$$

Solution 3.5 Exploiting that $H|z\rangle = \frac{1}{\sqrt{2}}(|0\rangle + (-1)^z|1\rangle)$ and using linearity, we obtain

$$\begin{aligned} U_f(|x\rangle \otimes H|z\rangle) &= \frac{1}{\sqrt{2}}U_f|x\rangle|0\rangle + (-1)^z\frac{1}{\sqrt{2}}U_f|x\rangle|1\rangle = \frac{1}{\sqrt{2}}|x\rangle|f(x)\rangle + (-1)^z\frac{1}{\sqrt{2}}|x\rangle|1\oplus f(x)\rangle \\ &= |x\rangle \otimes \frac{1}{\sqrt{2}}(|f(x)\rangle + (-1)^z|1\oplus f(x)\rangle) = (-1)^{zf(x)}|x\rangle \otimes H|z\rangle \end{aligned}$$

where the last equality is by a case-by-case analysis: first the case $z = 0$, and then the cases $z = 1 \wedge f(x)=0$ and $z = 1 \wedge f(x)=1$. More elegantly, but less straightforward, one can observe that $U_f(|x\rangle \otimes |\psi\rangle) = |x\rangle \otimes X^{f(x)}|\psi\rangle$ for any $|\psi\rangle \in \mathcal{S}(\mathbb{C}^2)$ and recycle that $X = HZH$ to then conclude that

$$U_f(|x\rangle \otimes H|z\rangle) = |x\rangle \otimes H^{f(x)}Z^{f(x)}H^{f(x)}H|z\rangle = |x\rangle \otimes HZ^{f(x)}|z\rangle = (-1)^{zf(x)}|x\rangle \otimes H|z\rangle\,,$$

where the second equality is by considering the cases $f(x) = 0$ and $f(x) = 1$ separately.

Vice versa,

$$\begin{aligned} V_f(|x\rangle \otimes H|y\rangle) &= \frac{1}{\sqrt{2}}V_f|x\rangle|0\rangle + (-1)^y\frac{1}{\sqrt{2}}V_f|x\rangle|1\rangle = \frac{1}{\sqrt{2}}|x\rangle|0\rangle + (-1)^y(-1)^{f(x)}\frac{1}{\sqrt{2}}|x\rangle|1\rangle \\ &= |x\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle + (-1)^{y+f(x)}|1\rangle) = |x\rangle \otimes H|y\oplus f(x)\rangle\,. \end{aligned}$$

Alternatively, and somewhat more directly, from the first derivation we can read out that $(\mathbb{I}\otimes H)U_f(\mathbb{I}\otimes H) = V_f$. Hence

$$V_f(|x\rangle \otimes H|y\rangle) = (\mathbb{I}\otimes H)U_f|x\rangle|y\rangle = |x\rangle \otimes H|y\oplus f(x)\rangle\,.$$

Solution 3.6 Applying X^xZ^z to the first qubit turns $|\Phi^+\rangle$ into

$$(X^xZ^z\otimes\mathbb{I})|\Phi^+\rangle = \frac{1}{\sqrt{2}}\big(X^xZ^z|0\rangle\otimes|0\rangle + X^xZ^z|1\rangle\otimes|1\rangle\big) = \frac{1}{\sqrt{2}}\big(|x\rangle|0\rangle + (-1)^z|x\oplus 1\rangle|1\rangle\big)\,.$$

Applying *CNOT*, controlled by the second qubit, results in the state

$$\frac{1}{\sqrt{2}}\big(|x\rangle|0\rangle + (-1)^z|x\rangle|1\rangle\big) = |x\rangle\otimes\frac{1}{\sqrt{2}}(|0\rangle + (-1)^z|1\rangle) = |x\rangle\otimes H|z\rangle\,,$$

and thus by measuring the first qubit in the computational and the second qubit in the Hadamard basis, Bob recovers x and z.