Solutions to Exercise Set 1

Solution 1.1 $\langle\varphi| = [2-i, 1+3i]$, and thus

$$\langle\varphi|\varphi\rangle = [2-i, 1+3i]\begin{bmatrix} 2+i \\ 1-3i \end{bmatrix} = (2-i)(2+i) + (1+3i)(1-3i) = 15$$

and

$$|\varphi\rangle\langle\varphi| = \begin{bmatrix} 2+i \\ 1-3i \end{bmatrix}[2-i, 1+3i] = \begin{bmatrix} 5 & -1+7i \\ -1-7i & 10 \end{bmatrix}.$$

And we see that indeed $\mathrm{tr}(|\varphi\rangle\langle\varphi|) = 5 + 10 = 15$, and that $|\varphi\rangle\langle\varphi|$ is Hermitian.

Solution 1.2 For the first claim, note that for every vector $|e_j\rangle$ from the basis $\{|e_i\rangle\}_{i\in I}$ it holds that

$$\sum_i |e_i\rangle\langle e_i||e_j\rangle = \sum_i |e_i\rangle\langle e_i|e_j\rangle = |e_j\rangle\,.$$

Thus, $\sum_i |e_i\rangle\langle e_i|$ acts as identity on the basis vectors, and thus must be the identity $\mathbb{I}$.

Towards the second claim, let $\{|e_i\rangle\}_{i\in I}$ be an arbitrary family of vectors with $\sum_i |e_i\rangle\langle e_i| = \mathbb{I}$, and let $|e_j\rangle$ be any vector from the family. Then,

$$0 = \Big(\sum_i |e_i\rangle\langle e_i| - \mathbb{I}\Big)|e_j\rangle = \sum_i |e_i\rangle\langle e_i|e_j\rangle - |e_j\rangle = \sum_{i\neq j} |e_i\rangle\langle e_i|e_j\rangle + |e_j\rangle(\langle e_j|e_j\rangle - 1)\,.$$

In case the $|e_i\rangle$'s are linearly independent, the linear combination on the right hand side must have vanishing coefficients; thus, $\langle e_i|e_j\rangle = 0$ for all $i \neq j$ and $\langle e_j|e_j\rangle = 1$. In case the $|e_i\rangle$'s have norm 1, we apply $\langle e_j|$ from the left to the above equality to obtain

$$0 = \sum_{i\neq j} \langle e_j|e_i\rangle\langle e_i|e_j\rangle = \sum_{i\neq j} |\langle e_i|e_j\rangle|^2\,,$$

where we exploited that $\langle e_j|e_i\rangle = \overline{\langle e_i|e_j\rangle}$, and it follows that $\langle e_i|e_j\rangle = 0$ for every $i \neq j$. Finally, the equality $|\varphi\rangle = \sum_{i\in I} |e_i\rangle\langle e_i||\varphi\rangle = \sum_{i\in I} |e_i\rangle\langle e_i|\varphi\rangle$ shows that $\{|e_i\rangle\}_{i\in I}$ spans all of $\mathcal{H}$. Thus, it forms an orthonormal basis.

Solution 1.3 $X^2 = Y^2 = Z^2 = \mathbb{I}$ follow from working out these squares. Similarly for the other identities: by working out the product XY we obtain

$$XY = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix} = iZ\,,$$

and so $-iXYZ = \mathbb{I}$ follows from the self-inverseness of Z. $iZYX = \mathbb{I}$ can be shown similarly, or by observing that the above implies that $\mathbb{I} = \mathbb{I}^\dagger = i(XYZ)^\dagger = iZ^\dagger Y^\dagger X^\dagger = iZYX$. For the terms with the Hadamard operators, also here it is a straightforward calculation to see that $HXH = Z$, and thus $HZH = X$, while $HYH = -Y$.

Solution 1.4 Copying the definition of Z, we have that $Z|0\rangle = |0\rangle$ and $Z|1\rangle = -|1\rangle$. As for X, we get that

$$X|\pm\rangle = \frac{1}{\sqrt{2}}(X|0\rangle \pm X|1\rangle) = \frac{1}{\sqrt{2}}(|1\rangle \pm |0\rangle) = \begin{cases} |+\rangle \\ -|-\rangle \end{cases}.$$

Thus, in both cases, the corresponding eigenvalues are ± 1.

By solving the characteristic polynomial, we see that the eigenvalues are ± 1 here as well. Alternatively, using that $Y^2 = \mathbb{I}$ it follows that the eigenvalues (if existent) must lie in $\{\pm 1\}$. Setting $|\psi\rangle = |0\rangle + \omega|1\rangle$ and demanding that

$$\pm(|0\rangle + \omega|1\rangle) = \pm|\psi\rangle \overset{!}{=} Y|\psi\rangle = Y|0\rangle + \omega Y|1\rangle = i|1\rangle - \omega i|0\rangle\,,$$

we see that $\omega = \pm i$ satisfies the equation. Thus, $\frac{1}{\sqrt{2}}(|0\rangle \pm i|1\rangle)$ are the respective normalized eigenvectors (which are unique up to the phase) to the eigenvalues ± 1.

Solution 1.5 It is clear that the zero matrix $\mathbf{0}$ is Hermitian, and that $A \pm B$ is Hermitian if A and B are: $(A \pm B)^\dagger = A^\dagger \pm B^\dagger = A \pm B$. Finally and crucially, for Hermitian A and $\lambda \in \mathbb{R}$ (but not for general $\lambda \in \mathbb{C}$, unless $A = \mathbf{0}$)

$$(\lambda A)^\dagger = \bar{\lambda} A^\dagger = \bar{\lambda} A = \lambda A\,.$$

The space of Hermitian 2×2-matrices is given by matrices of the form

$$A = \begin{bmatrix} d & a + bi \\ a - bi & e \end{bmatrix}$$

for $a, b, d, e \in \mathbb{R}$, which also shows that the dimension of the space is 4. Indeed, for A to be Hermitian, the diagonal elements need to be real and the off-diagonals complex conjugates of each other, and this is also sufficient. But now, any such matrix can be written as

$$A = aX - bY + \frac{1}{2}(d - e)Z + \frac{1}{2}(d + e)\mathbb{I}\,,$$

as can be easily verified. Finally, given that the space has dimension 4, it follows that $\mathbb{I}, X, Y$ and Z are linearly independent.

Solution 1.6 We can easily read out the p_i's for the computational basis as

$$p_0 = \left|\frac{1}{2}\right|^2 = \frac{1}{4} \qquad \text{and} \qquad p_1 = \left|\frac{\sqrt{3}}{2}\right|^2 = \frac{3}{4}\,.$$

The probabilities for the Hadamard basis we can get as

$$p_+ = \big|\langle +|\varphi\rangle\big|^2 = \left|\frac{1}{2\sqrt{2}}\big(\langle 0| + \langle 1|\big)\big(|0\rangle + \sqrt{3}|1\rangle\big)\right|^2 = \left|\frac{1+\sqrt{3}}{2\sqrt{2}}\right|^2 = \frac{4 + 2\sqrt{3}}{8}$$

and

$$p_- = \big|\langle -|\varphi\rangle\big|^2 = \left|\frac{1}{2\sqrt{2}}\big(\langle 0| - \langle 1|\big)\big(|0\rangle + \sqrt{3}|1\rangle\big)\right|^2 = \left|\frac{1-\sqrt{3}}{2\sqrt{2}}\right|^2 = \frac{4 - 2\sqrt{3}}{8}\,.$$

Solution 1.7 Fix an orthonormal basis $\{|e_i\rangle\}_{i \in I}$, and consider arbitrary fixed indices $i, j \in I$. For $|\varphi\rangle = |e_i\rangle + |e_j\rangle$ and $|\varphi'\rangle = |e_i\rangle + i|e_j\rangle$, we see then that $0 = \langle\varphi|A|\varphi\rangle = \langle e_i|A|e_j\rangle + \langle e_j|A|e_i\rangle$ and $0 = \langle\varphi'|A|\varphi'\rangle = i\langle e_i|A|e_j\rangle - i\langle e_j|A|e_i\rangle$, where for the latter we used that $\langle\varphi'| = \langle e_i| - i\langle e_j|$. It now follows that $\langle e_i|A|e_j\rangle = 0$. Since $i \in I$ was arbitrary, we get that $A|e_j\rangle = 0$; indeed, only the 0-vector is orthogonal to a basis. Finally, since $j \in I$ was arbitrary, we then get $A = 0$.

For the second part, we note that for any $|\varphi\rangle \in \mathcal{H}$

$$\langle\varphi|(A - A^\dagger)|\varphi\rangle = \langle\varphi|A|\varphi\rangle - \langle\varphi|A^\dagger|\varphi\rangle = \overline{\langle\varphi|A|\varphi\rangle} - \langle\varphi|A^\dagger|\varphi\rangle = \langle\varphi|A^\dagger|\varphi\rangle - \langle\varphi|A^\dagger|\varphi\rangle = 0\,,$$

where the second equality used that $\langle\varphi|A|\varphi\rangle \in \mathbb{R}$. Thus, by the above, $A - A^\dagger = 0$.