Introduction to Quantum Computing
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Exercise Set 1

Exercise 1.1 ☺ Let $|\varphi\rangle = \begin{bmatrix} 2+i \\ 1-3i \end{bmatrix} \in \mathbb{C}^2$. Compute $\langle\varphi|\varphi\rangle \in \mathbb{C}$ and $|\varphi\rangle\langle\varphi| \in \mathcal{L}(\mathbb{C}^2) = \mathbb{C}^{2\times 2}$. Verify that $\mathrm{tr}(|\varphi\rangle\langle\varphi|)$, as sum of the diagonal elements, equals $\langle\varphi|\varphi\rangle$, and that $(|\varphi\rangle\langle\varphi|)^\dagger = |\varphi\rangle\langle\varphi|$.

Exercise 1.2 😐 Show that $\sum_{i\in I} |e_i\rangle\langle e_i| = \mathbb{I}$ holds for any orthonormal basis $\{|e_i\rangle\}_{i\in I}$ of $\mathcal{H}$. Also show that if $\{|e_i\rangle\}_{i\in I}$ are arbitrary vectors that are linearly independent or have norm 1, then $\sum_{i\in I} |e_i\rangle\langle e_i| = \mathbb{I}$ implies that they form an orthonormal basis of $\mathcal{H}$.

Exercise 1.3 ☺ Verify that the Pauli operators satisfy $X^2 = Y^2 = Z^2 = -iXYZ = iZYX = \mathbb{I}$. Also, work out HXH, HYH and HZH, where H is the Hadamard operator.

Exercise 1.4 ☺ Show that $\{|0\rangle, |1\rangle\}$ are eigenvectors of Z (i.e., $Z|i\rangle = \lambda_i|i\rangle$ for $i \in \{0,1\}$ and appropriate $\lambda_0, \lambda_1 \in \mathbb{C}$), and $\{|+\rangle, |-\rangle\}$ are eigenvectors of X.
Find the eigenvectors of Y?

Exercise 1.5 😐 Show that for any integer $d > 0$, the set of Hermitian $d \times d$-matrices $A \in \mathcal{L}(\mathbb{C}^d)$ forms a vector space over $\mathbb{R}$ (but not over $\mathbb{C}$). Show that the matrices $\mathbb{I}, X, Y, Z \in \mathcal{L}(\mathbb{C}^2)$ form a basis of the $\mathbb{R}$-vector-space of Hermitian 2×2-matrices.

Exercise 1.6 ☺ Consider the qubit state $|\varphi\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle \in \mathbb{C}^2$. What are the probabilities p_0 and p_1 to observe 0 and 1 when measuring $|\varphi\rangle$ in the computational basis $\{|0\rangle, |1\rangle\}$? What are the probabilities p_+ and p_- to observe the two possible outcomes "+" and "−" when measuring $|\varphi\rangle$ in the Hadamard basis $\{|+\rangle, |-\rangle\}$?

Exercise 1.7 🥷 As usual, let $\mathcal{H}$ be a finite-dimensional, complex Hilbert space, i.e., $\mathcal{H} = \mathbb{C}^d$, and let $A \in \mathcal{L}(\mathcal{H})$ be arbitrary. Show that if $\langle\varphi|A|\varphi\rangle = 0$ for all $|\varphi\rangle \in \mathcal{H}$ then $A = 0$.
Hint: It is sufficient to show that the assumption implies that $\langle e_i|A|e_j\rangle = 0$ for all $i, j \in I$ and some orthonormal basis $\{|e_i\rangle\}_{i\in I}$ of $\mathcal{H}$. Also, it is helpful to realize that the statement is not true in case $\mathcal{H}$ is a *real* Hilbert space.
Use the above to show that if $\langle\varphi|A|\varphi\rangle \in \mathbb{R}$ for all $|\varphi\rangle \in \mathcal{H}$ then A is Hermitian, i.e., $A^\dagger = A$.

Legend: ☺ = easy, 😐 = medium difficult, 🥷 = tricky.