Interpolation on Sparse Gauss-Chebyshev Grids in Higher Dimensions

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ABSTRACT

In this paper, we give a unified approach to error estimates for interpolation on sparse Gauß–Chebyshev grids for multivariate functions from Besov–type spaces with dominating mixed smoothness properties. The error bounds obtained for this method are almost optimal for the considered scale of function spaces.

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1. INTRODUCTION

The Gauß–Chebyshev knots are quite often used as interpolation points for functions given on the interval [−1, 1]. Beside the well–studied polynomial interpolation, one can investigate interpolation by splines adapted to these special nodes.

Using generalized translates, the so–called Chebyshev–shifts, the interpolation of univariate functions on Gauß–Chebyshev knots can be seen as interpolation by translates. Shift–invariant spaces corresponding to such translates and their wavelet analysis are described in detail in [11]. Forming tensor products yields interpolation on full Gauß–Chebyshev grids. This is a reasonable choice for fairly small dimensions only.

For higher dimension, interpolation on sparse grids, using essentially less points, is much more appropriate. It can be realized by $j$–th order blending. Some recent papers use this fact for determining the quadrature error for smooth functions [6, 9, 10] or for investigating interpolatory wavelets for sparse Gauß–Chebyshev grids [15]. Interpolation and approximation on sparse grids has been fairly well investigated for periodic functions [5, 12, 14]. It is closely related to hyperbolic approximation (see e.g. [17]).

Error estimates for interpolation on sparse Gauß–Chebyshev grids were given up to now only for functions from Sobolev–type spaces [16] and for polynomial interpolation of functions with bounded mixed derivatives measured in the maximum norm [1].
In the present paper, we give a unified approach to error estimates for interpolation on sparse Gauß–Chebyshev grids for functions from certain Besov–type spaces to be introduced in Section 3.

Therefore, we can adapt the concept of Strang–Fix conditions for this special situation, see Section 4. Furthermore, we use essentially the properties of the multivariate function spaces which can be represented as tensor products of the univariate function spaces provided with uniform crossnorms. In this way, we propose interpolation methods which are almost optimal for a wide range of Besov–type spaces.

2. Notation
We denote our reference interval by \( I := [-1, 1] \) and the Chebyshev weight by \( w(x) := (1 - x^2)^{-1/2} \) \( x \in (-1, 1) \). Let \( L^2_w(I) \) be the weighted Hilbert space of all measurable functions \( f : I \rightarrow \mathbb{R} \), with
\[
\int_I f(x)^2 w(x) \, dx < \infty.
\]

For \( f, g \in L^2_w(I) \), the corresponding inner product is given by
\[
\langle f, g \rangle := \frac{2}{\pi} \int_I f(x)g(x)w(x) \, dx.
\]

By \( \Pi_n \), we denote the set of all real valued polynomials of degree at most \( n \) restricted on \( I \). Furthermore, let \( T_n \in \Pi_n \) be the Chebyshev polynomials
\[
T_n(x) := \cos(n \arccos x).
\]

They form a complete orthogonal basis
\[
\langle T_k, T_\ell \rangle = \begin{cases} 
2 & \text{for } k = \ell = 0, \\
1 & \text{for } k = \ell \neq 0, \\
0 & \text{otherwise},
\end{cases}
\]

of \( L^2_w(I) \). With the help of the Chebyshev coefficients
\[
a_k[f] := \langle f, T_k \rangle , \quad f \in L^2_w(I), \; k \in \mathbb{N}_0,
\]

we characterize the Wiener algebra
\[
A(I) := \left\{ f \in L^2_w(I) ; \sum_{k \in \mathbb{N}_0} |a_k[f]| < \infty \right\}
\]
of functions with an absolutely summable Fourier–Chebyshev series. Here and in the sequel, we use the notation $\sum'$ if we halve the first term and $\sum''$ where both the first and the last term in the sum are halved. By $G_N$, we denote the grid of the Gauß–Chebyshev nodes

$$G_N := \{g_k := \cos k \frac{\pi}{N}; \ k = 0, \ldots, N\}.$$  

The discrete Chebyshev coefficients are given as

$$a_k^N[f] := \frac{2}{N} \sum_{\ell=0}^{N} f(g_\ell) T_k(g_\ell), \quad k = 0, \ldots, N.$$  

For functions $f \in A(I)$, one can prove the aliasing formula

$$a_k^N[f] = \sum_{\ell \in \mathbb{N}_0} a_{2\ell N+k}[f] + a_{2(\ell+1)N-k}[f], \quad k = 0, \ldots, N.$$  

### 3. Function Spaces

We want to interpolate multivariate functions. Their smoothness will be measured in the scale of Besov–type spaces. In this section, we give a definition of these function spaces via the summability of their Fourier–Chebyshev series. The Chebyshev coefficient of an $n$–variate function is given in the usual way as

$$a_k[f] = \langle f, T_{k_1} \otimes T_{k_2} \otimes \cdots \otimes T_{k_n}\rangle, \quad \text{for } k \in \mathbb{N}_0^n.$$  

Furthermore, we need the index sets

- $Q_0^n = \{0\}$,
- $Q_j^n = \{k \in \mathbb{N}_0^n; k_r < 2^j, r = 1, \ldots, n\}$ \small
- \small $\setminus \{k \in \mathbb{N}_0^n; k_r < 2^{j-1}, r = 1, \ldots, n\}.$

**Definition 1** Let $1 \leq q \leq \infty$ and $s \geq 0$. Then we define the isotropic Besov–type space $B_{2,q,w}^s(I^n)$ as

$$B_{2,q,w}^s(I^n) := \left\{ f \in L_w^2(I^n) ; \| f \|_{B_{2,q,w}^s(I^n)} \right\}$$

$$= \left( \sum_{j=0}^{\infty} 2^{jsq} \left( \sum_{k \in Q_j^n} a_k^2[f] \right)^{q/2} \right)^{1/q} < \infty$$

for $q < \infty$ and

$$B_{2,\infty,w}^s(I^n) := \left\{ f \in L_w^2(I^n) ; \| f \|_{B_{2,\infty,w}^s(I^n)} \right\}$$

$$= \sup_{j \in \mathbb{N}_0} 2^{js} \left( \sum_{k \in Q_j^n} a_k^2[f] \right)^{1/2} < \infty,$$

respectively.
For the definition of the spaces of functions with dominating mixed smoothness properties, we put the index sets

\[ P_j^n = Q_{j_1}^1 \times Q_{j_2}^1 \times \cdots \times Q_{j_n}^1, \quad j \in \mathbb{N}_0^n, \]

and denote the inner product in \( \mathbb{N}_0^n \) by \( j \cdot r := j_1 r_1 + j_2 r_2 + \cdots + j_n r_n \).

**Definition 2** Let \( 1 \leq q \leq \infty \) and \( r \in \mathbb{R}_+^n \). Then the Besov–type space \( S_{2,q,w}^r(I^n) \) of \( n \)-variate functions with dominating mixed smoothness properties is defined as

\[
S_{2,q,w}^r(I^n) := \left\{ f \in L_w^2(I^n) ; \| f \|_{S_{2,q,w}^r(I^n)} = \left( \sum_{j \in \mathbb{N}_0^n} 2^{j \cdot r} \left( \sum_{k \in P_j^n} a_k^2[f] \right)^{q/2} \right)^{1/q} < \infty \right\}
\]

for \( q < \infty \) and

\[
S_{2,\infty,w}^r(I^n) := \left\{ f \in L_w^2(I^n) ; \| f \|_{S_{2,\infty,w}^r(I^n)} = \sup_{j \in \mathbb{N}_0^n} 2^{j \cdot r} \left( \sum_{k \in P_j^n} a_k^2[f] \right)^{1/2} < \infty \right\},
\]

respectively.

By construction, it holds that

\[
B_{2,2,w}^0(I^n) = L_w^2(I^n) \quad \text{and} \quad S_{2,2,w}^0(I^n) = L_w^2(I^n).
\]

(3.1)

Furthermore, we have the imbeddings

\[
S_{2,q,w}^{s_i}(I^n) \hookrightarrow B_{2,q,w}^{s_i}(I^n) \hookrightarrow S_{2,q,w}^{s_o}(I^n)
\]

(3.2)

between isotropic and dominating mixed smoothness spaces with \( s_i = (s, s, \ldots, s) \) and \( s_o = (s/n, s/n, \ldots, s/n) \).

The Besov spaces of \( n \)-variate functions with dominating mixed smoothness properties can be characterized as tensor products

\[
B_{2,q,w}^{s_1}(I) \otimes_\lambda B_{2,q,w}^{s_2}(I) \otimes_\lambda \cdots \otimes_\lambda B_{2,q,w}^{s_n}(I) = S_{2,q,w}^{s}(I^n)
\]

(3.3)

of the corresponding univariate Besov spaces (equivalent norms). Here, the norm \( \lambda \) which was used for the completion of the algebraic tensor product is the injective tensor norm for \( 1 \leq q < \infty \) and a certain modification thereof for \( q = \infty \), which can be proved in the same manner as in the periodic case cf. [13]. These norms have the main advantage to be uniform crossnorms, cf. [8, 13]. In particular, this (together
with (3.1)) means that for \( n \) operators \( P_r \in \mathcal{L}(B_{2,q,w}^s(I^n), L_w^2(I^n)) \), their tensor product operator given by
\[
P(f_1 \otimes f_2 \otimes \cdots \otimes f_n) := (P_1 \otimes P_2 \otimes \cdots \otimes P_n)(f_1 \otimes f_2 \otimes \cdots \otimes f_n) := P_1(f_1) \otimes P_2(f_2) \otimes \cdots \otimes P_n(f_n)
\]
is bounded, i.e., \( P \in \mathcal{L}(S_{2,q,w}^s B(I^n), L_w^2(I^n)) \), and its norm can be estimated as
\[
\|P \|_{\mathcal{L}(S_{2,q,w}^s B(I^n), L_w^2(I^n))} \leq C \prod_{r=1}^n \|P_r \|_{\mathcal{L}(B_{2,q,w}^s(I), L_w^2(I))},
\]
with some constant \( C \) independent of \( P \). Note that for Hilbert spaces the tensor norms are uniform crossnorms by construction. In case of tensor products of Banach spaces, this has to be proved for each example separately, see [8, 13].

Because of the imbeddings
\[
B_{2,q,w}^s(I^n) \hookrightarrow B_{2,\infty,w}^s(I^n),
S_{2,q,w}^s B(I^n) \hookrightarrow S_{2,\infty,w}^s B(I^n),
\]
for \( q < \infty \), in the following sections, we give the error estimates for the most interesting case \( q = \infty \) only.

4. Univariate Interpolation by Generalized Translates

As a preparation for \( n \)-variate interpolation on sparse grids, we need to describe the interpolation method for univariate functions first. To this end, we concentrate our investigations on interpolation by generalized translates of a single function.

The Chebyshev shift (cf. [4, 11]) \( s_h f \) of a function \( f \) is defined by
\[
(s_h f)(x) := \frac{1}{2} f \left( xh - \frac{1}{w(x)w(h)} \right) + \frac{1}{2} f \left( xh + \frac{1}{w(x)w(h)} \right), \quad x \in I.
\]
For interpolation, we use the special shifts \( \sigma_k := s_{g_k}, k = 0, \ldots, N, \) into the Gauß–Chebyshev nodes. The Chebyshev shift effects the Chebyshev coefficients \( a_k[\sigma_n f] = T_k(g_n)a_k[f] \) in the same multiplicative way as the usual shift effects the Fourier coefficients of periodic functions [11]. We assume to know a modified Lagrange function \( \Lambda_N \in A(I) \) satisfying
\[
\sigma_k \Lambda_N(g_k) = \frac{1}{\varepsilon_k} \delta_{k,\ell}, \quad k = 0, \ldots, N,
\]
with
\[
\varepsilon_k := \begin{cases} 
\frac{1}{2} & \text{for } k = 0, N, \\
1 & \text{for } k = 1, \ldots, N - 1.
\end{cases}
\]
For the construction of such a Lagrange function, we refer to [16]. Then, we can write the corresponding interpolation operator interpolating in the Gauß–Chebyshev nodes as

\[ L_N f = \sum_{k=0}^{N} f(g_k) \sigma_k \Lambda_N. \]

In order to characterize the approximation properties of the modified Lagrange function, we impose conditions on the decay of the Chebyshev coefficients of \( \Lambda_N \), see [16]. They are the pendant for the interval of the strong cardinal Strang–Fix conditions [7] and the periodic Strang–Fix conditions [3].

**Definition 3** The Lagrange function \( \Lambda_N \in A(I) \) satisfies the Strang–Fix conditions (for Gauß–Chebyshev grids) of order \( m > 0 \) if for all \( k = 0, \ldots, N \) the inequalities

\[
\left| 1 - \frac{N}{2} a_k [\Lambda_N] \right| \leq b_0 k^m N^{-m}, \\
\left| \frac{N}{2} a_{2\ell N + k} [\Lambda_N] \right| \leq b_{2\ell} k^m N^{-m}, \quad \ell \in \mathbb{N}, \\
\left| \frac{N}{2} a_{2(\ell+1)N-k} [\Lambda_N] \right| \leq b_{2\ell+1} k^m N^{-m}, \quad \ell \in \mathbb{N}_0,
\]

hold for some sequence \( \{b_{\ell}\}_{\ell \in \mathbb{N}_0} \in l_2(\mathbb{N}_0) \).

With the help of these quite general conditions we obtain the following estimate for the univariate interpolation.

**Theorem 4** Let \( m > s > 1/2 \). Let the Lagrange function \( \Lambda_N \in A(I) \) satisfy the Strang–Fix conditions (4.1) of order \( m \). Then there exists a constant \( C \) (independent of \( N \)) such that

\[ \| f - L_N f \|_{L^2_w(I)} \leq C N^{-s} \| f \|_{B^s_{2,\infty,w}(I)}, \]

for all \( f \in B^s_{2,\infty,w}(I) \).

**Proof:** Throughout this proof, we denote by \( C \) a constant independent of \( N \). The value of \( C \) may differ even within the same equation.
The Chebyshev coefficients of the interpolant to the function $f \in A(I)$ can be computed by using aliasing as

$$a_{2\ell N+k}[L_N f] = \sum_{r \in \mathbb{N}_0} (a_{2r N+k}[f] + a_{2(r+1)N-k}[f]) \frac{N}{2} a_{2\ell N+k}[\Lambda_N],$$

$$a_{(2\ell+1)N+k}[L_N f] = \sum_{r \in \mathbb{N}_0} (a_{(2r+1)N+k}[f] + a_{(2r+1)N-k}[f]) \frac{N}{2} a_{(2\ell+1)N+k}[\Lambda_N],$$

for $k = 0, \ldots, N-1, \ell \in \mathbb{N}_0$.

First we prove the assertion for polynomials $f \in \Pi_{N-1}$. Using the Strang–Fix conditions, we obtain

$$\| f - L_N f \|_{L^2_w(I)}^2 \leq \sum_{k \in \mathbb{N}_0} (a_k[f] - a_k[L_N f])^2 \leq \sum_{k=0}^{N} \left( a_{2\ell N+k}[f] - \sum_{r \in \mathbb{N}_0} (a_{2r N+k}[f] + a_{2(r+1)N-k}[f]) \frac{N}{2} a_{2\ell N+k}[\Lambda_N] \right)^2 + \left( a_{(2\ell+1)N+k}[f] - \sum_{r \in \mathbb{N}_0} (a_{(2r+1)N+k}[f] + a_{(2r+1)N-k}[f]) \frac{N}{2} a_{(2\ell+1)N+k}[\Lambda_N] \right)^2 = \sum_{k=0}^{N-1} a_k^2[f] \left( \left( 1 - \frac{N}{2} a_k[\Lambda_N] \right)^2 + \sum_{\ell \in \mathbb{N}_0} \left( \frac{N}{2} a_{2\ell N+k}[\Lambda_N] \right)^2 + \left( \frac{N}{2} a_{2\ell N-k}[\Lambda_N] \right)^2 \right) \leq \sum_{k=0}^{N-1} a_k^2[f] k^{2m} N^{-2m} \sum_{\ell \in \mathbb{N}_0} b_{\ell}^2.$$

Let $2^{r-1} \leq N < 2^r$. Then

$$\sum_{k=0}^{N-1} a_k^2[f] k^{2m} \leq \sum_{\ell=0}^{r} \sum_{k \in Q_1^\ell} k^{2m} 2^{-2\ell s} 2^{2\ell s} a_k^2[f] \leq C \sum_{\ell=0}^{r} 2^{2(m-s)\ell} 2^{2\ell s} \sum_{k \in Q_1^\ell} a_k^2[f] \leq C N^{2(m-s)} \| f \|_{B_{2,\infty,w}^s(I)}^2.$$

This means that, for polynomials $f \in \Pi_{N-1}$, we proved

$$\| f - L_N f \|_{L^2_w(I)} \leq C N^{-s} \| f \|_{B_{2,\infty,w}^s(I)}. \quad (4.2)$$

Now, we consider the general case $f \in B_{2,\infty,w}^s$. Because of $s > 1/2$ it holds that $f \in A(I)$. Therefore, interpolation is well-defined and aliasing is allowed. Let $S_{N-1}$
denote the \((N - 1)\)st Fourier–Chebyshev partial sum. The Strang–Fix conditions applied with \(1/2 < s' < s\) and the Cauchy–Schwarz inequality yield
\[
\|L_N(f - S_{N-1}f) \mid L_w^2(I)\|^2 \leq \sum_{k \in \mathbb{N}_0} a_k^2[L_N(f - S_{N-1}f)]
\]
\[
\leq CN^{-2s} \|\{b_t\} \mid \ell_2(\mathbb{N}_0)\|
\times \sum_{k=0}^{N} k^{2s'} \left[ \left( \sum_{r \in \mathbb{N}_0} (2r N + k)^{-s'} (2r N + k)^s a_{2r N+k}[f - S_{N-1}f] \right)^2 \right.
+ \left[ \left( \sum_{r \in \mathbb{N}_0} ((2r + 1) N - k)^{-s'} ((2r + 1) N + k)^s a_{(2r + 1) N+k}[f - S_{N-1}f] \right)^2 \right.
+ \left( \sum_{r \in \mathbb{N}_0} (2r + 1) N - k)^{-s'} ((2r + 1) N - k)^s a_{(2r + 1) N-k}[f - S_{N-1}f] \right)^2
\]
\leq CN^{-2s} \|f - S_{N-1}f \mid H_w^{s'}(I)\|^2 \sup_{k=0,\ldots,N} k^{2s'} \sum_{r \in \mathbb{N}_0} (2r N + k)^{-2s'} + (2r + 1) N - k)^{-2s'} + ((2r + 1) N + k)^{-2s'} + ((2r + 1) N - k)^{-2s'},
\]
where \(H_w^{s'}(I)\) denotes the Sobolev–type space with the norm
\[
\|f \mid H_w^{s'}(I)\|^2 := \sum_{k \in \mathbb{N}_0} (1 + k^2)^{s'} a_k^2[f],
\]
as in [2]. Since \(s' > 1/2\), we have
\[
\sup_{k=0,\ldots,N} \left( \frac{k}{N} \right)^{2s'} \sum_{r \in \mathbb{N}_0} \left( \frac{2r}{N} + \frac{k}{N} \right)^{-2s'} + \left( \frac{2r}{N} + 1 - \frac{k}{N} \right)^{-2s'}
+ \left( 2r + 1 + \frac{k}{N} \right)^{-2s'} + \left( 2r + 1 - \frac{k}{N} \right)^{-2s'} \leq C < \infty
\]
and hence
\[
\|L_N(f - S_{N-1}f) \mid L_w^2(I)\| \leq CN^{-s'} \|f - S_{N-1}f \mid H_w^{s'}(I)\|.
\]
One proves easily that for \(s > s'\) it holds that
\[
\|f - S_{N-1}f \mid H_w^{s'}(I)\| \leq CN^{s'-s} \|f \mid B_{2,\infty,w}^s(\mathcal{I})\|.
\]
(4.3)
With this, it follows
\[
\|L_N(f - S_{N-1}f) \mid L_w^2(I)\| \leq CN^{-s} \|f \mid B_{2,\infty,w}^s(\mathcal{I})\|.
\]
(4.4)
Now, our results (4.2), (4.4) and (4.3) applied with \( s' = 0 \) imply

\[
\| f - L_N f \mid L^2_w(I) \| \\
\leq \| f - S_{N-1} f \mid L^2_w(I) \| + \| S_{N-1} f - L_N (S_{N-1} f) \mid L^2_w(I) \| \\
+ \| L_N (f - S_{N-1} f) \mid L^2_w(I) \| \\
\leq C N^{-s} \| f \mid B^s_{2,\infty,w}(I) \|.
\]

This proves the theorem.

5. **Examples**

The interpolation on Gauß–Chebyshev grids is closely related to periodic interpolation on equidistant grids. The function \( \mathcal{M} := \Lambda_N(\cos \cdot)/2 \) is an even periodic fundamental interpolant on the grid \( \{ \frac{2\pi k}{2N} : k = -N, \ldots, N-1 \} \). If \( \mathcal{M} \) satisfies the periodic Strang–Fix conditions of order \( m \) (see [3, 12]) with the constants \( \{ d_\ell \}_{\ell \in \mathbb{Z}} \) then \( \Lambda_N \) satisfies the Strang–Fix conditions for Gauß–Chebyshev grids of order \( m \) with the constants \( b_{2\ell} = \pi^m d_\ell \) and \( b_{2\ell+1} = \pi^m d_{\ell+1}, \ell \in \mathbb{N}_0 \).

In this way, one obtains that the interpolatory scaling functions of the multiresolution analysis for a bounded interval described in [11] fulfil Strang–Fix conditions of certain order.

So the fundamental interpolant of the transformed B–spline of even order \( r \) satisfies Strang–Fix conditions of order \( r \). The de la Vallée Poussin means of Chebyshev polynomials also described in [11] are fundamental interpolants and satisfy Strang–Fix conditions of arbitrary order \( m \).

The constants \( \{ d_\ell \}_{\ell \in \mathbb{Z}} \) for the corresponding periodic functions can be found in [3, 12].

6. **\( n \)-variate Interpolation on Sparse Grids**

We now want to consider interpolation on sparse grids. Therefore, we choose \( N_0 \in \mathbb{N} \) and set \( N_j := N_0 2^j \) and \( L_0 = 0, L_j := L_{N_j-1}, j \in \mathbb{N} \). Furthermore, we assume the imbeddings \( \text{Im} L_j \subset \text{Im} L_{j+1}, j \in \mathbb{N} \). Therefore, the interpolation operators form a chain, i.e., \( L_j L_{j+1} = L_{j+1} L_j = L_j \). The corresponding Lagrange functions \( \Lambda_{N_j} \) have to satisfy the Strang–Fix conditions with the same sequence of constants \( \{ b_\ell \}_{\ell \in \mathbb{N}_0} \). For our examples in Section 5, both assumptions are fulfilled.

Then, the interpolation operator on a sparse grid is the \( j \)-th order Boolean sum (\( j \)-th order blending operator), cf. [5], Chap. 1,

\[
B^n_j := \bigoplus_{|r|=j} L_{r_1} \otimes L_{r_2} \otimes \cdots \otimes L_{r_n},
\]
with \( j \geq n, |r| = r_1 + r_2 + \cdots + r_n \) and \( A \oplus B := A + B - AB \). The Boolean sum can be rewritten in terms of ordinary sums (see [18])

\[
B^n_j = \sum_{j-n+1 \leq |r| \leq j} (-1)^{|r|} \binom{n-1}{j-|r|} L_{r_1} \otimes L_{r_2} \otimes \cdots \otimes L_{r_n}.
\]

It interpolates on the sparse grid

\[
\bigcup_{|r|=j} \mathcal{G}_{r_1} \times \mathcal{G}_{r_2} \times \cdots \times \mathcal{G}_{r_n}.
\]

These grids are also nested. The number \( N_G = N_G(j, n) \) of interpolation nodes in the sparse grid belonging to \( B^n_j \) can be estimated as (see [9]) as

\[
N_G \leq C_n j^{n-1} 2^j.
\]

The Boolean sum operators form a chain, i.e., \( B^n_j B^n_{j+1} = B^n_{j+1} B^n_j = B^n_j \).

**Theorem 5** Let \( m > s > 1/2 \), \( s = (s, s, \ldots, s) \). Let \( \Lambda_{N_j} \) satisfy the Strang–Fix conditions of order \( m \) with the same sequence of constants \( \{b_i\}_{i \in \mathbb{N}_0} \). The corresponding interpolation operators form a chain. Then there exists a constant \( C \) (independent of \( j \)) such that

\[
\|f - B^n_j f| L^2_w(I^n)\| \leq C j^{n-1} N^{-s}_j \|f| S^s_{2,\infty,w} B(I^n)\|
\]

for all \( f \in S^s_{2,\infty,w} B(I^n) \).

**Proof:** Let \( I_n \) denote the natural imbedding \( S^s_{2,\infty,w} B(I^n) \hookrightarrow L^2_w(I^n) \). From Theorem 4, we obtain the estimates

\[
\|I_1 \mid \mathcal{L}(B^s_{2,\infty, w}(I), L^2_w(I))\| = 1,
\]

\[
\|I_1 - L_r \mid \mathcal{L}(B^s_{2,\infty, w}(I), L^2_w(I))\| \leq C 2^{-sr},
\]

\[
\|L_r - L_{r-1} \mid \mathcal{L}(B^s_{2,\infty, w}(I), L^2_w(I))\| \leq D 2^{-sr}.
\]

With \( E := C \left( \max\{2^s, D\} \right) \) and the uniform crossnorm property (3.4) of the underlying Besov–type spaces of functions with dominating mixed smoothness properties, one proves easily (see [18]) that

\[
\|I_n - B^n_j \mid \mathcal{L}(S^s_{2,\infty, w} B(I^n), L^2_w(I^n))\| \leq E \left( \frac{j}{n-1} \right) 2^{-sj}.
\]

This proves the theorem.
Corollary 6. Under the assumptions of Theorem 5, the error of interpolation on the sparse grid can be estimated in terms of the number of grid points as

\[ \| f - B^n_j f \|_{L^2_w(I^n)} \leq C N_G^{-s} (\log N_G)^{(s+1)(n-1)} \| f \|_{S^n_{2,\infty,w}B(I^n)}. \]

Remark 1. The interpolation on sparse grids is almost optimal (up to logarithmic factors).

One can do better by substituting the interpolation operators in the definition of \( B^n_j \) by the Fourier–Chebyshev partial sum operators. This would yield the hyperbolic Fourier–Chebyshev sum \( S_{U^n_j} \) with all Chebyshev polynomials which have their degree contained in the hyperbolic cross

\[ U^n_j := \bigcup_{|r| \leq j} P^n_r, \]

with the index sets \( P^n_r \) as used in Definition 2. As usual (see [17]), one can estimate

\[ \| f - S_{U^n_j} f \|_{L^2_w(I^n)} \leq C 2^{-2js} \| f \|_{S^n_{2,\infty,w}B(I^n)}. \]

But, of course, here one has to use general linear information instead of function values only.

Remark 2. Under the assumptions of Theorem 5 one can easily prove the estimate

\[ \| f - (L_j \otimes L_j \otimes \cdots \otimes L_j) f \|_{L^2_w(I^n)} \leq C N^{-s}_j \| f \|_{S^n_{2,\infty,w}B(I^n)} \]

for all \( f \in S^n_{2,\infty,w}B(I^n) \) using the tensor product interpolation on \( N_T = N^n_j \) grid points. One gets the same order of approximation already for the functions \( f \in B^n_{2,\infty,w}(I^n), m > s > n/2 \) from the corresponding isotropic Besov–type space. In terms of grid points, we have an error of \( O(N_T^{-s/n}) \) in both cases.

This means that for functions with dominating mixed smoothness properties, the interpolation on sparse grids is essentially better suited than the tensor product construction.

For functions from isotropic spaces, we would lose some approximation order in sparse grid interpolation and obtain with (3.2) an error of order \( O(j^{n-1}N^{-s/n}_j) = O((\log N_G)^{(s/n+1)(n-1)}N^{-s/n}_G) \) only. So, for functions not providing dominating mixed smoothness, the interpolation on sparse grids is less suited. But still it is only by a logarithmic term worse than the full grid.

Remark 3. With the help of the Strang–Fix conditions, we obtain error estimates for a wide variety of possible sparse grid constructions, including for instance splines or polynomials, which are almost optimal for functions from all the Besov–type spaces \( S^n_{2,\infty,w}B(I^n) \) up to the order of the Strang–Fix conditions (which can be seen to be infinity for polynomials).
References


