Graph limits meet Markov chains

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Notions of graph convergence and limits

Dense graphs: local (left-) convergence

Limit objects: graphons

Borgs-Chayes-L-Sós-Vesztergombi; Razborov

\[ W: [0,1]^2 \rightarrow [0,1], \]

symmetric, measurable

L - Szegedy

Extending graph theory to graphons

Connectivity, matchings, automorphisms, extremal graphs,…
Notions of graph convergence and limits

Bounded degree: local convergence

Limit objects: involution-invariant distributions on rooted graphs

Benjamini - Schramm

All?
Notions of graph convergence and limits

Bounded degree: local-global convergence

Bollobás - Riordan

Limit objects: graphings (bounded degree Borel-measurable graphs with a measure-preserving property)

Hatami-L-Szegedy

Extending graph theory to graphings

Matchings, flows, expansion, edge-coloring,...
What about inbetween?

Limit of:  
- hypercubes?  
- incidence graphs of finite projective planes?  
- stars?  
- 1-subdivisions of complete graphs?

$L^p$-convergence $\rightarrow$ $L^p$-graphon  
Borgs, Chayes, Cohn, Zhao

Scaled convergence $\rightarrow$ graphonizing  
Frenkl

Shape convergence $\rightarrow$ s-graphon  
Kunszenti-Kovács, L, Szegedy

Action convergence $\rightarrow$ graphop  
Backhausz, Szegedy
(Borel) sigma-algebra + random node + random edge

Random walk / Markov chain
Basic setup

\( \mathcal{A} \): standard Borel sigma-algebra (e.g. Borel sets of \([0,1]\))

\( J \): its underlying set \((J=\bigcup \mathcal{A})\)

\( \mathcal{M}(\mathcal{A}) \): set of finite signed measures on \( \mathcal{A} \) (Banach space)

\( \mu^* \): flip coordinates in \( \mu \in \mathcal{M}(\mathcal{A} \times \mathcal{A}) \)

\( \text{Symmetric measure: } \mu = \mu^* \)

\( \mu^1(A) = \mu(J \times A), \mu^2(A) = \mu(A \times J) \): marginals of \( \mu \)
Markov chain: random variables \((w^0, w^1, w^2, \ldots)\) such that 
\(w^{i+1}\) depends only on \(w^i\)

Markov kernel: \((J, \mathcal{A}, (P_u))\), \(\mathcal{A}\) is a (Borel) sigma-algebra on \(J\), 
\(\forall u \in J: P_u\) probability measure on \(\mathcal{A}\), 
\(P_u(A)\) measurable function of \(u\).

Markov space: \((J, \mathcal{A}, \eta)\), \(\mathcal{A}\) is a (Borel) sigma-algebra on \(J\), 
\(\eta\) is a probability measure on \(\mathcal{A}^2\), \(\eta^1 = \eta^2\). 
(\(\eta\) is symmetric \(\iff\) time reversible chain)
Markov chains, schemes and spaces

Markov kernel + starting distribution $\Leftrightarrow$ Markov chain

Markov kernel + stationary distribution $\Leftrightarrow$ Markov space

Stationary distribution: $\pi(X) = \int_{J} P_{u}(X) \, d\pi(u) = \eta^{1} = \eta^{2}$

Ergodic circulation: $\eta(A \times B) = \int_{A} P_{u}(B) \, d\pi(u)$

Markov space + node distribution $\Leftrightarrow$ s-graphon, graphop
Graphons (bounded or unbounded) and graphings

Orthogonality spaces

Are Markov spaces rich enough to allow nontrivial generalization of graph theory?
Sampling and subgaph density
Flow theory
Random walks
Expanders and spectra
Cut distance, counting lemma
Regularity partitions
Subgraph densities in (dense) graphs

\[ \text{Hom}(F,G) = \{ \text{adjacency preserving maps } V(F) \rightarrow V(G) \} \]

\[ t(F,G) = \frac{\text{Hom}(F,G)}{|V(G)|^{|V(F)|}} = P(\text{random } V(F) \rightarrow V(G) \text{ preserves edges}) \]

\[ t^*(F,G) = \frac{t(F,G)}{t(K_2,G)^{|E(F)|}} \]

**Sidorenko–Simonovits Conjecture:**

\[ F \text{ bipartite} \Rightarrow t^*(F,G) \geq 1 \ (\forall G) \]
Subgraph densities in graphons

Normalize: \( t(K_2, W) = \int_{[0,1]^2} W(x, y) \, dx \, dy = 1 \quad \forall x \)

\( t^*(F, W) = t(F, W) = \int_{[0,1]^{V(F)}} \prod_{ij \in E(F)} W(x_i, x_j) \, dx \)

\( W^F(x) = \prod_{ij \in E(F)} W(x_i, x_j) : \text{density function of a measure} \)

\text{on homomorphisms } F \to W. \)
Important example: orthogonality graph

orthogonality graph $H_d$

$\eta$: uniform distribution on orthogonal pairs

homomorphism of $G$ into $H_d$

$\iff$

orthonormal representation of the complement of $G$ in $\mathbb{R}^d$
Important example: orthogonality graph

homomorphism of $G$ into $H_d$

$\iff$

orthonormal representation

of the complement of $G$ in $\mathbb{R}^d$

density? „random copy”?  

$C_3$: prob. measure: trivial  
density: nontrivial  

$C_4$: trouble!  

$C_5$: nontrivial
Important example: orthogonality graph

\[ t^*(T,G) = 1 \quad (T \text{ tree}) \]

\[ t^*(K_3,H_3) = \frac{2}{\pi}, \quad t^*(K_3,H_4) = \frac{\pi}{4}, \quad t^*(K_3,H_5) = \frac{8}{3\pi}, \ldots \]

\[ t^*(C_4,H_3) = \infty, \quad t^*(C_4,H_4) = \frac{2}{3\pi^2}, \ldots \]
Important example: orthogonality graph

$G$ has an orthonormal rep in $\mathbb{R}^d$ in general position

$\iff G$ is $(n-d)$-connected

$\iff \bar{G}$ contains no complete bipartite subgraph on $d+1$ nodes

any $d$ vectors are linearly independent

L-Saks-Schrijver
Important example: orthogonality graph

$G$ has a homomorphism into $H_d$ in general position

$\Leftrightarrow G$ contains no complete bipartite subgraph on $d+1$ nodes

- map $G \rightarrow \mathbb{R}^d$ sequentially, each node uniform on the sphere orthogonal to previous neighbors;

- show that distribution of this map is independent absolutely continously on the order;

- figure out Radon-Nikodym derivatives.
Subgraph measures in Markov spaces

Markov space: \( M=(J, \mathcal{A}, \eta) \), \( \mathcal{A} \) is a (Borel) sigma-algebra on \( J \), \( \eta \) is a symmetric probability measure on \( \mathcal{A}^2 \).

\( G=(V,E) \): simple graph

No general notion of homomorphisms \( G \rightarrow M \)

- edge set \( \llll \) edge measure \( \eta \)
- Hom set \( \llll \) homomorphism measure \( \eta^G \) on \( J^V \)
Axioms for subgraph measures

(i) Normalization: \( \eta^K_1 = \pi, \ \eta^K_2 = \eta \)

(ii) Decreasing: marginal of \( \eta^G \) on \( S \subseteq V \) is abs. continuous w.r.t. \( \eta^{G[S]} \)

(iii) Markovian: \( U, W \subseteq V \), no edge between \( U \setminus W \) and \( W \setminus U \) \( \Rightarrow \)

for almost all \( z \in J^{U \setminus W}, \left( \eta^{G[U \cup W]} \mid z \right) = \left( \eta^{G[U]} \mid z \right) \times \left( \eta^{G[W]} \mid z \right) \)
$y$: random point from $\pi$

$x_1, \ldots, x_k$: independent Markov chain steps from $y$

$\sigma_k$: joint distribution of $(x_1, \ldots, x_k)$

$k$-loose: $\sigma_k$ absolutely continuous w.r.t. $\pi^k$

$k$-loose for all $k$

1-loose but not 2-loose

2-loose but not 3-loose
\((\mathcal{U}, \mathcal{B}, \eta)\): \(k\)-loose Markov space. Then \(\eta^G\) is well defined for graphs of girth \(\geq 5\) and degrees \(\leq k\), and normalized, decreasing and Markovian.

Two approaches: **generalizing sequential mapping**

approximation by graphons
Circulations

flow condition

\[ \sum_{j \in N_+(i)} f(ij) = \sum_{j \in N_-(i)} f(ji) \]

\[ \sum_{i} \sum_{j} f(ij) = \sum_{i} \sum_{j} f(ji) \]

circulation: \( \alpha \in M(\mathcal{A} \times \mathcal{A}) \), \( \alpha^1 = \alpha^2 \)

s-t flow: \( \alpha \in M(\mathcal{A} \times \mathcal{A}) \), \( \alpha^1 - \alpha^2 = \nu \cdot (\delta_s - \delta_t) \)

Markov space: circulation \( \eta \geq 0 \)

with \( \eta(\mathcal{J} \times \mathcal{J}) = 1 \)

Measure on sets of edges
For two measures $\varphi, \psi \in M(A \times A)$ there exists a circulation $\alpha$ such that $\varphi \leq \alpha \leq \psi$

iff $\varphi \leq \psi$ and $\varphi(X \times X^c) \leq \psi(X^c \times X)$ for every $X \in A$. 

Diagram:

- $X$ and $X^c$
- $X \times X^c$
- $X^c \times X$
Flow theory

Natural generalizations of:

- Max-Flow-Min-Cut;
- decomposition of flows into paths;
- minimum cost flow/circulation theorem;
- integrality of potentials;
- multicommodity flows.
Multicommodity flows (finite case)

Want: \( \{f_{st} : s,t \in V\} \)

\( f_{st} \): s-t flow of value \( \sigma_{st} \) 1.

feasible multicommodity flow

\[ \sum_{s,t} \sigma_{st} f_{st}(ij) \leq \psi_{ij} \quad \forall ij \in E \]

undirected case: \( \sigma_{st} = \sigma_{ts} \)

\[ \psi_{ij} = \psi_{ji} \]
Let $G = (V, E)$ and $\sigma, \psi : E \to \mathbb{R}_+$. There exists a feasible multicommodity flow $(f_{st} : s, t \in V)$ iff for every metric $d$ on $V$

$$\sum_{s,t} \sigma_{st}d(s,t) \leq \sum_{s,t} \psi_{st}d(s,t).$$

Iri, Shahroki-Matula
Multicommodity flow:

- symmetric measure ("demand") \( \sigma \in \mathcal{M}(A \times A) \);
- symmetric measure ("capacity") \( \psi \in \mathcal{M}(A \times A) \);
- family \( \{f_{st} : s,t \in J\} \) of s-t flows of value 1

Want: feasible multicommodity flow \( F = ( f_{st} : s,t \in J) \) s.t. \( \forall S \in A^2 \)

\[ \int_{J \times J} f_{xy}(S) \, d\sigma(x,y) \leq \psi(S). \]
"Conjecture". Let $\sigma, \psi \in M(A \times A)$, symmetric, $\sigma, \psi \geq 0$. There exists a feasible multicommodity flow

$$
\iff

\int_{J \times J} g \, d\sigma \leq \int_{J \times J} g \, d\psi
$$

for every bounded measurable metric $g$ on $J$. 

Multicommodity flows (measure case)
D bounded linear functional on $M(A \times A)$ is metrical:
(a) $D(\mu) = 0 \ \forall \mu$ concentrated on the diagonal $\Delta = \{(x, x)\}$;
(b) $D(\mu) = D(\mu^*) \ \forall \mu$;
(c) $D(\kappa^{12}) + D(\kappa^{23}) \geq D(\kappa^{13}) \ \forall \kappa \in M(A^3), \ \kappa \geq 0$.

\[ \kappa^{12}(A \times B) = \kappa(A \times B \times J), \]
\[ \kappa^{23}(A \times B) = \kappa(J \times A \times B), \]
\[ \kappa^{13}(A \times B) = \kappa(A \times J \times B). \]
For every metrical \( D : \mathcal{A} \times \mathcal{A} \rightarrow \mathbb{R} \), and every \( \phi, \theta \), 0, there is a bounded semimetric \( \mu : \mathbb{R} \rightarrow \mathbb{R} \) such that
\[
D(\phi) \leq \mu(\phi) \leq D(\theta).
\]
Conjecture:
For every metrical \( D : \mathcal{M}(\mathcal{A} \times \mathcal{A}) \rightarrow \mathbb{R} \), and every \( \psi \geq 0 \), there is a bounded semimetric \( \mu : \mathcal{M}(\mathcal{A} \times \mathcal{A}) \rightarrow \mathbb{R} \) such that
\[
D(\phi) \leq \mu(\phi) \leq D(\theta) \quad \forall \phi, \theta \in \mathcal{M}(\mathcal{A} \times \mathcal{A})
\]
Example: For a bounded semimetric \( g : J \times J \rightarrow \mathbb{R} \), let
\[
D(\phi) = \int_{J \times J} g \, d\phi
\]
Let \( \sigma, \psi \in M(A \times A) \), symmetric.

\[ \forall \varepsilon > 0 \text{ there is a multicommodity flow } F \]

for demands \( \sigma \) with \( \| (\varphi_F - \psi)_+ \|_{tv} < \varepsilon \)

\[ \iff \]

\[ D(\sigma) \leq D(\psi) \text{ for every metrical } D. \]
Thank you, this is all for today!
Important example: orthogonality graph

orthonormal representation:

\[ i \in V \implies v_i \in \mathbb{R}^d \]

- \( u_i^T u_j = 0 \) \( (\forall ij \not\in E) \)
- \( |u_i| = 1 \)