Graph limits meet Markov chains

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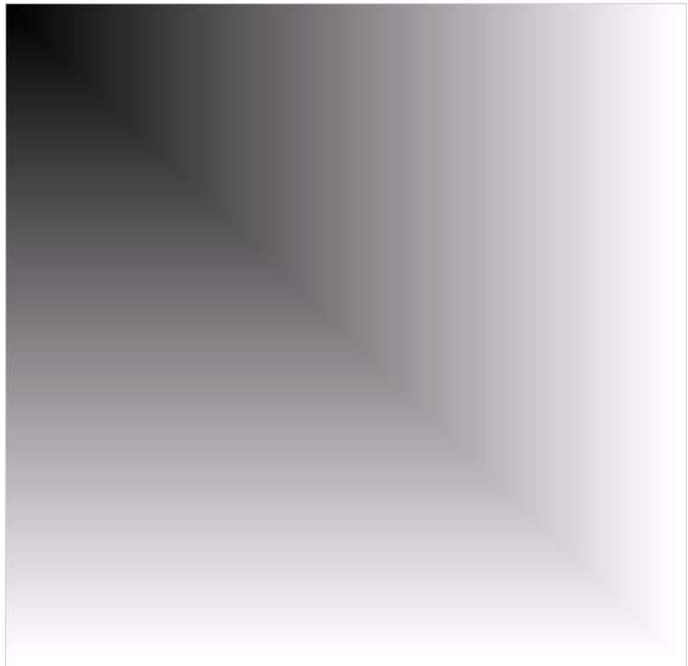
Notions of graph convergence and limits

Dense graphs: local (left-) convergence

Borgs-Chayes-L-Sós-Vesztergombi; Razborov

Limit objects: graphons



$W: [0,1]^2 \to [0,1]$,
symmetric, measurable

L - Szegedy

Extending graph theory to graphons

Connectivity, matchings, automorphisms, extremal graphs,...

Notions of graph convergence and limits

Bounded degree: local convergence

Limit objects: involution-invariant distributions on rooted graphs

Benjamini - Schramm

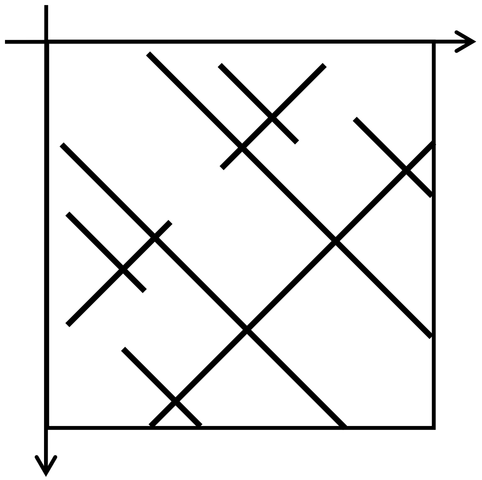
All?

Notions of graph convergence and limits

Bounded degree: local-global convergence

Bollobás - Riordan

Limit objects: graphings (bounded degree Borel-measurable graphs with a measure-preserving property)



Hatami-L-Szegedy

Extending graph theory to graphings

Matchings, flows, expansion, edge-coloring,...

What about inbetween?

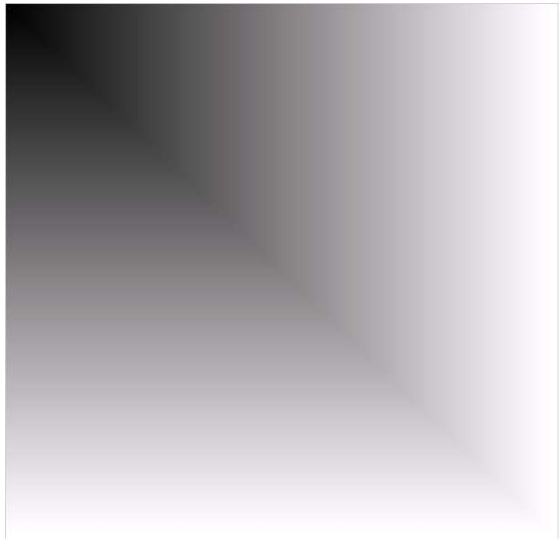
Limit of: *hypercubes?*
incidence graphs of finite projective planes?
stars?
1-subdivisions of complete graphs?

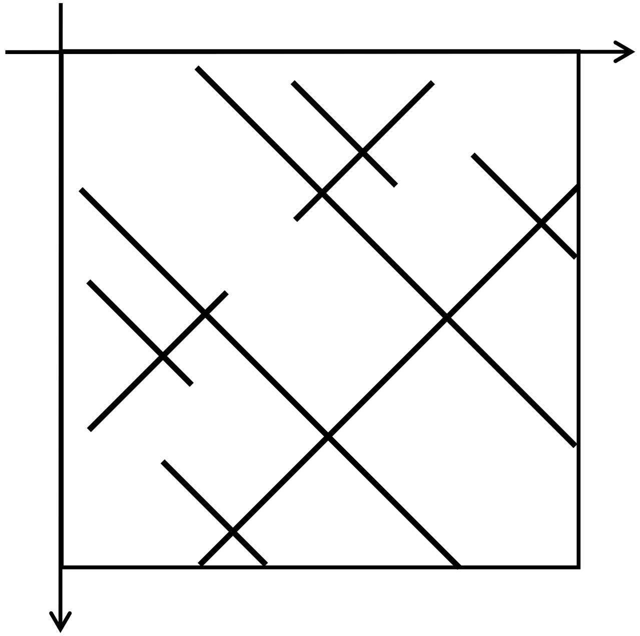
L^p-convergence	$\to$	L^p-graphon	Borgs, Chayes, Cohn, Zhao
Scaled convergence	$\to$	graphoning	Frenkl
Shape convergence	$\to$	s-graphon	Kunszenti-Kovács, L, Szegedy
Action convergence	$\to$	graphop	Backhausz, Szegedy

Common in limit structures

(Borel) sigma-algebra + random node + random edge

Random walk / Markov chain





Basic setup

$\mathcal{A}$: standard Borel sigma-algebra (e.g. Borel sets of [0,1])

J: its underlying set ($J=\cup \mathcal{A}$)

$M(\mathcal{A})$: set of finite signed measures on $\mathcal{A}$ (Banach space)

μ^*: flip coordinates in $\mu \in M(\mathcal{A} \times \mathcal{A})$

Symmetric measure: $\mu = \mu^*$

$\mu^1(A) = \mu(J \times A)$, $\mu^2(A) = \mu(A \times J)$: marginals of μ

Markov chains, schemes and spaces

Markov chain: random variables $(w^0, w^1, w^2,\dots)$ such that w^{i+1} depends only on w^i

Markov kernel: $(J, \mathcal{A}, (P_u))$, $\mathcal{A}$ is a (Borel) sigma-algebra on J,
$\forall u \in J$: P_u probability measure on $\mathcal{A}$,
$P_u(A)$ measurable function of u.

Markov space: $(J, \mathcal{A}, \eta)$, $\mathcal{A}$ is a (Borel) sigma-algebra on J,
η is a probability measure on $\mathcal{A}^2$, $\eta^1 = \eta^2$.
(η is symmetric $\Leftrightarrow$ time reversible chain)

Markov chains, schemes and spaces

Markov kernel + starting distribution $\Leftrightarrow$ Markov chain

Markov kernel + stationary distribution $\Leftrightarrow$ Markov space

Stationary distribution: $\pi(X) = \int_J P_u(X)\, d\pi(u) = \eta^1 = \eta^2$

Ergodic circulation: $\eta(A \times B) = \int_A P_u(B)\, d\pi(u)$

Markov space + node distribution $\Leftrightarrow$ s-graphon, graphop

Markov spaces: examples

Graphons (bounded or unbounded) and graphings

Orthogonality spaces

Are Markov spaces rich enough to allow nontrivial generalization of graph theory?

Graphs and Markov spaces

Sampling and subgaph density

Flow theory

Random walks

Expanders and spectra

Cut distance, counting lemma

Regularity partitions

Subgraph densities in (dense) graphs

Hom(F,G)={adjacency preserving maps $V(F)\to V(G)$}

$$t(F,G)=\frac{|\mathrm{Hom}(F,G)|}{|V(G)|^{|V(F)|}}=P(\text{random } V(F)\to V(G) \text{ preserves edges})$$

$$t^*(F,G)=\frac{t(F,G)}{t(K_2,G)^{|E(F)|}}$$

Sidorenko–Simonovits Conjecture:

$$F \text{ bipartite} \Rightarrow t^*(F,G)\geq 1 \quad (\forall G)$$

Subgraph densities in graphons

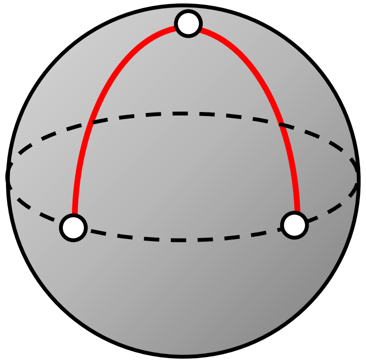
Normalize: $t(K_2,W) = \int_{[0,1]^2} W(x,y)\,dx\,dy = 1 \quad \forall x$

$$t^*(F,W) = t(F,W) = \int_{[0,1]^{V(F)}} \prod_{ij\in E(F)} W(x_i,x_j)\,dx$$

$W^F(x) = \prod_{ij\in E(F)} W(x_i,x_j)$: density function of a measure

on homomorphisms $F \to W$.

Important example: orthogonality graph



orthogonality graph H_d

η: uniform distribution

on orthogonal pairs

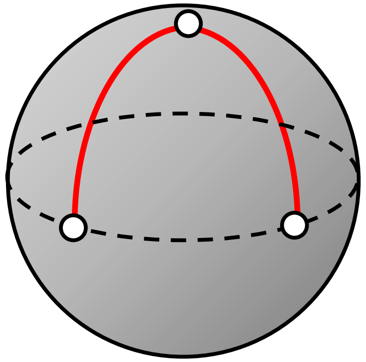
homomorphism of G into H_d

$\leftrightarrow$

orthonormal representation

of the complement of G in R^d

Important example: orthogonality graph



homomorphism of G into H_d

$\leftrightarrow$

orthonormal representation

of the complement of G in R^d

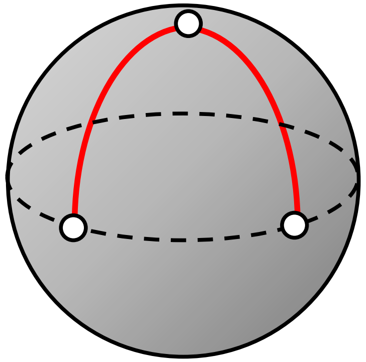
density? „random copy”?

C_3: prob. measure: trivial
density: nontrivial

C_4: trouble!

C_5: nontrivial

Important example: orthogonality graph



$$t^*(T,G) = 1 \quad (T \text{ tree})$$

$$t^*(K_3,H_3) = \frac{2}{\pi}, \qquad t^*(K_3,H_4) = \frac{\pi}{4}, \qquad t^*(K_3,H_5) = \frac{8}{3\pi}, \ldots$$

$$t^*(C_4,H_3) = \infty, \qquad t^*(C_4,H_4) = \frac{2}{3\pi^2}, \quad \ldots$$

Important example: orthogonality graph

G has an orthonormal rep in $\mathbb{R}^d$ in general position

$\Leftrightarrow G$ is $(n\text{-}d)$-connected

$\Leftrightarrow \overline{G}$ contains no complete bipartite subgraph on $d+1$ nodes

any d vectors are linearly independent

L-Saks-Schrijver

Important example: orthogonality graph

G has a homomorphism into H_d in general position

$\Leftrightarrow G$ contains no complete bipartite subgraph on d+1 nodes

- map $G \rightarrow \mathbb{R}^d$ sequentially, each node uniform on the sphere orthogonal to previous neighbors;
- show that distribution of this map ~~is independent~~ depends absolutely continously on the order;
- figure out Radon-Nikodym derivatives.

Subgraph measures in Markov spaces

Markov space: M=(J, $\mathcal{A}$, η), $\mathcal{A}$ is a (Borel) sigma-algebra on J,
η is a symmetric probability measure on $\mathcal{A}^2$.

$G=(V,E)$: simple graph

No general notion of homomorphisms $G \to M$

edge set <<< edge measure η

Hom set <<< homomorphism measure η^G on J^V

Axioms for subgraph measures

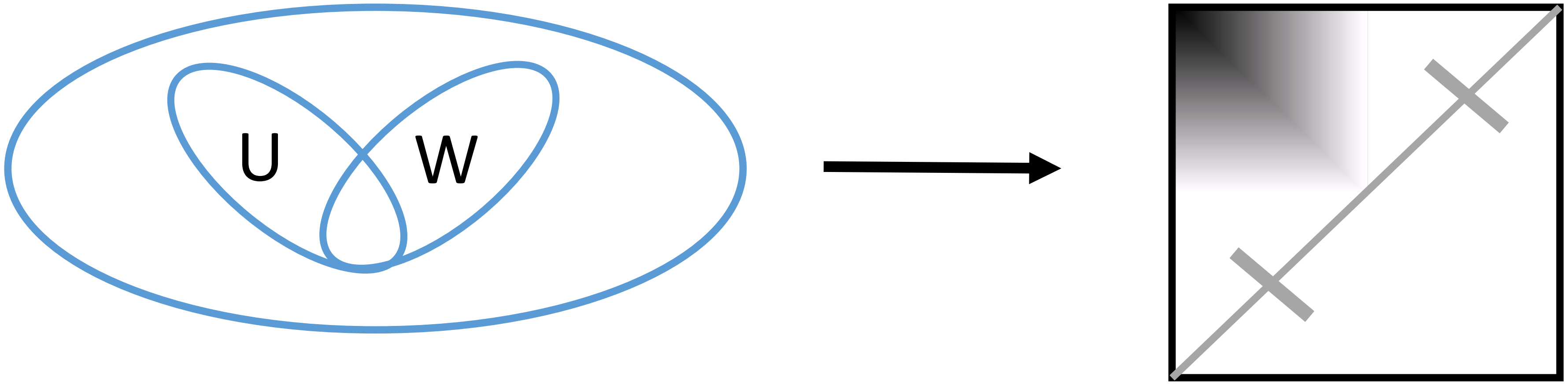
(i) Normalization: $\eta^{K_1} = \pi, \; \eta^{K_2} = \eta$

(ii) Decreasing: marginal of η^G on $S \subseteq V$ is abs. continuous w.r.t. $\eta^{G[S]}$

(iii) Markovian: $U, W \subseteq V$, no edge between $U \setminus W$ and $W \setminus U$ $\Rightarrow$

for almost all $z \in J^{U \cap W}$, $\left(\eta^{G[U \cup W]} \mid z\right) = \left(\eta^{G[U]} \mid z\right) \times \left(\eta^{G[W]} \mid z\right)$

G:



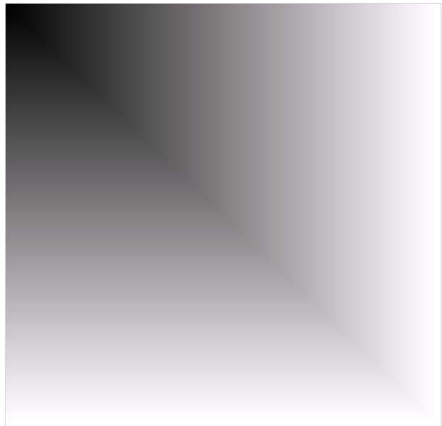
k-loose Markov spaces

y: random point from π

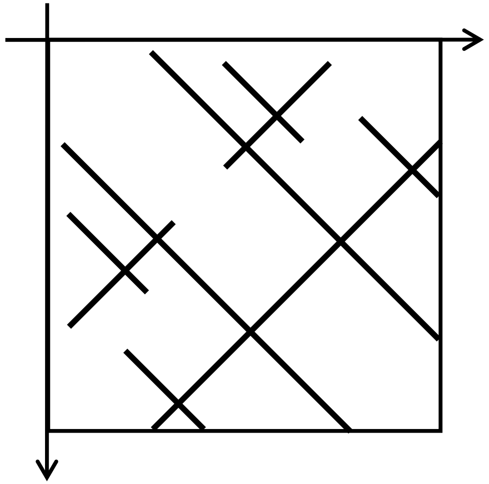
$x_1,\ldots,x_k$: independent Markov chain steps from y

σ_k: joint distribution of $(x_1,\ldots,x_k)$

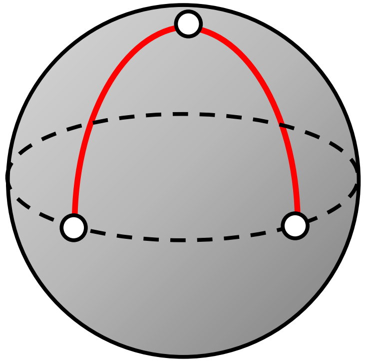
k-loose: σ_k absolutely continuous w.r.t. π^k



k-loose for all k



1-loose but not 2-loose



2-loose but not 3-loose

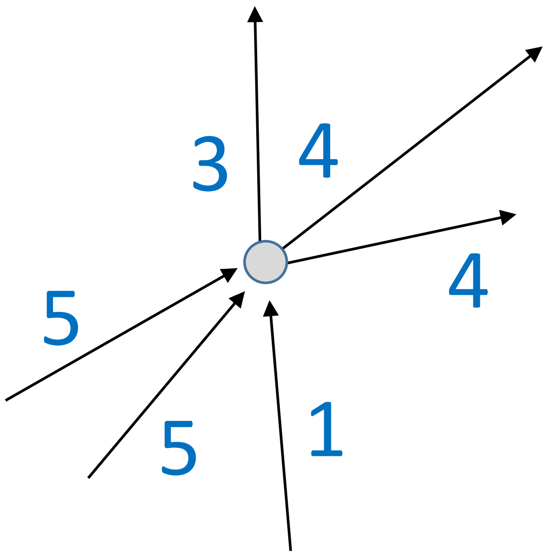
Subgraph measures in Markov spaces

$(J,\mathcal{B},\eta)$: k-loose Markov space. Then η^G is well defined for graphs of girth ≥ 5 and degrees $\leq k$, and normalized, decreasing and Markovian.

Two approaches: generalizing sequential mapping

approximation by graphons

Circulations



$$\sum_{j \in N_+(i)} f(ij) = \sum_{j \in N_-(i)} f(ji)$$

flow condition

circulation: $\alpha \in M(\mathcal{A} \times \mathcal{A})$, $\alpha^1 = \alpha^2$

s-*t* flow: $\alpha \in M(\mathcal{A} \times \mathcal{A})$, $\alpha^1 - \alpha^2 = v \cdot (\delta_s - \delta_t)$

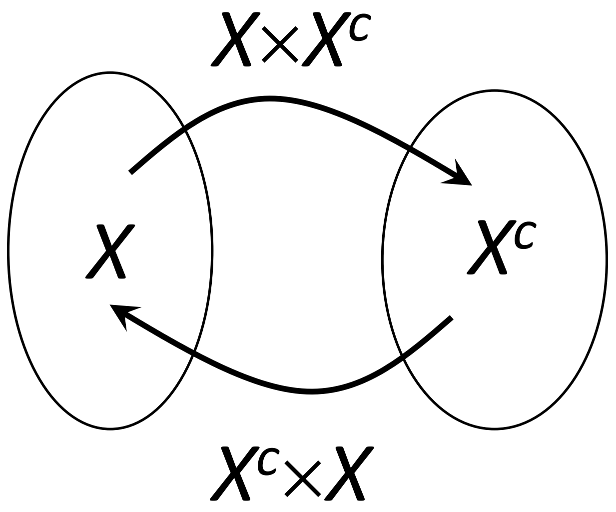
Markov space: circulation $\eta \geq 0$

with $\eta(J \times J) = 1$

Measure on sets of edges

Hoffman Circulation Theorem

For two measures $\varphi, \psi \in M(\mathrm{A} \times \mathrm{A})$
there exists a circulation α such that $\varphi \le \alpha \le \psi$
iff $\varphi \le \psi$ and $\varphi(X \times X^c) \le \psi(X^c \times X)$ for every $X \in \mathrm{A}$.

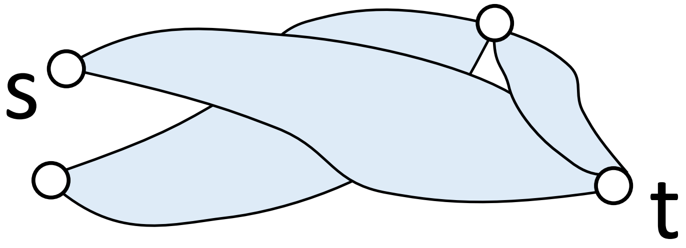


Flow theory

Natural generalizations of:

- Max-Flow-Min-Cut;
- decomposition of flows into paths;
- minimum cost flow/circulation theorem;
- integrality of potentials;
- multicommodity flows.

Multicommodity flows (finite case)



σ_{st}: demand $\forall s,t \in V$

ψ_{ij}: capacity $\forall ij \in E$

Want: $\{f_{st}: s,t \in V\}$

f_{st}: s-t flow of value ~~σ_{st}~~ 1.

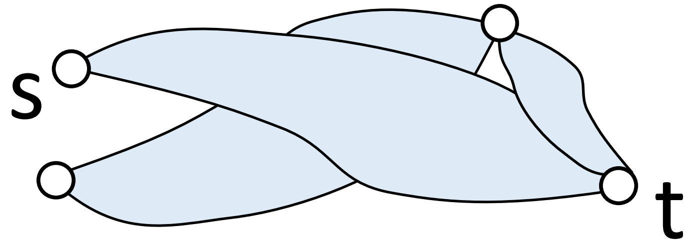
$$\sum_{s,t} \sigma_{st} f_{st}(ij) \leq \psi_{ij} \quad \forall ij \in E$$

feasible multicommodity flow

undirected case: $\sigma_{st} = \sigma_{ts}$

$\psi_{ij} = \psi_{ji}$

Multicommodity flows (finite case)



Let $G=(V,E)$ and $\sigma,\psi : E \to \mathbb{R}_+$. There exists a feasible multicommodity flow $(f_{st} : s,t \in V)$ iff for every metric d on V

$$\sum_{s,t} \sigma_{st} d(s,t) \leq \sum_{s,t} \psi_{st} d(s,t).$$

Iri, Shahroki-Matula

Multicommodity flows (measure case)

Multicommodity flow:

- symmetric measure („demand”) $\sigma \in M(\mathcal{A} \times \mathcal{A})$;
- symmetric measure („capacity”) $\psi \in M(\mathcal{A} \times \mathcal{A})$;
- family $\{f_{st}: s,t \in J\}$ of s-t flows of value 1

Want: feasible multicommodity flow $F = (f_{st} : s,t \in J)$ s.t. $\forall S \in \mathrm{A}^2$

$$\int_{J \times J} f_{xy}(S)\, d\sigma(x,y) \leq \psi(S).$$

Multicommodity flows (measure case)

"Conjecture". Let $\sigma, \psi \in M(\mathrm{A} \times \mathrm{A})$, symmetric, $\sigma, \psi \geq 0$.

There exists a feasible multicommodity flow

$$\Leftrightarrow$$

$$\int_{J \times J} g\, d\sigma \leq \int_{J \times J} g\, d\psi$$

for every bounded measurable metric g on J.

Metrical linear functionals

D bounded linear functional on $M(\mathrm{A}\times\mathrm{A})$ is metrical:

(a) $\mathrm{D}(\mu)=0\ \forall\mu$ concentrated on the diagonal $\Delta=\{(x,x)\}$;

(b) $\mathrm{D}(\mu)=\mathrm{D}(\mu^{*})\ \forall\mu$;

(c) $\mathrm{D}(\kappa^{12})+\mathrm{D}(\kappa^{23})\geq\mathrm{D}(\kappa^{13})\ \forall\kappa\in M(\mathrm{A}^{3}),\ \kappa\geq 0.$

$$\kappa^{12}(A\times B)=\kappa(A\times B\times J),$$
$$\kappa^{23}(A\times B)=\kappa(J\times A\times B),$$
$$\kappa^{13}(A\times B)=\kappa(A\times J\times B).$$

Metrical linear functionals

Example: $\mathrm{D}(\varphi) = \varphi(A \times A^c) + \varphi(A^c \times A)$

Example: For a bounded semimetric $g: J \times J \to \mathbb{R}_+$,

let $\mathrm{D}(\varphi) = \int_{J\times J} g \, d\varphi$

Conjecture:

For every metrical D: $M(\mathrm{A} \times \mathrm{A}$ [illegible] $\in M(\mathrm{A} \times \mathrm{A})$, $\psi \geq 0$,

there i[illegible] $\to \mathbb{R}_+$ such that

$\mathrm{D}(\varphi) = \int_{J\times J} g \, d\varphi \;\; \forall 0 \leq \varphi \leq \psi$.

True for graphons, graphings, …

Multicommodity flows (measure case)

Let $\sigma, \psi \in M(A \times A)$, symmetric.

$\forall \varepsilon > 0$ there is a multicommodity flow F

for demands σ with $\|(\varphi_F - \psi)_+\|_{tv} < \varepsilon$

$$\Leftrightarrow$$

$D(\sigma) \leq D(\psi)$ for every metrical D.

overload

Thank you, this is all for today!

Important example: orthogonality graph

orthonormal representation:

$$i \in V \quad \mapsto \quad v_i \in \mathbb{R}^d$$

- $u_i^T u_j = 0 \quad (\forall ij \notin E)$
- $|u_i| = 1$