# Excluding affine configurations over a finite field

Abel Prize Laureates Lectures

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Consider a homogeneous balanced system of linear equations:

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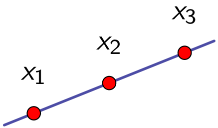
**Problem**

How large must $S \subseteq \mathbb{F}_q^n$ be to ensure a **non-trivial** solution $x = (x_1, \ldots, x_k)$ with $x_1, \ldots, x_k \in S$?

# Cap sets

$$x_1 - 2x_2 + x_3 = 0$$

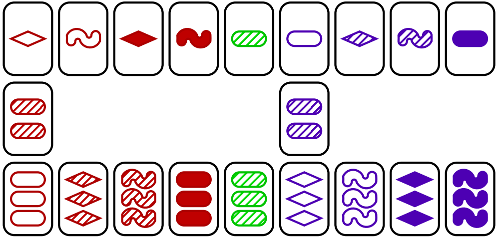



A cap set: subset $S \subseteq \mathbb{F}_3^n$ containing no non-trivial solution to $x_1 - 2x_2 + x_3 = 0$.

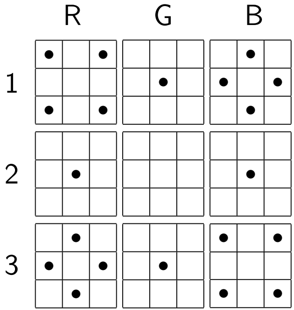
Equivalently: no (non-trivial) 3-term arithmetic progression (3AP).

## Cap set problem

What is the asymptotic growth of maximum size of a cap set in $\mathbb{F}_3^n$?



Pellegrino cap (1971)




## Online Encyclopedia of Integer Sequences: A090245

| $n$ | 1 | 2 | 3 | **4** | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| max cap size | 2 | 4 | 9 | **20** | 45 | 112 | 236 – 291 |
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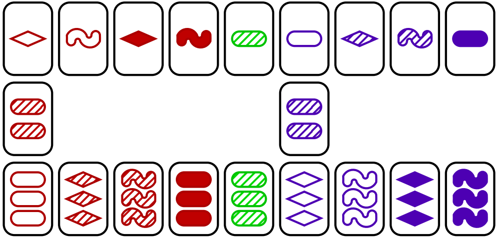

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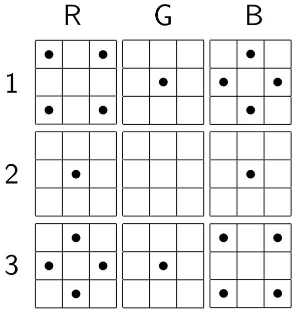


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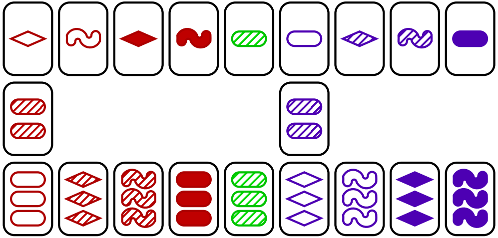

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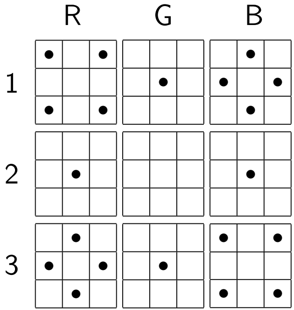


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- $f(n) = \Omega(2.217^n)$ [Edel, 2004]

# Motivation

## Arithmetic progressions

The cap set problem is a **toy model** for understanding arithmetic progressions in the integers

**Terence Tao**: *"Perhaps my favourite open question is the problem on the maximal size of a cap set"*

## Fast matrix multiplication

Possible schemes for fast matrix multiplication rely on large cap sets (e.g. Coppersmith-Winograd conjecture)

## Related to other problems in extremal combinatorics

e.g. Erdős-Szemerédi sunflower conjecture.

# Solution of the cap set problem

**Theorem (2016) [Ellenberg-G.]**

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- Coppersmith-Winograd conjecture is false
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  (not viable path for fast matrix multiplication)

- Proof builds upon work of Croot-Lev-Pach for 3APs in $(\mathbb{Z}/4\mathbb{Z})^n$.
  CLP lemma.
- Proof reformulated by Tao in terms of slice rank of tensors.
  Slice rank method.

# Slice rank method

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1k}x_k &= 0 \\ &\vdots \\ a_{m1}x_1 + \cdots + a_{mk}x_k &= 0 \end{aligned} \qquad (\star)$$

where $a_{ij} \in \mathbb{F}_q$. Variable vectors $x_j \in \mathbb{F}_q^n$.

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where $a_{ij} \in \mathbb{F}_q$. Variable vectors $x_j \in \mathbb{F}_q^n$.

**Theorem**

*Suppose that $S \subseteq \mathbb{F}_q^n$ contains no nontrivial solutions to $(\star)$.*
*If $k \geq 2m+1$, then $|S| \leq q^{(1-\delta)n}$ for some $\delta > 0$.*

Note: No (non-trivial) bound for $k \leq 2m$.

### Theorem

*Suppose that* $S \subseteq \mathbb{F}_q^n$ *contains no nontrivial solutions to* $(\star)$. *If* $k \geq 2m+1$ *then there is a* $\delta > 0$ *such that* $|S| \leq q^{(1-\delta)n}$.

Note: No (non-trivial) bound for $k \leq 2m$.

### Open problem 4APs

Let $p \geq 5$ prime. Is there a $\delta > 0$ such that the following holds.
If $S \subseteq \mathbb{F}_p^n$ has no (non-trivial) solutions to

$$\begin{aligned} x_1 - 2x_2 + \ x_3 \qquad &= 0 \\ x_2 - 2x_3 + x_4 &= 0 \end{aligned} \qquad (\star)$$

then $|S| \leq p^{(1-\delta)n}$ ?

# Non-degenerate solutions

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Slice rank method does not work (for $p > 3$)!

However, bounds $O(p^{(1-\delta)n})$ obtained by modifying/augmenting the slice rank method

Naslund (2020), Fox-Sauermann (2018), Sauermann (2021)

For which systems is there a $\delta > 0$ such that
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- Sauermann: all $m \times m$ minors nonzero and $k \geq 3m$.

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Note: to use the slice-rank method, we certainly need $k \geq 2m + 1$ and similarly for every implied system.

We call $(\star)$ tame if every implied system with $m'$ equalities uses $k' \geq 2m' + 1$ variables.

**Theorem**

*Suppose that $(\star)$ is tame. Consider subsets $S \subseteq \mathbb{F}_q^n$.*

*There is a $\delta > 0$ such that $|S| = \Omega(q^{(1-\delta)n})$ implies generic solutions to $(\star)$ in $S$ (for $n$ large enough).*

# Proof sketch 1/5 (setup)

- Restrict to the 'worst' case: $k = 2m + 1$.
  Goal: Show: $|S| = \Omega(q^{(1-\delta)n})$ implies generic solutions in $S$
  generic $\equiv$ affine rank $m + 1$
- Induction on $r$:
  Assume: we get solutions of affine rank $r < m + 1$, but not $r + 1$.
  Goal: obtain a contradiction.

Important tool is super saturation.

**Proposition (Super saturation)**

*Let $0 < \delta' < \delta$. There is a constant $c > 0$ such that the following holds.*

*Suppose:* $|S| = \Omega(q^{(1-\delta)n})$ *implies solutions of affine rank* $\geq r$ *(for $n$ large)*

*Then:* $|S| = \Omega(q^{(1-\delta')n})$ *implies* $\Omega(q^{nr - c\delta' n})$ *solutions of affine rank* $\geq r$

# Proof sketch 2/5 (polynomials)

The solutions to $(\star)$ can be modeled by a low-degree polynomial.
Let $f : \underbrace{S \times \cdots \times S}_{k \text{ times}} \to \{0,1\} \subseteq \mathbb{F}_q$ be the indicator function of the solution set.

Then

$$f(x_1, \ldots, x_k) = \prod_{i=1}^{m} \prod_{\ell=1}^{n} \left[1 - (a_{i1}x_{1\ell} + \cdots + a_{ik}x_{k\ell})^{q-1}\right],$$

a polynomial of degree $mn(q-1)$. $x_j = (x_{j1}, \ldots, x_{jn})$

Note: $\deg(f) = \frac{m}{2m+1} \cdot$ maximum possible degree (recall that $k = 2m+1$).

# Proof sketch 3/5 (Using tameness)

Tameness of $(\star)$ implies (by matroid union theorem):

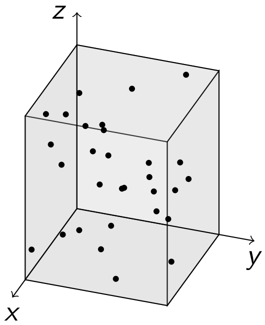
If $(x_1, \ldots, x_{2m+1})$ is a solution of affine rank $r$, there exist disjoint $I, J \subseteq \{1, \ldots, 2m+1\}$ of size $r$ such that $\{x_i : i \in I\}$ and $\{x_i : i \in J\}$ are affinely independent.

Assume:

- all solutions have affine rank $r$
- can always take $I = \{1, \ldots, r\}$ and $J = \{r+1, \ldots, 2r\}$.

Rename:

- $x = (x_1, \ldots, x_r)$
- $y = (x_{r+1}, \ldots, x_{2r})$
- $z = (x_{2r+1}, \ldots, x_{2m+1})$



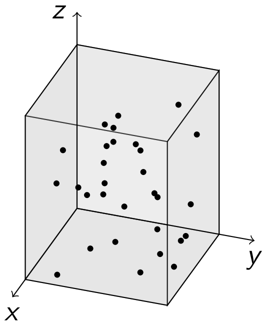
# Proof sketch 4/5 (constructing low rank matrix, CLP lemma)

Let $g : S^{2m+1-2r} \to \mathbb{F}_q$ be random function such that

$$\sum_{z \in S} g(z) z^\alpha = 0 \quad \text{for all monomials } z^\alpha \text{ of degree } |\alpha| \le (q-1)n \cdot (2m+1-2r) \cdot \frac{m}{2m+1}$$

Compress $f$ to a function $M : S^{2r} \to \mathbb{F}_q$:

$$M(x, y) = \sum_z f(x, y, z) g(z)$$



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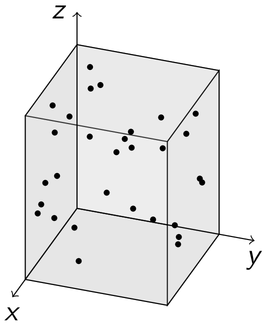
Then $M$ has low degree: $\deg(M) \leq (q-1)n \cdot 2r \cdot \frac{m}{2m+1}$.




Can view $M$ as a $|S|^r \times |S|^r$-matrix.

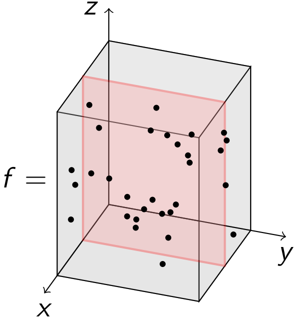
Croot-Lev-Pach lemma: $M$ has small rank.

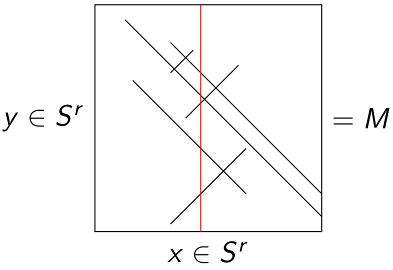
# Proof sketch 5/5 (structure solution set implies high rank)

Matrix $M$ satisfies:

- Bounded number of non-zeroes in each row/column.
- Total number of non-zeroes is $\Omega(q^{nr-\epsilon n})$ (by supersaturation).

Conclusion: $M$ has high rank ($\Omega(q^{nr-\epsilon n})$). Contradiction!




□

Thank you!

# CLP lemma

## CLP lemma

Let $f \in \mathbb{F}_q[x_1, \ldots, x_n, y_1, \ldots, y_n]$ be a polynomial of degree $d$.
Then the $q^n \times q^n$-matrix

$$M_{a,b} = f(a_1, \ldots, a_n, b_1, \ldots, b_n)$$

has rank $\leq 2 \times$ the number of monomials $x^\alpha$, where
$\alpha \in \{0, \ldots, q-1\}^n$ and $|\alpha| := \alpha_1 + \cdots + \alpha_n \leq d/2$.

## Proof.

Write

$$f = \sum_{|\alpha| \leq d/2} x^\alpha f_\alpha(y) + \sum_{|\beta| \leq d/2} y^\beta g_\beta(x)$$

for certain $f_\alpha$ and $g_\beta$.
Each term $x^\alpha f_\alpha(y)$ and each term $y^\beta g_\beta(x)$ corresponds to a rank 1 matrix (outer product of two vectors). □