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# Optimal control and stories we tell about climate change economics

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**Workshop on Modern Applications of Control Theory and Reinforcement Learning, Centrum Wiskunde & Informatica (CWI), 20-21 May 2025**

# Key points to be discussed in the lecture

- Formalism of the optimal control theory
- Key principles of the economics of climate change (at the global level): cost-benefit analysis, time discounting, and optimization over long time horizons
- How assumptions shape the story we tell using models, how assumptions are chosen to construct a particular story (in several examples)
- Theoretical methods to solve optimal control problems, when they can deliver an analytical solutions, may allow to craft stronger stories than numerical simulations*

    *This is potentially changing with the development of ML/AI
- The specific examples presented in this talk are intended solely to illustrate general points. There is no need to fully understand each example in detail.

# Optimal control: Key facts

**Definition**:
Optimal control is the process of determining a control (policy) for a dynamic system such that a certain performance criterion (objective function) is optimized—typically minimized cost or maximized utility—over a given time horizon.

**Core Components**:

1. **State variables**: Describe the system's current condition (e.g., position, temperature).
2. **Control variables**: Inputs or actions that influence the system (e.g., force, investment).
3. **Dynamics**: Differential equations that govern how the state evolves over time.
4. **Objective function**: A functional (usually integral over time) that quantifies the cost or benefit of a trajectory and control.

objective (goal) function

$$\int_0^T g\big(t, x(t), u(t)\big)\mathrm{d}t \to \min$$

$$\dot{x}(t) = f\big(t, x(t), u(t)\big), x(0) = x_0$$

state variable

control variable

$u(\cdot) \in \Omega$ functional class (piece-wise continuous or measurable functions)

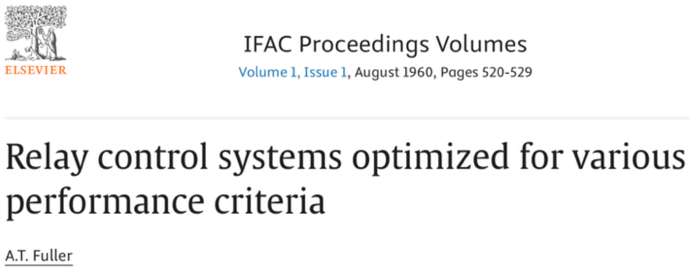
$u(t) \in U$ geometric constraints on controls

# Example

**Fuller problem:**

$$\int_0^{\infty} x^2(t)\mathrm{d}t \to \min$$

$$\ddot{x}(t) = u(t)$$
$$x(0) = x_0,\ \dot{x}(0) = v_0$$
$$x(\infty) = 0,\ \dot{x}(\infty) = 0$$
$$u(t) \in [-1,1]$$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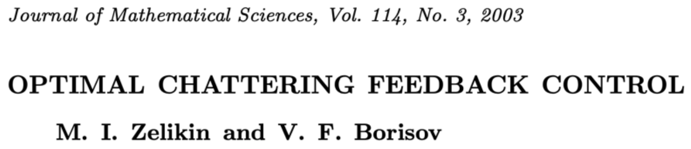
IFAC Proceedings Volumes
Volume 1, Issue 1, August 1960, Pages 520-529

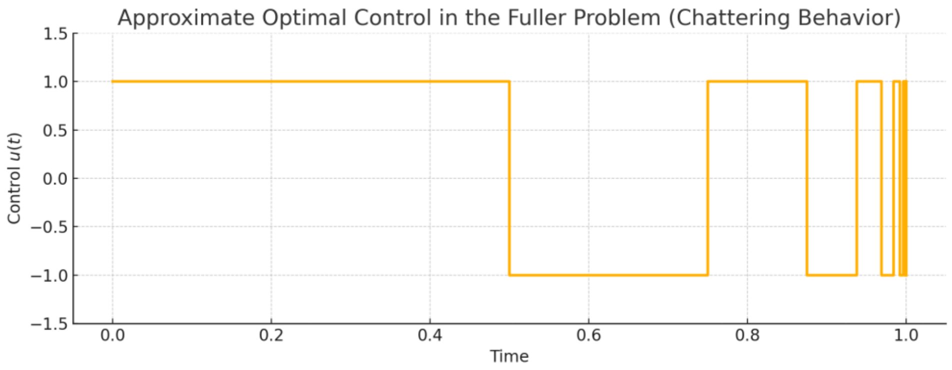
Relay control systems optimized for various performance criteria

A.T. Fuller


*Journal of Mathematical Sciences, Vol. 114, No. 3, 2003*

OPTIMAL CHATTERING FEEDBACK CONTROL

M. I. Zelikin and V. F. Borisov




**Interpretation:** The goal is to steer a mechanical object (a point particle) to a given state with a minimal error

**Optimal solution:** A *bang-bang* control with a *countable number* of switches between +1 and -1 which occur in a progressively faster sequence (chattering phenomenon).
This solution is both non-intuitive and requries using a class of measurable functions as controls

# Optimal control: Key facts

**Methods to solve:**

1. **Pontryagin's Maximum Principle**:
   A foundational result providing necessary conditions for optimality, it introduces the **Hamiltonian** and **adjoint variables** (costate), analogous to Lagrange multipliers in constrained optimization.

2. **Dynamic Programming**:
   Another key method based on Bellman's Principle of Optimality, which leads to the **Hamilton–Jacobi–Bellman (HJB) equation**—a partial differential equation characterizing the value function of the control problem.

**3. Numerical Methods**:
Solving optimal control problems often requires discretization and numerical optimization, including:

- Direct methods
- Indirect methods (solving necessary conditions)

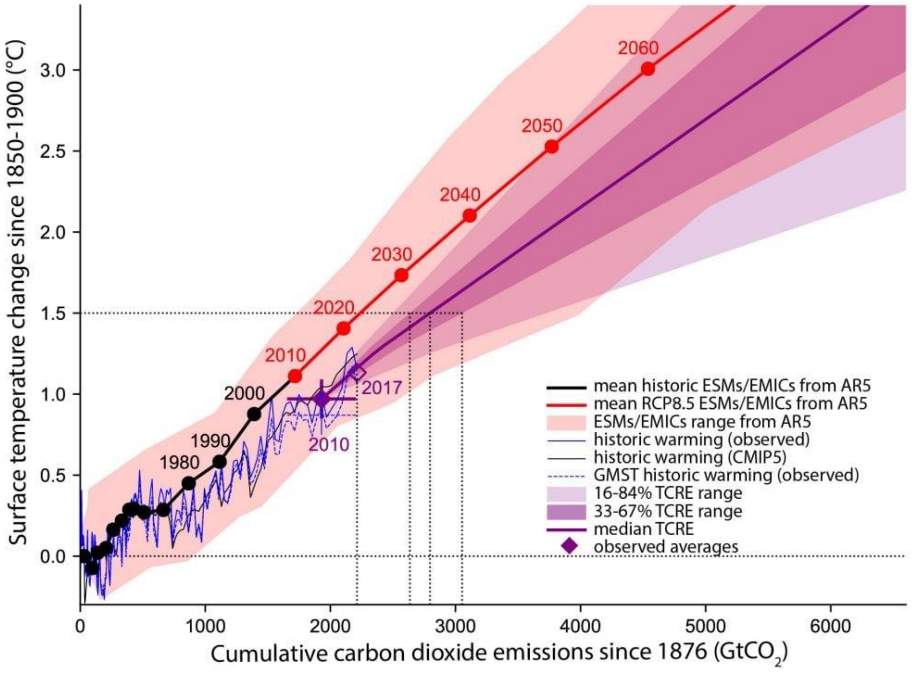
# Economics of climate change: Key postulates

**At the global level:**

- Anthropogenic greenhouse gas (GHG) emissions drive global warming, which in turn triggers adverse effects such as sea level rise, extreme weather events, etc. Climate outcomes are deeply uncertain and some damages may be irreversible (e.g., ice sheet collapse, biodiversity loss).
- Inter-temporal trade-offs: Climate change involves long time horizons — emissions today affect the climate for centuries.
- Discounting (how we value the future relative to the present) is crucial. Small changes in the **social discount rate** drastically affect optimal policy (e.g., how much we invest in mitigation now).

**At the national and local level:**

- Climate is a global good, which makes climate action prone to free-riding because the benefits of reduced emissions are shared globally while the costs are borne locally—giving each country an incentive to under-contribute, hoping others will act instead.

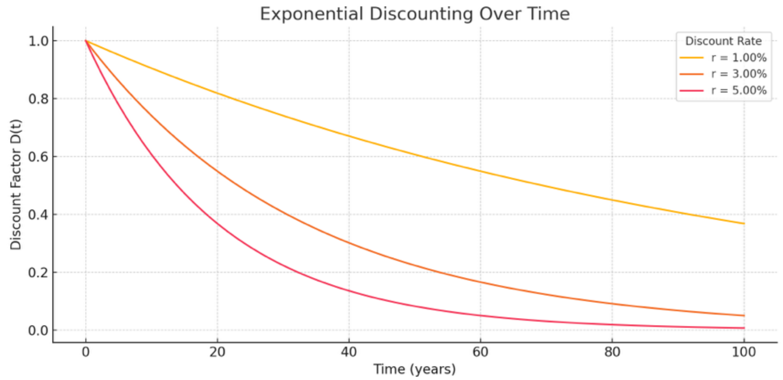



Source: IPCC, AR5

In this talk, we focus on the global level. An economic perspective is only one possible perspective for tackling climate change

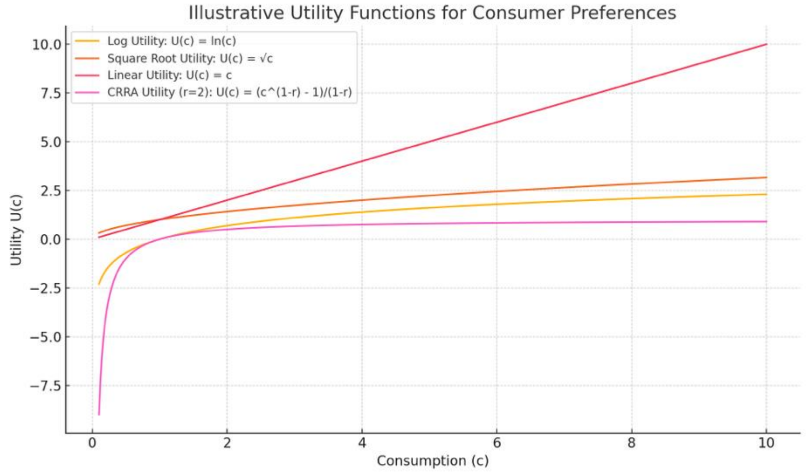
# Economics of climate change: Key postulates

- Cost-benefit analysis: At the global level, the economics of climate change views the challenge of reducing GHG emissions as a problem of optimizing social welfare over a time horizon long enough to capture the full impacts (damages) of climate change.




$$\int_0^T e^{-rt} L(t) u\big(c(t)\big) \mathrm{d}t \rightarrow \max$$

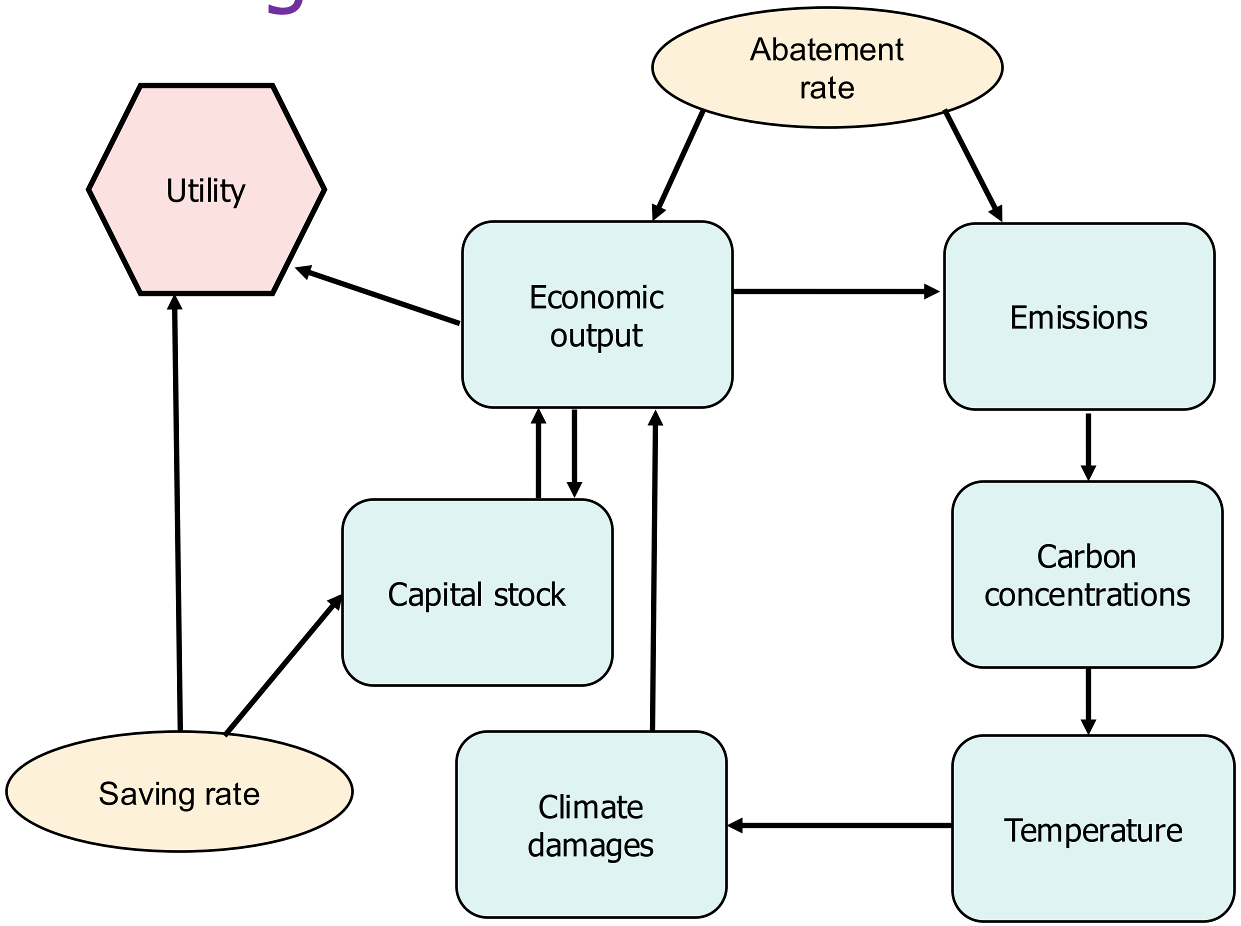
- Exponential discounting $e^{-rt}$ ensures time-consistency of optimal solutions
- $u(\cdot)$ stands for the utility of consumption (a concave function ensuring a decreasing return to scale of any additional unit of consumption)
- $c(\cdot)$ is consumption of a 'representative agent'
- $L(\cdot)$ is the population size



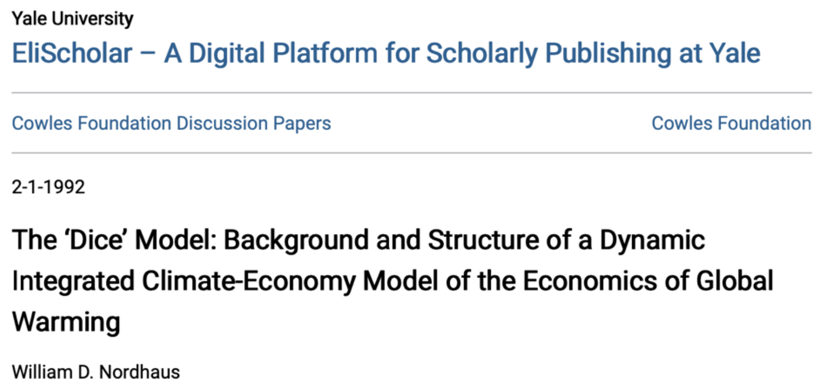

- Cost-benefit analysis equalizes the marginal benefit and the marginal cost

$$B(x) - C(x) \rightarrow \max \Rightarrow B'(x) = C'(x)$$

# DICE: A flagship model of the economics of climate change




An optimal control problem with six dynamic equations and two controls (consumption rate and abatement rate)


Yale University
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Cowles Foundation Discussion Papers | Cowles Foundation

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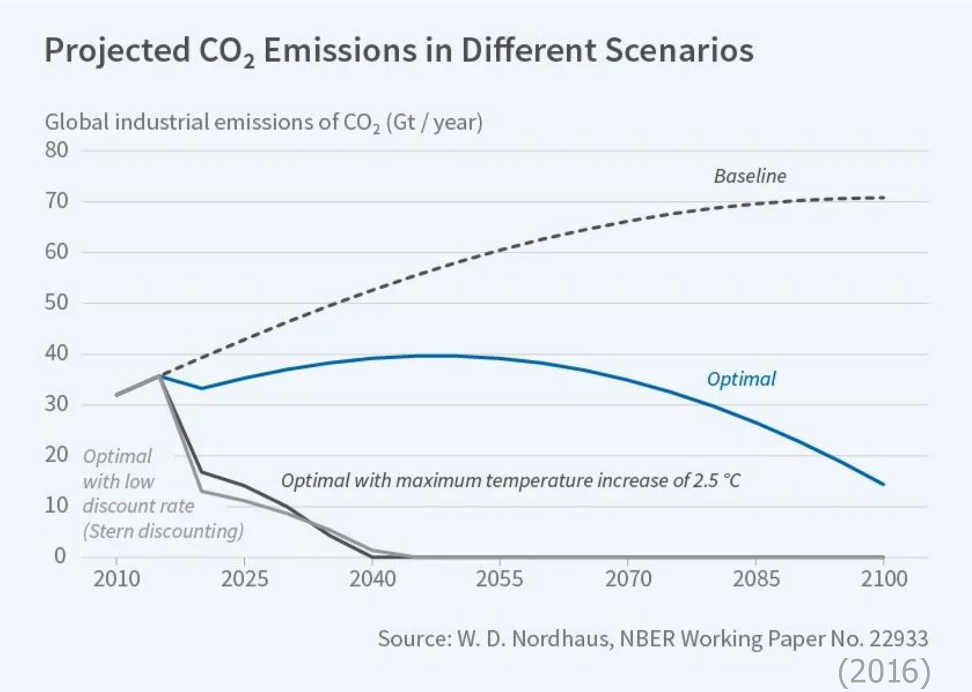
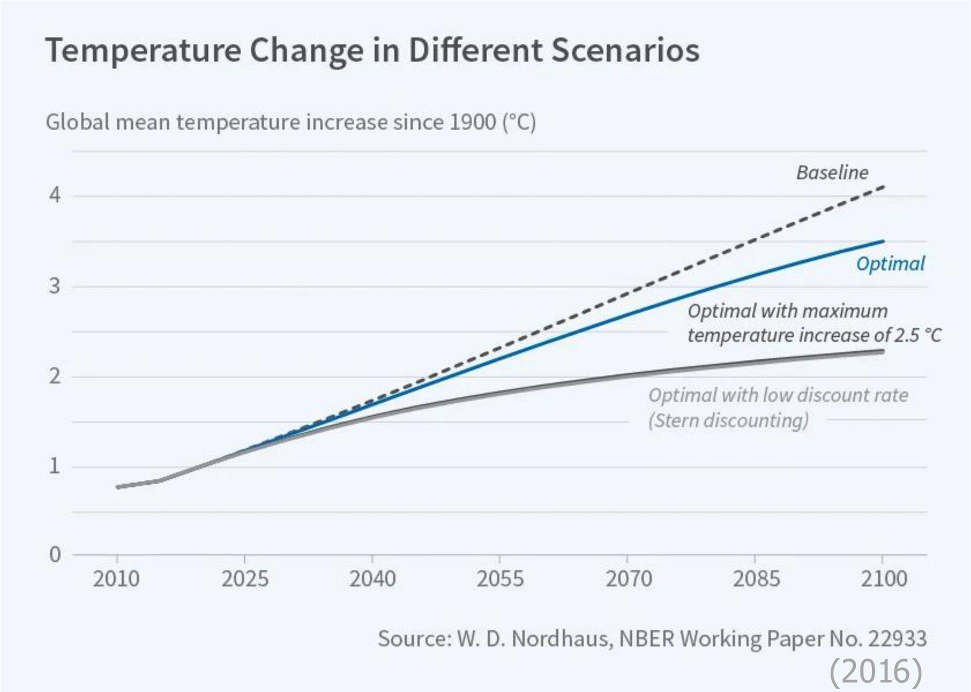
The 'Dice' Model: Background and Structure of a Dynamic Integrated Climate-Economy Model of the Economics of Global Warming

William D. Nordhaus

# DICE: A flagship model of the economics of climate change



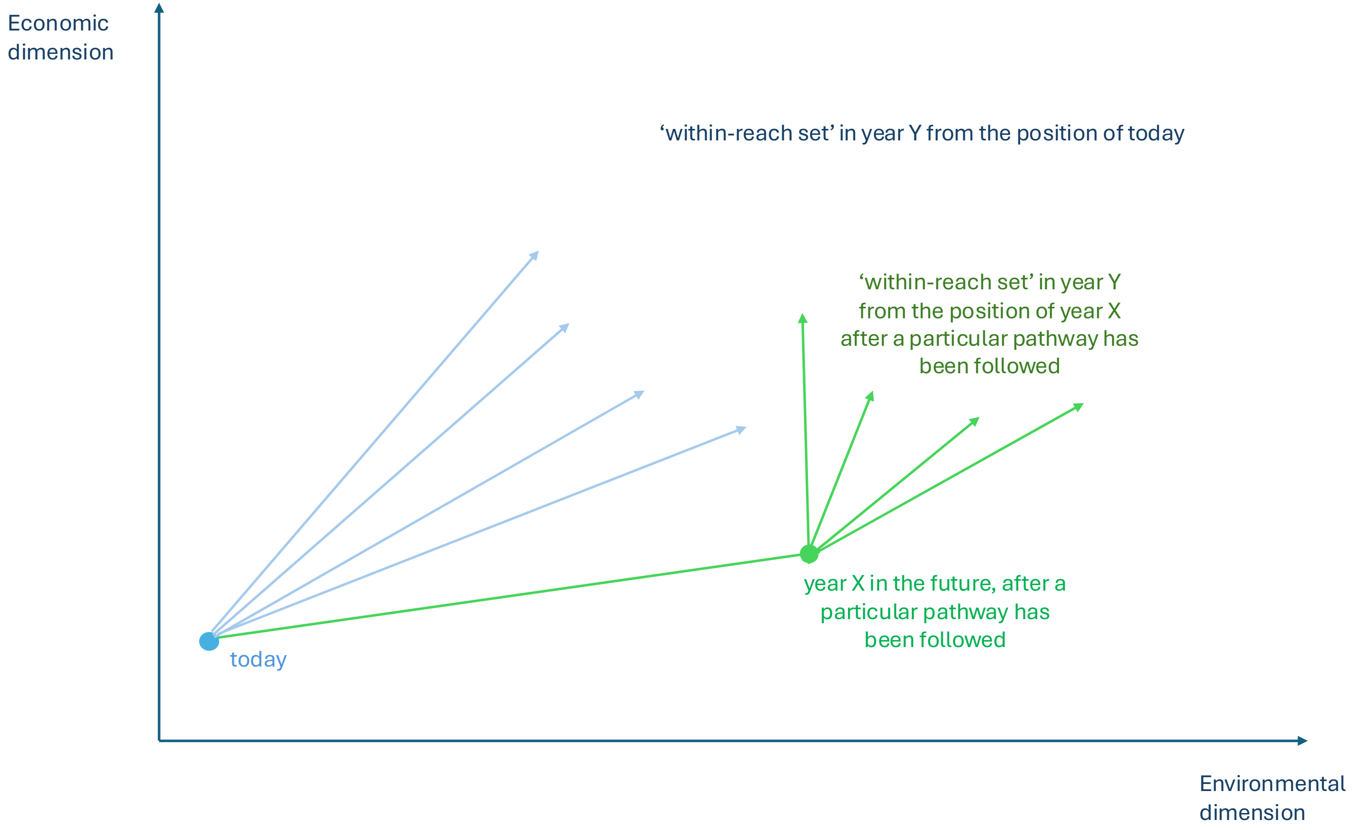



# Three examples of IIASA research

# Example 1: Reachable sets in DICE model

# Within-reach sets

- The Brundtland Report's definition requires that sustainable development must meet "the needs of the present without compromising the ability of future generations to meet their own needs" (1987).
- A within-reach set (reachable set, attainability domain) in year Y comprises all system states, to which the system can move as a result of the application of various feasible controls (policies) between today and year Y.
- After a certain control has been applied between today and year X, the within-reach set typically shrinks.



# Within-reach sets in the DICE model

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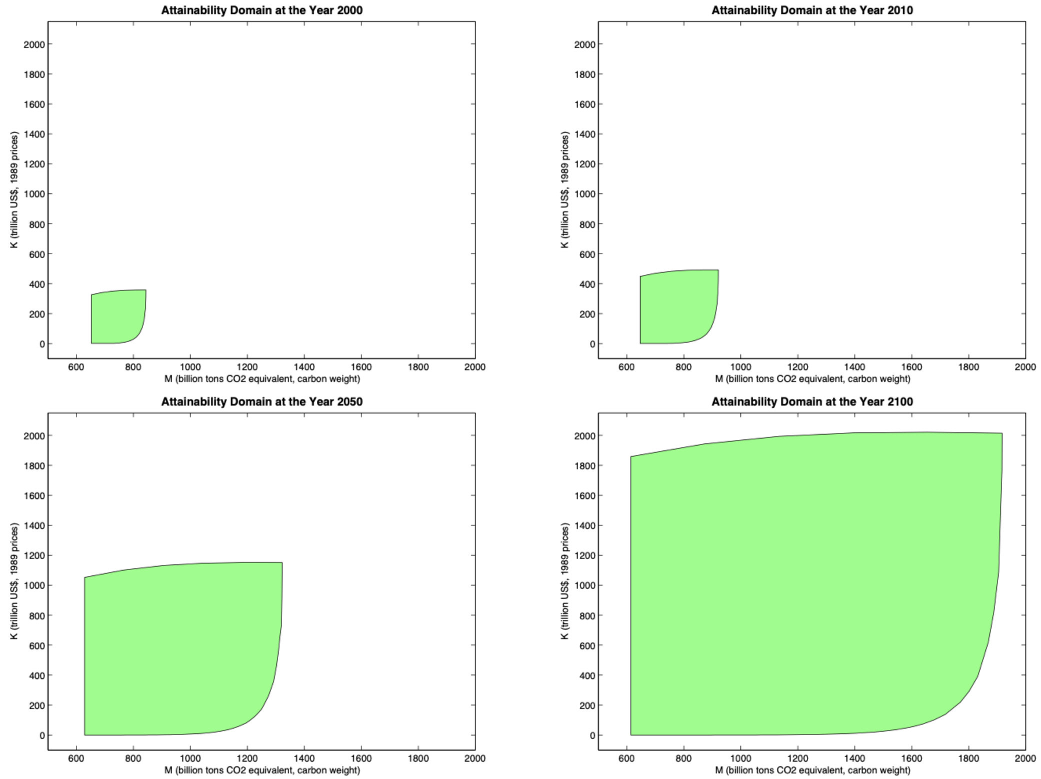
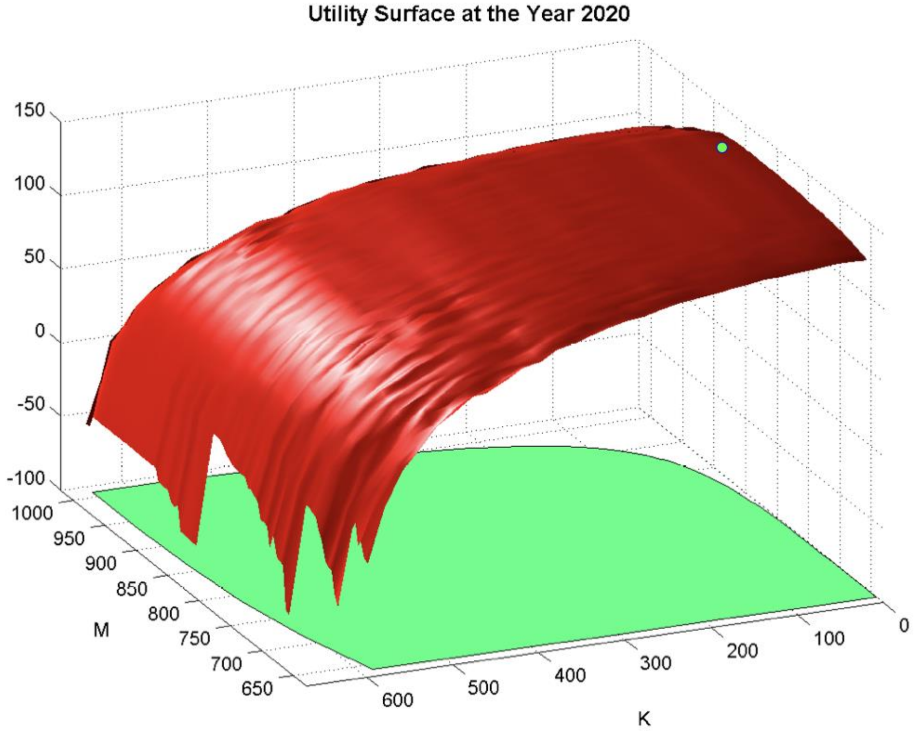
**Interim Report** **IR-05-049**

**Attainability Analysis of the DICE Model**

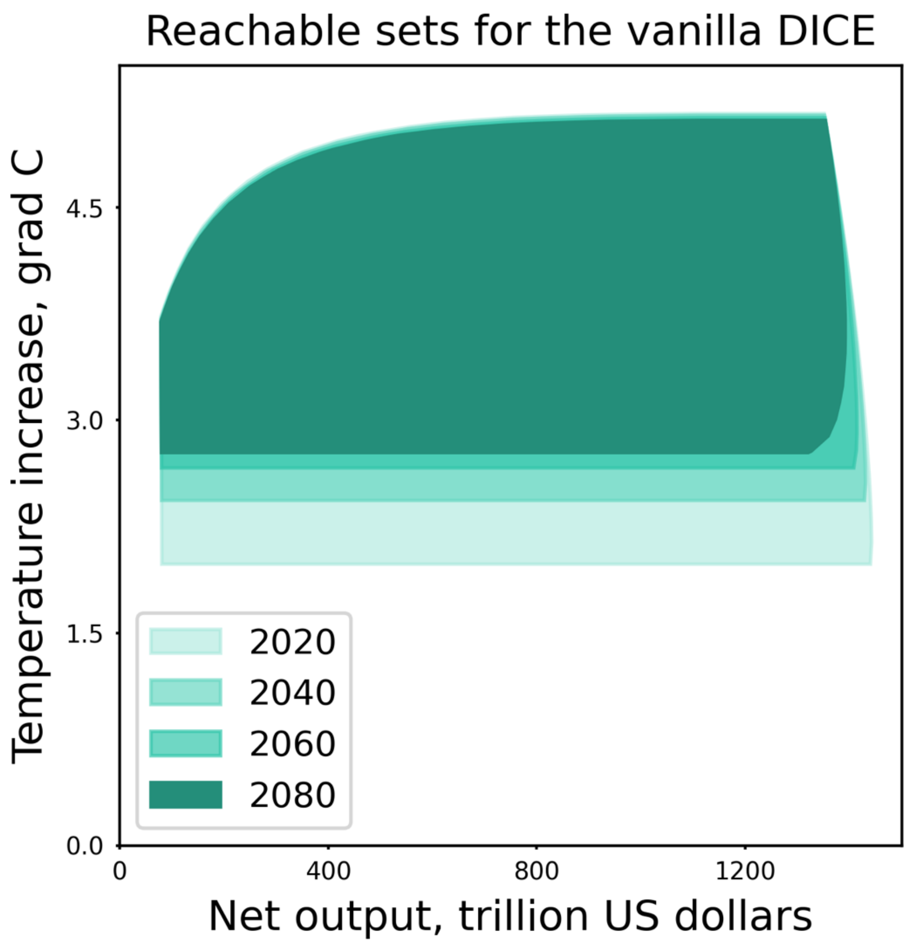
*Alexey Smirnov (asmirnov@cs.msu.su)*




DICE version from: Nordhaus, W.D., Boyer, J., Warming the World — Economic Models of Global Warming, MIT Press, 2001

# Within-reach sets in the DICE model




- Application of business-as-usual policies leads to shrinkage of the within-reach sets
- Work in progress

DICE version from: Nordhaus, W. Projections and Uncertainties about Climate Change in an Era of Minimal Climate Policies. *Am. Econ. J. Econ. Policy* 10, 333–360 (2018).

# Example 2: Two agents in DICE

# Two-agent version of the DICE model

- Work in progress.
- The model differentiates between capital-owning and wage-dependent households (i.e., two representative households instead of one as in the original DICE).
- Motivation: Empirical evidence suggests that households at different income levels perceive their responsibilities for funding climate change mitigation differently. This observation becomes even more relevant, given the widening of income gap between wealthier and poorer segments of society.
- Capital-owning households possess all capital and receive wages. They invest in capital growth a part of their income and consume the remaining part. Income of wage-dependent households relies on wages only and they consume their full income.

Socially optimal solution (social planner's problem)

$$\int_0^T e^{-rt}\Big(L_{CO}(t)u(c_{CO}(t)) + L_{WD}(t)u(c_{WD}(t))\Big)\mathrm{d}t \to \max$$

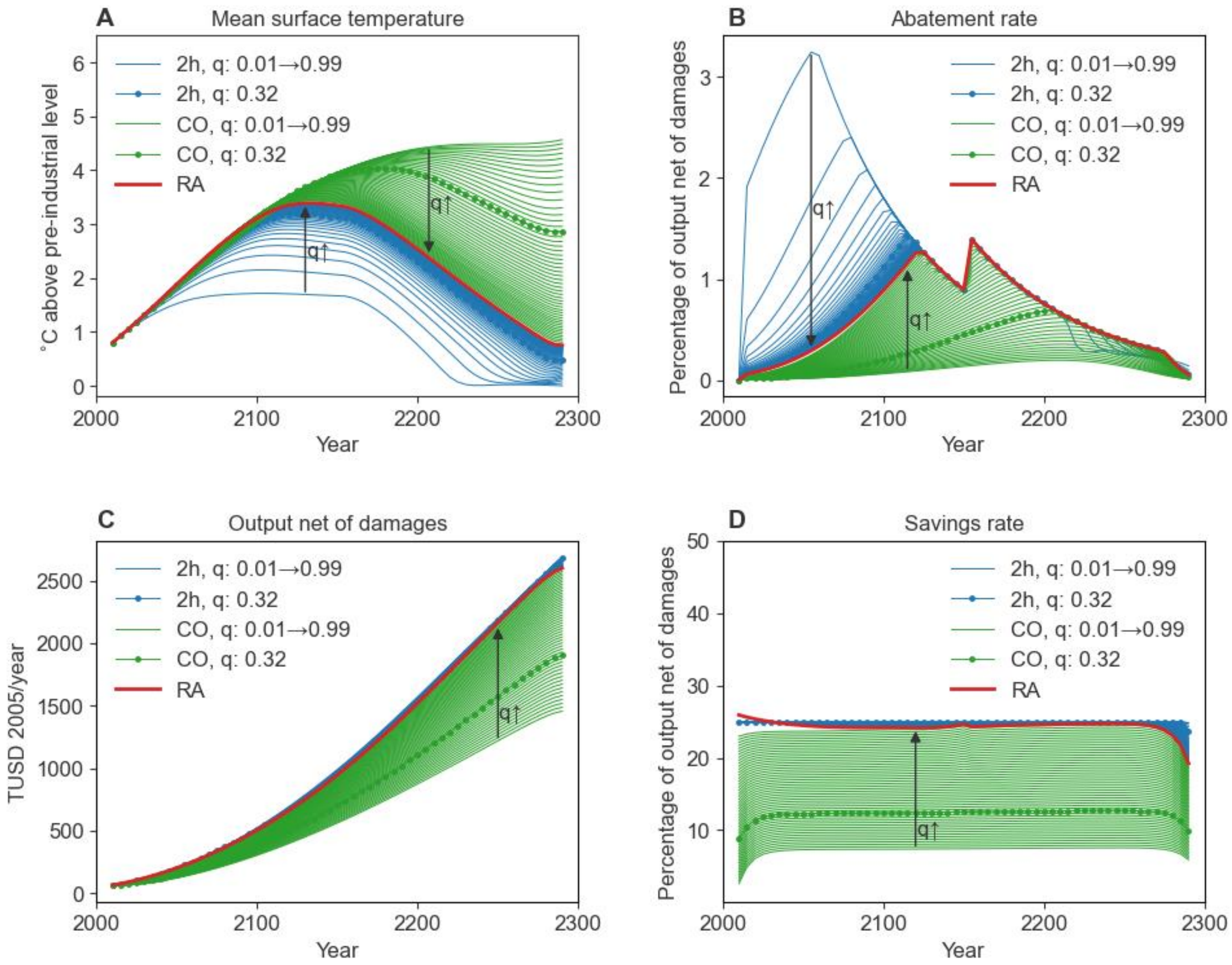
population of capital owners

population of wage-dependent households

Solution optimal for capital owners

$$\int_0^T e^{-rt}L_{CO}(t)u(c_{CO}(t))\mathrm{d}t \to \max$$

# Two-agent version of the DICE model

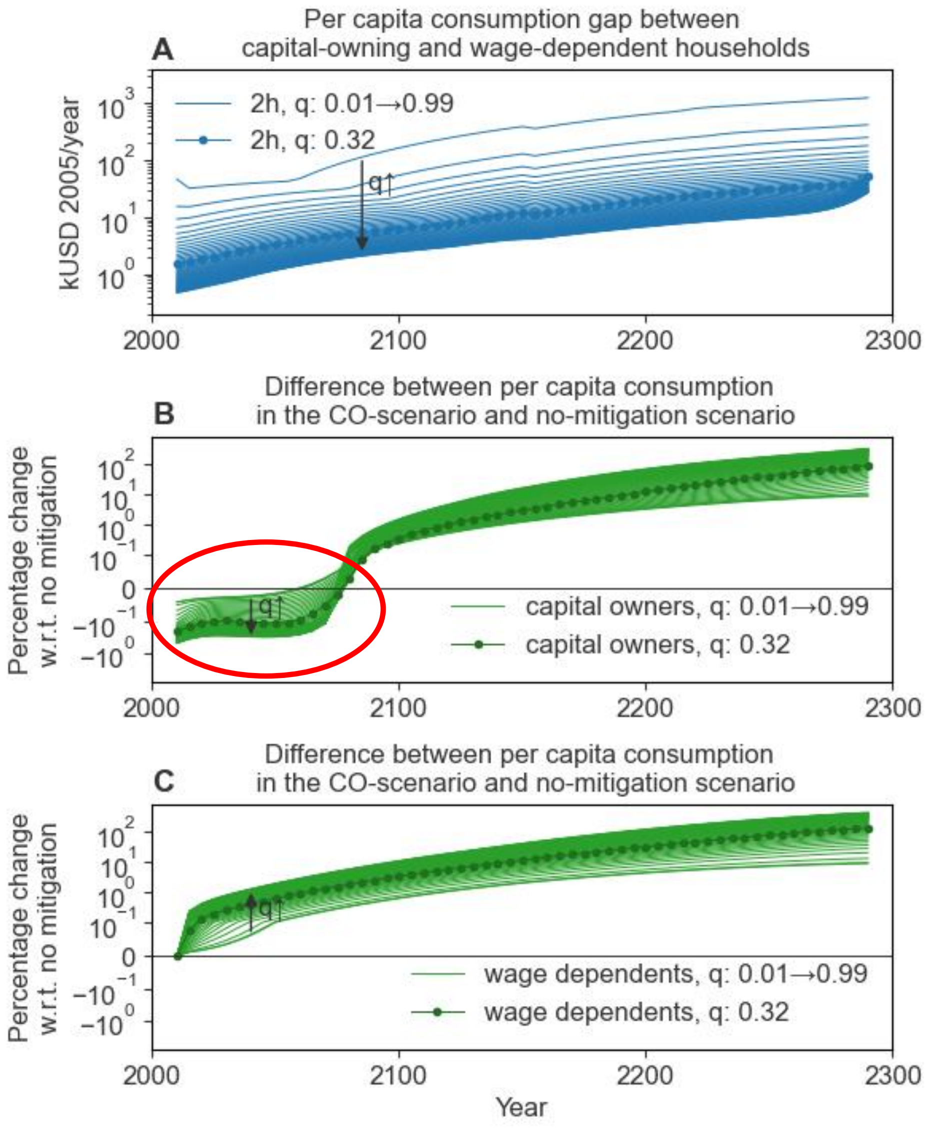



RA: regular DICE
2h: social planner’s problem
CO: capital owners’ problem

q: share of capital owners in the economy (between 0 and 1)

- Economic variables are insensitive to capital dispersion in the social planner’s problem.
- In the capital owners’ problem, higher capital dispersion ($q \rightarrow 1$) leads to higher saving rate and economic growth.
- In the social planner’s problem, optimal abatement rate is higher with higher capital concentration and low warming is optimal. This is because of their low weight in the social welfare function.
- In the capital owners’ problem, optimal abatement is lower than that in the DICE model and temperatures are higher due to free riding.

# Two-agent version of the DICE model




RA: regular DICE
2h: social planner’s problem
CO: capital owners’ problem

q: share of capital owners in the economy (between 0 and 1)

- Capital owners face intertemporal ‘unfairness’ because optimal mitigation requires earlier generations to reduce their consumption, while later generations benefit from significantly higher consumption levels compared to a no-mitigation scenario.
- Wage-dependent households do not face intertemporal ‘unfairness’ and with higher dispersion of capital enjoy higher consumption gains when capital owners optimize their abatement.

# Example 3: Two sectors in an environmental economics model

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Journal of Mathematical Economics
Volume 98, January 2022, 102554


## Optimal transition to greener production in a pro-environmental society

Sergey Orlov, Elena Rovenskaya

# Two-sector model

- $K_G(\cdot)$ and $K_B(\cdot)$ stand for the 'green' and 'brown' capital stocks
- $Y_B(t) = A_B K_B(t)$ and $Y_G(t) = A_G K_G(t)$ are outputs of the 'green' and 'brown' sectors (AK production function)
- Brown capital is more productive than the green capital: $A_B > A_G$
- The total output then becomes

$$Y(t) = A_G K_G(t) + A_B K_B(t)$$

- The social planner distributes total output $Y(\cdot)$ between consumption $C(\cdot)$, investment $I_G(\cdot)$ in the green sector, and investment $I_B(\cdot)$ in the brown sector:

$$Y(t) = C(t) + I_G(t) + I_B(t)$$

- We obtain the following dynamic system

$$\dot{K}_G(t) = I_G(t) - \delta_G K_G(t),\ K_G(0) = K_{G0}$$

$$\dot{K}_B(t) = I_B(t) - \delta_B K_B(t),\ K_B(0) = K_{B0}$$

where $K_{G0} > 0$ and $K_{B0} > 0$ determine the initial respective capital stocks; parameters $\delta_G > 0$ and $\delta_B > 0$ are the respective depreciation rates.

# Two-sector model

- We assume the following social planner's instantaneous utility function

$$U(t) = \ln C(t) + \omega \ln Y_G(t)$$

where

$\ln C(\cdot)$ is the utility of consumption

$\ln Y_G(\cdot)$ is an 'amenity value' of green production (e.g., a political will). In this model there are no damages hence environmental quality is an externality

$\omega \geq 0$ is a weight coefficient

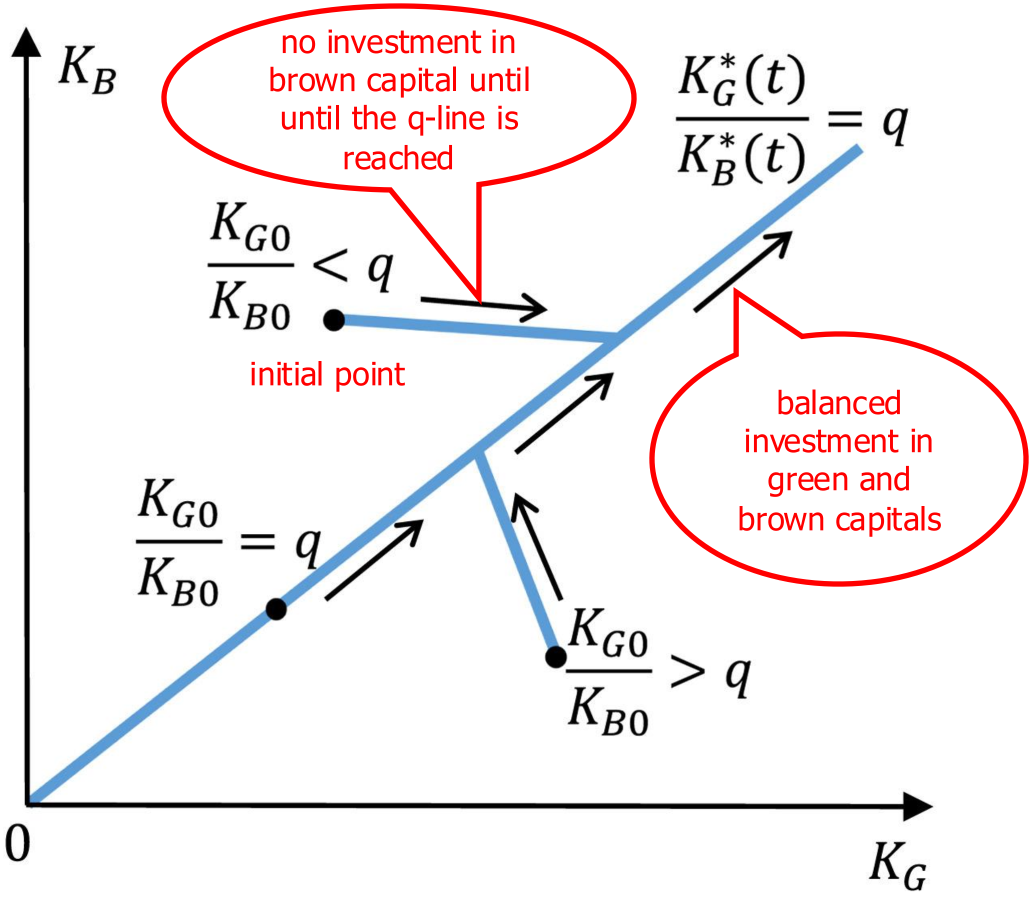
- We consider the model over an infinite time horizon and introduce the total utility as

$$J = \int_0^\infty e^{-rt} U(t)\, \mathrm{d}t$$

where $r > 0$ is a discount rate.

# Two-sector model

**Optimal solution**
**(a theorem with a rigorous proof using Pontryagin's maximum principle)**




Notations:

- Let $D = D_B - D_G$ be the difference between augmented productivities of the brown and green capitals respectively; here $D_B = A_B - \delta_B > 0$ and $D_G = A_G - \delta_G > 0$
- Let $q = \left(\frac{D}{\omega r} + \frac{D-r}{r}\right)^{-1}$

Assumptions:

(A1) The augmented productivity of the brown sector exceeds the augmented productivity of the green sector at least by the discount factor: $D > r$;

(A2) The productivity of the green sector is greater than the discount factor: $A_G > r$.


- The optimal control theory relies on the description of dynamical systems using differential equations with time-varied controls. Because of the time dimension, optimal controls can be non-intuitive even in rather simple models.
- Theoretical methods to solve optimal control problems exist but their application is limited to 'stylized' models.
- Stylized models are a powerful basis to inform stories as they focus on key processes and omit many details (allowing to "see the forests for the trees").
- While climate change mitigation is a very complex matter, scientists have developed numerous rather stylized cost-benefit models to illustrate and analyze key tradeoffs involved. Some of these models have been used to inform real-world policies, e.g., the value of carbon tax.
- When developing new models, it is essential to aim for the simplest model necessary to address the research question—no simpler.

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# Thank you for your attention!
# Questions? Comments?

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