THE (R,S)-MODEL

Probably the most widely-used strategy, either implicity or explicitly, is the periodic review replenishment strategy without lot-sizing, the (R,S)-strategy. The structure of the strategy is quite simple and intuitively appealing. Basically the strategy boils down to:
Order such an amount of items that the sum of physical stock and items on order (including the presently determined one) minus backorders is enough to cover demand from now until receipt of the order, which will be placed at the vendor one review period from now.

On purpose we have stated the strategy as generally as possible. For particular cases we elaborate on a more precise formulation below. What should be explained more clearly is the word "enough". This will turn out to depend on either service level constraints or cost considerations.

The (R,S)-strategy presumes flexibility of the supplier, since no lot-sizes are prescribed. On the other hand, stock is only reviewed at equi-distant points in time. The latter holds for almost all computer-based inventory management systems. The review period typically varies from one hour, to daily, to monthly. Whether the flexibility-assumption on the supplier holds true depends largely on the review period. Typically, the shorter the review period, the less likely the supplier is able to satisfy demand in any quantity. As always there is a trade-off between flexibility in lot-sizing and flexibility in monitoring of stock. This topic will be discussed in chapter 8, where we compare the most well-known inventory management strategies.

This chapter is organized as follows. In section 3.1. we discuss the stationary demand model. We derive expressions for service measures P_1 and P_2 defined in chapter 1. In section 3.2. we derive expressions for the average physical stock. It will be shown that these expressions depend on the nature of the demand process as well as on the moments at which stock is monitored. In section 3.3. we discuss the dynamic demand case.

3.1. Service measures

In this section we assume that demand is stationary, i.e.

$$P\{D_i^R \leq d\} = F_D(d), d \geq 0, i \geq 0$$

where D_i^R equals the demand during [iR,(i+1)R].
Also we assume that the lead time L_i needed to replenish stock after an order has been initiated at time iR is distributed according to $F_L(.)$,

$$P\{L_i \leq t\} = F_L(t), t \geq 0.$$

Hence lead times are assumed to be stationary as well.
For this particular case the (R,S)-strategy can be formulated as follows:

At each review moment order such an amount that the inventory position, i.e. the physical stock plus items on order minus backorders, immediately after the review moment equals S, the order-up-to-level.

It is clear that a fixed order-up-to-level S suffices in this case, since demand is from a probabilistic point of view the same at each review moment.

We find the following important relation between the inventory position and the net stock,

$$X(iR+L_i) = S-D[iR, iR+L_i] \tag{3.1}$$

where

$X(t)$:= net stock at time t, $t \geq 0$

$Y(t)$:= inventory position at time t, $t \geq 0$

$D[t,s]$:= demand during $[t,s]$, $0 \leq t \leq s$

This relation can be motivated as follows. Without loss of generality we may assume that i=0. Suppose that after ordering at time 0 no demand occurs in $(0,L_0)$. Assuming orders do not overtake it is easy to see that at time L_0 all orders placed before or at time 0 have arrived at the stock keeping facility to replenish stock. Hence all items on order, which were part of the inventory position S immediately after time 0 are at the stock keeping facility at time L_0. Assuming no demand occurred in $[0,L_0]$ the net stock now equals S.

However, demand $D[0,L_0]$ indeed occurred in $[0,L_0]$. This reduces the net stock at time L_0 from S to $S-D[0,L_0]$. This proves (3.1).

Since no orders arrive until R+L, we also have

$$X(R+L_1-0) \ = \ S-D(0,R+L_1-0],$$

where $R+L_1-0$ denotes the point in time an infinitesimal time before $R+L_1$. More general, we claim that

$$X((i+1)R+L_{i+1}-0) \ = \ S-D[iR,(i+1)R+L_{i+1}-0] \qquad (3.2)$$

Equations (3.1) and (3.2) play a key role in the derivation of expressions for service levels under various service criteria.

P_1-service measure

We first consider the P_1-service measure,

P_1 = the probability of no stockout during a replenishment cycle.

We define a stockout as follows. A stockout is the event that the net stock drops from a positive value to a negative value.
It is easy to see that a necessary condition for the occurrence of a stock-out during a replenishment cycle is that the net stock is less than or equal to zero at the end of a replenishment cycle. Considering the replenishment cycle $[L_0,R+L_1]$ we have

$$\text{A stockout occurs during } [L_0,R+L_1] \Rightarrow X[R+L_1-0] \leq 0$$

Yet the converse is not true, i.e.

$$X[R+L_1-0]\leq 0 \Rightarrow \textit{A stockout occurs during } [L_0,R+L_1]$$

The key to this statement lies in the phrase "drops from a positive value to a negative value". If $X[R+L_1-0]\leq 0$, then there is still the possibility that the net stock did not drop from a positive value to a negative value during $[L_0,R+L_1]$. This is the case when the net stock was already negative at the beginning of the replenishment cycle or equivalently $X(L_0)\leq 0$! Only when $X(L_0)>0$ and $X(R+L_1-0)\leq 0$ we have a stockout occurring during $[L_0,R+L_1]$. Hence

$$X(R+L_1-0)\leq 0 \textit{ and } X(L_0)>0 \Leftrightarrow \textit{A stockout occurs during } [L_0,R+L_1]$$

This implies that

$$P_1 = 1-P\{X(R+L_1-0)\leq 0,\ X(L_0)>0\} \tag{3.3}$$

Let us elaborate on the r.h.s. of (3.3)

$$P\{X(R+L_1-0)\leq 0,\ X(L_0)>0\}$$

$$= P\{X(R+L_1-0)\leq 0\} - P\{X(R+L_1-0)\leq 0,\ X(L_0)\leq 0\}$$

$$= P\{X(R+L_1-0)\leq 0\} - P\{X(L_0)\leq 0\}$$

$$+ P\{X(R+L_1-0)>0,\ X(L_0)\leq 0\}$$

Now we know from (3.1) and (3.2) that

$$X(L_0) = D(L_0,R+L_1) + X(R+L_1-0)$$

Note that we assume that $L_0 \leq R+L_1$, since orders cannot overtake. If $X(R+L_1\text{-}0)>0$ then certainly $X(L_0)>0$. Hence

$$P\{X(R+L_1-0)>0,\ X(L_0)\leq 0\} = 0$$

Thus we find

$$P_1 = 1-P\{X(R+L_1-0)\leq 0\} + P\{X(L_0)\leq 0\}$$

Using (3.1) and (3.2) we find

$$P_1 = 1-P\{D(0,R+L_1)\geq S\} + P\{D[0,L_0]\geq S\} \tag{3.4}$$

Equation (3.4) differs from the equation (7.31) in Silver and Peterson [1985]. Implicitly it is assumed there that the probability that $X(L_0)$ is negative is negligible. In general this is not true for (R,S)-models. Even with moderately varying demand during the lead time there is a considerable probability of $X(L_0)$ being negative when $P_1\leq 0.90$, say.

To compute P_1 we have to make assumptions about the demand during $[0,L_0]$ and the demand during $[0,R+L_1]$. We assume that demand over a finite time interval is gamma-distributed. This assumption has been empirically verified by several authors and proves to be quite applicable. This also follows from our investigations. Assuming that L_0 and L_1 only take values on the set $\{kR/k\varepsilon N\}$ it is readily verified, that

$$E[D[0,L_0]] = E[L]\ E[D] \tag{3.5}$$

$$\sigma^2(D(0,L_0)) = E[L]\sigma^2(D) + \sigma^2(L)E^2[D] \tag{3.6}$$

$$E[D[0,R+L_1]] = (R+E[L])E[D] \tag{3.7}$$

$$\sigma^2[D(0,R+L_1)] = (R+E[L])\sigma^2(D) + \sigma^2(L)E^2[D] \tag{3.8}$$

Fitting a gamma distribution to these first two moments of $D[0,L_0]$ and $D(0,R+L_1)$ yields an expression for P_1. There is something peculiar about the P_1-measure. Suppose S equals zero. In that case P_1 equals 1, suggesting a perfect system. Of course, this is not true. Yet no stockouts are registered, since the net stock never becomes positive. Our conclusion is that the P_1-measure is not a proper service-measure. To circumvent this problem we define

$P_1^{'}$:= the probability that the net stock immediately before the end of a replenishment cycle is positive.

Hence

$$P_1^{\prime} = 1-P\{D(0,R+L_1)\geq S\} \tag{3.9}$$

Equation (3.9) is identical to (7.31) in Silver and Peterson (7.31). To obtain the order-up-to-level S for some value of $P_1^{'}$ we may apply the PDF-method described in chapter 2, since P_1=0 when S=0 and $P_1^{'}$=1 when S=∞. Note that the PDF-method cannot be applied to P_1, because P_1 is not monotone increasing.

A service measure related to P_1 and $P_1^{'}$ is the ready rate or fill rate, i.e. the fraction of time the net stock is positive. It turns out that finding expressions for this service measure involves a more intricate analysis, where explicit assumptions about the nature of the demand process are necessary.

P_2-measure

We first address the P_2-service measure defined as

P_2 := the long-run fraction of demand satisfied directly from stock on hand.

Note that P_2 relates to the demand over a long period in time. It can be shown that it suffices to consider a replenishment cycle only, i.e.

P_2 = the fraction of demand satisfied directly from stock on hand during a replenishment cycle.

Let us define

$B(s,t)$:= the amount of demand backordered during (s,t).

Considering the replenishment cycle $(L_0,R+L_1)$

$$P_2 = 1-\frac{E[B(L_0,R+L_1)]}{E[D(L_0,R+L_1)]}$$

We need an expression for $B(L_0,R+L_1)$. Towards this end we distinguish between three cases.

(i) $X(R+L_1-0)>0$

In this case no demand is backordered during $(L_0,R+L_1)$. Hence

$$B(L_0,R+L_1)=0$$

(ii) $X(R+L_1-0)<0$, $X(L_0)\geq 0$

In this case an amount of $-X(R+L_1-0)$ is backordered.

$$B(L_0,R+L_1) = -X(R+L_1-0)$$

(iii) $X(R+L_1-0)<0$, $X(L_0)<0$

In this case all demand during $(L_0,R+L_1))$ is backordered.

$$B(L_0,R+L_1) = D(L_0,R+L_1)$$

$$= (X(L_0)-X(R+L_1-0)$$

The expression for $B(L_0,R+L_1)$ can be combined into a single expression,

$$B(L_0,R+L_1) = X^-(R+L_1-0)-X^-(L_0),$$

where

$$x^- = \max(0,-x)$$

Since

$$X(L_0) = S-D[0,L_0]$$

$$X(R+L_1-0) = S-D[0,R+L_1),$$

we have

$$B(L_0,R+L_1) = (D(0,R+L_1)-S)^+ - (D[0,L_0]-S)^+,$$

where

$$x^+ = \max(0,x)$$

Hence

$$P_2 = 1-\frac{E[(D(0,R+L_1)-S)^+] - E[(D[0,L_0]-S)^+]}{E[D(L_0,R+L_1)]} \tag{3.10}$$

In Silver and Peterson [1985] equation (7.32) gives an expression for $E[B(L_0,R+L_1)]$, namely

$$E[B(L_0,R+L_1)] = E[(D(0,R+L_1)-S)^+]$$

Again we note that this expression is erroneous and this error considerably impacts the value of S which is calculated given some value of P_2. It is easy to see that the expression above does not yield a proper service measure since $E[(D(0,R+L_1)-S)^+]$ may exceed $E[D(L_0,R+L_1)]$, yielding a negative value for P_2.

To calculate the value of P_2 given a value of S one may again fit gamma distributions to $D(0,R+L_1)$ and $D[0,L_0]$ and use some fast numerical scheme as given in Appendix B. Also, one may fit a mixture of Erlang distributions, which yields almost identical results (cf. Tijms [1986]).

To calculate the value of S given a value of P_2 the PDF-method is applicable, as is shown in De Kok [1990]. Let

$\gamma(S) := P_2(S)$

and let X_γ denote the random variable distributed according to $\gamma(.)$. Then along the lines of section 2.6. we find

$$E[X_\gamma] = \frac{1}{2E[D(L_0,R+L_1]]} (E[D^2[0,R+L_1)] - E[D^2[0,L_0]]) \tag{3.11}$$

$$E[X_\gamma^2] = \frac{1}{3E[D(L_0,R+L_1]]} (E[D^3[0,R+L_1)] - E[D^3[0,L_0]]) \tag{3.12}$$

Assuming $D[0,R+L_1)$ and $D[0,L_0]$ are gamma distributed we have

$$E[D^3[0,R+L_1)] = (1+c_{R+L}^2)(1+2c_{R+L}^2)E^3[D[0,R+L_1)] \tag{3.13}$$

$$E[D^3[0,L_0]] = (1+c_L^2)(1+2c_L^2)E^3[D[0,L_0]] \tag{3.14}$$

where

$$C_{R+L}^2 = \frac{\sigma^2(D[0,R+L_1))}{E^2[D[0,R+L_1)]} \;,\; C_L^2 = \frac{\sigma^2(D[0,L_0])}{E^2(D[0,L_0])} \tag{3.15}$$

Hence C_{R+L}^2, C_L^2 denote the squared coefficient of variation of the demand during a review period plus its consecutive lead time and the demand during a lead time, respectively. Together with

$$E[D^2[0,R+L_1]] = \sigma^2(D(0,R+L_1)) + E^2(D(0,R+L_1))$$

$$E[D^2[0,L_0]] = \sigma^2(D[0,L_0]) + E^2[D[0,L_0]]$$

we can compute $E[X_\gamma]$ and $E[X_\gamma^2]$ from (3.11)-(3.15). Next we fit a gamma distribution $\hat{\gamma}(.)$ to $E[X_\gamma]$ and $E[X_\gamma^2]$. Then we can compute the order-up-to-level S satisfying a P_2-value β from

$$S \cong \hat{\gamma}^{-1}(\beta) \tag{3.16}$$

where $\hat{\gamma}^{-1}$ is the inverse of the incomplete gamma function (cf. chapter 2). Using the inversion scheme (2.77)-(2.81) we can calculate S.

<u>Fill rate</u>

It was noted that the P_1-service measure shows undesirable behaviour, since $P_1(S)$ is not monotone increasing in S. The P_1'-service measure suffers from the fact that it relates to specific points in time only, instead of relating to the long-run behaviour of the system. Therefore we consider the so-called *fill rate* $\hat{P}_1$ defined by:

$\hat{P}_1$:= the long-run fraction of time the stock on hand is positive.

In the literature it is often assumed that $\hat{P}_1$ is equal to P_2. Basically these authors implicitly assume that during periods of negative net stock the net stock decreases linearly with the average demand rate. This is not true in general, as is shown below.

The $\hat{P}_1$-measure differs from the P_1, P_1'- and P_2-measures in that the latter three service measures are completely determined by the distribution of the demand during the lead time and its preceding review period, whereas $\hat{P}_1$ depends on the way the inventory process is monitored. Let us explain this by an example. Suppose we only register the weekly stock depletion. In that case we cannot account for a stockout during the week. Hence the time interval during which the stock on hand is zero is always a multiple of a week. In case we register each stock depletion, i.e. continuous monitoring, we can indeed account for stockouts during the week. In the continuous monitoring situation the time interval during which the stock on hand is zero is longer than in the periodic monitoring situation, just because of lack of information. Since $\hat{P}_1$ is directly related to these time intervals during which the stock on hand is zero, we find that $\hat{P}_1$ depends on the monitoring policy used.

Note that we distinguish between the review policy, which defines the points in time where reordering is allowed, and the monitoring policy, which defines the points in time at which the stock level is registered by the administration department, say.

In view of the above we need a more detailed description of the demand process to distinguish between the periodic monitoring case (the discrete time case) and the perpetual monitoring case.

<u>Demand process for the discrete time case</u>

Define T by

T := the time between two consecutive moments in time at which the net stock is registered

and define $\{D_n\}$ by

D_n := the demand during $(nT,(n+1)T]$, $n \geq 0$.

We assume that $\{D_n\}$ is a series of initial inventory demand random variables with pdf $F_D(.)$.

We assume that $R=rT$, $r\geq 1$.

Demand process for the perpetual monitoring case

Let $\{A_n\}$ and $\{D_n\}$ be defined as

A_1 := the time at which the first customer arrives after time 0.

A_n := the time between the arrival of the $(n-1)^{th}$ and n^{th} customer, $n\geq 2$.

D_n := the demand of the n^{th} customer $n\geq 1$.

We assume that $\{A_n\}$ and $\{D_n\}$ are independent and $\{A_n, n\geq 2\}$ and $\{D_n\}$ are series of initial inventory demand random variables.

These definitions will hold throughout the remainder of this monograph.

Now let us return to the derivation of an expression for the $\hat{P}_1$-measure, the fill rate. Clearly we must distinguish between the perpetual monitoring case and the discrete time case. We first discuss the latter case.

Discrete time case

Define the random variable $T^+(S)$ by

$T^+(S)$:= time the net stock is positive during the replenishment cycle $(L_0, R+L_1]$.

Based on results from renewal reward theory we claim that

$$\hat{P}_1(S) = \frac{E[T^+(S)]}{R},$$

where $\hat{P}_1(.)$ denotes the fill rate. Let us derive an expression for $E[T^+(S)]$.

First of all we assumed that the lead times $\{L_k\}$ only take values in $\{nT \mid n\varepsilon\mathbb{N}\}$. Hence at time L_0 and $R+L_1$ a customer arrived. This implies that in the discrete time case we can apply the expression for $E[T^+(x,t)]$ given in section (2.3). Recall that

$T^+(x,t)$:= the time the inventory is positive during $(0,t]$, given an initial inventory position $x\geq 0$.

Applying standard probabilistic arguments we find

$$E[T^+(S)] = \int_0^\infty \int_0^S E[T^+(S-x,t)]\, dF_{D(0,L_0]\mid R+L_1-L_0=t}(x)\ dF_{R+L_1-L_0}(t)$$

From equation (2.51) we find

$$E[T^{+}(S)] = \int_0^{\infty}\int_0^{S} T \left\{ M(S-x) - \int_0^{S-x} M(S-x-y)\, dF_{D(0,t]}(y) \right\} dF_{D(0,L_0]|R+L_1-L_0=t}(x) \; dF_{R+L_1-L_0}(t)$$

Here

$$M(x) = \sum_{n=0}^{\infty} F_D^{n*}(x)$$

The above equation can be rewritten after some algebra into

$$E[T^{+}(S)] = T \left\{ \int_0^{S} M(S-x)\, dF_{D(0,L_0]}(x) - \int_0^{S} M(S-x)\, dF_{D(0,R+L_1]}(x) \right\}$$

Let us rewrite $D(0,R+L_1]$ as

$D(0,R+L_1] = D(0,R] + D(R,R+L_1]$

Since R=rT we have that

$P\{D(0,R] \leq x\} = F_D^{r*}(x)$

Hence

$$E(T^{+}(S)] = T \left\{ \int_0^{S} M(S-x)\, dF_{D(0,L_0]}(x) - \int_0^{S}\int_0^{S-x} M(S-x-y)\, dF_D^{r*}(y) \; dF_{D(R,R+L_1]}(x) \right\}$$

After applying the equation

M*F(x) = M(x)-1

r times we find

$$E[T^+(S)] = T\left\{\int_0^S M(S-x)\,dF_{D(0,L_0]}(x) - \int_0^S M(S-x)\,dF_{D(R,R+L_1]}(x) + \sum_{n=0}^{r-1}\int_0^S F^{n*}(S-x)\,dF_{D(R,R+L_1]}(x)\right\}$$

$$= T\sum_{n=0}^{r-1}\int_0^S F^{n*}(S-x)\,dF_{D(R,R+L_1]}(x)$$

$$= T\sum_{n=0}^{r-1} F_D^{n*} * F_{D(R,R+L_1]}(S)$$

Since $F_D^{n*}(x) = P\{D(R\text{-}nT,R]\leq x\}$ we finally find

$$E[T^+(S)] = T\sum_{n=0}^{r-1} P\{D(R-nT,R+L_1]\leq S\}$$

$$= T\sum_{n=1}^{r} P\{D(nT,rT+L_1]\leq S\},$$

where we used the fact that R=rT. Then we have the following simple expression for $\hat{P}_1(S)$,

$$\hat{P}_1(S) = \frac{1}{r}\sum_{n=1}^{r} P\{D(nT,rT+L_1]\leq S\}, \tag{3.17}$$

which indeed differs considerably from the expression for P_2 given by (3.10).

Again it is helpful to apply the PDF-method to be able to apply the standard routines, when determining the order-up-to-level S, such that $\hat{P}_1(S)$ equals α, say.

Define

$\gamma(x) = \hat{P}_1(x)$

and X_γ is a random variable with pdf $\gamma(.)$. Then we find

$$E[X_\gamma] = \frac{1}{r}\sum_{n=1}^{r} E[D[nT, rT+L_1]]$$

$$E[X_\gamma^2] = \frac{1}{r}\sum_{n=1}^{r} E[D^2[nT, rT+L_1]]$$

Under the assumptions made when deriving (3.5)-(3.8) we obtain

$$E[X_\gamma] = (E[L]+½(r-1))E[D_T] \tag{3.18}$$

$$E[X_\gamma^2] = (E[L]+½(r-1))\sigma^2(D_T) + (\sigma^2(L)+E^2[L]+(r-1)E[L]+\frac{1}{6}(r-1)(2r-1))E^2[D] \tag{3.19}$$

Next we fit a gamma distribution $\hat{\gamma}(.)$ to the first two moments of X_γ and use the approximate inversion scheme from section 2.4 to solve for S in

$S \approx \hat{\gamma}^{-1}(x)$

<u>Perpetual monitoring case</u>

As in the discrete time case we have

$$\hat{P}_1(S) = \frac{E[T^+(S)]}{R},$$

where $\hat{P}_1(.)$ and $T^+(.)$ are defined above. To obtain an expression for $E[T^+(S)]$ we assume that the APIT-assumption applies to both L_0 and $R+L_1$. Hence the beginning and the end of the replenishment cycle are considered to be arbitrary points in time, so that the time until the first customer arrives after L_0 and $R+L_1$ is distributed according to $\tilde{A}_1$, with

$$P\{\tilde{A}_1 \leq t\} = \frac{1}{E[A]} \int_0^t (1-F_A(y))\,dy$$

As in the discrete time case we find an expression for $E[T^+(S)]$ via the random variable $T^+(x,t)$,

$$E[T^+(S)] = \int_0^\infty \int_0^S E[T^+(S-x,t)]\,dF_{D(0,L_0]|R+L_1-L_0=t}(x) \quad dF_{R+L_1-L_0}(t)$$

Since we apply the APIT-assumption to L_0 and $R+L_1$, we find from (2.53)

$$\begin{aligned} E(T^+(S)] = {} & \frac{(c_A^2-1)}{2} E[A] \;\{F_{D(0,L_0]}(S) - F_{D(0,R+L_1]}(S)\} \\ & + E[A] \left(\int_0^S M(S-x)\,dF_{D(0,L_0]}(x) \right. \\ & \left. - \int_0^S M(S-x)\,dF_{D(0,R+L_1]}(x) \right), \end{aligned} \tag{3.20}$$

with

$$M(x) = \sum_{n=0}^{\infty} F_D^{n*}(x)$$

In general the above expression for $E[T^+(S)]$ is intractable. To obtain a practically applicable expression for $\hat{P}_1(S)$ we apply the PDF method again. Note that

$$\lim_{S\to\infty} \hat{P}_1(S) = 1 \ , \ \hat{P}_1(0) = 0$$

Hence $\hat{P}_1(.)$ is a pdf of some random variable X_γ. We define $\gamma(.)$ by

$\gamma(x) = \hat{P}_1(x) \ , \ x \geq 0$

As before we have

$$E[X_\gamma] = \int_0^\infty (1-\gamma(x))\,dx$$

$$E[X_\gamma^2] = 2\int_0^\infty x(1-\gamma(x))\,dx$$

Let us first derive an expression for $E[X_\gamma]$. By definition of $\gamma(.)$ we have

$$E[X_\gamma] = \int_0^\infty \left(1 - \frac{E[T^+(x)]}{R} \right) dx$$

After considerable algebra using (2.34) and (2.35) we obtain

$$E[X_\gamma] = \frac{E[A]}{R}\,\frac{1}{2E[D]}\,(E[D^2(0,R+L_1]] - E[D^2(0,L_0]]) - \frac{(c_A^2 + c_D^2)}{2}\,E[D] \qquad (3.21)$$

$$E[X_Y^2] = \frac{E[A]}{R} \frac{1}{3E[D]} (E[D^3(0,R+L_1]] - E[D^3(0,L_0]])$$
$$- \frac{(c_A^2 + c_D^2)}{2} \frac{E[A]}{R} (E[D^2(0,R+L_1]] - E[D^2(0,L_0]]) \tag{3.22}$$
$$+ \frac{1}{6} (1+c_D^2)(1+3c_D^2)E^2[D]$$

Note that only in case E[D], c_A^2 and c_D^2 small, that (3.21) and (3.22) are close to (3.11) and (3.12), respectively. In that case $\hat{P}_1(.)$ and $P_2(.)$ are almost identical. In general $\hat{P}_1(.)$ and $P_2(.)$ differ, as follows from (3.10) and (3.20).

3.2. Physical stock

As with the $\hat{P}_1$-measure the value of the mean physical stock depends on the monitoring or registration policy. Therefore we must again distinguish between the discrete time case and the perpetual monitoring case.

3.2.1. The discrete time case

Throughout this subsection we again assume that the inventory management system monitors stock at the beginning of time intervals [nT,(n+1)T], $n \varepsilon \mathbb{N}$. We assume that $R = rT$, $r \varepsilon \mathbb{N}$. We assume that demand during time interval (nT,(n+1)T) is distributed according to F_D, independent of n. Demands during disjunct time intervals are mutually independent. Let D be the generic random variable associated with F_D. As before it can be shown that the long-run mean stock can be computed from the mean stock during an arbitrary replenishment cycle.

Note that

$X^+(t)$ = physical stock at time t, $t \geq 0$.

We want to have an expression for

$E[X^+]$:= long-run mean physical stock.

We implicitly assume that both L_0 and L_1 are an integral number times T. It is easy to see that

$$E[X^+] = \frac{E\left[\int_{L_0}^{R+L_1} X^+(t)\,dt\right]}{R}$$

To obtain an expression for $E[\int_{L_0}^{R+L_1} X^+(t)dt]$ we can apply the general result from section 2.3., in which an expression is given for the expected surface below the graph, depicting the depletion of a fixed amount of stock by a compound renewal demand process during a fixed time interval. Recall the definition of H(x,t),

$$H(x,t) \;:=\; \left[\; \int_0^t X^+(y)\,dy \;\middle|\; X(0)=x \;\right]$$

Since $X(L_0) = S\text{-}D[0,L_0]$ we have

$$E[X^+(S)] \;=\; \frac{1}{R}\left\{ \int_0^\infty \int_0^S E[H(S-x,t)]\,dF_{D[0,L_0]|R+L_1-L_0=t}(x)\;\;dF_{R+L_1-L_0}(t) \right\} \tag{3.23}$$

where

$$F_{D(t,s)}(x) \;=\; P\{D(t,s)\le t\}$$

$$F_{R+L_1-L_0}(t) \;=\; P\{R+L_1-L_0\le t\}$$

Note that in (3.23) we explicitly show the dependence of $E[X^+]$ on S.

From equation (2.56) we know that

$$E[H(x,t)] = \int_0^x (x-y)\ dM(y) - \int_0^x \int_0^{x-y} (x-y-z)\ dM(z)\ dF_{D(0,t)}(y) \tag{3.24}$$

where

M(x) := renewal function associated with the demand during time period [nT,(n+1)T].

After quite some algebra we obtain from (3.23) and (3.24)

$$\begin{aligned} E[X^+(S)] &= \frac{1}{r} \sum_{n=1}^{r} \int_0^S (S-y)\, dF_{D[nT,\, rT+L_1]}(y) \\ &= \frac{1}{r} \sum_{n=1}^{r} E[(S-D[nT,rT+L_1])^+] \end{aligned} \tag{3.25}$$

Hence $E[X^+(S)]$ involves similar expressions as needed to compute the P_2-value associated with a given order-up-to-level S.

To obtain some more intuition for equation (3.25) let us consider the case of constant lead time L. In that case the replenishment cycle consist of exactly r consecutive time intervals of length T. At the beginning of time interval L+nT the net stock equals S-D[0,L+nT]. Hence the physical stock at time L+nT equals $(S-D[0,L+nT])^+$. The mean physical stock at an arbitrary time kT is found by averaging the mean physical stock at times L+nT, $0 \leq n \leq r-1$. Hence

$$E[X^+(S)] = \frac{1}{r} \sum_{n=0}^{r-1} E[(S-D([0,L+nT])^+] \tag{3.26}$$

Let us substitute m=r-n in (3.25) and $L_1 \equiv L$. Then we obtain

$$E[X^+(S)] = \frac{1}{r} \sum_{n=0}^{r-1} E[S-D((r-n)T, rT+L))^+] \qquad (3.27)$$

Now note that the interval [(r-n)T,rT+L] has a length nT+L. Therefore

$$D((r-n)T, rT+L]) \;\; d \;\; D[0, L+nT]$$

Hence (3.26) and (3.27) are identical and therefore (3.25) may be interpreted as averaging over the mean stock at N consecutive points in time.

For practical purposes expression (3.25) may involve too much computational effort. To circumvent this problem we rewrite (3.25) as follows

$$E[X^+(S)] = \frac{1}{r} \sum_{n=1}^{r} \left\{ \int_0^\infty (S-y)\, dF_{D[nt, rT+L_1]}(y) + \int_S^\infty (y-S)\, dF_{D[nT, rT+L_1]}(y) \right\}$$

$$= S - \frac{1}{r} \sum_{n=1}^{r} E[D[nT, rT+L_1] + \frac{1}{r} \sum_{n=1}^{r} \int_S^\infty (y-S)\, dF_{D[nT, rT+L_1]}(y)$$

Since $E[D[nT,rT+L_1]]=((r-n)T+E[L_1])E[D]$

$$E[X^+(S)] = S - E[L_1]E[D] - ½(r-1)E[D_T] + \frac{1}{r} \sum_{n=1}^{r} \int_S^\infty (y-S)\, dF_{D(nT, rT+L_1)}(y) \qquad (3.21)$$

If S is large then

$$\frac{1}{r} \sum_{n=1}^{r} \int_S^\infty (y-s)\, dF_{D(nT, rT+l_1]}(y) \approx 0$$

and

$$E[X^+(S)] \approx S-E[L_1]E[D] - \tfrac{1}{2}(r-1)E[D] \tag{3.28}$$

Note the difference between (3.28) and (7.33) in Silver and Peterson [1985],

$$E[X^+] \approx S-E[L_1]E[D] - \tfrac{1}{2}\, rE[D] \qquad (7.33\ \text{S+P})$$

due to the fact that we use a step-function approximation instead of linear interpolation.

If T↓0 (3.28) and (7.33 S+P) lead to the same result.

Let us return to (3.27). In case S is not very large (3.28) and (7.33 S+P) will yield poor results. From (3.27) we can derive a simple expression applying the PDF-method again.

Define ζ(.) by

$$\zeta(x) := \frac{1}{r}\sum_{n=1}^{r}\int_{x}^{\infty}(y-x)\,dF_{D(nT,\,rT+L_1]}(x)$$

Hence

$$E[X^+] = S-E[L_1]E[D] - \tfrac{1}{2}(r-1)E[D] + \zeta(S) \tag{3.29}$$

We have that ζ(.) is monotone decreasing and

$\zeta(0) = \tfrac{1}{2}(r-1)E[D] + E[L_1]E[D]$

$\zeta(\infty) = 0$

Define γ(.) by

$$\gamma(x) = 1 - \frac{\zeta(x)}{\zeta(0)}$$

Then $\gamma(.)$ is a probability distribution function with associated random variable X_γ. After some algebra we obtain

$$E[X_\gamma] = \frac{1}{\zeta(0)} \frac{1}{2r} \sum_{n=1}^{r} \{(E[L]+r-n)\sigma^2(D)+\sigma^2(L)E^2[D] + (E[L]+r-n)^2 E^2[D]\}$$

$$E[X_\gamma^2] = \frac{1}{\zeta(0)} \frac{1}{3r} \sum_{n=1}^{r} (1-c_{r-n+L}^2)(1+2c_{r-n+L}^2)E^3[D](r-n+E[L])^3$$

where

$$c_{r-n+L}^2 = \frac{\sigma^2(D[nT,rT+L_1])}{E^2(D[nT,rT+L_1])}$$

and

$$\sigma^2(D[nT,rT+L_1]) = (E[L]+r-n)\sigma^2(D)+\sigma^2(L)E^2[D]$$

The expression for $E[X_\gamma]$ might be further elaborated, the expression for $E[X_\gamma^2]$ has to be evaluated term by term. In practical cases $r \leq 31$, since in the worst case T is one day and R is a month.

Next we fit a Gamma distribution $\hat{\gamma}(.)$ to $E[X_\gamma]$ and $E[X_\gamma^2]$ along the lines sketched in section (2.5.). Then we end up with the following result.

$$E[X^+(S)] \simeq S - \zeta(0)\hat{\gamma}(S) \tag{3.30}$$

An improvement of the simple equation (3.28) is obtained from the following reasoning. Assume linearly decreasing demand during $[L_0,R+L_1]$. Since the physical stock equals $(S-D[0,L_0])^+$ at time L_0 and $(S-D[0,R+L_1])^+$ at time $R+L_1-0$ we may approximate $E[X^+]$ by

$$E[X^+(S)] \approx ½(E[(S-D[0,L_0])^+]+E[(S-D(0,R+L_1))^+]) \tag{3.31}$$

$$= S-E[L]E[D_T]-\tfrac{1}{2}rE[D]+\tfrac{1}{2}(E[(D[0,L]-S)^+]+E[(D([0,R+L_1]-S)^+])$$

Note that the last two terms in (3.31) have already been computed, when determining the P_2-value given the order-up-to-level S.

The equations (3.25), (3.28), (3.30) and (3.31) all provide approximations for $E[X^+]$.

3.2.2. The perpetual monitoring case

In this subsection we assume that the inventory management systems registers each depletion of stock. We apply the same definitions associated with the demand process, when deriving an expression for the fill rate $\acute{P}_1$.

To obtain an expression for $E[X^+]$ we again employ the expression for H(x,t) for the case of a compound renewal demand process as derived in section 2.3. Remember

H(x,t) := surface between the physical stock $X^+(t)$ and the time-axis given that $X^+(0)=x\geq 0$.

For the case of a compound renewal demand process we have an approximation for E[H(x,t)],

$$E[H(x,t)] \simeq (E[A_1]-E[A])\left(x - \int_0^x (x-y)\ dF_{D(0,t]}(y) \right) \tag{3.32}$$

$$+ E[A]\left(\int_0^x (x-y)\ dM(y) - \int_0^x \int_0^{x-y} (x-y-z)\ dM(z)\ dF_{D(0,t]}(y) \right),$$

with M(.) the renewal function associated with $F_D(.)$. Equation (3.32) is exact for the case of a compound Poisson demand process.

Applying the definition of H(x,t) we obtain an expression for $E[X(.)^+]$, the mean physical stock.

$$E[X^+(S)] \approx \frac{1}{R} \int_0^\infty \int_0^S E[H(S-y,t)] \; dF_{D(0,L_0]}(y) \; dF_{R+L_1-L_0}(t) \tag{3.33}$$

Substitution of (3.32) and (3.33) and some algebra yields

$$\begin{aligned} E[X^+(S)] \simeq \; & \frac{1}{R} \; (E[A_1]-E[A]) \left(\int_0^S (S-y) \; dF_{D(0,L_0]}(y) \; - \int_0^S (S-y)\, dF_{D(0,R+L_1]}(y) \right) \\ & + \frac{E[A]}{R} \left(\int_0^S \int_0^{S-y} (S-y-z) \; dM(z) \; dF_{D(0,L_0]}(y) \right. \\ & \left. - \int_0^S \int_0^{S-y} (S-y-z) \; dM(z) \; dF_{D(0,R+L_1]}(y) \right) \end{aligned} \tag{3.34}$$

Now we insert the asymptotic expansions for the two-fold integrals on the right hand side of (3.34), which are given by theorem (2.11),

$$\lim_{x \to \infty} \int_0^x \int_0^{x-y} (x-y-z) \; dM(z) \; dF_{D(0,L_0]}(y) - (a_2 x^2 + a_1 x + a_0) \; = \; 0$$

$$\lim_{x \to \infty} \int_0^x \int_0^{x-y} (x-y-z) \; dM(z) \; dF_{D(0,R+L_1]}(y) \; - \; (b_2 x^2 + b_1 x + b_0) \; = \; 0$$

$$a_2 = \frac{1}{2E[D]} \quad a_1 = \left(\frac{E[D^2]}{2E^2[D]} - \frac{E[D(0,L_0]]}{E[D]} \right)$$

$$a_0 = \frac{E[D^2(0,L_0]]}{2E[D]} - \frac{E[D^3]}{6E^2[D]} + \frac{E^2[D^2]}{4E^3[D]} - \frac{E[D(0,L_0]]E[D^2]}{2E^2[D]} \tag{3.35}$$

$$b_2 = a_2 \quad b_1 = a_1 - \frac{E[D(0,R+L_1]]}{E[D]} + \frac{E[D(0,L_0]]}{E[D]}$$

$$b_0 = \frac{E[D^2(0,R+L_1]]}{2E[D]} - \frac{E[D^2(0,L_0]]}{2E[D]} - (E[D(0,R+L_1]] - E[D(0,L_0]]) \frac{E[D^2]}{2E^2[D]} + a_0 \tag{3.36}$$

This yields

$$\begin{aligned} E[X^+(S)] = {} & \frac{1}{R} \left(\frac{c_A^2-1)}{2} E[A] \left(\frac{E[D]R}{E[A]} + \int_S^\infty (y-S)\, dF_{D(0,L_0]}(y) \right.\right. \\ & \left. - \int_S^\infty (y-S)\, dF_{D(0,R+L_1]}(y) \right) \\ & + \frac{E[A]}{R} \left(\int_0^S \int_0^{S-y} (S-y-t)\, dM(z)\, dF_{D(0,L_0]}(y) - (a_2S^2+a_1S+a_0) \right. \\ & \left. - \int_0^S \int_0^{s-y} (S-y-t)\, dM(z)\, dF_{D(0,L_0]}(y) + b_2S^2+b_1S+b_0 \right) \\ & + \frac{E[A]}{R} ((a_1-b_1)S + (a_0-b_0)) \end{aligned} \tag{3.37}$$

Equation (3.37) can be rewritten into the following approximation

$$E[X^{+}(S)] \simeq S - \left(E[L] + \frac{R}{2} \right) \frac{E[D]}{E[A]}$$

$$+ \frac{E[A]}{R} \frac{(c_A^2 - 1)}{2} \left(\int_S^{\infty} (y-S) \ dF_{D(0,L_0]}(y) - \int_S^{\infty} (y-S) \ dF_{D(0,R+L_1]}(y) \right) \tag{3.38}$$

$$+ \frac{E[A]}{R} \left(\int_0^S \int_0^{S-y} (S-y-z) \ dM(z) \ dF_{D(0,L_0]}(y) - (a_2 S^2 + a_1 S + a_0) - \int_0^S \int_0^{S-y} (S-y-z) \ dM(z) \ dF_{D(0,R+L_1]}(y) + b_2 S^2 + b_1 S + b_0 \right)$$

Define the function E[B(.)] by

$$E[B(S)] = E[X^{+}(S)] - \left(S - \left(E[L] + \frac{R}{2} \right) \frac{E[D]}{E[A]} \right) \tag{3.39}$$

It follows from the fact that $E[X^{+}(0)] = 0$ that

$$E[B(0)] = \left(E[L] + \frac{R}{2} \right) \frac{E[D}{E[A]}$$

It follows from (3.37) and (3.38) that

$$\lim_{S \to \infty} E[B(S)] = 0$$

It will be shown in section 3.3. that E[B(.)] is monotone decreasing in S. Therefore we can apply the PDF-method to E[B(S)]/E[B(0)]. Define γ(.) by

$$\gamma(x) = 1 - \frac{E[B(x)]}{E[B(0)]}, \quad x \ge 0 \tag{3.40}$$

and let X_γ be the random variable with pdf $\gamma(.)$. Then it follows from Theorem (2.12) and (2.13) and after considerable algebra that

$$\begin{aligned} E[X_\gamma] = \frac{1}{E[B(0)]} \Bigg\{ & \frac{(c_A^2-1)}{4} \frac{E[A]}{R} (E[D^2(0,L_0]] - E[D^2(0,R+L_1]]) \\ & + \frac{E[A]}{R} \Bigg((E[D^3(0,R+L_1]] - \frac{E[D^3(0,L_0]])}{6E[D]} \\ & + (E[D^2(0,L_0]] - E[D^2(0,R+L_1]]) \frac{E[D^2]}{4E^2[D]} \\ & + (E[D(0,R+L_1]] - E[D(0,L_0]]) \left(\frac{E^2(D^2]}{4E^3[D]} - \frac{E[D^3]}{6E^2[D]} \right) \Bigg) \Bigg\} \end{aligned} \tag{3.41}$$

$$\begin{aligned} E[X_\gamma^2] = \frac{1}{E[B(0)]} \Bigg\{ & \frac{(c_A^2-1)}{6} \frac{E[A]}{R} (E[D^3(0,L_0]] - E[D^3(0,R+L_1]]) \\ & + \frac{E[A]}{R} \Bigg((E[D^4(0,R+L_1]] - \frac{E[D^4(0,L_0]])}{12E[D]} \\ & - (E[D^3(0,R+L_1]] - E[D^3(0,L_0]]) \frac{E[D^2]}{6E[D]} \\ & + (E[D^2(0,R+L_1]] - E[D^2(0,L_0]]) \left(\frac{E^2(D^2]}{4E^3[D]} - \frac{E[D^3]}{8E^2[D]} \right) \\ & + (E[D(0,R+L_1]] - E[D(0,L_0]]) \left(\frac{E[D^3E[D^3]}{3E^3[D]} - \frac{E[D^4]}{12E^2[D]} - \frac{E^3[D^2]}{4E^4[D]} \right) \Bigg) \Bigg\} \end{aligned} \tag{3.42}$$

The first two moments of $D(0,L_0]$ and $D(0,R+L_1]$ are given by

$$E[D(0,L_0]] \simeq \frac{E[L]}{E[A]} E[D] \tag{3.43}$$

$$E[D^2(0,L_0)] \simeq \left(\frac{E[L^2]}{E^2[A]} + \frac{E[L]}{E[A]} (c_A^2+c_D^2) + \frac{1-c_A^4}{6} \right) E^2[D] \tag{3.44}$$

$$E[D(0,R+L_1)] \simeq \frac{(R+E[L])}{E[A]} E[D] \tag{3.45}$$

$$E[D^2(0,R+L_1)] \simeq \left(\frac{E[L^2] + 2RE[L] + R^2}{E^2[A]} + \frac{(R+E[L])}{E[A]} (c_A^2+c_D^2) + \frac{1-c_A^4}{6} \right) E^2[D] \tag{3.46}$$

Define γ(.) as the gamma distribution with its first two moments equal to $E[X_\gamma]$ and $E[X_\gamma^2]$. Then we claim that

$$\hat{\gamma}(x) \simeq \gamma(x)$$

and therefore

$$E[X^+(S)] \simeq S - \left(E[L] + \frac{R}{2} \right) \frac{E[D]}{E[A]} \hat{\gamma}(S) \tag{3.47}$$

Since

$$\lim_{S\to\infty} \hat{\gamma}(S) = 1$$

we find for S large

$$E[X^+(S)] \simeq S - \left(E[L] + \frac{R}{2} \right) \frac{E[D]}{E[A]} \tag{3.48}$$

3.3. Average backlog

Another important performance measure often considered is the long-run average backlog. Define the P_3-measure by

P_3 := the long-run average shortage at an arbitrary point in time.

Note that the P_3-measure is not dimensionless. To get a dimensionless measure one may divide $P_3(S)$ by the average demand per unit time, yet this yields a measure, which does not necessarily takes values between 0 and 1.

We want to have an expression for the long-run average backlog. We first note that this is equivalent to the average backlog during a replenishment cycle.

Discrete time case

In section 2.3. we derived an expression for

B(x,t) := the cumulative backlog in [0,t], when the net stock at time 0 equals x.

It was found for the discrete time case (($\sigma^2(A)=0$) in chapter 2) that

$$E[B(x,t)] = \begin{cases} E[H(x,t)] + \tfrac{1}{2}t(t-1)E[D] - xt & x>0 \\ \tfrac{1}{2}t(t-1)E[D] - xt & x\leq 0 \end{cases} \tag{3.49}$$

It is easy to see that

$$P_3(S) = \frac{1}{\sigma}\int_0^\infty\int_0^\infty\int_0^\infty E[B(S-y,R+t-s)]\,dF_{D[0,s]}(y)\,dF_{L_1|L_0=s}(t)\,dF_{L_0}(s)$$

Here we take into account that in general L_1 depends on L_0, since orders cannot overtake each other.

Next we substitute (3.49) into the above equation. This yields

$$
\begin{aligned}
P_3(S) \;=\; & \frac{1}{r}\int_0^{\infty}\int_0^{\infty}\int_0^{S} E[H(S-y;R+t-s)]\,dF_{D[0,s]}(y)\,dF_{L_1|L_0=s}(t)\,dF_{L_0}(s) \\
& + \frac{E[D]}{2r}\int_0^{\infty}\int_0^{\infty} (R+t-s)(R-1+t-s)\,dF_{L_1|L_0=s}(t)\,dF_{L_0}(s) \\
& - \frac{1}{r}\int_0^{\infty}\int_0^{\infty}\int_0^{\infty} (S-y)(R+t-s)\,dF_{D[0,s]}(y)\,dF_{L_1|L_0=s}(t)\,dF_{L_0}(s)
\end{aligned}
$$

After some intricate algebra we obtain

$$P_3(S) \;=\; E[X^+(S)]-S+E[D]\left(E[L]+\frac{(r-1)}{2}\right) \tag{3.50}$$

Hence once we know the physical stock it is only a matter of simple algebra to obtain the average backlog. Since it is intuitively clear that $P_3(S)\rightarrow 0$ if $S\rightarrow\infty$, we have an alternative proof of the fact that

$$\lim_{S\rightarrow\infty} S-E[X^+(S)] \;=\; \left(\; E[L]+\frac{(r-1)}{2}\;\right)\; E[D] \qquad \text{(cf. 3.22)}$$

We emphasize that, though (3.50) is intuitively appealing, the result in itself is not trivial. An alternative derivation of (3.54) is as follows. By definition we have

$$E[Y(S)] \;=\; E[X^+(S)]+E[0(S)]-P_3(S) \tag{3.51}$$

where

E[0(S)] := the expected amount on order.

E[Y(S)] := the expected inventory position.

Let us first derive an expression for E[Y(S)]. By the definition of $H_c(t)$ in section 2.3. we find

$$E[Y(S)] = S - \frac{1}{R} H_c(R)$$

This yields

$$E[Y(S)] = S-½(r-1)E[D] \tag{3.52}$$

Next we focus on E[0(S)]. Suppose that for each batch ordered at the supplier we pay the supplier $1 per item in the batch per time unit on order. Since the average batch size equals rE[D] and each batch is on order for on average E[L] time units, each batch pays $rE[D]E[L]. Since each r^{th} time unit a batch is ordered at the supplier, we pay $E[D]E[L] per time unit. On the other hand the supplier receives at a particular point in time $1 for each item that is on order. Hence the supplier receives $E[0] per time unit. Then it follows that

$$E[0] = E[L]E[D] \tag{3.53}$$

Substituting (3.52) and (3.53) into (3.51) and rearranging terms yields (3.50).

<u>The compound renewal demand case</u>

In the compound renewal demand case we again find an expression for $P_3(S)$ by relating it to the physical stock. Towards this end we again apply the above arguments starting from the equation (3.51), i.e.

$$P_3(S) = E[X^+(S)]+E[0(S)]-E[Y(S)]$$

Then we obtain from the cost arguments

$$E[0] = E[D]\frac{E[L]}{E[A]}$$

An approximate expression for E[Y(S)] is derived from equation (2.68) and the fact that $E[Y(S)] = S - \frac{1}{R}H_c(R)$, yielding

$$E[Y(S)] \approx S - \tfrac{1}{2}R\frac{E[D]}{E[A]}$$

Thus we find

$$P_3(S) \approx E[X^+(S)] - \left(S - (E[L] + \frac{R}{2})\frac{E[D]}{E[A]}\right) \tag{3.54}$$

This yields the asymptotic result that

$$\lim_{S\to\infty}(S - E[X^+]) \approx \left(E[L] + \frac{R}{2} \right)\frac{E[D]}{E[A]}$$

This concludes our discussion of the average backlog. Most important result of this section is that the P_3-measure can be related to the physical stock.

3.4. Conclusions concerning the stationary model

In sections 3.1. to 3.3. we discussed stationary (R,S)-models. Either we assumed that demand occurred at discrete equidistant points in time or we assumed a compound renewal demand process. The analysis needed to obtain expressions for the most important performance measures is sometimes cumbersome, yet the expressions themselves turn out to be such that simple routines can be applied to do the calculations.

An important aspect of the results is, that their final form is more or less standardized. To be precise, with each performance measure we associated a random variable X_γ of which we determined the first two moments. Application of the PDF-method then yields either S as a function of the performance measure or the performance measure as a function of S. This unification enables to apply standard procedures for the PDF-method. Only the first two moments of X_γ differ for each performance measure. In the next chapters we show that this holds not only within the framework of a particular inventory model, but also across <u>all</u> basic inventory models. The benefit of this is clear.

The computational procedures are so simple and fast that they can be applied in inventory management systems dealing with a large number of items. The complexity involved is similar to the complexity involved in the routines of IBM-Impact [1968], which is widely used in practice. However, the routines are more robust and more transparent to those who have some knowledge of inventory management modes.

In chapter 7 we employ the results obtained in the preceding chapters to compare the (R,S)-model with other inventory management models in terms of costs. In particular we consider linear holding costs and fixed order costs. The holdings costs are derived from the average physical stock, the order costs depend on the review period R. Shortage costs are included implicitly by the condition to achieve a target service level. In principle shortage costs might be obtained from the expression involved in the derivation of the P_2- and P_3-measure, yet in practice it is often hard to obtain unit shortage costs. Therefore we prefer a service level approach. Once the cost associated with the (R,S)-policy satisfying the service level constraint are known, we might use the expression for the P_2- and P_3-measures to obtain the implicit shortage costs assumed. This can be done by taking the shortage cost per unit (per unit time, resp.) as a variable and determine the value of this variable for which the (R,S)-policy found is cost-optimal.

Finally a word on the mathematical rigour. Since Hadley and Whitin [1963] there has hardly been a mathematical rigorous treatment of the basic models, assuming Hadley and Whitin did the job. Hopefully, it is clear that the preceding sections provided substantial generalizations of the results of Hadley and Within. In particular we relaxed their assumptions, respectively stating, that the physical stock is positive immediately after an arrival and that lead times are independent random variables. The latter is not necessary, whereas the first assumption is evidently not realistic for (R,S)-models. The PDF-method copes with the problem of more complicated expressions, when relaxing the first assumption. The results obtained in Hadley and Whitin for the physical stock are claimed to be good approximations, yet computational results show that this is not true for demand processes of today. These observations apply to all models discussed in this monograph.

3.5. Dynamic demand

When applying inventory models in practical situations one of the first assumptions that has to be discarded is the assumption of stationary demand. In practice demand shows trend, seasonality, incidents, and other patterns that may well be explained and are time-dependent. As will be shown in subsequent chapters this causes considerable problems, when we want to derive inventory management policies, which take into account these phenomena, yielding e.g. the required customer service or the required physical stock. However, if we assume that (R,S)-policies are applied, then things do not further complicate at all. This follows from the expressions derived in the preceding chapters.

In the sequel we focus on the derivation of order-up-to-levels in a dynamic environment, such that a target P_2-service level is achieved.

As before we define D[t,s] as

D[t,s] := demand during the time interval [t,s].

At each review moment kR we have to take only one decision: How much to order. We relate this decision to the P_2-service level as follows.

$P_2(k)$:= the fraction of demand satisfied directly from stock on hand during the replenishment cycle $[kR+L_k,(k+1)R+L_{k+1}]$,

where L_k is the lead time of the delivery ordered at time kR.

From the analysis in section 3.1. we find that the P_2-service level associated with an order-up-to-level S_0 assumed at the review at time 0 can be written as

$$P_2(S_0) = 1 - \frac{E[(D[0,R+L_1]-S_0)^+] - E[(D[0,L_0]-S_0)^+]}{E[D(L_0,R+L_1)]} \tag{3.55}$$

Since we have dynamic demand we cannot apply (3.5)-(3.8). We have to forecast demand during the time intervals $(0,L_0)$ and $(L_0,R+L_1)$. There are several options here available:

a. Expert estimates
b. Time series analysis
c. A combination of a. and b.

Expert estimates

The dynamics of demand stem from a lot of sources. Often it is hard to distinguish deliberate actions like advertising campaigns and discounts from statistical fluctuations. Typically one needs an expert opinion from salesmen or product managers to get some idea of the impact of the deliberate actions. The problem is that these experts are not used to quantify these forecasts in terms of both an expected increase or decrease and some measure of uncertainty, like a standard deviation or minimum and maximum increase. I would like to emphasize here that this is a fundamental problem. It involves cultural change to solve it. One must not expect that mathematical techniques like time series analysis can filter out future actions based on historic data. In a rapidly changing market as is the case today, one has to assess this problem again and again.

Time series analysis

Assuming that we have quantified the effect of the deliberate actions and other a-typical incidents we may well apply time series analysis to historic data to find all more or less "random" demand fluctuations. It is beyond the scope of this monograph to go into detail about forecasting based on extrapolation or intrapolation. A very nice paper on forecasting techniques, which in fact discusses both expert estimation and mathematical techniques, is Chambers et al. [1971].

Some practical observations should be discussed. First of all it appears to be relatively easy to find more or less deterministic phenomena like trend and seasonality. Hence it remains to forecast the effect of the statistical fluctuations superposed on the already known components of the forecast. It is important to note that this does not mean that a more or less stationary process remains, on which our standard results from the previous sections can be applied. Usually, the magnitude of the statistical fluctuations depend on the magnitude of the aggregate forecast obtained from expert estimates and time series analysis.

In principle these effects can be derived from e.g. Box-Jenkins models. However, it appears that the added value of these kind of sophisticated technique is marginal when comparing the performance of these techniques with simpler ones like (double) exponential smoothing, when the latter ones are applied by a professional.

We therefore conclude that one needs to combine both expert opinions and rather simple mathematical techniques. The simplicity of the mathematical technique has its price. The human component involved in these techniques almost completely determine the performance. Hence forecasting is 90% human activity.

Practical considerations

Another important observation is that we advocate a direct forecast of $D(0,R+L_1)$ and $D(0,L_0)$, instead of a forecast of these random variables based on e.g. daily or weekly demand. Typically the latter approach needs assumptions like independence and stationarity when calculating forecast errors. Though a complete mathematical model cannot be analyzed when dropping these assumptions, in practice it is as easy to obtain direct forecasts of $D(0,R+L_1)$ and $D(0,L_0)$ from historic data as weekly demand, say. It is a matter of proper data handling. By doing so we can incorporate any possible dependencies and irregularities in the demand process. The impact on existing forecast systems is immense, since they are typically forecasting demand during calendar periods. The widely-used IBM-Impact, e.g., assumes independent demand during consecutive periods.

Calculating the dynamic order-up-to-level

Let us assume that we have obtained a forecast. Then we rewrite the random variables $D(0,R+L_1)$ and $D(0,L_0)$ as follows

$$D(0,R+L_1) = D^F(0,R+L_1) + \varepsilon(0,R+L_1)$$

$$D(0,L_0) = D^F(0,L_0) + \varepsilon(0,L_0)$$

Here $D^F(0,R+L_1)$ and $D^F(0,L_0)$ are forecasts and therefore known constants. The deviation from the forecast is given by $\varepsilon(0,R+L_1)$ and $\varepsilon(0,L_0)$, which are random variables.

Due to the nature of forecasting it is often assumed that the forecast errors $\varepsilon(0,R+L_1)$ and $\varepsilon(0,L_0)$ are normally distributed. Here some comments are in order. In a lot of situations forecasting schemes are applied, which produce as an output the standard deviation or mean absolute deviation of demand itself, instead of a standard deviation or MAD of the difference between the actual outcome and the outcome of some model. In the first case one must not apply normal distribution at all. We advise to use gamma distributions. In the second case it is quite natural to apply the normal distributions provided that the standard deviation of the forecast error is not too large. Let us explain this more carefully. Define

$$c := \frac{\sigma(\varepsilon(0,R+L_1))}{D^F(0,R+L_1)}$$

Assuming an unbiased forecast, c is the coefficient of variation of $D(0,R+L_1)$. If $\varepsilon(0,R+L_1)$ is normally distributed, there is a possibility of negative demand in our model.

$$P\{D(0,R+L_1)<0\} = P\{\varepsilon(0,R+L_1)<-D^F(0,R+L_1)\}$$

After elementary calculus we find

$$P\{D(0,R+L_1)<0\} = \Phi(-\frac{1}{c})$$

Suppose we want $P\{D(0,R+L_1)<0\}<0.05$. Then we find that $c<0.5$. For values of c exceeding 0.5, our model does not fit.

A more robust approach is as follows. Again assume that $D^F(0,R+L_1)$ and $\sigma(\varepsilon(0,R+L_1))$ are known. Now assume that

$\varepsilon(0,R+L_1)+D^F(0,R+L_1)$ *is gamma distributed*.

Then $P\{D(0,R+L_1)<0\}=0$ and for small values of c the two models almost coincide because of the central limit theorem. Moreover, this approach again unifies results. We can apply the algorithms developed in section 3.1. to find the appropriate value of S_0.

The approach sketched above is not mathematically rigorous. Usually the value of $\sigma(\varepsilon(0,R+L_1))$ is derived from some model assuming normally distributed forecast errors. Yet the robustness of the suggested model, as well as the similarities in case the normal distribution provides a good fit compensate for this.

Step 1 Determine $D^F(0,R+L_1)$, $D^F(0,L_0)$ using a combination of expert estimates and mathematical techniques.

Step 2 Use mathematical techniques to determine $\sigma(\varepsilon(0,R+L_1))$ and $\sigma(\varepsilon(0,L_0))$.

Step 3 Compute S_0 from (3.58) using the PDF-method, assuming gamma distributions.

For the stationary demand case we derived expression for the mean physical stock. In the dynamic demand case this hardly makes sense due to the dynamics. In that case it is preferred to use the expected net stock immediately before replenishment moments. These are easily obtained from the analysis.

Expected net stock immediately before $R+L_1$

$$= S_0 - D^F(0, R+L_1)$$

If we need an estimate of the average stock during the replenishment cycle (L_0, R+L_1) we suggest to use:

Expected mean physical stock during (L_0,R+L_1)

$$\simeq S_0 - D^F(0, L_0) - \frac{(D^F(0, R+L_1) - D^F(0, L_0))}{2}$$

Any exact mathematical analysis is not possible.

Typically, the Box-Jenkins method provides an estimate of the standard deviation of the forecast error, assuming white noise. Hence we have

$$D^F - D \underline{d} N(0, \sigma^2)$$

Let us suppose that we want to calculate a constant S such that

$$P\{D>S\} = \alpha$$

Using our forecasting results this is equivalent to

$$P\{D^F-(D^F-D)>S\}=\alpha$$

or

$$P\{D^F-D<D^F-S\}=\alpha$$

We assume that the standard deviation σ is proportional to D^F, which is quite reasonable from a practical point of view. Hence

$$D^F-D \underline{d} N(0, C^2D_F^2)$$

This concludes our discussion of the (R,S)-model. The (R,S)-model is probably the most widely-used inventory management policy. We discussed at length the stationary model and generalized results for the stationary model to the dynamic demand model, when deriving an expression for the P_2-measure. Practitioners may justly argue that the dynamic demand case is the only relevant one. Yet the results obtained for the stationary demand case can well be applied to obtain insight and to set initial parameter values. As the system evolves in time the logic for dynamic demand should be used and the system should collect data, such that feedback and adjustments lead to better performances. In terms of forecasting and data handling a lot needs to be done, especially one needs to focus on lead time demand itself.

In the next chapters, we discuss other inventory management policies. In chapter 7 the (R,S)-model is compared in terms of costs with the other policies.