THE (R,b,Q)-MODEL

The (R,b,Q)-strategy applies to situations where decisions are made periodically, once a week, say and order procurement costs are too high to allow for an (R,S)-strategy. The (R,b,Q)-policy is applied implicitly in many MRP-packages, where fixed lot sizes are used and a time phased order point determines the order (or explosion) moments.

The (R,b,Q)-strategy is described as follows:

Stock is reviewed every R^{th} time unit. If at a review moment the inventory position is below b, then an integral multiple of Q is ordered, such that the inventory position is raised to a value between b and b+Q.

The analysis of the (R,b,Q)-model is quite similar to the analysis of the (b,Q)-model. This chapter is organized as follows. In section 6.1. we describe the model in more detail. In section 6.2. expressions for the P_2-measure and the fill rate are derived. In section 6.3. we discuss the mean physical stock and the mean backlog.

6.1. Model description

We consider two instances of the (R,b,Q)-model. First we describe the discrete time situation, where depletion of stock is registered at equidistant points in time and secondly, we describe the situation, where depletion of stock is registered after each customer arrival. The latter system is a so-called real time inventory management system, the former system operates in a batch-mode.

I: The discrete time situation

We agree upon a time unit, a day, say, at the end of which we collect data about stock depletion during the time unit, as well as arrivals of replenishments during that time unit. Next we decide about the review period, i.e. how may time units elapse between decision epochs, at which we may order an amount at the supplier. Let R be the review period duration, R is an integral number of time units. Then decisions about when and how much to order are governed by the (R,b,Q)-policy.

Due to the fact that during a time unit replenishments may arrive, while stock is also depleted, we must agree upon the way we define disservice and shortages. Indeed, it differs if the replenishment arrives at the beginning of the time unit or at the end of it. We assume the following pessimistic way of processing the data about replenishments and stock depletions.

We assume that a replenishment arrives at the end of a time unit.

As in chapter 3 we describe the demand process by $\{D_n\}$, with

D_n := demand during time unit n.

$\{D_n\}$ is a sequence of i.i.d. random variables. Furthermore we have a sequence of lead times $\{L_k\}$, which are identically distributed and are such that orders cannot overtake. Each lead time is an integral number of time units.

II: The compound renewal situation

In this case we assume that customers arrive according to a compound renewal demand process. The sequence of interarrival times $\{A_n\}$ form a renewal process. The same holds for the demands per customer $\{D_n\}$. The lead times $\{L_k\}$ are identically distributed and orders cannot overtake. In this case we do not encounter problems concerning the processing of inventory transactions, since each transaction is processed individually.

6.2. The service measures

We want to determine the reorder level b, such that for a given value of Q a target service level is achieved. As before we restrict ourselves to the P_2-measure and the $\dot{P}_1$-measure.

P_2-measure

We derive an expression for the P_2-measure for any demand process. Consider an order cycle, i.e. the time between two consecutive order moments. We define the random variables σ_1, D_R and $U_{R,i}$, i=0,1 as:

σ_1 := the point in time at which the inventory position drops below b for the first time after time 0.

D_R := demand during (0,R].

$U_{R,0}$:= the undershoot of b at time 0.

$U_{R,1}$:= the undershoot of b at time σ_1.

Then

$$D_R = \sum_{n=1}^{R} D_n \tag{6.1}$$

$$\begin{aligned} D(0,\sigma_1] &= b+Q-U_{R,0}-(b-U_{R,1}) \\ &= Q-U_{R,0}+U_{R,1} \end{aligned} \tag{6.2}$$

Note that we implicitly assume that only one batch of size Q is ordered. Therefore we must assume that

$Q >> E[D_R]$.

It turns out that the results derived even hold for $Q \approx E[D_R]$, yet from a mathematical point of view the above assumption is necessary.

When we compare the evolution of the inventory position for the (R,b,Q)-model with that for the (b,Q)-model we see that D_R in the (R,b,Q)-model plays the role of the demand per customer in the (b,Q)-model. Then we apply the approximation for the undershoot in the (b,Q)-model to $U_{R,i}$,

$$P\{U_{R,i} \leq x\} = \frac{1}{E[D_R]} \int_0^x (1-F_{D_R}(y))\,dy \tag{6.3}$$

Next we consider the replenishment cycle (L_0, σ_1+L_1], where

L_0 := lead time of order initiated at time 0.

L_1 := lead time of order initiated at time σ_1.

As for the (b,Q)-model we can derive the following expression for the P_2-measure:

$$P_2(b,Q) = 1-\frac{\{E[(D_{\sigma_1,\sigma_1+L_1]}+U_{R,1}-b)^+]-E[(D_{0,L_0]}+U_{R,0}-(b,Q))^+]\}}{E[D(0,\sigma_1]]} \tag{6.4}$$

Since (6.4) is identical to (4.4) we can apply all the results in section (4.1) in order to obtain an expression for P_2(b,Q), which is based on the PDF-method. Without going into further detail we claim that

$$P_2(b,Q) \simeq \hat{\gamma}(b,Q), \tag{6.5}$$

where $\hat{\gamma}$ is the gamma distribution with its first two moments $E[X_\gamma]$ and $E[X_\gamma^2]$ given by

$$E[X_\gamma] = E[D(0,L_0]+U_{R,0}]+\frac{1}{2}Q \tag{6.6}$$

$$E[X_Y^2] = E[(D(0,L_0]+U_{R,0})^2]+Q\ E[D(0,L_0]+U_{R,0}] + \frac{Q^3}{3} \tag{6.7}$$

Equations (6.6) and (6.7) are the equivalent of (4.7) and (4.8), respectively. From (6.1) and (6.3) we derive that

$$E[U_{R,0}] = \frac{E[D_R^2]}{2E[D]} \tag{6.8}$$

$$E[U_{R,0}^2] = \frac{E[D_R^3]}{3E[D_R]} \tag{6.9}$$

Now we distinguish between the discrete time case and the compound renewal case.

Case I: Discrete time case

We assume that D_R is gamma distributed. This yields

$$E[D_R] = R\ E[D] \tag{6.10}$$

$$E[D_R^2] = R\ \sigma^2(D) + R^2\ E^2[D] \tag{6.11}$$

$$E[D_R^3] = (1+c_{D_R}^2)(1+2c_{D_R}^2)E^3[D_R], \tag{6.12}$$

with C_{D_R} the coefficient of variation of D_R, which can be derived from (6.10) and (6.11).

The first two moments of $D(0,L_0]$ are given by equations (3.5) and (3.6), which are repeated below.

$$E[D(0,L_0]] = E[L]E[D] \tag{6.13}$$

$$E[D^2(0,L_0]] = E[L]\sigma^2(D) + E[L^2]E^2[D] \tag{6.14}$$

Case II: Compound renewal case

We again assume that D_R is gamma distributed, such that (6.12) holds. As in section 3.2.2. we make the following assumption about review moments and replenishment moments.

From the point of view of the arrival process, the review moments and replenishment moments are arbitrary points in time.

Then we can apply (3.43) and (3.44) to yield

$$E[D_R] \simeq \frac{R}{E[A]} E[D] \tag{6.15}$$

$$E[D_R^2] \simeq \left(\frac{R^2}{E^2[A]} + \frac{R}{E[A]} (c_A^2+c_D^2) + \frac{1-c_A^4}{16} \right) E^2[D] \tag{6.16}$$

$$E[D(0,L_0]] \simeq \frac{E[L]}{E[A]} E[D] \tag{6.17}$$

$$E[D^2(0,L_0]] \simeq \left(\frac{E[L^2]}{E^2[A]} + \frac{E[L]}{E[A]} (c_A^2+c_D^2) + \frac{(1-c_A^4)}{16} \right) E^2[D] \tag{6.18}$$

For both cases we have the required expressions to calculate $E[X_\gamma]$ and $E[X_\gamma^2]$ and it is routine to apply the PDF-method.

$\hat{P}_1$-measure

The $\hat{P}_1$-measure yields more complicated mathematics than the P_2-measure as the reader must have noticed in the preceding chapters. We need to have a close look at the demand process and the evolution of the net stock in time. We immediately must distinguish between the different demand processes described in section 6.1. We first consider the discrete time model.

Case I: The discrete time model

To obtain results for the $\hat{P}_1$-measure in this case we proceed similar to the analysis preceding equation (3.19) for the mean physical stock. In chapter 2 we defined the function $T^+(x,t)$ by

$T^+(x,t)$:= the expected time the net stock is positive during (0,t], given the net stock at time 0 is $x \geq 0$.

Then equation (2.51) tells us that

$$E[T^+(x,t)] = M(x) - \int_0^x M(x-y)\, dF_{D(0,t]}(y) \qquad (6.19)$$

The net stock at the beginning of replenishment cycle $(L_0,\sigma_1+L_1]$ equals $b+Q-U_{0,R}-D(0,L_0]$. Conditioning on the net stock at time L_0 we find

$$E[T^+(b,Q)] = \int_0^{b+Q} M(b+Q-y)\, dF_{U_{0,R}+D(0,L_0]}(y) - \int_0^{b+Q} M(b+Q-y)\, dF_{U_{0,R}+D(0,\sigma_1+L_1]}(y)$$

Since

$$U_{0,R} + D(0,\sigma_1+L_1] = Q+U_{1,R} + D(\sigma_1,\sigma_1+L_1]$$

we find

$$E[T^+(b,Q)] = \int_0^{b+Q} M(b+Q-y)\,dF_{U_{0,R}+D(0,L_0]}(y) - \int_0^{b} M(b-y)\,dF_{U_{1,R}+D(\sigma_1,\sigma_1+L_1]}(y) \tag{6.20}$$

Let us take a close look at the time interval $(\sigma_1\text{-}R_1,\sigma_1]$. At some time $\sigma_1+T_U\text{-}R$ in $(\sigma_1\text{-}R_1,\sigma_1]$ the inventory position drops below by an amount U_1, say. Then it is clear that

$$U_{1,R} = U_1 + \sum_{n=T_U+1}^{R} D_n \tag{6.21}$$

The undershoot U_1 is the undershoot in the continuous review (b,Q)-model with demand per customer D_n. Hence

$$P\{U_1 \le x\} \simeq \frac{1}{E[D]} \int_0^x (1-F_D(y))\,dy \tag{6.22}$$

Furthermore it can be shown that

$$P\{T_U=t\} = \frac{1}{R} \qquad t=1,...,R\ , \tag{6.23}$$

which is intuitively appealing. Equation (6.23) tells us that the level b is undershot at any time in $(\sigma_1\text{-}R,\sigma_1]$ with equal probability.

Define the random variable W by

$$W := \sum_{n=T_U+1}^{R} D_n$$

Then (6.20) can be rewritten as

$$E[T^+(b,Q)] = \int_0^{b+Q}\int_0^{b+Q-y} M(b+Q-y-z)\,dF_U(z)\,dF_{W+D(0,L_0]}(y) - \int_0^{b}\int_0^{b-y} M(b-y-z)\,dF_U(z)\,dF_{W+D(0,L_0]}(y)$$

Now we apply the identity

$$\int_0^x M(x-y)\,dF_U(y) = \frac{x}{E[D]} \qquad x \geq 0 \ ,$$

with U distributed according to (6.22) to obtain

$$\begin{aligned} E[T^+(b,Q)] &= \int_0^{b+Q} \frac{(b+Q-y)}{E[D]}\ dF_{W+D(0,L_0]}(y) - \int_0^{b} \frac{(b-y)}{E[D]}\ dF_{W+D(0,L_0]}(y) \\ &= \frac{Q}{E[D]} - \frac{1}{E[D]}\left(\int_b^{\infty} (y-b)\,dF_{W+D(0,L_0]}(y) - \int_{b+Q}^{\infty} (y-(b+Q))\,dF_{W+D(0,L_0]}(y) \right) \end{aligned} \tag{6.24}$$

By definition we have that

$$\hat{P}_1(b,Q) = \frac{E[T^+(b,Q)]}{E[\sigma_1]}$$

Thus we find the following expression for $\hat{P}_1(b,Q)$,

$$\hat{P}_1(b,Q) = 1-\frac{1}{Q}\left(\int_b^{\infty}(y-b)\,dF_{W+D(0,L_0]}(y) - \int_{b+Q}^{\infty} (y-(b+Q))\,dF_{W+D(0,L_0]}(y) \right) \tag{6.25}$$

We can alternatively write (6.25) as

$$\hat{P}_1(b,Q) = 1 - \frac{E[(W+D(0,L_0]-b)^+] - E[(W+D(0,L_0]-(b+Q))^+]}{Q}$$

Note the remarkable resemblance of the above equation with equation (6.4) for the P_2-measure. Therefore we can proceed along the same lines as in the derivation of the first two moments of the gamma fit of P_2(b,Q).

So let X_γ be the random variable associated with $\hat{P}_1$(b,Q). Then we have

$$E[X_\gamma] = E[D(0,L_0]+W] + \frac{1}{2}Q \tag{6.26}$$

$$E[X_\gamma^2] = E[(D(0,L_0]+W)^2] + Q\, E[D(0,L_0]+W] + \frac{Q^2}{3} \tag{6.27}$$

It remains to find an expression for the first two moments of W. Recall that

$$W = \sum_{n=T_u+1}^{R} D_n$$

Since $\{D_n\}$ independent of T_u this yields

$$E[W] = (R-E[T_U])E[D] \tag{6.28}$$

$$E[W^2] = (R-E[T_U])\sigma^2(D) + (R^2-2RE[T_U]+E[T_U^2])E^2[D] \tag{6.29}$$

The problem of finding E[W] and E[W^2] has been reduced to finding E[T_U] and E[T_U^2]. These follow from (6.23).

$$E[T_U] = \frac{(R+1)}{2} \tag{6.30}$$

$$E[T_U^2] = \frac{1}{6}R(2R+1) \tag{6.31}$$

Equations (6.26)-(6.31) enable us to compute E[X_γ] and E[X_γ^2]. Fitting a gamma distributed $\hat{\gamma}(.)$ to $P_1(b,Q)$ we find

$$\hat{P}_1(b,Q) \simeq \hat{\gamma}(b+Q) \qquad b \geq -Q$$

and the service level equation

$$\hat{P}_1(b^*,Q) = \alpha$$

can be approximately solved by

$$b^* \simeq \hat{\gamma}^{-1}(\alpha)-Q$$

This concludes the analysis of the discrete time model.

Case II: The compound renewal demand model

As in the discrete time case we start with an approximation for $E[T^+(x,t)]$ derived in chapter 2. Equation (2.53) states that in the compound renewal case

$$E[T^+(x,t)] \simeq (E[\tilde{A}]-E[A])(1-F_{D(0,t]}(x)) + E[A](M(x) - \int_0^x M(x-y)\,dF_{D(0,t]}(y))$$

The net stock at the beginning of replenishment cycle $(L_0,\sigma_1+L_1]$ equals again $b+Q-U_{0,R}-D(0,L_0]$ and therefore we find

$$\begin{aligned} E[T^+(b,Q)] \simeq\ & (E[\tilde{A}]-E[A])(F_{D(0,L_0]+U_{0,R}}(b+Q) \\ & - F_{D(0,\sigma_1+L_1]+U_{0,R}}(b,Q) \\ & + E[A]\left(\int_0^{b+Q} M(b+Q-y)\,dF_{D(0,L_0]+U_{0,R}}(y)\right. \\ & \left. - \int_0^{b+Q} M(b+Q-y)\,dF_{D(0,\sigma_1+L_1]+U_{0,R}}(y)\right) \end{aligned}$$

Since

$$P\{D(0,\sigma_1+L_1]+U_{0,R} \leq x\} = P\{D(\sigma_1,\sigma_1+L-1]+U_{1,R} \leq x-Q\} \qquad x \geq Q$$

and $D(\sigma_1,\sigma_1+L_1]+U_{1,R}$ is identically distributed to $D(0,L_0]+U_{0,R}$, we find

$$E[T^+(b,Q)] \simeq (E[\tilde{A}]-E[A])\;(F_{D(0,L_0]+U_{0,R}}(b+Q)-F_{D(0,L_0]+U_{0,R}}(b))$$
$$+\;E[A]\left(\int_0^{b+Q} M(b+Q-y)\,dF_{D(0,L_0]+U_{0,R}}(y) - \int_0^{b} M(b-y)\,dF_{D(0,L_0]+U_{0,R}}(y)\right) \tag{6.32}$$

As in the discrete time case we express the periodic review undershoot $U_{0,R}$ in terms of the customer undershoot of level b, U_0. Towards this end we define

T_U := the time at which the level b is undershoot by the demand of a customer, σ_1-R$\leq T_U \leq \sigma_1$.

We conjecture the following for Q sufficiently large.

$$P\{T_U \leq t\} = \frac{t}{R} \qquad 0\leq t\leq R \tag{6.33}$$

T_U and U_0 are independent.

It can be shown that this conjecture holds asymptotically for Q→∞ and compound Poisson demand. For arbitrary arrival processes the conjecture was verified empirically by computer simulation. Define N(t) by

N(t) := the number of customers arriving in (0,t], given that at time t a customer arrived.

Then we have the following relation between $U_{0,R}$ and U_0,

$$U_{0,R} = U_0 + \sum_{n=1}^{N(R-T_U)} D_n \tag{6.34}$$

Define the random variable W as

$$W := \sum_{n=1}^{N(R-T_U)} D_n$$

Then it follows that

$$D(0,L_0] + U_{0,R} = U_0 + W + D(0,L_0]$$

Convolving M(.) with U_0 in (6.32) yields

$$E[T^+(b,Q)] \simeq (E[\tilde{A}]-E[A])(F_{D(0,L_0]+U_{0,R}}(b+Q)-F_{D(0,L_0]+U_{0,R}}(b)) + E[A]\left(\int_0^{b+Q}\frac{(b+Q-y)}{E[D]}dF_{W+D(0,L_0]}(y) - \int_0^{b}\frac{(b-y)}{E[D]}dF_{W+D(0,L_0]}(y)\right),$$

which can be rewritten into

$$E[T^+(b,Q)] \simeq \frac{(c_A^2-1)}{2}E[A](F_{D(0,L_0]+U_{0,R}}(b+Q) - F_{D(0,L_0]+U_{0,R}}(b)) + E[A]\left(\frac{Q}{E[D]} - \left(\int_b^{\infty}\frac{(y-b)}{E[D]}dF_{W+D(0,L_0]}(y) - \int_{b+Q}^{\infty}(y-(b+Q))dF_{W+D(0,L_0]}(y)\right)\right) \tag{6.35}$$

Dividing (6.35) by $E[\sigma_1]$ we find

$$\hat{P}_1(b,Q) \simeq \frac{(c_A^2-1)}{2}\frac{E[D]}{Q}\left(F_{D(0,L_0]+U_{0,R}}(b+Q) - F_{D(0,L_0]+U_{0,R}}(b)\right)$$
$$+ 1-\frac{1}{Q}\left(\int_b^\infty (y-b)\,dF_{W+D(0,L_0]}(y) - \int_{b+Q}^\infty (y-(b+Q))\,dF_{W+D(0,L_0]}(y)\right) \tag{6.36}$$

Equation (6.36) is well suited for application of the PDF-method. Applying by now standard arguments we find

$$E[X_Y] = \int_{-Q}^\infty (1-\hat{P}_1(x,Q))\,dx$$
$$= \frac{(c_A^2-1)}{2}E[D] + E[W+D(0,L_0]] + \frac{Q}{2} \tag{6.37}$$

$$E[{}_Y^2] = 2\int_{-Q}^\infty (x+Q)(1-\hat{P}_1(x,Q))\,dx$$
$$= -\frac{(c_A^2-1)}{2}(Q+2E[D(0,L_0]+U_{0,R}])$$
$$+ E[(W+D(0,L_0])^2] + Q\,E[W+D(0,L_0]]$$
$$+ \frac{Q^2}{3} \tag{6.38}$$

The only information still lacking are the first two moments of W. It has been conjectured that T_U is homogeneously distributed on (0,R). Therefore

$$E[N(R-T_U)] = \frac{1}{R}\int_0^R E[N_A(t)]\,dt \tag{6.39}$$

$$E(N^2(R-T_U)] = \frac{1}{R}\int_0^R E[N_A^2(t)]dt \ , \tag{6.40}$$

where $N_A(.)$ is the renewal process associated with $\{A_n\}$.

Application of renewal theoretic results reveals that (cf. (2.26) and (2.27))

$$\lim_{x\to\infty} \int_0^x E[N(t)]dt - \left(\frac{x^2}{2E[A]} + \left(\frac{E[A^2]}{2E[A]} - 1 \right) x + \frac{E^2[A^2]}{4E^3[A]} - \frac{E[A^3]}{6E^2[A]} \right) = 0 \tag{6.41}$$

$$\lim_{x\to\infty} \int_0^x E[N^2(t)]dt - \left(\frac{x^3}{3E^2[A]} + \left(\frac{E[A^2]}{E^3[A]} - \frac{3}{2E[A]} \right) x^2 + \left(\frac{3E^2[A^2]}{2E^4[A]} - \frac{2E[A^3]}{3E^3[A]} - \frac{3E[A^2]}{2E^2[A]} + 1 \right) x + \frac{E[A^4]}{6E^3[A]} - \frac{E[A^2]E[A^3]}{E^4[A]} + \frac{E^3[A^2]}{E^5[A]} - \frac{E[A^3]}{2E^2[A]} - \frac{3E^2[A^2]}{4E^3[A]} \right) = 0 \tag{6.42}$$

Assuming R >> E[A] we find

$$\int_0^R E[N(t)]dt \simeq \frac{R^2}{2E[A]} + \left(\frac{E[A^2]}{2E[A]} - 1\right)R + \frac{E^2[A]}{4E^3[A]} - \frac{E[A^3]}{6E^2[A]} \tag{6.43}$$

$$\int_0^R E[N^2(t)]dt - \left(\frac{R^3}{3E^2[A]} + \left(\frac{E[A^2]}{E^3[A]} - \frac{3}{2E[A]} \right) R^2 \right.$$

$$+ \left(\frac{3E^2[A^2]}{2E^4[A]} - \frac{2E[A^3]}{3E^3[A]} - \frac{3E[A^2]}{2E^2[A]} + 1 \right) R \tag{6.44}$$

$$\left. + \frac{E[A^4]}{6E^3[A]} - \frac{E[A^2]E[A^3]}{E^4[A]} + \frac{E^3[A^2]}{E^5[A]} - \frac{E[A^3]}{2E^2[A]} - \frac{3E^2[A^2]}{4E^3[A]} \right) = 0$$

and assuming gamma distributed interarrival times,

$$E[N(R-T_U)] \simeq \frac{R}{2E[A]} + \frac{(c_A^2-1)}{2} + \frac{1}{R}\frac{(1-c_A^4)}{12}E[A] \tag{6.45}$$

$$E[N^2(R-T_U)] \simeq \frac{R^2}{3E^2[A]} + \left(c_A^2-\frac{1}{2}\right)\frac{R}{E[A]} + \frac{1}{6}(c_A^2-2)(c_A^2-1) - \frac{1}{12R}(1-c_A^4)E[A] \tag{6.46}$$

Once we know $E[N(R-T_U)]$ and $E[N^2(R)]$, it is an easy matter to calculate $E[W]$ and $E[W^2]$ from

$$E[W] = E[N(R-T_u)]E[D] \tag{6.47}$$

$$E[W^2] = E[N(R-T_U)]\sigma^2(D) + E[N^2(R-T_U)]E^2[D] \tag{6.48}$$

Note that the assumption of R>>E[A] is not unrealistic. Indeed, if we use a periodic review policy it does not make sense to have a review frequency higher than the arrival frequency. In that case reviews triggered by customer arrivals are more economic. In that case we use the standard (b,Q)-model.

This concludes the analysis of the service measures P_2 and $\dot{P}_1$. For both measures we have derived approximations based on the PDF-method. It remains to validate the approximations. Results of the validation are given in chapter 8.

6.3. Physical stock and backlog

As has been shown in the preceding chapters the mean physical stock depends on the way the inventory transactions are processed. The discrete time model assumes batch processing of the inventory transactions. This implies that the administrative stock is constant during the day, say and updated daily. This also implies an overestimation of the actual stock. The smaller the time between inventory updates, the smaller the bias of the estimation. This situation is modelled in the discrete time model. Hence the discrete time model yields an overestimate of the physical stock.

The compound renewal case describes on line processing of inventory transactions. In that case the administrative stock equals the actual stock. Hence the mean physical stock is properly estimated by the continuous monitoring model.

As with the $\hat{P}_1$-measure we must distinguish between the discrete time model and the compound renewal model. For both models we derive approximate expressions based on renewal-theoretic results.

Case I: <u>The discrete time model</u>

For the discrete time model we can exploit results from chapter 2, which have already been used in chapter 3 for the (R,S)-model. More specifically, the starting point for our analysis is the function K(x,t) defined as

H(x,t) := the expected surface between the net stock and the zero level during (0,t], given that at time 0 the net stock equals $x \geq 0$.

Note that t should be a multiple of the time unit.

The function H(x,t) has been studied in chapter 2 and equation (2.56) with E[A] identical to one time unit tells us that

$$K(x,t) = \int_0^x (x-y)\,dM(y) - \int_0^x \int_0^{x-y} (x-y-z)\,dM(z)\,dF_{D(0,t]}(y) \tag{6.49}$$

Then by conditioning on the net stock at time L_0 we find that

$$E[X^+(b,Q)] \simeq \frac{1}{E[\sigma_1]} \int_0^\infty \int_0^{b+Q} K(b+Q-y,t)\,dF_{D(0,L_0]+U_{0,R}}(y)\;\; dF_{(\sigma_1,L_1-L_0]}(t) \tag{6.50}$$

Substitution of (6.49) in (6.50) and some algebra yields

$$E[X^+(b,Q)] \simeq \frac{1}{E[\sigma_1]} \left\{ \int_0^{b+Q} \int_0^{b+Q-y} (b+Q-y-z)\,dM(z)\,dF_{D(0,L_0]+U_{0,R}}(y) \right.$$
$$\left. - \int_0^{b+Q} \int_0^{b+Q-y} (b+Q-y-z)\,dM(z)\,dF_{D(0,\sigma_1,L_1]+U_{0,R}}(y) \right\}$$

As in the analysis preceding equation (6.25) we note that

$$U_{0,R} = U_0 + W$$

with W defined below (6.23) and

$$D(0,\sigma_1+L_1] + U_{0,R} = Q + D(\sigma_1,\sigma_1+L_1] + U_{1,R}$$

Substituting these results into the above approximation for $E[X^+(b,Q)]$ yields

$$E[X^+(b,Q)] \simeq \frac{1}{E[\sigma_1]} \left\{ \int_0^{b+Q} \frac{(b+Q-y)^2}{2E[D]} dF_{W+D(0,L_0]}(y) - \int_0^{b} \frac{(b-y)^2}{2E[D]} dF_{W+D(0,L_0]}(y) \right\}$$

Using $E[\sigma_1] = Q/E[D]$ we find after some algebra

$$E[X^+(b,Q)] \simeq b+\frac{Q}{2}-E[W+D(0,L_0]] + \frac{1}{Q}\left\{ \int_0^{\infty} \frac{(y-b)^2}{2} dF_{W+D(0,L_0]}(y) - \int_{b+Q}^{\infty} \frac{(y-(b+Q))^2}{2} dF_{W+D(0,L_0]}(y) \right\} \tag{6.51}$$

Equation (6.51) is by now standard for further evaluation. Before doing so we relate $E[X^+(b,Q)]$ to $E[B(b,Q)]$, the average backlog. This relation has already been derived in chapter 3. We repeat the arguments here for the reader's convenience.

Assume the stock keeping facility pays the supplier \$1 per purchased product per time unit this product is on order with the supplier. Then per order on average \$ E[L].Q is paid, assuming Q is large compared to the undershoot of the reorder level b. Since on average every Q/E[D] time units a batch of Q products is ordered at the supplier, the average payment per unit time equals

$$E[L].Q \ / \ (Q/E[D]) = E[D]E[L].$$

On the other hand, the supplier receives on average \$ E[O] per time unit, where

E[O] := the average amount on order.

Therefore

$$E[O] = E[D]E[L].$$

The basic equation determining the inventory position tells us that

$$E[Y] = E[X^+(b,Q)] + E[O] - E[B(b,Q)]$$

and thus

$$E[B(b,Q)] = E[X^{+}(b,Q)] + E[D]E[L] - E[Y]$$

We need an expression for E[Y]. From the analysis in Hadley and Whitin [1963] it can be derived that

the inventory position at review moments is homogeneously distributed between b and b+Q.

Consider an arbitrary review cycle (0,R). At time 0 the inventory position equals x. Then it follows from the expression for the complementary holding cost given by (2.67) that the average inventory position during a review cycle with initial inventory position x equals x-½(R-1)E[D]. Conditioning on the homogeneously distributed initial inventory position yields

$$E[Y] = b+\frac{1}{2}Q - \frac{1}{2}(R-1)E[D]$$

This finally yields

$$E[B(b,Q)] = E[X^{+}(b,Q)] + E[D]E[L] - b-\frac{1}{2}Q + \frac{1}{2}(R-1)E[D] \tag{6.52}$$

Let us now reconsider (6.51). We know that

$$E[D(0,L_0]] = E[L]E[D]$$

$$E[W] = \frac{1}{2}(R-1)E[D]$$

and thereby

$$E[X^{+}(b,Q)] = b+\frac{Q}{2} - \frac{1}{2}R(-1)E[D] - E[L]E[D] + \frac{1}{Q}\left\{ \int_{b}^{\infty} \frac{(y-b)^2}{2} dF_{W+D(0,L_0]}(y) - \int_{b+Q}^{\infty} \frac{(y-(b+Q))^2}{2} dF_{W+D(0,L_0]}(y) \right\} \tag{6.53}$$

Then it follows from (6.52) and (6.53) that

$$E[B(b,Q)] = \frac{1}{Q}\left\{ \int_{b}^{\infty} \frac{(y-b)^2}{2} dF_{W+D(0,L_0]}(y) - \int_{b+Q}^{\infty} \frac{(y-(b+Q))^2}{2} dF_{W+D(0,L_0]}(y) \right\} \tag{6.54}$$

For the case of b<-Q we directly obtain

$$E[B(b,Q)] = E[W+D(0,L_0]] - b - \frac{Q}{2} \qquad b \leq -Q \tag{6.55}$$

$$E[B(-Q,Q)] = \frac{Q}{2} + E[W+D(0,L_0]] \tag{6.56}$$

Define γ(.) by

$$\gamma(x) = 1 - \frac{E[B(x-Q,Q)]}{E[B(-Q,Q)]} \qquad x \geq 0 \tag{6.57}$$

Then $\gamma(.)$ is a probability distribution function. Let X_γ be the random variable which has a gamma distribution $\hat{\gamma}(.)$ with the same first two moments as $\gamma(.)$. Then

$$E[X_\gamma] = \frac{\frac{Q^2}{6} + E\left[\frac{W+D(0,L_0]]}{2}Q + \frac{E[(W+D(0,L_0])^2]}{2}}{\frac{Q}{2} + E[W+D(0,L_0]]} \tag{6.58}$$

$$E[X_\gamma^2] = \left\{ \frac{Q^3}{12} + \frac{E[W+D(0,L_0]]}{3}Q^2 + E[(W+D(0,L_0])^2]\frac{Q}{2} + \frac{E[(W+D(0,L_0])^3]}{3} \right\} \Big/ \left\{ \frac{Q}{2} + E[W+D(0,L_0]] \right\} \tag{6.59}$$

Once we determined $\hat{\gamma}(.)$ from (6.58) and (6.59) we can approximate E[B(b,Q)] and E[X^+(b,Q)] by

$$E[B(b,Q)] \simeq \begin{cases} \left(\frac{Q}{2} + E[W+D(0,L_0]])(1-\hat{\gamma}(b+Q) \right) & b \geq -Q \\ -b-\frac{Q}{2} + E[W+D(0,L_0]] & b<-Q \end{cases} \tag{6.60}$$

$$E[X^+(b,Q)] = \begin{cases} b+Q - \left(\frac{Q}{2} + E[W+D(0,L_0]] \right) \hat{\gamma}(b+Q) & b \geq -Q \\ 0 & b<-Q \end{cases} \tag{6.61}$$

Case II: <u>The compound renewal model</u>

As in the case of an arrival process with constant interarrival times our starting point for our analysis is an expression for the function H(x,t). For the present case of a compound renewal arrival process an approximation for H(x,t) is given by (cf. 2)

$$H(x,t) \simeq (E[\tilde{A}]-E[A]) \left(x - \int_0^x (x-y)\, dF_{D(0,t]}(y) \right) + E[A] \left(\int_0^x (x-y)\, dM(y) - \int_0^x \int_0^{x-y} (x-y-)\, dM(z)\, dF_{D(0,t]}(y) \right) \tag{6.62}$$

We condition on the net stock at the start of the replenishment cycle $(L_0,\sigma_1+L_1]$, leading to an expression for $E[X^+(b,Q)]$,

$$E[X^+(b,Q)] = \frac{1}{E[\sigma_1]} \int_0^\infty \int_0^{b+Q} H(b+Q-y,t)\, dF_{U_{0,R}+D(0,L_0]}(y)\, dF_{\sigma_1+L_1-L_0}(t) \tag{6.63}$$

We substitute (6.62) into (6.63) and after application of some probabilistic arguments we obtain

$$\begin{aligned} E[X^+(b,Q)] \simeq \frac{1}{E[\sigma_1]} \Bigg\{ & (E[\tilde{A}]-E[A]) \left(\int_0^{b+Q} (b+Q-y)\, dF_{U_{0,R}+D(0,L_0]}(y) - \int_0^{b+Q} (b+Q-y)\, dF_{U_{0,R}+D(0,\sigma_1+L_1]}(y) \right) \\ & + E[A] \left(\int_0^{b+Q} \int_0^{b+Q-y} (b=Q-y-)\, dM(z)\, dF_{U_{0,R}+D(0,L_0]}(y) - \int_0^{b+Q} \int_0^{b+Q-y} (b+Q-y-z)\, dM(z)\, dF_{U_{0,R}+D(0,\sigma_1+L_1]}(y) \right) \Bigg\} \end{aligned}$$

Applying the by now standard arguments concerning $D(0,\sigma_1+L_1]$ and $U_{0,R}$ we find after some algebra

$$E[X^+(b,Q)] \simeq \frac{(c_A^2-1)}{2}\frac{E[A]}{Q}\left\{\int_0^{b+Q}(b+Q-y)\,dF_{U_{0,R}+D(0,L_0]}(y) - \int_0^{b}(b-y)\,dF_{U_{0,R}+D(0,L_0]}(y)\right\}$$
$$+\frac{E[D]}{Q}\left(\int_0^{b+Q}\frac{(b+Q-y)^2}{2E[D]}dF_{W+D(0,L_0]}(y) - \int_0^{b}\frac{(b-y)^2}{2E[D]}dF_{W+D(0,L_0]}(y)\right) \tag{6.64}$$

where W is defined in section 2, when deriving an expression for the $\hat{P}_1$-measure.

The second term on the right hand side of (6.64) is identical to the expression for $E[X^+(b,Q)]$ for the discrete time case given above by equation (6.51). Hence we apply the same transformation rules. The first term on the right hand side of (6.64) can also be rewritten by writing the integral from 0 to b+Q as the difference between the integral from 0 to ∞ and the integral from b+Q to ∞. This yields

$$E[X^+(b,Q)] \simeq \frac{(c_A^2-1)}{2}E[D] - \frac{(c_A^2-1)}{2}\frac{E[D]}{Q}\left\{\int_b^{\infty}(y-b)\,dF_{U_{0,R}+D(0,L_0]}(y) - \int_{b+Q}^{\infty}(y-(b,Q))\,dF_{U_{0,R}+D(0,L_0]}(y)\right\}$$
$$+ b+\frac{Q}{2} - E[W+D(0,L_0]]$$
$$+\frac{1}{Q}\left\{\int_b^{\infty}\frac{(y-b)^2}{2}dF_{W+D(0,L_0]}(y) - \int_{b+Q}^{\infty}\frac{(y-(b+Q))^2}{2}dF_{W+D(0,L_0]}(y)\right\} \tag{6.65}$$

Instead of fitting distribution to $U_{0,R}+D(0,L_0]$ and $W+D(0,L_0]$ explicitly calculating the integrals, we apply the PDF-method to the mean backlog. This can be done since we have an explicit relation between the mean physical stock and the mean backlog.

$$E[X^+(b,Q)] = E[Y(b,Q)] - E[D]\frac{E[L]}{E[A]} + E[B(b,Q)] \tag{6.66}$$

Equation (6.66) has been derived in exactly the same way as its equivalent in the discrete time model.

To obtain an expression for $E[Y(b,Q)]$ we consider an arbitrary review cycle (0,R). We assume that review moments are arbitrary moments in time from the point of view of the arrival process. We further assume that at time 0 the inventory position equals x. To calculate the average inventory position in (0,R), we divide the expected area between the x-level and the inventory position by R and subtract this from x. In chapter 2 we have already analyzed this expected area and found that this is approximately equal to $\frac{1}{2}R^2E[D]/E[A]$. Hence the average inventory position in (0,R) equals x-½R/E[A]E[D]. Now it follows from the analysis in Hadley and Whitin that the inventory position at the beginning of an arbitrary review cycle is homogeneously distributed on (b,b+Q). This yields

$$E[Y(b,Q)] \simeq b+\frac{1}{2}Q - \frac{R}{2E[A]}E[D]$$

and so

$$E[X^+(b,Q)] \simeq b+\frac{Q}{2} - \frac{R}{2E[A]}E[D] - E[D]\frac{E[L]}{E[A]} + E[B(b,Q)] \tag{6.67}$$

We reconsider (6.65). It follows from (6.45) and (6.47) that

$$E[W] \simeq \left(\frac{R}{2E[A]} + \frac{(c_A^2-1)}{2} + \frac{(1-c_A^4)}{12R} E[A] \right) E[D]$$

From (6.17) we know that

$$E[D(0,L_0]] \simeq \frac{E[L]}{E[A]} \; E[D]$$

Then (6.65) becomes

$$\begin{aligned} E[X^+(b,Q)] = {} & b+\frac{Q}{2} - \frac{E[L]}{E[A]}E[D] - \frac{R}{2E[A]}E[D] - \frac{(1-c_A^4)}{12}\frac{E[A]}{R}E[D] \\ & -\frac{(c_A^2-1)}{2}\frac{E[D]}{Q}\left\{ \int_b^\infty (y-b)\,dF_{U_{0,R}+D(0,L_0]}(y) \right. \\ & \qquad \left. - \int_{b+Q}^\infty (y-(b+Q))\,dF_{U_{0,R}+D(0,L_0]}(y) \right\} \\ & +\frac{1}{Q}\left\{ \int_b^\infty \frac{(y-b)^2}{2}dF_{W+D(0,L_0]}(y) - \int_{b+Q}^\infty \frac{(y-(b+Q))^2}{2}dF_{W+D(0,L_0]}(y) \right\} \end{aligned} \tag{6.68}$$

Comparison of (6.67) and (6.68) suggest that

$$E[B(b,Q)] = \frac{(1-c_A^4)}{12}\frac{E[A]}{R}E[D] - \frac{(c_A^2-1)}{2}\frac{E[D]}{Q}\left\{\int_b^\infty (y-b)\,dF_{U_{0,R}+D(0,L_0]}(y) - \int_{b+Q}^\infty (y-(b+Q))\,dF_{U_{0,r}+D(0,L_0]}(y)\right\} + \frac{1}{Q}\left\{\int_b^\infty \frac{(y-b)^2}{2}dF_{W+D(0,L_0]}(y) - \int_{b+Q}^\infty \frac{(y-(b+Q))^2}{2}dF_{W_+D(0,L_0]}(y)\right\}$$

This, however, is inconsistent with lim E[B(b,Q)]=0. This inconsistency is caused by the approximations for E[W] and H(x,t). On the other hand, assuming that R >> E[A] we may assume that

$$\frac{(1-c_A^4)}{12}\frac{E[A]}{R} \text{ negligable.}$$

Therefore we suggest to approximate the mean backlog by

$$E[B(b,Q)] = \frac{1}{Q}\left\{\int_b^\infty \frac{(y-b)^2}{2}dF_{W+D(0,L_0]}(y) - \int_{b+Q}^\infty \frac{(y-(b+Q))^2}{2}dF_{W+D(0,L_0]}(y)\right\} - \frac{(c_A^2-1)}{2}\frac{E[D]}{Q}\left\{\int_b^\infty (y-b)\,dF_{U_{0,R}+D(0,L_0]}(y) - \int_{b+Q}^\infty (y-(b+Q))\,dF_{U_{0,R}+D(0,L_0]}(y)\right\} \quad (6.69)$$

It follows from (6.69) (as well as from (6.68)), that

$$E[B(b,Q)] = E[W+D(0,L_0]] - \frac{Q}{2} - b - \frac{(c_A^2-1)}{2}E[D] \qquad b\le -Q \quad (6.70)$$

An expression for E[B(b,Q)] for b≥-Q is derived from application of the PDF-method.

Let γ(.) be the pdf defined by

$$\gamma(x) = 1-\frac{E[B(x-Q,Q)]}{E[B(-Q,Q)]}, \quad x\ge 0$$

Let X_γ be the random variable X_γ with pdf γ(.). Then the first two moments of X_γ are given by

$$E[X_\gamma] = \left\{ \frac{Q^2}{6} + \frac{E[W+D(0,L_0)]}{2}Q + \frac{E[(W+D(0,L_0))^2]}{2} - \frac{(c_A^2-1)}{2}E[D]\left(\frac{Q}{2} + E[W+D(0,L_0)] \right) \right\} / \left\{ \frac{Q}{2} + E[W+D(0,L_0)] - \frac{(c_A^2-1)}{2}E[D] \right\} \tag{6.71}$$

$$E[X_\gamma^2] = \left\{ \frac{Q^3}{12} + \frac{E[W+D(0,L_0)]}{3}Q^2 + E[(W+D(0,L_0))^2]\frac{Q}{2} + \frac{E[(W+D(0,L_0))^3]}{3} - \frac{(c_A^2-1)}{2}E[D]\left(\frac{Q^2}{3} + E[W+D(0,L_0)]Q + E[(W+D(0,L_0))^2] \right) \right\} \left\{ \frac{Q}{2} + E[W+D(0,L_0)] - \frac{(c_A^2-1)}{2}E[D] \right\} \tag{6.72}$$

Fitting the gamma distribution $\hat{\gamma}(.)$ to $E[X_\gamma]$ and $E[X_\gamma^2]$ we have the following approximation for E[B(b,Q)] for $b \geq -Q$,

$$E[B(b,Q)] \simeq \left(\frac{Q}{2} + E[W+D(0,L_0)]\right)(1-\hat{\gamma}(b+Q)) \;,\; b \geq -Q \tag{6.73}$$

Substituting this approximation into (6.67) yields

$$E[X^{+}(b,Q)] \simeq b+\frac{Q}{2} - \frac{R}{2E[A]}E[D] - E[D]\frac{E[L]}{E[A]} + \left(\frac{Q}{2} + E[W+D(0,L_0]])(1-\hat{\gamma}(b+Q) \right) , \quad b \geq -Q \tag{6.74}$$

This completes the analyses of the (R,b,Q)-model. The analysis turned out to be quite similar to that of the (b,Q)-model. Main differences are caused by the undershoots of b during the review period, which leads to the introduction of the random variable W.