REORDER POINT STRATEGY WITH FIXED ORDER QUANTITY

Probably the mostly addressed inventory management policy in the literature is the continuous review (b,Q)-policy. The (b,Q)-policy operates as follows.

As soon as the inventory position drops below the reorder point b, an amount equal to an integral number times Q is ordered at the supplier, such that the inventory position after ordering is between b and b+Q.

When we compare the (b,Q)-policy with the (R,S)-policy we observe that the (b,Q)-policy provides flexibility with respect to the order moment, yet it lacks the flexibility of the (R,S)-policy with respect to the order size. As mentioned before the more flexibility one has with respect to the order moment and order size, the less inventory is needed to provide some service. For the moment it is unclear which policy performs best given some particular situations. This discussion is postponed until chapter 8.

The structure of this chapter is similar to that of chapter 3. We discuss the stationary demand model first, where we concentrate on service measures. In section 4.2. we derive expressions for the mean physical stock. Section 4.3. is devoted to the average backlog. In section 4.4. a numerically elegant scheme for computing a cost-optimal policy is given.

4.1. Stationary demand and service measures

To describe the model situation we distinguish between the customers, the stock keeping facility and the supplier. We assume that the demand process is a compound renewal process.

D := demand per customer.

A := interarrival time customers.

The stockkeeping facility executes a (b,Q)-policy. The supplier delivers an order after a lead time L. D, A and L are random variables, of which the first two moment are known.

We want to obtain expressions for the $\hat{P}_1$-measure, the fill rate, and the P_2-service measure. Recall that

$\hat{P}_1$:= long-run fraction of time the net stock is positive.

P_2 := long-run fraction of demand delivered directly from stock on hand.

First we concentrate on the P_2-measure.

Assume that at time 0 the inventory position drops below b by an amount U_0. Then an amount Q is ordered at the supplier assuming $b+Q-U_0>b$. Then after some stochastic time σ_1 the inventory position again drops below b by an amount U_1, which initiates another order of size Q.

Let us consider the replenishment cycle (L_0,σ_1+L_1). At time L_0 the amount Q ordered at time 0 arrives at the stockkeeping facility. All previous orders have arrived and hence immediately after time L_0 the physical inventory equals the inventory position at time 0 minus the demand during $[0,L_0]$. At time σ_1+L_1 the next order arrives and the physical stock has further decreased to $b+Q-U_0$ minus the demand during $[0,\sigma_1+L_1]$.

Along the same lines as the derivation of 3.10 we find

$$P_2 = 1 - \frac{E[(D[0,\sigma_1+L_1]-(b+Q-U_0))^+]-E[(D[0,L_0]-(b+Q-U_0))^+]}{E[D]L_0,\sigma+L_1]]} \tag{4.1}$$

The expression for P_2 involves the demand during the interval $[0,\sigma_1+L_1]$. The problem is that σ_1 is a random variable endogenous to the model and not known beforehand. We circumvent this problem as follows. The demand during $[0,\sigma_1+L_1]$ can be rewritten.

$$D[0,\sigma_1+L_1] = D[0,\sigma_1]+D[\sigma_1,\sigma+L_1]$$

The second term is the demand during the lead time L_1, and we will derive expressions for the first two moments of this random variable in a few moments. The first term is rewritten as follows. $D[0,\sigma_1]$ is the difference between the inventory position at time 0 and the inventory position immediately before σ_1,

$$\begin{aligned} D[0,\sigma_1] &= b+Q-U_0-(b-U_1) \\ &= Q+U_1-U_0 \end{aligned}$$

Substituting these results we find

$$P_2 = 1-\frac{E[(D[\sigma_1,\sigma_1+L_1]+U_1-b)^+] - E[(D[0,L_0]+U_0-(b+Q))^+]}{Q} \tag{4.2}$$

We know that $D[\sigma_1,\sigma_1+L_1]$ and $D[0,L_0]$ are identically distributed, with known first two moments. Also U_1 is independent of $D[\sigma_1,\sigma_1+L_1]$ and U_0 is independent of $D[0,L_0]$. It remains to find expressions for the first two moments of U_0 and U_1. These expressions are obtained using the following assumption.

Q is sufficiently large to guarantee that

$$P\{b+Q-U_0>b\}=1 \ \ and \ \ Q>>E[D]$$

Now note that the difference between $b+Q-U_0$ and the inventory position at customer arrival epochs constitute a renewal process with interrenewal time D. If Q>>E[D] it has been shown in De Kok [1987] that the undershoot of the reorder level b is distributed according to the stationary residual lifetime associated with D, i.e.

$$P\{U_1 \le x\} \simeq \frac{1}{E[D]} \int_0^x (1-F_D(y))\,dy, \quad x \ge 0 \tag{4.3}$$

where

$$F_D(y) = P\{D \le y\}, \quad y \ge 0$$

Assuming that (4.3) is correct, U_1 is independent of b and Q and hence U_0 is also independent of b and Q and distributed according to (4.3). We conclude that (4.2) can be rewritten as

$$P_2 = 1 - \frac{E[(Z-b)^+] - E[(Z-(b+Q))^+]}{Q} \tag{4.4}$$

Z is a generic random variable, for which holds,

$$Z \; d \; D[0, L_0] + \hat{U}_1$$

and

$$P\{\hat{U}_1 \le x\} = \frac{1}{E[D]} \int_0^x (1-F_D(y))\,dy$$

We emphasize that (4.4) is an approximation. Extensive experimentation shows that (4.4) performs extremely well, even for values of Q smaller than E[D]. Before providing insight into this phenomenon we elaborate on (4.4) to obtain an algorithm based on the PDF-method. We remark that (4.4) can be applied directly by fitting some tractable distribution to the first two moments of Z and then calculate P_2 for a given value of b and Q.

Application of the PDF-method

As in chapter 3 we note that (4.4) is particularly suited for application of the PDF-method, since $P_2(b)$ is a pdf as a function of b for a given value of Q. Indeed,

$$P_2(b) = \begin{cases} 0 & b<-Q \\ 1-\frac{1}{Q}\ \{E[Z]-b - E[(Z-(b+Q))^+]\} & -Q\le b<0 \\ 1-\frac{1}{Q}\ \{E[(Z-b)^+] - E[(Z-(b+Q))^+]\} & b\ge 0 \end{cases} \qquad (4.5)$$

Define $\gamma(.)$ by

$$\gamma(x) = P_2(x-Q), \quad x\ge 0 \qquad (4.6)$$

Then $\gamma(.)$ is the pdf of some non-negative random variable X_γ, i.e.

$$P\{X_\gamma \le x\} = \gamma(x), \quad x\ge 0$$

We must compute the first two moments of X_γ. The first moment of X_γ is derived as follows.

First we write $E[X_\gamma]$ as

$$E[X_\gamma] = \int_0^\infty (1-\gamma(x))\,dx$$

Substituting (4.5) and (4.6) we obtain

$$E[X_\gamma] = \frac{1}{Q}\left\{\int_0^Q (E[Z]-(x-Q) - \int_x^\infty (y-x)\,dF_Z(y))\,dx \right.$$
$$\left. + \int_Q^\infty \left(\int_{x-Q}^\infty (y-(x-Q))\,dF_Z(y) - \int_x^\infty (y-x)\,dF_Z(y) \right) dx\right\}$$

Rearranging terms in the above equation we find

$$E[X_\gamma] = E[Z] + ½Q - \frac{1}{Q}\int_0^\infty \int_x^\infty (y-x)\,dF_Z(y)\,dx$$
$$+ \frac{1}{Q}\int_Q^\infty \int_{x-Q}^\infty (y-(x-Q))\,dF_z(y)\,dx$$

Substituting w=x-Q in the last term we find

$$E[X_\gamma] = E[Z] + ½Q \tag{4.7}$$

In a similar fashion we obtain

$$E[X_\gamma^2] = E[Z^2] + E[Z]Q + \frac{Q^2}{3} \tag{4.8}$$

Next we fit a gamma distribution $\hat{\gamma}(.)$ to the first two moments of X_γ. Suppose we want to solve the following equation for b*,

$$P_2(b^*, Q) = \beta$$

Then b^* can be found by

$$b^* \simeq \hat{\gamma}^{-1}(\beta) - Q \tag{4.9}$$

We still have not given the first two moments of Z. These can be computed from the following set of equations and the independence of $D[0,L_0]$ and $\mho_1$ (cf. (2.37) and (2.38)).

$$E[D(0,L_0)] = \left(\frac{E[L]}{E[A]} + \frac{E[A^2]}{2E^2[A]} - 1 \right) E[D] \tag{4.10}$$

$$\sigma^2(D[0,L_0]) = \frac{E[L]}{E[A]}\sigma^2(D) + \frac{E[L]}{E[A]} c_A^2 E^2[D] + \sigma^2(L)\frac{E^2[D]}{E^2[A]} + \frac{(C_A^2-1)}{2}\sigma^2(D) + \frac{(1-c_A^4)}{12}E^2[D] \tag{4.11}$$

$$E[\hat{U}_1] = \frac{E[D^2]}{2E[D]} \tag{4.12}$$

$$E[\hat{U}_1^2] = \frac{E[D^3]}{3E[D]} \tag{4.13}$$

Exact analysis and synthesis

The analysis has been approximative, since we assumed that each order consisted of one Q only and Q was considerably larger than E[D]. In Hadley and Whitin [1963] a rigorous treatment has been given of the (b,Q)-model. The essential result obtained there is.

The inventory position immediately after an arrival of a customer is homogeneously distributed in the interval (b,b+Q).

This result holds for both discrete and continuous demand distributions.

This result can be exploited as follows. With each arrival epoch a pseudo-replenishment cycle can be associated, since

each arrival epoch is a potential order moment.

Assume a customer arrives at time 0. Let Y_0 denote the inventory position immediately after time 0. Sample a lead time L_0 from the probability distribution function of L, the generic lead time. Then at time L_0 the net stock equals Y_0-E[0,L_0]. The pseudo-replenishment cycle lasts until the potential arrival of the next order. This order is initiated at time A_1, the first interarrival time and, if initiated, arrives at time A_1+L_1. The net stock immediately before A_1+L_1 equals Y_0-D[0,A_1+L_1]. Then we find an alternative (exact) expression for $P_2(b,Q)$,

$$P_2(b,Q) = 1-\frac{[(D[0,A_1+L_1]-Y_0)^+] - E[(D[0,L_0]-Y_0)^+]}{E[D[(L_0,A_1+L_1]]}$$

It is easy to see that

$$E[D[L_0,A_1+L_1]] = E[D],$$

$$D[0,A_1+L_1] = D_1+D[A_1,A_1+L_1],$$

where D_1 is the demand of the customer arriving at A_1. Then we can further elaborate on these expressions applying the Hadley and Whitin result, that Y_0 is homogeneously distributed between b and b+Q.

$$P_2(b,Q) = 1-\frac{1}{QE[D]}\left\{\int_b^{b+Q}(E[(D[0,A_1+L_1]-x)^+] - E[(D[0,L_0]-x)^+])\,dx\right\} \tag{4.14}$$

Define Z_1 and Z_2 by

$Z_1 := D_1 + D[A_2,A_1+L_1]$
$Z_2 := D[0,L_0]$

Note that

$P\{Z_1 \leq z\} = P\{D + Z_2 \leq z\}$.

Letting $F_i(.)$ denote the pdf of Z_i we obtain after some algebra

$$P_2(b,Q) = 1-\frac{1}{QE[D]}\left\{\int_b^{\infty}½(x-b)^2dF_{Z_1}(x) - \int_{b=Q}^{\infty}½(x-(b+Q))^2\ dF_{Z_1}(x)\right.$$
$$\left. - \int_b^{\infty}½(x-b)^2dF_{Z_2}(x) - \int_{b+Q}^{\infty}½(x-(b+Q))^2dF_{Z_2}(x)\right\} \tag{4.15}$$

Fitting tractable pdf's to Z_1 and Z_2, e.g. mixtures of Erlang distributions, we can calculate $P_2(b,Q)$ for given b and Q. We might compare the approximation resulting from (4.4) with the approximations resulting from (4.15): We stress the fact that, though (4.15) is exact, any result obtained from this equation is inevitably an approximation, because of the intractability of the exact distribution of Z_1 and Z_2. The approximation is caused by the two- or three-moment fit used.

The computations involved with (4.15) are considerably more complex than the computations involved with (4.4). Since the approximations resulting from direct application of (4.4) perform well we advise to apply (4.4) instead of (4.15). Another comment is in order here. In the derivation of (4.15) we implicitly assume that the sequence of pseudo-lead times do not include overtaking lead times. This is quite restrictive, since the time between the initiation of the pseudo-lead times may be small compared to the lead times themselves, so that overtaking might occur frequently when lead times are stochastic. With large Q even for stochastic lead times overtaking of the real lead times hardly occurs. Therefore the analysis yielding (4.4) is applicable to that case. Apparently the assumptions leading to (4.4) and (4.15) yield results that are applicable, even when the assumptions are not valid. We now provide insight into the robustness of (4.4), for all Q in spite of the fact that the derivation of (4.4) is based on Q>>E[D].

We observe that (4.15) is fit for application of the PDF-method as well. Let us define $\hat{\gamma}(.)$ by

$\hat{\gamma}(x) = P_2(x-Q,Q)$, $x \geq 0$,

where $P_2(b,Q)$ is given by (4.15). Then $\hat{\gamma}(.)$ is a pdf of some random variable $X_{\hat{\gamma}}$. For application of the PDF-method the first two moments of $X_{\hat{\gamma}}$ are required. Along the usual lines we obtain after some algebra,

$$E[X_{\hat{\gamma}}] = ½Q + ½\left(\frac{E[Z_1^2] - E[Z_2^2]}{E[D]} \right)$$

$$E[X_{\hat{\gamma}}^2] = \frac{1}{3}Q^2 + Q\left(\frac{E[Z_1^2] - E[Z_2^2]}{2E[D]} \right) + \left(\frac{E[Z_1^3] - E[Z_2^3]}{3E[D]} \right)$$

Next we substitute $Z_1 = D + Z_2$. This yields

$$E[X_{\hat{\gamma}}] = \frac{1}{2}Q + E[Z_2] + \frac{1}{2}\frac{E[D^2]}{E[D]} \tag{4.16}$$

$$E[X_{\hat{\gamma}}^2] = \frac{1}{3}Q^2 + Q\left(E[Z_2] + \frac{1}{2}\frac{E[D^2]}{E[D]} \right) + \frac{E[D^3]}{3E[D]} + 2\frac{E[D^2]}{2E[D]}E[Z^2] + E[Z_2^2] \tag{4.17}$$

Let us return to equations (4.7) and (4.8), which give the first two moments of X_γ associated with approximation (4.4) of $P_2(b,Q)$.

$$E[X_\gamma] = \frac{1}{2}Q + E[Z]$$

$$E[X_\gamma^2] = E[Z^2] + E[Z]Q + \frac{Q^2}{3}$$

From the definition of Z and Z_2 we find

$Z = Z_2 + \tilde{U}_1$

and hence

$$E[X_\gamma] = \frac{1}{2}Q + E[Z_2] + E[\tilde{U}_1] \tag{4.18}$$

$$\begin{aligned} E[X_\gamma^2] = & \frac{1}{3}Q^2 + Q(E[Z_2] + E[\tilde{U}_1]) \\ & + (E[Z_2^2] + 2E[Z_2]E[\tilde{U}_1] + E[\tilde{U}_1^2]) \end{aligned} \tag{4.19}$$

Substituting the first two moments of $\tilde{U}_1$ we find that (4.16) and (4.17) are identical to (4.18) and (4.19), respectively! Hence application of the PDF-method to either (4.4) or (4.14) leads to exactly the same results. Assuming the PDF-method performs well (which is true), we thereby have an explanation for the robustness of approximation (4.4).

It is interesting to note that as $Q \rightarrow \infty$, $\gamma(.)$ becomes an uniform distribution on (0,Q). This can be shown by the use of Laplace-Stieltjes transforms. This implies that for Q large, $Q \geq 20E[D]$, say, we must fit a uniform distribution to $\gamma(.)$. This yields even a simpler scheme than given in section 2.6.

Another feasible approach for Q large and b>0 is to approximate $P_2(b,Q)$ by

$$P_2^\infty(b,Q) \simeq 1 - \frac{1}{Q} E[(Z-b)^+]$$

and apply the PDF-method to $\hat{\gamma}(.)$, with

$$\hat{\gamma}(x) = \frac{P_2^\infty(x,Q) - P_2^\infty(0,Q)}{1 - P_2^\infty(0,Q)} = 1 - \frac{E[(Z-x)^+]}{E[Z]}$$

It remains to show that the PDF-method performs well applied to the P_2-measure in the (b,Q)-model. The results of extensive simulations are depicted in table 4.1.

The average order size

In the definition of the (b,Q)-strategy we stated that upon decreasing below b the inventory position is increased by a multiple of Q, such that the inventory position immediately after ordering is between b and b+Q. If Q is large compared with E[D], then the probability that two or more minimal batches of size Q are ordered, is negligible. However, in present day's industry there is a trend towards smaller batch sizes in order to have frequent replenishments on Just In Time basis. This is not only a difference in terms of magnitude of Q. The batch size Q gets a completely different function: Q is no longer an economic lot size, which is determined based on cost considerations. From now on Q is a transportation batch which size is based on material handling considerations. Q is a pallet, a box or a truck load. Typically, the batch size Q based on cost consideration, like the EOQ in the deterministic inventory management model, is much larger than the batch size Q based on material handling considerations and other logistic notions, such as throughput time and pipeline stock.

This discussion poses a problem. In most literature it is assumed that the order size equals Q. This no longer holds for small values of Q. Is it possible to get some exact expression or accurate approximation for the order size distribution? This is indeed possible along the following lines.

Recall that Y_0 is the inventory position immediately after an arrival of a customer. Y_0 is homogeneously distributed on (b,b+Q). The next customer arriving at the stock keeping facility causes an undershoot of the reorder level b if $D>Y_0-b$.

Let us denote the undershoot by U. Then the probability distribution function of U is given by

$$P\{U\geq x\} = \frac{\int_0^Q P\{D\geq x+w\}\,dw}{\int_0^Q P\{D\geq w\}\,dw} \tag{4.20}$$

Note that taking limits for $Q\downarrow 0$ and $Q\to\infty$ we have

$$P\{U(0)\geq x\} = P\{D\geq x\}$$

$$P\{U(\infty)\geq x\} = \frac{1}{E[D]}\int_x^\infty P\{D\geq w\}\,dw,$$

which is consistent: If Q=0 then the (b,Q)-model becomes a lot for lot ordering model. The undershoot is identical to the order size, which in turn is equal to the demand size. If $Q\to\infty$ then our approximation $\tilde{U}_1$ of U(.) is exact and consistent with the above result.

Let us concentrate on the batch ordered. This batch is a multiple K of Q, where K is a random variable. It follows that

$$K = k \Leftrightarrow (k-1)Q \leq U(Q) < kQ \qquad k=1,2,...$$

Define the order size Q̂ by

Q̂ = K.Q

We want to have an expression for E[Q̂]. It suffices to derive an expression for E[K]. We proceed as follows.

$$\begin{aligned} E[K] &= \sum_{k=1}^{\infty} k \ P\{K=k\} \\ &= \sum_{k=1}^{\infty} k[P\{U(Q) \geq (k-1)Q\} - P\{U(Q) \geq kQ\}] \\ &= \sum_{k=0}^{\infty} P\{U(Q) \geq kQ\} \end{aligned}$$

Next we substitute (4.20) into the above equation.

$$E[K] = \frac{\sum_{k=0}^{\infty} \int_0^Q P\{D \geq kQ+w\} dw}{\int_0^Q P\{D \geq w\} dw}$$

The numerator is further elaborated on and we end up with the remarkable result that

$$E[K] = \frac{E[D]}{\int_0^Q P\{D \geq w\} dw}$$

and therefore

$$E[\hat{Q}] = \frac{Q\ E[D]}{\int_0^Q P\{D \geq w\}\, dw} \tag{4.21}$$

The nominator can routinely be evaluated fitting a mixture of Erlang distributions to D. To check consistency we again take limits for $Q \downarrow 0$ and $Q \rightarrow \infty$. It follows that

$$\lim_{Q \downarrow 0} E[\hat{Q}] = E[D]$$

$$\lim_{Q \rightarrow \infty} \frac{E[\hat{Q}] - Q}{Q} = 0,$$

It turns out that higher moments of $\hat{Q}$ cannot be written as simple formulas like (4.21). We therefore restrict to the first moment of $\hat{Q}$, only.

<u>The fill rate</u>

Another practically useful service measure we discuss is the fill rate. Recall that the fill rate $\hat{P}_1$ is defined by

$\hat{P}_1$:= the long-run fraction of time the net stock is positive.

As in section 3.1. it turns out that a derivation of an expression for the fill rate is considerably more complex than a derivation of an expression for the P_2-measure. Only for the special case of deterministic interarrival times and constant lead times we find a simple expression along the following lines. The inventory position immediately after an arrival epoch equals Y_0. As before let L_0 be the pseudo-lead time associated with the present arrival. At the moment of the pseudo-replenishment the net stock equals Y_0-D[0,L_0] and remains constant until the next pseudo-replenishment, which is at A+L_0, since both A and L_0 are constant. Then it is easy to see that

$$\hat{P}_1(b,Q) = P\{Y_0 - D(0,L_0] > 0\} \tag{4.22}$$

This equation is not valid for stochastic interarrival times and/ or stochastic lead times, due to the fact that the net stock is no longer constant during replenishment cycles.

For the general case we must do a more intricate analysis, finally yielding again tractable expressions. The basis for our analysis is the real replenishment cycle. As in the derivation of (4.4) we assume that Q is large enough to assume that the undershoot U is distributed as the stationary residual lifetime associated with demand D, i.e.

$$P\{U\leq x\} = \frac{1}{E[D]} \int_0^x (1-F_D(y))\,dy$$

Suppose that a batch of size Q is ordered at the supplier at time 0. The lead time of this order is L_0. The next order is initiated at time σ_1 and arrives at time σ_1+L_1.

The random variable $T^+(b,Q)$ is defined as

$T^+(b,Q)$:= time the net stock is positive during the replenishment cycle $(L_0,\sigma+L_1]$.

Then we can express the fill rate $\hat{P}_1(b,Q)$ as follows

$$\hat{P}_1(b,Q) = \frac{E[T^+(b,Q)]}{E[\sigma_1]} \tag{4.23}$$

We need expressions for $E[\sigma_1]$ and $E[T^+(b,Q)]$. We first consider $E[\sigma_1]$. The random variable σ_1 is determined by the sum of the interarrival times associated with the customers arriving between 0 and σ_1. The number of customers arriving is completely determined by the inventory position at time 0 and the demands of the arriving customers. Translating this into formulas, define

N := the number of customers arriving in $(0,\sigma_1]$.

A_n := n^{th} interarrival time , $n\geq 1$.

D_n := demand of n^{th} customer arriving after 0 , $n\geq 1$.

It follows that

$$\sigma_1 = \sum_{n=1}^{N} A_n$$

We assumed that $\{D_n\}$ and $\{A_n\}$ are independent. Thus N is independent of A_n, since N is completely determined by $\{D_n\}$ and the inventory position at time 0. This implies that

$$E[\sigma_1] = E[N]\ E[A]$$

We need an expression for E[N]. Let us consider the total demand during $(0,\sigma_1]$. It is clear that

$$D(0,\sigma_1] = \sum_{n=1}^{N} D_n$$

Applying the mathematical concept of stopping times we obtain

$$E[D(0,\sigma_1]] = E[N]E[D]$$

On the other hand $D(0,\sigma_1]$ equals the difference between the inventory immediately after 0 and at time σ_1. Hence

$$D(0,\sigma_1] = b+Q-U_0-(b-U_1)$$

Assuming Q>>E[D] we have that $E[U_0]\approx E[U_1]$ and therefore $E[D(0,\sigma_1]]=Q$

Combining the above equations we find

$$E[\sigma_1] = \frac{Q}{E[D]}\ E[A], \tag{4.24}$$

which is intuitively clear.

An expression for $E[T^+(b,Q)]$ requires a more intricate analysis. We resort to a result obtained in chapter 2. Define $T^+(x)$ as follows

$T^+(x,t)$:= the time the net stock is positive during (0,t], given the net stock at time 0 equals x, $x\geq 0$.

Here it is assumed that both time 0 and time t are arbitrary points in time from the point of view of the arrival process $\{A_n\}$, i.e. assumption (B) holds for both time 0 and t. Of course $T^+(x,t)$ depends on $\{D_n\}$ and $\{A_n\}$. In chapter 2 we derived an approximation for $E[T^+(x,t)]$, which is repeated for the reader's convenience

$$\begin{aligned} E[T^+(x,t)] &= (E[\hat{A}]-E[A])(1-F_{D(0,t]}(x)) \\ &+ E[A]\left(M(x) - \int_0^x M(x-y)\, dF_{D(0,t]}(x) \right),\quad x\geq 0, \end{aligned} \tag{4.25}$$

where $E[\hat{A}]$ is the stationary residual lifetime associated with the renewal process $\{A_n\}$ (cf. 2.53).

Conditioning on $\sigma_1+L_1-L_0$, U_0 and $D(0,L_0]$ we can express $E[T^+(b,Q)]$ in terms of $E(T^+(x,t)]$,

$$E[T^+(b,Q)] = \int_0^\infty \int_0^{b+Q} E[T^+(b+Q-y,t)]\, dF_{U_0+D(0,L_0]\,|\,\sigma_1+L_1-L_0=t}(y)\, dF_{\sigma_1+L_1-L_0}(t) \qquad (4.26)$$

Substituting (4.25) into (4.26) we find

$$E[T^+(b,Q)] = \left(E[\hat{A}]-E[A]\int_0^\infty\int_0^{b+Q} (1-F_{D(0,t]}(b+Q-y)\right) dF_{U_0+D(0,L_0]\,|\,\sigma_1+L_1-L_0=t}(y)\, dF_{\sigma_1+L_1-L_0}(t)$$

$$+\ E[A]\int_0^\infty\int_0^{b+Q} \left(M(b+Q-y)-\int_0^{b+Q-y} M(b+Q-y-z)\, dF_{D(0,t]}(z)\right) dF_{U_0+D(0,L_0]\,|\,\sigma_1+L_1-L_0}(y)\, dF_{\sigma_1+L_1-L_0}(t)$$

Using standard probabilistic arguments we can further simplify this equation to

$$E[T^+(b,Q)] = (E[\hat{A}]-E[A])(F_{U_0+D(0,L_0]}(b+Q) - F_{U_0+D(0,\sigma_1+L_1]}(b+Q)$$

$$+\ E[A]\left(\int_0^{b+Q} M(b+Q-y)\, dF_{U_0+D(0,L_0]}(y) - \int_0^{b+Q} M(b+Q-y)\, dF_{U_0+D(0,\sigma_1+L_1]}(y)\right)$$

$$(4.27)$$

Now note that

$$\begin{aligned} U_0+D(0,\sigma_1+L_1] &= U_0+D(0,\sigma_1]+D(\sigma_1,\sigma_1+L_1] \\ &= U_0+(b+Q-U_0-(b-U_1))+D(\sigma_1,\sigma_1+L_1] \\ &= U_1+Q+D(\sigma_1,\sigma_1+L_1] \end{aligned}$$

Hence

$$P\{U_0+D(0,\sigma_1+L_1]\geq x\} = P\{U_1+D(\sigma_1,\sigma_1+L_1]\leq x-Q\},\ x\geq Q \qquad (4.28)$$

Substitution of (4.28) into (4.27) yields

$$E[T^+(b,Q)] = (E[\hat{A}]-E[A])(F_{U_0+D(0,L_0]}(b+Q)-F_{U_1+D(\sigma_1,\sigma_1+L_1]}(b)) + E[A]\left(\int_0^{b+Q} M(b+Q-y)\,dF_{U_0+D(0,L_0]}(y) - \int_0^{b} M(b-y)\,dF_{U_1+D(\sigma_1,\sigma_1+L_1]}(y)\right) \tag{4.29}$$

The first term on the rhs of (4.29) can be routinely evaluated by fitting mixtures of Erlang distributions or a gamma distribution to the first two moments of U_0+D(0,L_0]. Of course U_1+D[σ_1,σ_1+L_1] is identically distributed as U_0+D(0,L_0]. The second term on the rhs of (4.29) can be simplified considerably.

In general the renewal function M(.) cannot be given explicitly. At first sight the second term on the rhs of (4.29) seems intractable, since it involves M(.). Here we are rescued by another basic result from renewal theory. Let U be the stationary residual lifetime associated with M(.). In this case U is associated with {D_n}. Then we have the following fundamental result.

$$\int_0^{x} M(x-y)\,dF_U(y) = \frac{x}{E[D]},\ x\geq 0 \tag{4.30}$$

Let us consider the first integral in the second term on the rhs of (4.29).

$$\int_0^{b+Q} M(b+Q-y)\,dF_{U_0+D(0,L_0]}(y) = \int_0^{b+Q}\int_0^{b+Q-y} M(b+Q-y-z)\,dF_{U_0}(z)\,dF_{D(0,L_0]}(y)$$

The above equation is just using the fact that $F_{U0+D(0,L_0]}$ is the convolution of F_{U0} and $F_{D(0,L_0]}$. Then (4.30) tells us that

$$\int_0^{b+Q} M(b+Q-y)\, dF_{U_0+D(0,L_0]}(y) = \int_0^{b+Q} \frac{(b+Q-y)}{E[D]}\, dF_{D(0,L_0]}(y) \tag{4.31}$$

Our final result for $E[T^+(b,Q)]$ combines (4.29) and (4.31) to yield

$$\begin{aligned} E[T^+(b,Q)] &= (E[\hat{A}]-E[A]\,(F_{Z_1}(b+Q)-F_{Z_1}(b)) \\ &\quad + E[A]\left(\int_0^{b+Q} \frac{(b+Q-y)}{E[D]}\, dF_{Z_2}(y) - \int_0^{b} \frac{(b-y)}{E[D]}\, dF_{Z_2}(y) \right) \end{aligned} \tag{4.32}$$

Then the fill rate $\hat{P}_1$ can be derived from

$$\begin{aligned} \hat{P}_1(b,Q) &= \frac{E[D]}{Q} \left\{ \frac{E[\tilde{A}]-E[A]}{E[A]}\, (F_{Z_1}(b+Q)-F_{Z_1}(b)) \right. \\ &\quad \left. + \frac{1}{Q} \int_0^{b+Q} (b+Q-y)\, dF_{Z_2}(y) - \int_0^{b} (b-y)\, dF_{Z_2}(y) \right\} \end{aligned} \tag{4.33}$$

In several publications in the literature it is assumed that the fill rate $\hat{P}_1$ can be expressed in terms of the P_2-measure along the following lines. Suppose the average shortage at the end of a replenishment cycle is E[B]. Since the average demand rate is E[D]/E[A] it follows that the average time that the stock was negative during the replenishment cycle equals E[B]E[A]/E[D]. We approximately have

$$P_2 \simeq 1-\frac{E[B]}{Q}$$

$$\begin{aligned} \hat{P}_1 &\simeq 1-\frac{E[B]\,E[A]}{E[D]} \Big/ \frac{Q}{E[D]}\; E[A] \\ &= 1-\frac{E[B]}{Q} \end{aligned}$$

Hence $\hat{P}_1 = P_2$. It is clear that these arguments are erroneous. Putting the expressions for $\hat{P}_1$ and P_2 alongside the difference is apparent.

$$\begin{aligned} \hat{P}_1(b,Q) &= \frac{E[D]}{Q}\; \frac{E[\tilde{A}]-E[A]}{E[A]}\, (F_{Z_1}(b+Q)-F_{Z_1}(b)) \\ &\quad + \frac{1}{Q}\left(\int_0^{b+Q} (b+Q-y)\, dF_{Z_2}(y) - \int_0^{b} (b-y)\, dF_{Z_2}(y) \right), \quad b \geq -Q \end{aligned}$$

$$P_2(b,Q) = \frac{1}{Q}\left(\int_0^{b+Q} (b+Q-y)\,dF_{Z_1}(y) - \int_0^{b} (b-y)\,dF_{Z_1}(y) \right), \quad b \geq -Q$$

We observe that equality holds if (E[Ã]-E[A])E[A] is negligible and also the undershoot of the reorder level b is negligible. The latter condition ensures that $Z_1 \stackrel{d}{=} Z_2$. This holds in the case of demand at high rate and incremental demand per customer. It is clear that for this case the heuristic arguments are valid. Also if demand is compound Poisson with constant demand, equality holds.

Let us apply the PDF-method to $\hat{P}_1(b,Q)$. As usual let γ(.) be defined as

$$\gamma(x) = \hat{P}_1(x-Q,Q), \quad x \geq 0$$

Then γ(.) is the pdf of some random variable X_γ. Without going into detail we claim that the first two moments of X_γ are given by

$$E[X_\gamma] = E[Z_2] + \frac{1}{2}Q - \frac{(E[\tilde{A}]-E[A])}{E[A]}E[D] \qquad (4.34)$$

$$E[X_Y^2] = E[Z_2^2] + 2E[Z_2]Q + \frac{Q^2}{3} - \frac{(E[\tilde{A}]-E[A])}{E[A]}E[D]\ (Q+2E[Z_1]) \qquad (4.35)$$

Then $\hat{\gamma}(.)$ is the gamma distribution with its first two moments given by (4.34) and (4.35). Then we can solve for b* in

$\hat{P}_1(b^*,Q) = \alpha$

by

$b^* \simeq \hat{\gamma}^{-1}(\alpha)-Q$

for given value of Q.

Conclusions

This concludes our discussion of the service measures. We have shown that some intricate mathematical analysis yields tractable results for both the P_2- and the $\hat{P}_1$-measure. The PDF-method provides the routine calculations to solve for appropriate reorder levels in large scale systems, such as warehouses for service parts and purchase systems. We explained the fact that the $\hat{P}_1$-measure is in general not the same as the P_2-measure as is often claimed in the literature. We gave intuition for what situations $\hat{P}_1$ and P_2 are approximately the same.

Unlike most of the literature we discussed the general case of compound renewal processes. We think this is appropriate, since in most cases the demand process does not originate from a large number of independent customers, which leads to Poissonian arrivals, nor are interarrival times constant, leading to discrete time models. It is clear that assuming compound Poisson demand, where the interarrival are non-erratic, the target service levels are exceeded.

4.2. The mean physical stock

Using the expressions for the service measures derived in the last section we are able to compute the reorder level b satisfying a service level constraint given some order quantity Q. Another important performance criteria is the average physical stock needed to maintain the required service. In the literature (cf. Silver and Peterson [1985]) usually an interpolation rule is applied to determine the average net stock. Assuming backorders are negligible the average net stock equals the mean physical stock. The resulting approximation for $E[X^+]$ is given by

$$E[X^+] \simeq b+\frac{1}{2}Q - E[D(0,L_0]] - E[U_0]$$

Substituting approximations for $E[D(0,L_0]]$ and $E[U_0]$ from (4.14)-(4.17) we obtain

$$E[X^+] \simeq b + \frac{1}{2}Q - \left(\frac{E[L]}{E[A]} + \frac{(c_A^2+c_D^2)}{2} \right) E[D] \tag{4.36}$$

Here c_A and c_A denote the coefficient of variation of the interarrival time and demand per customer, respectively.

We attempt a more rigorous mathematical approach. We consider the replenishment cycle $(L_0,T_1+L_1]$. As in section 3.2.2. we make the following assumption.

From the point of view of the renewal process all replenishment moments are arbitrary points in time.

This assumption enables us to apply basic results from chapter 2 concerning renewal theory.

Assume for the moment that x equals the net stock at time L_0 and t equals the length of the replenishment cycle $(L_0,T_1+L_1]$. Furthermore assume that \$1 is paid per time unit for each item in stock. Then equation (2.60) gives an approximation for the amount $E[H(x,t)]$ paid during the replenishment cycle $(L_0,T_1+L_1]$,

$$\begin{aligned} E[H(x,t)] \simeq\ & (E[\tilde{A}]-E[A])\left(x-\int_0^x (x-y \right) dF_{D(0,t]}(y) \\ & + E[A]\left(\int_0^x (x-y)\, dM(y) - \int_0^x \int_0^{x-y} (x-y-z)\, dM(z)\, dF_{D(0,t]}(y) \right) \end{aligned} \tag{4.37}$$

From the analysis in section 3.1. we know that the net stock at time L_0 equals $b+Q-U_0-D(0,L_0]$. Conditioning on the net stock position at time L_0 and the length of the replenishment cycle we find an approximation for $E[H(b,Q)]$, the average amount paid during a replenishment cycle.

$$E[H(b,Q)] \simeq \int_0^\infty \int_0^{b+Q} E[H(b+Q-y,t)]\, dF_{U_0+D(0,L_0]/\sigma_1+L_1-L_0=t} dF_{\sigma_1+L_1-L_0}(t) \tag{4.38}$$

Substitution of (4.37) into (4.38) yields after careful probabilistic analysis

$$
\begin{aligned}
E[H(b,Q)] \simeq\ & (E[\tilde{A}]-E[A])\left(\int_0^{b+Q} (b+Q-y)\, dF_{U_0+D(0,L_0]}(y) \right. \\
& \left. - \int_0^{b+Q} (b+Q-y)\, dF_{U_0+D(0,\sigma_1+L_1]} \right) \\
& + E[A]\left(\int_0^{b+Q} \int_0^{b+Q-y} (b+Q-y-z)\, dM(z)\, dF_{U_0+D(0,L_0]}(y) \right. \\
& \left. - \int_0^{b+Q} \int_0^{b+Q-y} (b+Q-y-z)\, dM(z)\, dF_{U_0+D(0,\sigma_1+L_1]} \right)
\end{aligned}
\tag{4.39}
$$

At first sight (4.39) seems intractable, due to the occurrence of M(.) and σ_1. We observe that, as in the derivation of an expression for the fill rate, M(.) only occurs in conjunction with $F_{U0}(.)$ (cf. 4.29). From renewal theory we learn that

$$
\begin{aligned}
& \int_0^{x} \int_0^{x-y} (x-y-z)\, dM(z)\, dF_U(y) = \\
& = \int_0^{x} (x-y)\, d(M*F_U)(y) \\
& = \frac{1}{2E[D]}\ x^2, \quad x \geq 0
\end{aligned}
\tag{4.40}
$$

Furthermore we know that

$$
U_0 + D(0,\sigma_1+L_1] = Q + U_1 + D(\sigma_1,\sigma_1+L_1] \tag{4.41}
$$

Combining (4.39), (4.40) and (4.41) we find a remarkable simple expression for E[X(b,Q)],

$$
\begin{aligned}
E[H(b,Q)] = {} & (E[\tilde{A}]-E[A])\left(\int_0^{b+Q} (b+Q-y)\, dF_{U_0+D(0,L_0]}(y) \right. \\
& \left. - \int_0^{b} (b-y)\, dF_{U_1+D(\sigma_1,\sigma_1+L_1]}(y) \right) \\
& + E[A]\left(\int_0^{b+Q} \frac{(b+Q-y)^2}{2E[D]}\, dF_{D(0,L_0]}(y) \right. \\
& \left. - \int_0^{b} \frac{(b-y)^2}{2E[D]}\, dF_{D(\sigma_1,\sigma_1+L_1]}(y) \right)
\end{aligned}
\tag{4.42}
$$

For a compound Poisson demand process the first term of the rhs of (4.42) vanishes due to the fact that $E[\tilde{A}]$ equals E[A]. For non-Poisson interarrival times this term can be rewritten into

$$\int_0^{b+Q} (b+Q-y)\,dF_{U_0+D(0,L_0]}(y) \;-\; \int_0^{b} (b-y)\,dF_{U_1+D(\sigma_1,\sigma_1+L_1]}(y)$$
$$= QP_2(b,Q) \tag{4.43}$$

For the second term on the rhs of (4.42) we derive an alternative expression along the same lines,

$$\int_0^{b+Q} \frac{(b+Q-y)^2}{2E[D]}\,dF_{D(0,L_0]}(y) \;-\; \int_0^{b} \frac{(b-y)^2}{2E[D]}\,dF_{D(\sigma_1,\sigma_1+L_1]}(y)$$
$$= \frac{bQ}{E[D]} + \frac{Q^2}{2E[D]} - \frac{QE[D(0,L_0]]}{E[D]} \tag{4.44}$$
$$- \int_{b+Q}^{\infty} \frac{(y-(b+Q))^2}{2E[D]}\,dF_{D(0,L_0]}(y) \;+\; \int_{b}^{\infty} \frac{(y-b)^2}{2E[D]}\,dF_{D(\sigma_1,\sigma_1+L_1]}(y)$$

For the moment we do not further elaborate on (4.42)-(4.44) for arbitrary values of b and Q. Let us first consider the case of high reorder levels b. The mean physical stock $E[X^+(b,Q)]$ is computed from

$$E[X^+(b,Q)] = \frac{E[H^+(b,Q)]}{E[\sigma_1]}$$
$$= \frac{E[H^+(b,Q)]}{E[A]\,Q}\,.\,E[D] \tag{4.45}$$

in conjunction with (4.42)-(4.44) for large values of b.

We have the following asymptotic results

$$\lim_{b\to\infty} \frac{E[D]}{E[A]\,Q}\;.\;QP_2(b,Q) = \frac{E[D]}{E[A]}$$

$$\lim_{b\to\infty}\left(\frac{E[D]}{E[A]Q}\ .\ (4.48)\ -\ \frac{b}{E[A]}\right) = \frac{Q}{2E[A]} - \frac{E[D(0,L_0]]}{E[A]}$$

Substituting these asymptotic results into (4.42) we find from (4.45)

$$\lim_{b\to\infty}\left(E[X^+(b,Q)]-b\right) = \frac{E[\tilde{A}]-E[A]}{E[A]}\ .\ E[D] + \frac{Q}{2} - E[D(0,L_0]]$$

This yields the following approximation for b large.

$$\begin{aligned} E[X^+(b,Q)] &\simeq b + \frac{Q}{2} - E[D(0,L_0] + \frac{(C_A^2-1)}{2}\ E[D] \\ &= b + \frac{Q}{2} - E[D]\frac{E[L]}{E[A]} \end{aligned} \tag{4.46}$$

Here we made use of approximation (4.14) for $E[D(0,L_0]]$. Approximation (4.46) provides an alternative to (4.36), based on a simple interpolation argument. Both approximations coincide only for incremental demand at high rate, i.e. E[D] small. We also note that (4.46) would be obtained from the interpolation arguments when ignoring undershoots of the reorder-level b as well as assuming a fairly constant demand rate (cf. Hadley and Whitin, p. 166, Silver and Peterson, p. 275). We thus find that ignoring the true stochasticity of the demand process yields an approximation, which is asymptotically exact for compound Poisson demand and performs quite well for non-Poisson interarrivals (as will be shown in the sequel), assuming the reorder level b is large. The present derivation of (4.46) has given true insight into why the widely-applied (and hardly ever motivated) interpolation approximation of the mean physical stock performs well!

We now derive an approximation for $E[X^+(b,Q)]$ for arbitrary values of b. Let us reconsider (4.42) and the auxiliary equations (4.43) and (4.44). Equation (4.43) involves $P_2(b,Q)$ for which we already found an accurate approximation by applying the PDF-method. Equation (4.44) is suited for application of the PDF-method as well. We define $\zeta(.)$ by

$$\zeta(x)\ := \int_x^\infty \frac{(y-x)^2}{2E[D]}\ dF_{D(\sigma_1,\sigma_1+L_1]}(y) - \int_{x+Q}^\infty \frac{(y-(x+Q))^2}{2E[D]}\ dF_{D(0,L_0]}(y)$$

Then it follows that

$$\zeta(-Q) = \frac{Q^2}{2E[D]} + Q\frac{E[D_L]}{E[D]}$$

$\zeta(\infty) = 0$.

Define $\gamma(.)$ by

$$\gamma(x) := 1 - \frac{\zeta(x-Q)}{\zeta(Q)}, \quad x \geq 0$$

Then $\gamma(.)$ is a pdf of some random variable X_γ. Applying the routines of the PDF-method we find

$$E[X_\gamma] = \frac{E[D^2(0,L_0]] + Q\ E[D(0,L_0]] + \frac{1}{3}Q^2}{Q + 2E[D(0,L_0]]} \tag{4.47}$$

$$E[X_\gamma^2] = \frac{\frac{2}{3}\ E[D^3(0,L_0]] + E[D^2(0,L_0]]Q + \frac{2}{3}\ E[D(0,L_0]]Q^2 + \frac{Q^3}{6}}{Q + 2E[D(0,L_0]]} \tag{4.48}$$

Until now we only needed the first two moments of $D(0,L_0]$. Equation (4.48) involves the third moment of $D(0,L_0]$. Instead of computing this third moment we assume that $D(0,L_0]$ is gamma distributed. Then we have

$$E[D^3(0,L_0]] \;=\; (1+C^2_{D(0,L_0]})(1+2c^2_{C(0,L_0]})E^3[D(0,L_0]] \qquad (4.49)$$

Here $c_{D(0,L0]}$ denotes the coefficient of variation of $D(0,L_0]$, which can be computed from (4.14) and (4.15).

Next we fit the gamma distribution $\hat{\gamma}(.)$ to $E[X_\gamma]$ and $E[X_\gamma^2]$ to get an approximation of $\gamma(.)$. Synthesis of all of the above yields

$$\begin{aligned} E[X^+(b,Q)] \;\simeq\; & \frac{(c_A^2-1)}{2}\; E[D]\;\; P_2(b,Q) \\ & + b+Q - (\frac{Q}{2} + E[D(0,L_0]])\;\hat{\gamma}(b+Q) \end{aligned} \qquad (4.50)$$

Note that (4.46) and (4.50) are consistent as should be expected when letting $b\to\infty$. Thus we found a simple-to-compute approximation for the mean physical stock, which considerably improves on the interpolation approximation (4.46) even for moderately variable demand.

4.3. Mean backlog

As in chapter 3 we derive an expression for the mean backlog, based on the relation between inventory position, net stock, pipeline stock and backorders. The random variables Y, X^+, O are defined as in section 3.3. The key equation to compute the P_3-measure, the long-run average backlog, is

$$P_3(b,Q) \;=\; E[X^+] \;+\; E[O] \;-\; E[Y]$$

The cost argument based to obtain E[O] applies here as well. Thus

$$E[O] \;=\; \frac{E[D]}{E[A]}E[L]$$

The mean physical stock is given by (4.50). The average inventory position is obtained from the exact result (cf. Hadley and Whitin [1963]) that the inventory position is homogeneous distributed between b and b+Q. Then an expression for $P_3(b,Q)$ follows.

$$P_3(b,Q) \;=\; E[X^+] \;+\; \frac{E[L]}{E[A]}.E[D] \;-\; b-\frac{1}{2}Q \qquad (4.51)$$

Substituting (4.46) into the above equation yields after rearranging terms

$$P_3(b,Q) = \left(\frac{Q}{2} + E[D]\frac{E[L]}{E[A]}) (1-\hat{\gamma}(b+Q)) - \frac{(c_A^2-1)}{2} E[D] (1-P_2(b,Q) \right) \tag{4.52}$$

Since both $\hat{\gamma}$(.) and P_2(.,Q) approach 1 as b→∞, we have consistently

$$\lim_{b\to\infty} P_3(b,Q) = 0$$

4.4. Cost considerations

We now have expressions for both the average physical stock and the average backlog. This enables us to comment on some conjectures in the literature about average-cost optimal (b,Q)-policies. Assume h and p are the holding cost per item per unit and the penalty cost per item backordered per unit time. Furthermore assume a fixed cost K per order. We want to solve the following problem

$$\underset{(b,Q)}{minimize}\; g(b,Q) = hE[X^+(b,Q)] + p\; P_3(b,Q) + K/E[\sigma_1] \tag{4.53}$$

i.e. minimize the average holding, ordering and penalty cost per time unit. The minimization involves taking partial derivatives of the above cost function with respect to b and Q. Equation (4.42) and relation (4.51) yield an expression for g(b,Q). Without going into details, we claim that the following results hold

$$\frac{\partial}{\partial b}\, g(b,Q) \simeq (h+p)\hat{P}_1(b,Q)-p \tag{4.54}$$

$$\frac{\partial}{\partial Q}\, g(b,Q) = (h+p)\left\{ \frac{E[X^+(b,Q)]}{Q} + \frac{(c_A^2-1)}{2}\frac{E[D]}{Q} F_{U_0+D(0,L_0]}(b+Q) + \frac{1}{Q}\int_0^{b+Q} (b+Q-y)\, dF_{D(0,L_0]}(y) \right\} - \frac{p}{2} - \frac{KE[D]}{E[A]Q^2} \tag{4.55}$$

We emphasize that (4.54) and (4.55) yield (accurate) approximate expressions, since we applied approximations for U_0 and U_1 and made the assumptions with respect to the replenishment moments. Yet assuming approximate exactness we can derive the following striking result from (4.54).

Minimization of average holding and penalty costs implies that the fill rate equals p/(p+h).

This is indeed striking since it the literature it is generally believed that the above result holds for the P_2-measure instead of for the fill rate. This is only true when P_2 and $\dot{P}_1$ are identical, i.e.,

(i) compound Poisson demand process with fixed demand per customer.

(ii) continuous demand.

In fact (ii) is the deterministic model. Case (i) is covered in Hadley and Whitin [1963], yet in the literature, e.g. in Silver and Peterson, this is erroneously generalized to arbitrary demand processes.

From (4.54) we might find b as a function of Q. Then the problem could be solved by finding a root of (4.55),

$$\frac{\partial}{\partial Q}\, g(b,Q)\,/_{b=b(Q)} = 0$$

Yet it is just as simple to minimize g(b(Q),Q) directly from (4.53), since g(b(Q),Q) is convex as a function of Q. Some standard approach might be used, which need not to be time consuming because of the simplicity of the approximations.

A straightforward approximate procedure is as follows.

(i) Let Q be the Economic Order Quantity.

$$Q = \sqrt{\frac{2KE[D]}{hE[A]}}$$

(ii) Determine b from (4.54).

This procedure yields reasonable results, since the average costs given a service level constraint are usually quite flat around the optimum order quantity.

4.4. Dynamic demand

As is pointed out in Silver and Peterson [1985] dropping the assumption of stationary demand dramatically impacts the analysis. In fact, the analysis does not go through at all in contrast with the analysis of the (R,S)-model. The main reason for this is that at decision moments, i.e. moments at which one might decide to order an amount Q, it is not clear when the next decision moment will be. In the (R,S)-model decision moments are R time units apart, no matter what the demand process is.

In Silver [1978] a heuristic analysis is given for the (s,S)-model with dynamic demand. It is based on a combination of the deterministic dynamic demand model, to which the Silver-Meal heuristic is applied, and a probabilistic analysis based on temporary stationarity.

First of all we have to determine decision moments in time. These might be any moment in time, where one is convinced of major changes in the demand process, causing previously calculated b and Q values to be wrong. Since this conviction is usually based on changes in either the demand per customer or the arrival rate, it seems reasonable to restrict decision moments to customer arrival moments.

Suppose we have prefixed the order quantity Q. The decision we have to make in this point in time is:

Do we order or don't we order an amount Q.

This decision can be made once we know the reorder level b.

Let us assume that we decide to set the reorder level equal to b.

Y(t) := inventory position at time t (now).

X(t) := net stock at time t.

-T := time elapsed since the last order has been initiated.

L_0 := lead time of last order initiated at time T.

L_1 := lead time of order at time 0 if initiated.

We consider the replenishment cycle $(-T+L_0,L_1]$. We suppose that if an order is initiated then Y(0) = b - U, where U is the undershoot of b. Then we can distinguish between the following cases.

(i) $-T + L_0 < 0$

In this case the replenishment cycle has started already. The net stock at time $-T+L_0$ is known and equals $X^+(-T+L_0]$. The projected net stock at time L_1 equals $Y(0)-D(0,L_1] = b-U-D(0,L_1]$. Hence

$$P_2 = 1 - \frac{E[(D(0,L_1]+U-b)^+] - X^-(-T+L-0)}{d(-T+L_0,0] + E[D(0,L_1]]}, \qquad (4.56)$$

where $d(-T+L_0,0]$ is the demand during $(-T+L_0,0]$ (which is known by now).

(ii) $-T + L_0 > 0$

In this case the replenishment cycle has not yet started. We know that $L_0>T$. Therefore we consider $\hat{L}_0$, defined by

$\hat{L}_0 = L_0 | L_0 > T$.

We then compute the P_2-measure from

$$P_2 = 1 - \frac{E[(D(0,L_1]+U-b)^+] - E[(D(0,\hat{L}_0] - (x(0)+Q))^+]}{E[D(\hat{L}_0,L_1]]} \tag{4.57}$$

Both (4.43) and (4.44) involve the same kind of expressions as obtained in the preceding sections. In both equations the decision variable b occurs in $E[(D(0,L_1]+U-b)^+]$. We rewrite (4.43) and (4.44) as follows

(i) $-T + L_0 < 0$

Solve for b in

$$E[(D(0,L_1]+U-b)^+] = X^-(-T+L_0)+(1-P_2)(d(-T+L_0,0]+E[D(0,L_1]] \tag{4.56$'$}$$

(ii) $-T + L_0 \geq 0$

Solve for b in

$$E[(D(0,L_1]+U-b)^+] = E[D(0,\hat{L}_0]-(X(0)+Q)^+]+(1-P_2)E[D(\hat{L}_0,L_1]] \tag{4.57$'$}$$

The PDF-method can be applied to equations of this type.

$$E[(D(0,L_1]+U-b)^+] = c \tag{4.58}$$

for some c>0.

Let ζ(x) be defined as

$\zeta(x) = E[(D(0,L_1]+U-x)^+], x \geq 0$

$\zeta(0) = E[D(0,L_1]+U]$

$\zeta(\infty) = 0$

Define γ(.) by

$$\gamma(x) = 1-\frac{\zeta(x)}{\zeta(0)}$$

Then γ(.) is the pdf of some random variable X_γ for which we have the following first two moments,

$$E[X_\gamma] = \frac{E[(D(0,L_1]+U)^2]}{2E(D(0,L_1]+U]}$$

$$E[X_\gamma^2] = \frac{E[(D(0,L_1]+U)^3]}{3E[D(0,L_1]+U]}$$

We assume that D(0,L_1]+U is gamma distributed to obtain an expression for $E[X_\gamma^2]$. Then (4.44) is solved as follows.

$$c > E[D(0,L_1]+U] \Rightarrow b = E[D(0,L_1]+U]-c$$

$$c < E[D(0,L_1]+U] \Rightarrow b = \hat{\gamma}^{-1}\left(1-\frac{c}{E(D(0,L_1]+U]}\right)$$

$\hat{\gamma}$ is the gamma distribution with has $E[X_\gamma]$ and $E[X_\gamma^2]$ as its first two moments.

We note that forecasts are needed for D(0,L_1] and D(0,$\hat{L}_0$]. D(0,L_1] can be derived from previously gathered data about lead time demand. D(0,$\hat{L}_0$] depends on $\hat{L}_0$, which in turn depends on L_0 and T. The first two moments of $\hat{L}_0$ might be derived from

$$E[\hat{L}_0] = \frac{\int_T^\infty (y-T)\,dF_{L_0}(y)}{1-F_{L_0}(T)}$$

$$E[\hat{L}_0^2] = \frac{\int_T^\infty (y-T)^2\,dF_{L_0}(y)}{1-F_{L_0}(T)}$$

and some convenient fit for $F_L(.)$, e.g. a gamma distribution. Then
0
the first two moments of $D(0,\hat{L}_0]$ might be derived from (4.14) and (4.15) assuming independent demand per customer and the known estimates of the first two moments of $D(0,L_1]$.

$$E[D(0,\hat{L}_0]] = E[D(0,L_1]] - (E[L_1] - E[\hat{L}_0])\frac{E[D]}{E[A]}$$

$$E^2(D(0,\hat{L}_0)) = \sigma^2(D(0,L_1)) - (E[L_1] - E[\hat{L}_0])(\sigma^2(D) + c_A^2\ E^2[D]) - (\sigma^2(L_1) - \sigma^2(\hat{L}_0))\frac{E^2[D]}{E^2[A]}$$

These equations only make sense for non-erratic lead times, since otherwise $E[\hat{L}_0] > E[L_1]$ and $\sigma^2(\hat{L}_0)\sigma^2(L_1)$. This would be conflicting with our assumption that orders do not overtake.

The above sketched procedure is truly dynamic. To reduce the computational burden one might decide to leave b constant unless demand and lead time information prescribe recalibration of b.

Another practical approach is to apply the stationary analysis to the latest demand and lead time information to find b. This approach is advocated by Silver and Peterson [1985] and performs well in practice.