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# Analysis of one product /one location inventory control models

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# 1. Introduction

This handout deals with stock control models for one product in one stock location, the $(s,nQ)$ model. We will derive expressions for performance measures most frequently used in practice. Thereby we observe the framework of concepts and notations defined by Silver, Pyke & Peterson [1998], SPP for short. Why a handout in addition to SPP? The reason is simple to explain: the formulae derived in SPP have a too limited validity. Extensions or adaptations of these formulae, in SPP presented via footnotes, appeared in practice essential for their applicability.

SPP is making assumptions, implicitly or explicitly, about the behaviour of the demand process, the order process and the delivery process. In this handout these assumptions first are made explicit, in order to determine the validity of the SPP formulae. Next we will replace the derivations of SPP by an analysis that is valid without restrictive assumptions. In this way we obtain formulae with a general applicability.

Furthermore, we distinguish between the derivation of formulae on the one hand and the numerical processing of the formulae on the other. In our opinion industrial engineers should be able to derive these formulae themselves. Such a derivation gives them insight in the mechanisms behind the control rules. Once the formulae have been derived, for an industrial engineer numerical processing becomes less interesting. For this reason the handout is accompanied by a spreadsheet. It enables numerical analysis of stock control models on the basis of the formulae derived. By analysing various operational situations, we obtain insight in the effects of uncertainty in demand process, supply performance, order costs, stock-keeping costs and the flexibility or speed of the delivery processes.

The structure of this handout is as follows. First we define the relevant parameters and variables, in Section 2. Next, in Section 3, we list the assumption made by SPP. In Section 4 we give a detailed analysis of the $(s,Q)$ model. Here the assumptions of SPP still play an important part. But in Section 5 we drop these assumptions one after the other, which leads to results that have a wide applicability to real-life problems. Then we analyse the $(R,S)$ model in a similar way (Section 6). Finally, in Section 7, we present the essential formulae of related models, namely the $(s,S)$, $(R,s,S)$ and $(R,s,Q)$ model, without giving the analysis.

# 2. Definitions

| | | |
|---|---|---|
| $X(t)$ | := | net stock at time $t$ |
| $X(t^-)$ | := | net stock just before time t → $X(t^-) = \lim_{t \uparrow t^-} X(t)$; |
| $Y(t)$ | := | inventory position at time $t$ |
| $D$ | := | demand per customer or demand per period |
| $D(t_1, t_2]$ | := | demand during the interval $(t_1, t_2]$, with $(t_1, t_2] = \{x \mid t_1 < \mathrm{x} \le t_2\}$ |
| $s$ | := | reorder point |
| $Q$ | := | order quantity |
| $\tau_i$ | := | $i^{th}$ replenishment order moment *after* time $t = 0$ ($i = 1,2,\ldots$) |
| $\tau_0$ | := | 0, time origin at which the first replenishment order is placed |
| $L_i$ | := | delivery time of the replenishment order placed at time $t = \tau_i$ ($i = 0,1,\ldots$) |
| $\upsilon$ | := | expected net stock immediately before the arrival of a replenishment order (safety stock) |
| $B(t_1, t_2]$ | := | demand backordered in $(t_1, t_2]$ |
| $\mathrm{x}^+$ | := | max (0, x) |
| $P\{\ldots\}$ | := | Probability {…} |
| $E[\ldots]$ | := | Expectation […] |
| $\sigma^2(\ldots)$ | := | Variance […] |

$\Phi(k)$ $\quad := \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\infty}^{k} \exp(-\frac{1}{2} z^2) dz$ .

$G(k)$ $\quad := E\left\{ (Z-k)^+ \right\} = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{k}^{\infty} (z-k) \exp\left( -\frac{1}{2} z^2 \right) dz$

$P_1$ : probability of not being out-of-stock just before a replenishment order arrives

$P_2$ : long-run fraction of total demand, which is being delivered from stock on hand (also known as fill-rate)

# 3. Assumptions

In SPP the following assumptions lie at the basis of all formulae derived in Chapters 7, 8 and 9:

(i) $D(t_1, t_2]$ has a normal distribution with expectation $(t_2 - t_1)\mu$ and variance $(t_2 - t_1)\sigma^2$.

(ii) At the moment of ordering the stock position is exactly equal to *s*.

(iii) Subsequent orders cannot overtake each other; so: an order placed later cannot arrive earlier.

(iv) Delivery times are constant and equal to $L$.

(v) The net inventory after arrival of an order is positive.

Implicitly there is another assumption that we are going to use too:

(vi) The reorder quantity is constant and equal to $Q$.

(vii) All demand which cannot be met immediately from stock is backordered.

In our derivations of expressions for $P_1$, $P_2$ and other performance characteristics we will only need assumption (iii) and (vii). For the manual computation of values for these expressions the other assumptions are very useful, since they enable the use of tables for functions associated with the normal distribution function. However, in most practical situations one or more assumptions are violated, so that we need a computer for the computation of $P_1$ and $P_2$. Fortunately, the expressions derived are relatively simple and are easily implemented in e.g. an Excel spreadsheet (cf. De Kok [2002]).

# 4. Analysis of the (*s*,*Q*) model

### 4.1 Performance measurement with $P_1$ (SPP, pages 266-268)

We start by defining the performance measure $P_1$:

(1) $P_1$ : = probability of *no* stock out just before the arrival of an order.

If we consider an order placed at $t = \tau_1$, i.e. the first order after $t = 0$, we can write

(2) $P_1 = P\left\{X\left(\left(\tau_1 + L_1\right)^-\right) \geq 0\right\}$.

This expression is easy to understand: at time $t = (\tau_1 + L_1)$ the order arrives. Just before that arrival, the net stock is $X\left((\tau_1 + L_1)^-\right)$, whereas $P_1$ is the probability that this quantity is not negative. Equation (2) gives us an expression for $P_1$. It is not very useful yet, because we do not know the probability distribution of $X\left((\tau_1 + L_1)^-\right)$. But we can rewrite $X\left((\tau_1 + L_1)^-\right)$ in terms of a known distribution, namely the probability distribution of the demand during the interval $(t_1, t_2]$. That demand equals $D(t_1, t_2]$, with $t_1$ en $t_2$ known constants, and according to assumption (i) it has a normal distribution.

### 4.2 The probability distribution of $X\left((\tau_1 + L_1)^-\right)$

To derive the probability density function of $X\left((\tau_1 + L_1)^-\right)$ we will study an inventory system during two replenishment cycles. We will start analysing the inventory system at a moment in time at which the first replenishment order is being placed. This moment in time is taken as the time origin. In other words, we start our time-scale at this point in time: time 0. The other moments in time, which will be used in our analysis, are:

- $L_0$, the delivery time of the order placed at time $0$
- $\tau_1$ , the moment the second replenishment order is placed
- $\tau_1 + L_1$, the moment the second replenishment order is delivered

These moments in time can be recognised in Figure 1**,** which shows the net inventory $X(t)$ as well as the inventory position $Y(t)$ as a function of time for a (s,Q)-controlled inventory system.

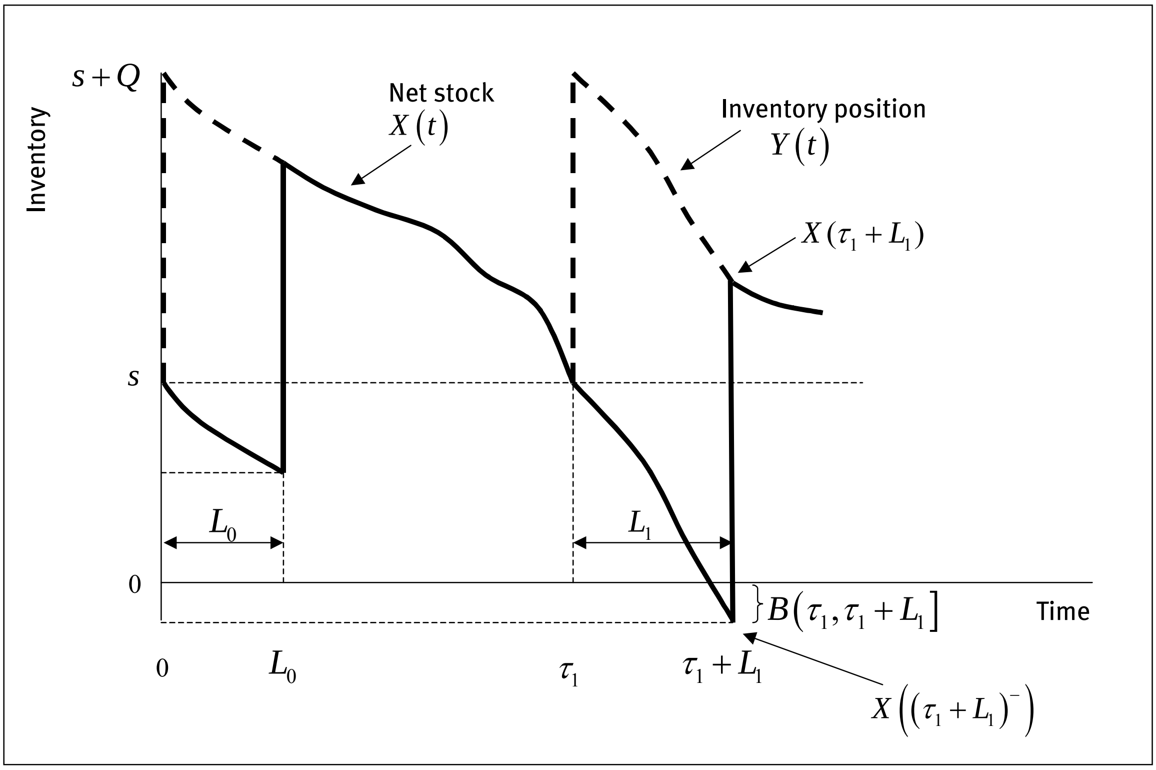


Figure 1 - The inventory in a (s,Q)-system as a function of time.

In order to derive the probability distribution function of $X\left((\tau_1+L_1)^-\right)$, we first state the general 'inventory balance equation'. For any $t_2 \geq t_1$ we have:

(3) $X(t_2) = X(t_1)$ plus all orders arriving in $(t_1, t_2]$ minus all demand in $(t_1, t_2]$.

Next we note that all orders placed after $t = 0$ will arrive after time $t = L_0$ (assumption (iii)). Hence:

(4) all orders arriving in $(0, L_0]$ = all orders outstanding at time 0,

and so

(5) $X(L_0) = X(0)$ plus all outstanding orders at time 0 minus all demand in $(0, L_0]$

$$= Y(0) - D(0, L_0].$$

With assumption (ii) this yields:

(6a) $X(L_0) \; = s + Q - D(0, L_0].$

This is the inventory level *after* the arrival of the order quantity *Q*. So just before that arrival we have:

(6b) $X\left((L_0)^-\right) = s - D(0, L_0]$.

Intermezzo

Apparently, $X(L_0)$ and $X\left((L_0)^-\right)$ have a normal distribution if the demand $D$ is normally distributed. Then their mean value and variance are easily derived from those of $D(0, L_0]$. For $X(\tau_1 + L_1)$ the situation is more complicated.

We return to the performance measure $P_1$. Between $L_0$ and $\tau_1 + L_1$ no orders arrive, because of assumption (iii). So we have:

(7) $X\left((\tau_1 + L_1)^-\right) = X(L_0) - D(L_0, \tau_1 + L_1]$.

Using expression (6a) for $X(L_0)$ we get:

(8) $$\begin{aligned} X\left((\tau_1 + L_1)^-\right) &= s + Q - D(0, L_0] - D(L_0, \tau_1 + L_1] \\ &= s + Q - D(0, \tau_1 + L_1]. \end{aligned}$$

Here a new difficulty arises: $\tau_1$ and $L_1$ are random variables and so $\tau_1 + L_1$ is a random variable too, and we do not know the probability distribution of $D(0, \tau_1 + L_1]$. This problem can be solved in the following way. We rewrite $D(0, \tau_1 + L_1]$:

(9) $D(0, \tau_1 + L_1] = D(0, \tau_1] + D(\tau_1, \tau_1 + L_1]$.

The first term on the right hand side is the demand during a replenishment cycle:

(10) $D(0, \tau_1] = Y(0) - Y(\tau_1)$.

Assumption (ii) tells us that $Y(0) = s + Q$ and $Y(\tau_1) = s$, and so

(11) $D(0, \tau_1] = Q$.

Substitution of (9) and (11) into (8) yields:

(12) $X\left((\tau_1 + L_1)^-\right) = s - D(\tau_1, \tau_1 + L_1]$.

Returning to (2) and substituting (12) we obtain:

(13) $P_1 = P\left\{D(\tau_1, \tau_1 + L_1] \le s\right\}$.

Note that this formula has been derived without any assumption on the number of outstanding orders at the moment an order is placed, i.e. at $\tau_n$, with $n = 1, 2, ....$ In case there is no outstanding order at the moment an order is placed, equation (13) can easily be derived from a simple drawing: see Figure 2.

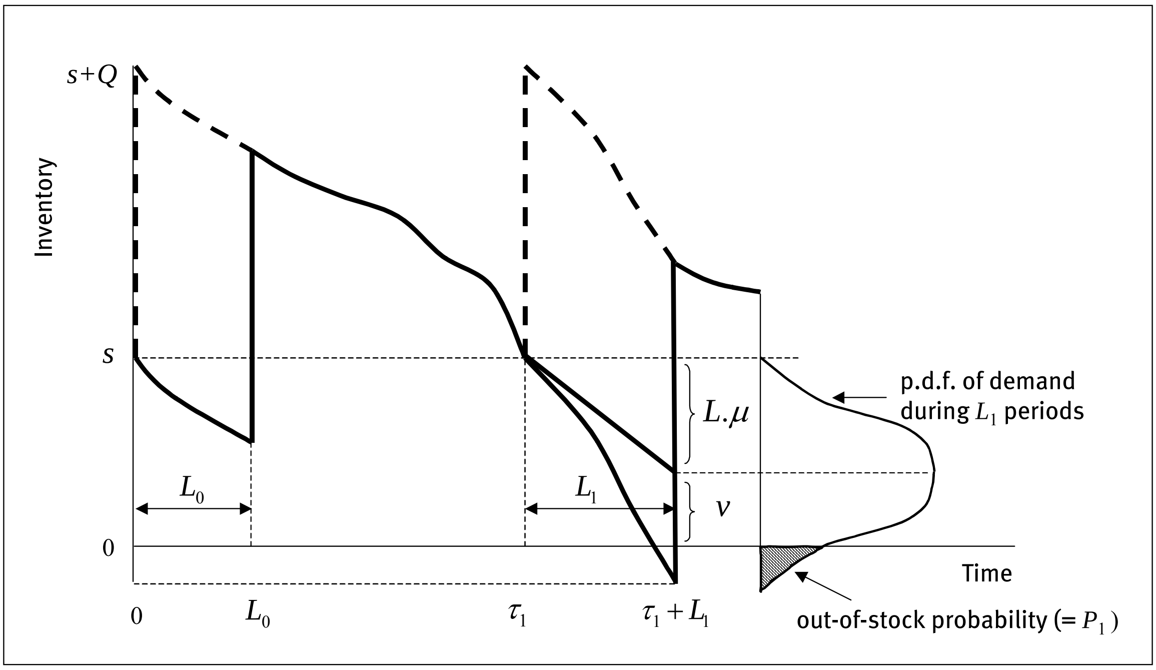


Figure 2 – The inventory level during the interval from $t = \tau_1$ until $t = \tau_1 + L_1$.
The service measure $P_1$ is equal to the probability that demand $D(\tau_1, \tau_1 + L_1)$ does not exceed the reorder level $s$.

More illustrations, showing how inventory level and number of outstanding orders, develop in time can be found in Appendices A and B.

### 4.3 Calculations for demand with a normal distribution

The reorder level s

For convenience we now drop the index of $\tau_1$ and $L_1$. Accepting assumptions (i) and (iv) we know that $D(\tau, \tau + L]$ has a normal distribution with mean $L\mu$ and variance $L\sigma^2$. Therefore we can write (13) as:

$$(14) \quad P_1 = P\left\{\frac{D(\tau,\tau+L]-L\mu}{\sigma\sqrt{L}} \le \frac{s-L\mu}{\sigma\sqrt{L}}\right\} = \Phi\left(\frac{s-L\mu}{\sigma\sqrt{L}}\right),$$

with $\Phi(.)$ the standard-normal probability distribution function.

Intermezzo

The function is related to expressions used by SPP by a simple equation: $\Phi(k) := 1 - p_{u\geq}(k)$.

If we require that $P_1 = \alpha$, then we have

(15) $$\Phi\left(\frac{s-L\mu}{\sigma\sqrt{L}}\right)=\alpha .$$

If we write this as $\Phi(k_\alpha)=\alpha$, we find

(16) $$\frac{s-L\mu}{\sigma\sqrt{L}}=k_\alpha .$$

This leads us an expression for the reorder level *s* (c.f. SPP, page 255):

(17) $$s = L\mu+k_\alpha\sigma\sqrt{L} .$$

So if $P_1 = \alpha$ is given, we can determine the corresponding value for $k_\alpha$. This number follows from $\alpha = \Phi(k_\alpha)$, by using a table or a computer program such as Excel, in which the function $\Phi^{-1}$ is available. Finally, if $L, \mu$ and $\sigma$ are known, we can substitute all these values into (17) and obtain a value for the reorder level *s*.

The safety stock

We define the safety stock ν as the expected net stock just before the arrival of an order. We have:

(18) $$\nu := E\left[X\left((\tau+L)^-\right)\right].$$

Using (12) we get:

$$\nu = E\left[s-D(\tau,\tau+L)\right]$$

$$= s-E\left[D(\tau,\tau+L)\right]$$

or

(19) $$\nu = s-L\mu = k_\alpha\sigma\sqrt{L} .$$

Because of this result, $k_\alpha$ is called the safety factor.

Numerical example (SPP, page 267)

Suppose we know that $\mu L = 58.3$ units and $\sigma\sqrt{L}$ = 13.1 units. We require a performance level $P_1$= 0.90. From a table for $\Phi(k_\alpha)$, or one for $p_{u\geq}(k)=1-\Phi(k_\alpha)$, we find $k_\alpha$ = 1.28 so that ν = 1.28x13.1 = 16.8 → 17 units and *s* = 58.3 + 16.8 = 75.1 → 76 units.

Average inventory

Consider the order placed at time 0 and the next one, placed at time $\tau_1$. They will arrive respectively at $t = L_0$ and $t = \tau_1 + L_1$. In between no orders arrive, because of assumption (iii). So for the average inventory during the replenishment cycle[2] we can write:

$$(20) \qquad E[X] = \frac{1}{2}\left(E[X(L_0)] + E\left[X\left((\tau_1 + L_1)^-\right)\right]\right).$$

We substitute (6a) and (12) into (20) and find:

$$(21) \qquad E[X] = \frac{1}{2}\left(s + Q - E\left[D(0, L_0)\right] + s - E\left[D(\tau_1, \tau_1 + L_1]\right]\right)$$

$$= \frac{1}{2}Q + s - \mu L\ .$$

Using (19) we get:

$$(22) \qquad E[X] = \frac{1}{2}Q + k_\alpha \sigma \sqrt{L}\ .$$

So on average the inventory equals $\frac{1}{2}Q$ plus the safety stock ν.

Numerical example (SPP, page 267), continued

Suppose again that $\mu L = 58.3$ units, $\sigma\sqrt{L}$ = 13.1 units, and the required performance level

$P_1$= 0.90. Then $E[X] = \frac{1}{2}Q + 17$ units.

## 4.4 Performance measurement with $P_2$ (SSP, page 268-269)

We start with two definitions:

(23) $P_2$ : = fraction of demand satisfied directly from the shelf,

and

(24) $P_2$ : = $1 -$ fraction of demand delivered as a backorder.

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[2] Note the difference between $E[X]$ and $E[X(t)]$; $E[X(t)]$ is the expected net stock in a particular period t, while $E[X]$ is the average net stock during the entire replenishment cycle.

In (23) and (24) the word "fraction" refers to the long-term behaviour of the stochastic processes involved, i.e. the demand process and the net inventory position. Mathematically we have then:

(25) $$P_2 = \lim_{t\to\infty}\left(1-\frac{B(0,t]}{D(0,t]}\right).$$

Since we are considering infinite time, $B(0,L_0]$ and $D(0,L_0]$ are not relevant for the long-term fraction. So we can also write:

(26) $$P_2 = \lim_{t\to\infty}\left(1-\frac{B(L_0,t]}{D(L_0,t]}\right).$$

Now we take $t$ very large and equal to $\tau_N + L_N$ with $N$ very large too. Then we get:

(27) $$\frac{B(L_0,t]}{D(L_0,t]} = \frac{B(L_0,\tau_N + L_N]}{D(L_0,\tau_N + L_N]} = \frac{\sum_{i=1}^{N} B(\tau_{i-1}+L_{i-1},\tau_i + L_i]}{\sum_{i=1}^{N} D(\tau_{i-1}+L_{i-1},\tau_i + L_i]},$$

with $\tau_0 = 0$. Substitution of (27) into (26) gives

(28) $$P_2 = \lim_{N\to\infty}\left(1-\frac{\sum_{i=1}^{N} B(\tau_{i-1}+L_{i-1},\tau_i + L_i]}{\sum_{i-1}^{N} D(\tau_{i-1}+L_{i-1},\tau_i + L_i]}\right)$$
$$=1-\frac{\lim_{N\to\infty}\sum_{i=1}^{N} B(\tau_{i-1}+L_{i-1},\tau_i + L_i]}{\lim_{N\to\infty}\sum_{i-1}^{N} D(\tau_{i-1}+L_{i-1},\tau_i + L_i]}.$$

We are allowed to make this step, because the series in (28) are bounded. $P_2$ can be rewritten as:

(29) $$P_2 = 1-\frac{\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N} B(\tau_{i-1}+L_{i-1},\tau_i + L_i]}{\lim_{N\to\infty}\frac{1}{N}\sum_{i-1}^{N} D(\tau_{i-1}+L_{i-1},\tau_i + L_i]}.$$

The argument $(\tau_{i-1}+L_{i-1},\tau_i + L_i]$ represents the $i^{th}$ replenishment cycle *after* t = 0. All replenishment cycles are stochastically identical. They start with the arrival of a replenishment quantity Q. Just *before* that arrival the net stock equals $X((\tau_i + L_i)^-) = s - D(\tau_i,\tau_i + L_i]$, in accordance with equation (6b) if $i = 0$ and with (12) if $i = 1$.

$X$ denotes the stock immediately *after* the arrival of the replenishment quantity $Q$. So the cycle $i$ begins with a net stock of size

$X(\tau_{i-1}+L_{i-1})=s+Q-D(\tau_{i-1},\tau_{i-1}+L_{i-1}]$, and ends with a net stock of size $X((\tau_i+L_i)^-)=s-D(\tau_i,\tau_i+L_i]$. According to the Law of Large Numbers, if $Z_i$ are identically distributed stochastic variables, then

$$\lim_{N\to\infty} \frac{1}{N}\sum_{i=1}^{N} Z_i = E[Z_1]$$

So we have

(30) $$P_2 = 1-\frac{E\left[B(L_0,\tau_1+L_1]\right]}{E\left[D(L_0,\tau_1+L_1]\right]}.$$

This means that

(31) $$P_2 = 1-\frac{\text{(the expected quantity backlogged in a replenishment cycle)}}{\text{(the expected demand in a replenishment cycle)}}$$

It is easy to see that:

(32) $$\begin{aligned} E\left[D(L_0,\tau_1+L_1]\right] &= E\left[D(0,\tau_1+L_1]-D(0,L_0]\right] \\ &= E\left[D((0,\tau_1]+D(\tau_1,\tau_1+L_1]-D(0,L_0]\right] \\ &= E\left[D(0,\tau_1]\right]+E\left[D(\tau_1,\tau_1+L_1]\right]-E\left[D(0,L_0]\right]. \end{aligned}$$

Using assumption (ii) we can now write:

(33) $$E[D(L_0,\tau_1+L_1]] = E\left[D(0,\tau_1]\right] = Q.$$

Note that this result has a general validity, because (33) describes an input-output balance equation: the average demand during a replenishment cycle should equal the average amount replenished during a replenishment cycle.

Finally, we have to find an expression for $E\left[B(L_0,\tau_1+L_1]\right]$, the expected quantity in backlog during the interval $(L_0,\tau_1+L_1]$. We consider three situations, namely:

(i) $X((\tau_1+L_1)^-)\geq 0$

(ii) $X((\tau_1+L_1)^-)<0$ and $X(L_0)\geq 0$

(iii) $X((\tau_1+L_1)^-)<0$ and $X(L_0)<0$.

Note that at time $L_0$ we consider the net stock *after* arrival of the order placed at $t=0$, whereas at time $\tau_1 + L_1$ the order placed at $t=\tau_1$ has not yet arrived. In situation (i) there is no backlog, in situation (ii) there is a backlog of size $-X((\tau_1+L_1)^-)$, and in situation (iii) all demand in $(L_0, \tau_1+L_1]$ is backlogged, i.e. $D\,(L_0,\tau_1+L_1]$. These three situations can be expressed in one formula:

(34) $$B(L_0,\tau_1+L_1] = (D(\tau_1,\tau_1+L_1]-s)^+ - (D(0,L_0]-(s+Q))^+ .$$

Intermezzo:

Before proving this formula, we generalise it (this generalised result will be used in paragraph 5). To do so, we copy two formulae from Section 4.2:

(12) $$X((\tau_1+L_1)^-) = s - D(\tau_1,\tau_1+L_1],$$

(6a) $$X(L_0) = s+Q-D(0,L_0].$$

With these formulae, we can also write (34) as

(34b) $$B(L_0,\tau_1+L_1] = \left(-X\left((\tau_1+L_1)^-\right)\right)^+ - \left(-X(L_0)\right)^+$$

We proof formula (34) by using equation (12) and (6a), mentioned in the Intermezzo above, and by looking at each of the three situations.

Situation (i): $X((\tau_1+L_1)^-) \geq 0$

Since $X(L_0) \geq X((\tau_1+L_1)^-) \geq 0$, we have

(35a) $$D(0,L_0] \leq s+Q$$

(35b) $$D(\tau_1,\tau_1+L_1] \leq s,$$

and so

(36a) $$(D(0,L_0]-(s+Q))^+ = 0$$

(36b) $$(D(\tau_1,\tau_1+L_1]-s)^+ = 0 .$$

Formula (34) then yields $B(L_0,\tau+L_1]=0$, as it should because in situation (i) there is no backlog. Conclusion: formula (34) correct for Situation (i).

Situation (ii): $X((\tau_1+L_1)^-) < 0$ and $X(L_0) \geq 0$

In equation (34) we substitute (12) and (6a). Then we get:

(37) $$B(L_0,\tau_1+L_1] = \left(-X\left((\tau_1+L_1)^-\right)\right)^+ - \left(-X(L_0)\right)^+$$

As $X((\tau_1 + L_1)^-) < 0$ we have $\left(-X\left((\tau_1 + L_1)^-\right)\right)^+ = -X\left((\tau_1 + L_1)^-\right)$. In the same way it follows from $X(L_0) \geq 0$ that $(-X(L_0))^+ = 0$. And so

(38) $\quad B(L_0, \tau_1 + L_1] = -X(\tau_1 + L_1)$.

This indeed is the backlog for situation (ii). Conclusion: equation (34) is correct for situation (ii).

Situation (iii): $\quad X((\tau_1 + L_1)^-) < 0$ and $X(L_0) < 0$.
We return to (37). Now this can be written as:

$$
\begin{aligned}
(39) \qquad B(L_0, \tau_1 + L_1] &= -X\left((\tau_1 + L_1)^-\right) - \left(-X(L_0)\right) \\
&= X(L_0) - X\left((\tau_1 + L_1)^-\right) \\
&= D(L_0, \tau_1 + L_1].
\end{aligned}
$$

In situation (iii) all demand in $(L_0, \tau_1 + L_1]$ is backlogged, i.e. equal to $D\,(L_0, \tau_1 + L_1]$. Conclusion: equation (34) is correct for situation (iii).

After this analysis, we can state that (34) is an expression for the backlog in $(L_0, \tau_1 + L_1]$ valid in all relevant situations. Now we can return to (30), the expression for $P_2$. Substitution of (33) and (34) gives:

(40) $\quad P_2 = 1 - \frac{1}{Q}\left(E\left[\left(D(\tau_1, \tau_1 + L_1] - s\right)^+\right] - E\left[\left(D(0, L_0] - (s+Q)\right)^+\right]\right)$.

With this expression we can calculate $P_2$, if we know $s$, $Q$ and the distribution of two stochastic variables: $D(0, L_0]$ and $D(\tau, \tau + L_1]$. These variables are distributed identically. Now using assumption (v), i.e. the net inventory after arrival of an order is positive, we have

(41) $\quad P\{D(0, L_0] \leq s + Q\} = 1$.

Therefore, the second expectation in (40) is zero and so

(42) $\quad P_2 = 1 - \frac{1}{Q} E\left[\left(D(\tau_1, \tau_1 + L_1] - s\right)^+\right]$.

## 4.5 Calculations for demand with a normal distribution

With assumptions (i) and (iv), $D(\tau_1, \tau_1 + L_1]$ is normally distributed with mean $\mu L$ and variance $\sigma^2 L$. So it makes sense to modify $P_2$ into:

$$(43) \quad \mathrm{P}_2 = 1 - \frac{\sigma\sqrt{L}}{Q} E\left[\left(\frac{D(\tau_1, \tau_1 + L_1] - \mu L}{\sigma\sqrt{L}} - \frac{(s - \mu L)}{\sigma\sqrt{L}}\right)^+\right]$$

$$= 1 - \frac{\sigma\sqrt{L}}{Q} E\left[\left(Z - \left(\frac{s - \mu L}{\sigma\sqrt{L}}\right)\right)^+\right].$$

Here $Z$ has a standard-normal distribution. Therefore, $Z$ is connected with two probability functions:

$$(44) \quad \Phi(x) := P\{Z \le x\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{1}{2}y^2\right) dy$$

and

$$(45) \quad G(k) := E\left[(Z - k)^+\right] = \frac{1}{\sqrt{2\pi}} \int_{k}^{\infty} (y - k) \exp\left(-\frac{1}{2}y^2\right) dy.$$

Next we define a quantity $k_\beta$:

$$(46) \quad k_\beta := \frac{s - \mu L}{\sigma\sqrt{L}}.$$

Then, also using (45), we can write (43) as:

$$(47) \quad \mathrm{P}_2 = 1 - \frac{\sigma\sqrt{L}}{Q} G(k_\beta).$$

If we are looking for a value of $s$ such that $\mathrm{P}_2 = \beta$, then

$$(48) \quad \beta = 1 - \frac{\sigma\sqrt{L}}{Q} G(k_\beta)$$

or

$$(49) \quad G(k_\beta) = \frac{(1 - \beta)Q}{\sigma\sqrt{L}}.$$

Now, if $\beta$, $\mu$, $\sigma, L$ and $Q$ are known, we can calculate the right-hand side of (49). It is a numerical value, say $C_1$. Then we have $G(k_\beta) = C_1$ and from a table we obtain a numerical value for $k_\beta$. Finally, with (46) we get the re-order level s:

$$(50) \quad s = \mu L + k_\beta \sigma\sqrt{L}.$$

There is an important difference with the safety factor $k_\alpha$ for the $P_1$ measure: $k_\beta$ depends for a given $\beta$ not only on $L$, $\mu$ and $\sigma$, but also on $Q$. If $Q$ increases, $k_\beta$ decreases and so do $s$ and the safety stock.

Numerical example 1

Like in Section 4.3 we suppose that $\mu L = 58.3$ units and $\sigma\sqrt{L} = 13.1$ units. We require a performance level $P_2 = 0.90$ and we take $Q = 10$ pieces. Then (48) gives $G(k_\beta) = 0.07634$.

From a table for $G(k_\beta)$ we find by interpolation $k_\beta = 1.045$, so that $s = 58.3 + 13.8 = 72.1 \rightarrow 73$ units. Apparently, in this situation the $P_2$-measure produces values for $s$ and the safety stock slightly lower than with the $P_1$-measure.

Numerical example 2 (SPP, page 269)

Now we have $\mu L = 50$ gallons and $\sigma\sqrt{L} = 11.4$ gallons. Management requires a $P_2$-level of 0.99, whereas $Q$ =200 gallons. Now (49) gives $G(k_\beta) = 0.175$. From the table we obtain $k_\beta = 0.58$ so that $s = 50 + 0.58\ (11.4) = 56.6 \rightarrow 57$ gallons.

# 5. Elimination of assumptions

In the previous chapters, our analysis was based on five assumptions, listed in Chapter 3. They made it easy to obtain results that have proven to be useful in practical situations until 1980. However, owing to changing conditions, in particular smaller order quantities and higher demand variability than in the past, the assumptions tend to lose their validity in modern practice. In this chapter we will remove them one after the other, in order to obtain results with a broader validity than those of SPP.

## 5.1 Assumption (i)

Assumption (i) says that demand has a normal distribution. It can easily be eliminated, for instance by assuming that $D(0,L_0]$ and $D(\tau,\tau+L_1]$ have the same gamma distribution. In many cases this turns out to be realistic. Then the problem is reduced to finding an expression for $E\left[D\left(0,L_0\right]\right]$ and one for $\sigma^2\left(D\left(0,L_0\right]\right)$, because they determine the parameters of the correct gamma distribution. Such expressions depend on the assumptions made about the demand process. In practice we can estimate the value of the variables $E\left[D\left(0,L_0\right]\right]$ and $\sigma^2\left(D\left(0,L_0\right]\right)$ by measuring demand during the delivery times of orders and subsequently calculating mean value and variance of the sample data. This approach has an important consequence: the expressions are only valid for the measured delivery times, so that these measurements lose their value as soon as the measurements change. We return to this subject in Section 5.4, where we elaborate the expressions for $E\left[D\left(0,L_0\right]\right]$ and $\sigma^2\left(D\left(0,L_0\right]\right)$ under different assumptions on the demand process and lead time distributions.

## 5.2 Assumption (ii)

According to assumption (ii), the stock position at the moment of ordering is *exactly* equal to $s$. This assumption is only valid if all customers order the same quantity, say $c$ units. Then it is obvious to take $Q$ as a multiple of $c$, so that at the moment of ordering the stock position indeed always equals $s$. In all more realistic cases, however, the stock position at the moment of ordering will be $s-U$, with $U$ a non-negative stochastic variable called the “undershoot”. See Figure 3.

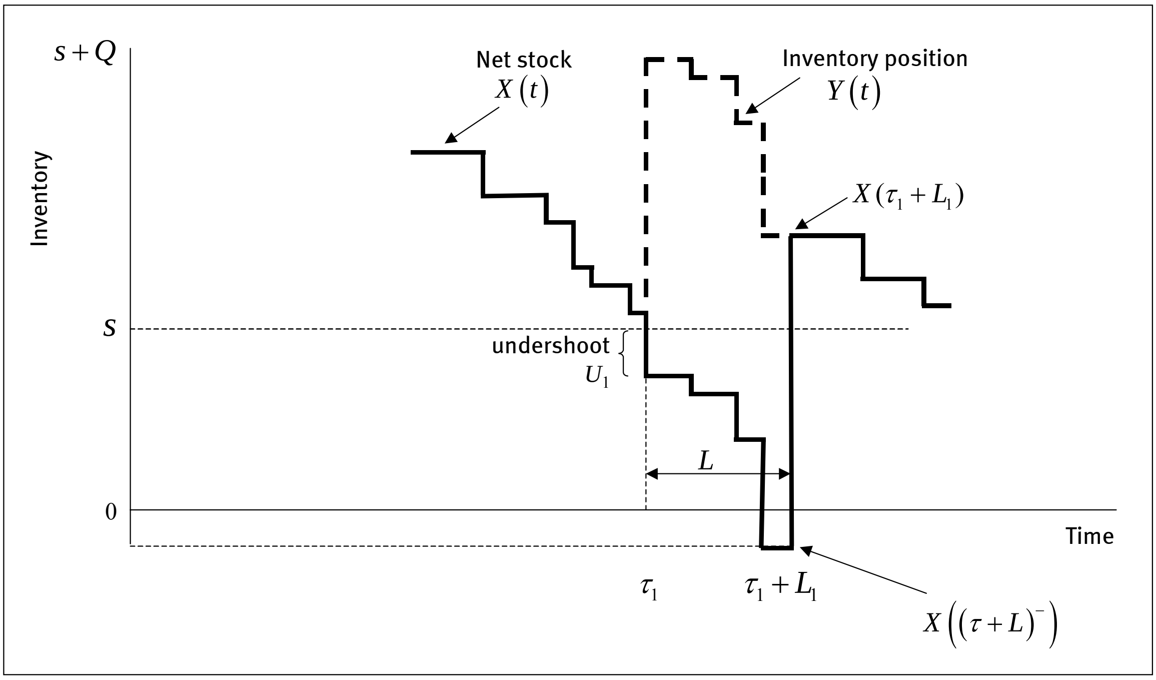



Figure 3 - The undershoot

If demand per customer is equal to $D^c$ we can find good approximations for $E[U]$ and $E\left[U^2\right]$, the mean value and the variance of the undershoot (see Tijms [1994]):

(50) $$E[U] \approx \frac{\sigma^2\left(D^c\right)+E^2\left[D^c\right]}{2E\left[D^c\right]}$$

(51) $$E\left[U^2\right] \approx \frac{E\left[\left(D^c\right)^3\right]}{3E\left[D^c\right]}.$$

In the formulae derived in the previous sections, we now need to account for the fact that the stock position equals $s-U$ at the moment of ordering. We can do so by means of the relations (cf. (6a) and (12)):

(52a) $X\left(L_0\right)=s+Q-U_0-D\left(0,L_0\right],$

(52b) $X\left(\left(\tau_1+L_1\right)^-\right)=s-U_1-D\left(\tau_1,\tau_1+L_1\right],$

where $U_0$ and $U_1$ are the undershoots at times 0 and $\tau_1$. After substitution of these relations the analysis can proceed in the same way as before. It starts by making an assumption about the probability distribution for $D\left(0,L_0\right]+U_0$, and taking the same one for $D\left(\tau_1,\tau_1+L_1\right]+U_1$. For a normal distribution of $D^c$, i.e. $D^c \sim N\left(\mu_c,\sigma_c\right)$ we get, with $cv=\sigma_c/\mu$ the coefficient of variation:

(53a) $E[U]=\frac{1}{2}\left(1+cv^2\right)\mu_c$

(53b) $E\left[U^2\right]=\frac{1}{3}\left(1+3cv^2\right)\mu_c^2$.

However, a gamma distribution is preferable because it is more realistic. So if

$$P\left\{D^c \leq x\right\}=\int_0^x \frac{\exp(-\lambda x)\lambda^\alpha x^{\alpha-1}}{\Gamma(\alpha)}dx,$$

with

$$\alpha=\frac{E^2\left[D^c\right]}{\sigma^2\left(D^c\right)} \text{ and } \lambda=\frac{\alpha}{E\left[D^c\right]},$$

equations (50) and (51) produce:

(54a) $E[U]=\frac{(\alpha+1)}{2\lambda}$

(54b) $E\left[U^2\right]=\frac{(\alpha+1)(\alpha+2)}{3\lambda^2}$.

Once we have determined the first two moments for $U$, we know the mean and variance of $U_1$ (note that var $(U_1)=E\left(U^2\right)-E^2(U)$). Since we also know the mean and variance of $D(\tau_1,\tau_1+L_1]$, we can add them to the mean and variance of $U_1$ to get the mean and variance of $D(\tau_1,\tau_1+L_1]+U_1$.
Next, we choose which pdf (normal, gamma or ...) we consider most appropriate for $D(\tau_1,\tau_1+L_1]+U_1$ and we fit this pdf to the mean and variance of $D(\tau_1,\tau_1+L_1]+U_1$.
Next, we have to combine the general formulas for $P_1$ (formula (2)), the safety stock (18), the average inventory (20), the backorders (34) and $P_2$ (40) with formulas (52a) and (52b) to take into account the undershoot. This results in the following formulas:

$(2')$ $P_1=P\left\{D(\tau_1,\tau_1+L_1]+U_1\leq s\right\}$

$(18')$ $v=s-E\left[D(\tau_1,\tau_1+L_1]\right]-E[U_1]$

$(21')$ $E[X]=s+\frac{Q}{2}-E\left[D(\tau_1,\tau_1+L_1]\right]-E[U_1]$

$(34b')$ $B(L_0,\tau_1+L]=\left(D(\tau_1,\tau_1+L_1]+U_1-s\right)^+-\left(D(0,L_0]+U_0-s-Q\right)^+$

$(40')$ $P_2=1-\frac{1}{Q}\left[E\left[\left(D(\tau_1,\tau_1+L_1]+U_1-s\right)^+\right]-E\left[\left(D(0,L_0]+U_0-s-Q\right)^+\right]\right]$

Given the pdf for $D(\tau_1,\tau_1+L_1]+U_1$, these formulas enable us to determine all relevant logistics variables.

Numerical analysis has revealed that it is indeed crucial to take $U$ into account. See, for instance, the Excel spreadsheet Classical Inventory Models (De Kok [2002]).

### 5.3 Assumption (iii)

Orders cannot overtake each other, according to assumption (iii). Elimination of this assumption is not sensible in the framework of our present model. So assumption (iii) has a general validity. Indeed, we order one product at one supplier. There is no reason whatsoever for the supplier to change the sequence in which he carries out the deliveries of the same product for the same customer. At most will the supplier combine orders to improve efficiency during production or transport.

### 5.4 Assumption (iv)

This assumption says that delivery times are constant and equal to $L$. If we assume furthermore that $L$ equals an integer number of periods, say $K$, we can find expressions for $E\left[D(0,L]\right]$ and $\sigma^2\left(D(0,L]\right)$ in the following way. Define:

(55) $D_k$ : = demand in period $k$,

then $D(0,L] = \sum_{k=1}^{K} D_k$ . Note that $L$ has the dimension "time" whereas $K$ is a dimensionless number. Suppose that $\{D_k\}$ are mutually independent and identically distributed stochastic variables. Then:

(56a) $$E\left[D(0,L]\right] = K\ E[D],$$

(56b) $$\sigma^2\left(D(0,L]\right) = K\sigma^2(D).$$

For convenience we define $L$ too as a *number* of periods. Then $L = K$ and

(57a) $$E[D(0,L]] = L\ E[D],$$

(57b) $$\sigma^2(D(0,L]) = L\ \sigma^2(D).$$

<u>Intermezzo</u>

In case assumption (i) is true, demand has a normal distribution with $E[D] = \mu$ and $\sigma^2(D) = \sigma^2$, so that equations (57) change into:

$$E\left[D(0,L]\right] = L\ \mu$$
$$\sigma^2\left(D(0,L]\right) = L\ \sigma^2 .$$

Next we drop the assumption that $L$ is constant, but we continue to assume that it equals an *integer* number of periods, say $K$. Then we can write

(58)
$$
\begin{aligned}
E\left[D(0,L)\right] &= E\left[\sum_{k=1}^{K} D_k\right] \\
&= \sum_{n=0}^{\infty} E\left[\sum_{k=1}^{n} D_k\right] P\{K=n\} \\
&= \sum_{n=0}^{\infty} nE[D]P\{K=n\} \\
&= E[D]\sum_{n=0}^{\infty} nP\{K=n\} \\
&= E[D]E[K]
\end{aligned}
$$

and

(59)
$$
\begin{aligned}
E\left[D^2(0,L)\right] &= E\left[\left(\sum_{k=1}^{K} D_k\right)^2\right] \\
&= \sum_{n=0}^{\infty} E\left[\left(\sum_{k=1}^{n} D_k\right)^2\right] P\{K=n\} \\
&= \sum_{n=0}^{\infty}\left(\sigma^2\left(\sum_{k=1}^{n} D_k\right) + E^2\left[\sum_{k=1}^{n} D_k\right]\right) P\{K=n\} \\
&= \sum_{n=0}^{\infty}\left(n\sigma^2(D) + n^2E^2[D]\right) P\{K=n\} \\
&= \sigma^2(D)\sum_{n=0}^{\infty} n\,P\{K=n\} + E^2[D]\sum_{n=0}^{\infty} n^2P\{K=n\} \\
&= \sigma^2(D)E[K] + E^2[D]E\left[K^2\right].
\end{aligned}
$$

Since $\sigma^2(D(0,L)) = E\left[D^2(0,L)\right] - E^2\left[D(0,L)\right]$ we find after substitution of (57) and (58):

(60)
$$
\begin{aligned}
\sigma^2(D(0,L)) &= E[K]\sigma^2(D) + E\left[K^2\right]E^2[D] - E^2[K]E^2[D] \\
&= E[K]\sigma^2(D) + \sigma^2[K]E^2[D].
\end{aligned}
$$

This formula can also be found in SPP, on page 283, whereby $K$ has been replaced by $L$. So in order to keep dimensions correct, it is essential to interpret $L$ as a *number* of periods and not as delivery *time*.

In case of stochastic delivery times, we first have to calculate $E[D(0,L]]$and $\sigma^2(D(0,L])$. Next we have to make an assumption about the probability distribution of $D(0,L]$. For normal distributions and using the $P_1$ measure, we obtain for the re-order level

(61) $$s = E[D(0,L]] + k\sigma(D(0,L]).$$

Instead of departing from information about demand per period, we also can base our analysis on $D^c$, demand per customer, and $A$, the time between the arrivals of customers. An often used assumption is $P\{A \le x\} = 1 - e^{-\lambda x},\ \ x \ge 0$. This means that A has an exponential distribution. That is equal to saying that the arrival process of customers is a Poisson process. For such an arrival process, the following expressions for $E\left[D(0,L]\right]$ and $\sigma^2\left[D(0,L]\right]$ can be derived:

(62a) $$E\left[D(0,L]\right] = \lambda E[L]E\left[D^c\right]$$

(62b) $$\sigma^2\left(D(0,L]\right) = \lambda E[L]E\left[D^{c2}\right] + \lambda^2\sigma^2(L)E^2\left[D^c\right].$$

It should be noted that these expressions have to be combined with formulae for $E[U]$and $E\left[U^2\right]$.

## 5.5 Assumption (v)

The net inventory after arrival of an order is positive, according to this assumption. This is realistic if $Q >> D(0,L_0]$. It only regards the expression for $P_2$. But an expression for $P_2$ has been derived already in Section 4.4:

(40) $$P_2 = 1 - \frac{1}{Q}\left(E\left[\left(D(\tau_1,\tau_1+L_1]-s\right)^+\right] - E\left[\left(D(0,L_0]-(s+Q)\right)^+\right]\right).$$

With formula (40), $P_2$ can be determined numerically under the assumption that $D(\tau_1, \tau_1 + L_1]$ and $D(0, L_0]$ have a normal or gamma distribution. Calculation of the reorder point $s$ for given $P_2 = \beta$ is also done numerically, by means of bisection or a similar method. Here we use the fact that $P_2$ is strictly ascending in $s$. So assumption (v) is only needed in order to use tables for the calculation of the reorder point. It should be emphasized here that assumption (v) is nowadays not valid due to the reduction of the replenishment batch size *Q*. It can easily be seen that formula (42) becomes negative as *Q* decreases, which of course should not be the case for a valid expression for a service measure like $P_2$.

Again we are able to take the undershoot into account in a simple way through the fact that the undershoot is independent of the time during the subsequent lead time. We only have to add the mean and variance of the undershoot to the mean and variance, respectively, of the demand during the lead time. Thereafter the expressions (40) and (42) can be applied.

# 6. Analysis of the (*R*,*S*) model

## 6.1 The $P_1$ measure

The $(R,S)$ model implies that after *each* period we reorder such a quantity that the inventory position becomes $S$. Then we find, analogous to the analysis of the (*s*,*Q*) model (see Section 4.1, equation (2)) that

(63) $$P_1 = P\left\{X\left(\left(\tau_1 + L_1\right)^-\right) \geq 0\right\}.$$

But now we have $\tau_1 = R$ and

(64a) $$X\left(L_0\right) = S - D\left(0, L_0\right]$$

(64b) $$X\left(R + L_1\right) = S - D\left(0, R + L_1\right],$$

so that

(65) $$P_1 = P\left\{D\left(0, R + L_1\right] \leq S\right\}.$$

With assumptions (i) and (iv) we find that

(66) $$S = \mu\left(R + L\right) + k\sigma\sqrt{R + L}.$$

In general we have:

(67) $$S = E\left[D\left(0, R + L_1\right]\right] + k\sigma\left(D\left(0, R + L_1\right]\right)$$

If $L = K*1$, i.e. if $L$ equals an integer and possibly stochastic number of periods, we get for demand that is mutually independent and identically distributed:

(68a) $$E\left[D\left(0, R + L_1\right]\right] = \left(R + E\left[K\right]\right)E\left[D\right]$$

(68b) $$\sigma^2\left(D\left(0, R + L_1\right]\right) = \left(E\left[K\right] + R\right)\sigma^2\left(D\right) + \sigma^2\left(K\right)E^2\left[D\right],$$

where $R$ is the *number* of periods that expresses the length of the review period. Like before, we can replace $K$ by $L$, if we ignore the dimension of $L$.

## 6.2 The $P_2$ measure

In the same way as for the $P_1$ measure, and using the expressions (64), we find

$$(69) \qquad P_2 = 1 - \frac{1}{E[D(0,R]]}\left(E\left[(D(R+L_1]-S)^+\right] - E\left[(D(0,L_0]-S)^+\right]\right).$$

The problem now is that assumption (v) is mostly not valid any more, unless $R >> L_0$ or $S >> D(0,L_0]$. In many cases this is not true. This means that in fact we cannot use the approximation that follows from SPP, equation (7.41), which ignores the last expectation in (69) and yields:

$$(70) \qquad P_2 \approx 1 - \frac{1}{E[D(0,R]]} E\left[(D[R+L_1]-S)^+\right].$$

Nevertheless, if we do ignore this restriction, then the assumptions (i) and (iv) lead to

$$(71) \qquad P_2 = 1 - \frac{\sigma\sqrt{L+R}}{\mu R} G\left(\frac{S-\mu(L+R)}{\sigma\sqrt{L+R}}\right).$$

If we use this formula to find $S$ for a given $\beta$ via

$$(72) \qquad G(k) = \frac{\mu R(1-\beta)}{\sigma\sqrt{L+R}},$$

we get a value for $S$ that is too high. It is also possible that we do not find a solution at all, since (71) becomes negative for small values of $R$. Once more we can say that the problem to use the correct expressions is only of a numerical nature. Such problems can easily be solved by computer software. We refer to the aforementioned Excel spreadsheet “Classical Inventory Models” (De Kok [2002]).

If we apply assumptions (i) and (v) to the correct formula (69), we obtain

$$(73) \qquad P_2 = 1 - \frac{1}{R\mu}\left(\sigma\sqrt{L+R}G\left(\frac{S-(L+R)\mu}{\sigma\sqrt{L+R}}\right) - \sigma\sqrt{L}G\left(\frac{S-L\mu}{\sigma\sqrt{L}}\right)\right).$$

It is clear that formula (73) can easily be applied using the tables for the function $G(...)$ in SPP if $S$ is given. The opposite, i.e. computing $G(.)$ given a target value for $P_2$, requires a computerized algorithm like bisection.

Finally we note that assumption (ii) is not relevant for the $(R,S)$ model.

# 7. Essential elements from the analyses of the (*s*,*S*), (*R*,*s*,*Q*) and (*R*,*s*,*S*) models

The essence of the analysis is contained in the expressions for $X(L_0)$ and $X(\tau_1 + L_1)$. Given these expressions and e.g. assumptions (i) and (iv), we can derive expressions for $P_1$, $P_2$ and $E[X]$. This approach has been implemented in the spreadsheet "Classical Inventory Models" (De Kok [2002]).

<u>The (*s*,*S*) model</u>
For this model we have, c.f. equations (64a) and (52b):

(74a) $X(L_0) = S - D(0, L_0],$

(74b) $X(\tau_1 + L_1) = s - U_1 - D(\tau_1, \tau_1 + L_1],$

where $U_1$ is the undershoot of the reorder level *s*.

<u>The (*R*,*s*,*Q*) model</u>
Similarly, c.f. equation (52):

(75a) $X(L_0) = s + Q - U_{0,R} - D(0, L_0],$

(75b) $X(\tau_1 + L_1) = s - U_{1,R} - D(\tau_1, \tau_1 + L_1].$

<u>The(*R*,*s*,*S*) model</u>
Finally, for this model we have:

(76a) $X(L_0) = S - D(0, L_0],$

(76b) $X(\tau_1 + L_1) = s - U_{1,R} - D(\tau_1, \tau_1 + L_1].$

In expressions (75) and (76), $U_{0,R}$ and $U_{1,R}$ are the undershoots in the periodic reorder models that are derived from the demands per review period. Under assumption (i), i.e. demand during intervals is normally distributed, we find:

(77a) $E[U_R] = \frac{1}{2}\left(1 + c^2\right)R\mu,$

(77b) $E\left[U_R^2\right] = \frac{1}{3}\left(1 + \frac{3c^2}{R}\right)R^2\mu^2.$

Again a gamma distribution is preferable because it is more realistic. So if

$$P\left\{D(0,R] \leq x\right\} = \int_0^x \frac{\exp\left(-\lambda x\right)\lambda^{\alpha} x^{\alpha-1}}{\Gamma\left(\alpha\right)} dx,$$

with

$$\alpha = \frac{E^2\left[D\left(0,R\right]\right]}{\sigma^2\left(D\left(0,R\right]\right)} \text{ and } \lambda = \frac{\alpha}{E\left[D\left(0,R\right]\right]},$$

we find

(78a) $$E\left[U_R\right] = \frac{\left(\alpha+1\right)}{2\lambda}$$

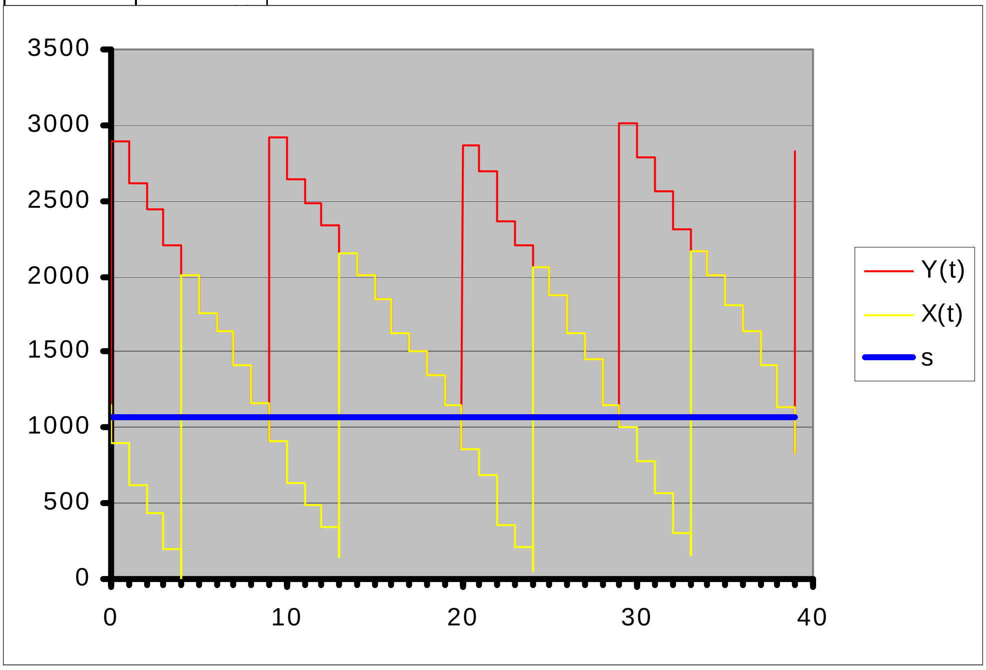
(78b) $$E\left[{U_R}^2\right] = \frac{\left(\alpha+1\right)\left(\alpha+2\right)}{3\lambda^2}.$$

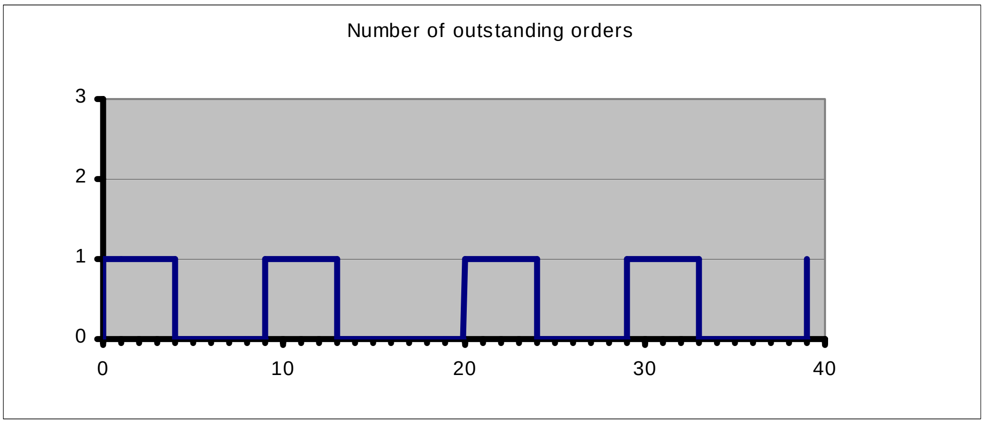
## References


- Tijms, H.C., 1994, “Stochastic Models: An Algorithmic Approach”, Wiley, Chichester.
- De Kok, A.G., 2002, “Classical Inventory Models: Student Version”, Excel spreadsheet, Studyweb, 1C230.

# Appendix A – Sample paths for the case of a single outstanding order

| | |
|---|---|
| R | 1 |
| E[D] | 200 |
| σ(D) | 50 |
| E[L] | 4 |
| Q | 2000 |



# Appendix B – Sample paths for the case of multiple outstanding orders

| | |
|---|---|
| R | 1 |
| E[D] | 200 |
| σ(D) | 50 |
| E[L] | 4 |
| Q | 500 |
| s | 1065 |