RENEWAL THEORY AND OTHER GROUND WORK

In this chapter we focus on a mathematical framework from which we derive most of the expressions for performance characteristics of the inventory management systems. This mathematical framework is called renewal theory and has proven to be useful in virtually any problem area, where stochasticity is involved. Furthermore we discuss the incomplete gamma distribution and its application to inventory management models and a powerful approximation method called the PDF-method.

The name "renewal theory" is derived from the basic problem to which it has been applied. Suppose you are a service engineer responsible for the maintenance of the illumination of some building. Your service strategy is to replace the one and only light bulb in this building as soon as the present one fails. All light bulbs are identical. Yet for one reason or another, the burning hours differ among the light bulbs. It is reasonable to assume that this is due to all kinds of unpredictable phenomena. Some tall guy from the administration department hits the bulb now and then, lightning occasionally increases current beyond the maximum current the light bulb can accommodate. A practical approach is to assume that the light bulbs are identical in a probabilistic sense: The probability that a light bulb burns longer than two months, say, is the same for any light bulb. Then it still may happen that the burning hours differ among light bulbs. The data about the burning hours of each light bulb give an insight into the characteristics of the probability distribution function of the burning hours. From that we can draw conclusions about:

- the number of replacements per year;
- preventive maintenance strategies;
- quality;
- etc.

To answer any practical question we need more background knowledge. This knowledge is provided by renewal theory. The only thing needed to "do" renewal theory is a probability distribution function (pdf) F(.), which is the pdf of the burning hours.

This chapter is not meant as an introduction to renewal theory. There are excellent text books dealing with that, e.g. Ross [1970], Tijms [1986], Cinlar [1975] and Feller [1971]. We introduce the basics of renewal theory in order to be able to derive some theorems, which are applied again and again in chapters 3 to 7.

This chapter is organized as follows. In section 2.1 we formally introduce the renewal process and all relevant random variables and functions. In section 2.2 we produce a number of useful limit theorems based on the so-called key renewal theorem. In section 2.3 we consider two renewal processes and their interactions. To be more specific, we consider a renewal process associated with demands of customers and a renewal process associated with the interarrival times of these customers. In section 2.4 we derive expressions for auxiliary functions associated with inventory holding costs and the like. Finally, in section 2.5 we discuss a method to compute the inverse of an incomplete gamma integral. This appears to be quite useful in the context of inventory management models. The same holds for the PDF-method defined in section 2.6.

2.1. Basics of renewal theory

Consider a sequence of i.i.d. random variables $\{X_n\}$ with pdf F(.), i.e.

$$P\{X_n \le x\} = F(x)$$

Define the cumulants $\{S_n\}$ by

$$S_n := \sum_{m=1}^{n} X_m$$

$$S_0 := 0$$

When X_n is interpreted as the burning time of the n^{th} light bulb, then S_n is the cumulative burning time of the first n light bulbs. The time S_n is called a renewal time. At time S_n a so-called renewal occurs.

We are interested in the characteristics of the random variable N(t), defined by

N(t) := the number of renewals in (0,t].

Then clearly

$P\{N(t) \geq n\} = P\{S_n \leq t\}$.

Hence

$P\{N(t)=n\} = F^{n*}(t) - F^{(n+1)*}(t)$,

where

$F^{n*}(.)$:= the n-fold convolution of F with itself.

$$F^{n*}(x) = \int_0^x F^{(n-1)*}(x-y)\,dF(y)$$

Let us derive expressions for the first two moments of N(t). First we consider E[N(t)]. The following sequence of equations is easily verified.

$$\begin{aligned} E[N(t)] &= \sum_{n=0}^{\infty} nP\{N(t)=n\} \\ &= \sum_{n=1}^{\infty} n(F^{n*}(t)-F^{(n+1)*}(t)) \\ &= \sum_{n=1}^{\infty} \sum_{m=1}^{n} (F^{n*}(t)-F^{(n+1)*}(t)) \\ &= \sum_{m=1}^{\infty} \sum_{n=m}^{\infty} (F^{n*}(t)-F^{(n+1)*}(t)) \\ &= \sum_{m=1}^{\infty} F^{m*}(t) \end{aligned}$$

In renewal theory a key role is played by the renewal function M(.) defined by

$$M(t) := \sum_{m=0}^{\infty} F^{m*}(t)$$

Apparently,

$$E[N(t)] = M(t)-1$$

We also express $E[N^2(t)]$ in terms of M(.).

$$\begin{aligned}
E[N^2(t)] &= \sum_{n=1}^{\infty} n^2 P\{N(t)=n\} \\
&= \sum_{n=1}^{\infty} n^2 (F^{n*}(t)-F^{(n+1)*}(t)) \\
&= \sum_{n=1}^{\infty} n(n+1) \ (F^{n*}(t)-F^{(n+1)*}(t)) \\
&\quad - \sum_{n=1}^{\infty} n(F^{n*}(t)-F^{(n+1)*}(t)) \\
&= 2 \sum_{n=1}^{\infty} \sum_{m=1}^{n} m(F^{n*}(t)-F^{(n+1)*}(t)) - (M(t)-1) \\
&= 2 \sum_{m=1}^{\infty} m \sum_{n=m}^{\infty} (F^{n*}(t)-F^{(n+1)*}(t)) - (M(t)-1) \\
&= 2 \sum_{m=1}^{\infty} \sum_{k=1}^{m} F^{m*}(t) - (M(t)-1) \\
&= 2 \sum_{k=1}^{\infty} \sum_{m=k}^{\infty} F^{m*}(t) - (M(t)-1) \\
&= 2 \sum_{k=1}^{\infty} \sum_{m=0}^{\infty} F^{(m+k)*}(t) - (M(t)-1) \\
&= 2 \sum_{k=1}^{\infty} F^{k*}(t) * \sum_{m=0}^{\infty} F^{m*}(t) - (M(t)-1) \\
&= 2 \ M{*}M(t) - 3 \ M(t)+1
\end{aligned}$$

Unfortunately an expression for M(t) is tractable for only special cases, like the case of exponential interrenewal time (burning time) with rate λ. In that case

$$M(t) = \lambda t+1 \tag{2.2}$$

For the general case we resort to approximations for M(t) and other expressions like (2.1) involving the renewal function. A key role in these approximations is played by the key renewal theorem.

Key Renewal Theorem

Let G(.), g(.) satisfy

$$G(x) = g(x) + \int_0^x G(x-y)\,dF(y) \tag{2.3}$$

Then

$$(i) \quad G(x) = \int_0^x g(x-y)\,dM(y) \tag{2.4}$$

$$(ii) \quad \lim_{x\to\infty} G(x) = \frac{1}{E[X]} \int_0^\infty g(y)\,dy \tag{2.5}$$

For a proof of (ii) we refer to Feller [1971]. Equation (i) is obtained by substitution of (2.4) into (2.3) and the uniqueness of the solution to (2.3).

Based on the key renewal theorem we obtain

$$\lim_{t\to\infty} M(t) - \left(\frac{t}{E[X]} + \frac{E[X^2]}{2E[X]} \right) = 0 \tag{2.6}$$

$$\lim_{t\to\infty} E[N(t)] - \left(\frac{t}{E[X]} + \frac{E[X^2]}{2E[X]} - 1 \right) = 0 \tag{2.7}$$

$$\lim_{t\to\infty} E[N^2(t)] - \left(\frac{t^2}{E^2[X]} + \left(\frac{2E[X^2]}{E^2[X]} - \frac{3}{E[X]} \right) t + \frac{3E^2[X^2]}{2E^4[X]} - \frac{2E[X^3]}{3E^3[X]} - \frac{3E[X^2]}{2E^2[X]} + 1 \right) = 0 \tag{2.8}$$

To give the flavour of the proofs of these kind of asymptotic results, of which many will follow in due course, we proof equation (2.8).

We assume that equation (2.2) has already been proven. Since for the exponential case we can prove from (2.1) that

$$E[N^2(t)] = \lambda^2 t^2 + \lambda t,$$

we expect that $E[N^2(t)]$ resembles a quadratic function as $t\to\infty$.

Let us assume that

$$\lim_{t\to\infty} E[N^2(t)] - (at^2+bt+c) = 0 \tag{2.9}$$

It suffices to show that constants a, b and c exist such that (2.9) holds. We define

$$G(x) := E[N^2(x)] - (ax^2+bx+c)$$

Some algebra reveals that

$$G(x) = 2M(x) - F(x) - 2 - 2aE[X]x + aE[X^2] - bE[X] - \int_x^\infty (a(x-y)^2+b(x-y)+c)\,dF(y) + \int_0^x G(x-y)\,dF(y)$$

Applying the Key Renewal Theorem, we obtain

$$\begin{aligned}\lim_{x\to\infty} G(x) &= \lim_{x\to\infty} \frac{1}{E[X]} \int_0^x \Big\{ 2M(y)-2-F(y)-2aE[X]y + aE[X^2] + bE[X] - \int_y^\infty (a(y-z)^2+b(y-z)+c)\,dF(z) \Big\}\,dy \\ &= \frac{1}{E[X]} \lim_{x\to\infty} \Big(2\int_0^x M(y)\,dy - 3x + \int_0^x (1-F(y))\,dy - aE[X]x^2 + (aE[X^2] - bE[X])x - \int_0^x \int_y^\infty (a(y-z)^2+b(y-z)+c)\,dF(z)\,dy \Big)\end{aligned} \tag{2.10}$$

In a similar fashion we derived that

$$\lim_{x\to\infty} \int_0^x M(y)\,dy - \left(\frac{x^2}{2E[X]} + \frac{E[X^2]x}{2E^2[X]} + \frac{E^2[X^2]}{4E^2[X]} - \frac{E[X^3]}{6E^2[X]} \right) = 0$$

Substitution of this result into (2.10) yields

$$\lim_{x\to\infty} G(x) = \frac{1}{E[X]}\left(x^2\left(\frac{1}{E[X]} - aE[X]\right) + x\left(\frac{E[X^2]}{E^2[X]} - 3 + aE[X^2] - bE[X]\right) + \frac{E^2[X^2]}{2E^3[X]} - \frac{E[X^3]}{3E^2[X]} + E[X] - \frac{aE[X^3]}{3} + \frac{bE[X^2]}{2} - cE[X]\right)$$

For (2.9) to hold we need

$$\frac{1}{E[X]} - aE[X] = 0$$

$$\frac{E[X^2]}{E^2[X]} - 3 + aE[X^2] - bE[X] = 0$$

$$\frac{E^2[X^2]}{2E^3[X]} - \frac{E[X^3]}{3E^2[X]} + E[X] - \frac{aE[X^3]}{3} + \frac{bE[X^2]}{2} - cE[X] = 0$$

Sequentially solving the above three equations yields (2.8), which completes our proof.

In the analysis of the inventory management systems we apply similar asymptotic results. These results are summarized in section (2.2).

Besides expressions for the moments of N(t) we frequently apply results for the so-called residual lifetime at time t, R(t), which is defined as

$$R(t) := s_{N(t)+1} - t \qquad (2.11)$$

The residual lifetime at time t is the time that elapses between t and the first renewal after time t. It follows from the definition of R(t) that

$$
\begin{aligned}
P\{R(t)>x\} &= \sum_{n=0}^{\infty} P\{S_n<t, S_{n+1}>t+x\} \\
&= \sum_{n=0}^{\infty} P\{S_n<t, X_{n+1}>t-S_n+x\} \\
&= \sum_{n=0}^{\infty} \int_0^t (1-F(t-s+x))\,dF^{n*}(s) \\
&= \int_0^t (1-F(t-s+x))\,dM(s)
\end{aligned}
\tag{2.12}
$$

Only for exponentially distributed burning times with mean 1/λ this yields a tractable expression, viz.

$$P\{R(t)>x\} = 1-e^{-\lambda x}, \qquad x>0,$$

which is a result of the memorylessness of the exponential distribution.

The Key Renewal Theorem can be applied to (2.12) to obtain the asymptotic residual lifetime distribution,

$$
\begin{aligned}
\lim_{t\to\infty} P\{R(t)>x\} &= \frac{1}{E[X]} \int_0^{\infty} (1-F(y+x))\,dy \\
&= \frac{1}{E[X]} \int_x^{\infty} (1-F(y))\,dy
\end{aligned}
\tag{2.13}
$$

In De Kok [1987] it has been made plausible that the asymptotic residual lifetime distribution provides an excellent approximation to the residual lifetime distribution even for moderate values of t. To be precise

$$P\{R(t)>x\} \simeq \frac{1}{E[X]} \int_x^\infty (1-F(y))\,dy, \qquad t \ge t_0 \tag{2.14}$$

The value of t_0 has been obtained from simulation experiments.

$$t_0 = \begin{cases} E[X] & c_X^2 \le 1 \\ \frac{3}{2} c_X^2 \; E[X] & c_X^2 > 1, \end{cases} \tag{2.15}$$

where c_X is the coefficient of variation of burning time X.

From (2.13) we can compute all moments of R(∞)

$$E[R(\infty)^k] = \frac{1}{E[X]} \frac{E[X^{k+1}]}{(k+1)} \tag{2.16}$$

We can relate the expected residual life to M(t) by (2.11).

$$\begin{aligned} E[R(t)] &= E\left[\sum_{n=1}^{N(t)+1} X_n \right] - t \\ &= E[N(t)+1]E[X]-t \\ &= M(t)E[X]-t \end{aligned}$$

Hence

$$M(t) = \frac{t}{E[X]} + \frac{E[R(t)]}{E[X]} \tag{2.17}$$

Since E[R(t)] approaches $E[X^2]/(2E[X])$ we find for t large

$$M(t) \simeq \frac{t}{E[x]} + \frac{E[X^2]}{2E^2[X]},$$

which is identical to (2.6). Unfortunately we cannot exploit similar relationships as easily as the above to obtain expressions for $E[N^2(t)]$ and other useful functions related to the renewal function. In that case we proceed along the lines of the derivation of the asymptotic expansion of $E[N^2(t)]$. We have found that for t approaching infinity the residual life at time t is distributed according to (2.13). We can enforce this result for any t by the following. Assume at time 0 that the residual life time is distributed according to (2.13). Then we define the delayed renewal process $\{\tilde{N}(t)\}$ as follows:

$S_0 = 0$

$$S_n \quad := \tilde{X}_1 + \sum_{m=2}^{n} X_n \qquad n \geq 1$$

$$\tilde{N}(t) \quad := \max\{n | S_n \leq t\} \qquad t \geq 0$$

$$\tilde{R}(t) \quad := S_{\tilde{N}(t)+1} - t \qquad t \geq 0$$

Hence $\tilde{N}t)$ is the number of renewals in (0,t] and $\tilde{R}(t)$ is the residual life at time t. We derive the distribution of $\tilde{R}(t)$ as follows.

$$\begin{aligned} P\{\tilde{R}(t)>x\} &= \sum_{n=0}^{\infty} P\{S_n<t,\ X_{n+1}>t-S_n+x\} \\ &= \sum_{n=1}^{\infty} \int_0^t (1-F(t-s+x))\,d(F^{(n-1)^*} * \tilde{F})(s) \\ &\quad + (1-\tilde{F}(t+x)) \\ &= \int_0^t (1-F(t-s+x))\,d(M*\tilde{F})(s) + (1-\tilde{F}(t+x)) \end{aligned} \qquad (2.18)$$

Here $\tilde{F}(t)$ is defined by

$$\tilde{F}(x) := \frac{1}{E[X]} \int_0^x (1-F(y))\,dy$$

Let us consider $M*\tilde{F}(.)$. By definition

$$M*\tilde{F}(x) = \frac{1}{E[X]} \int_0^x \sum_{n=0}^{\infty} F^{n*}(x-y)(1-F(y))\,dy$$

Interchange order of integration and summation and consider a single term of the summation. Application of partial integration yields

$$\begin{aligned}
&\int_0^x F^{n*}(x-y)(1-F(y))\,dy \\
&\quad= \int_0^x F^{n*}(y)\,dy(1-F(x)) + \int_0^x \int_0^y F^{n*}(x-z)\,dz dF(y) \\
&\quad= \int_0^x F^{n*}(y)\,dy(1-F(x)) + F(x) \int_0^x F^{n*} dy \\
&\qquad - \int_0^x \int_y^x F^{n*}(x-z)\,dz dF(y) \\
&\quad= \int_0^x F^{n*}(y)\,dy - \int_0^x \int_0^{x-y} F^{n*}(x-y-z)\,dz dF(y) \\
&\quad= \int_0^x F^{n*}(y)\,dy - \int_0^x \int_0^{x-z} F^{n*}(x-z-y)\,dF(y)\,dz \\
&\quad= \int_0^x F^{n*}(y)\,dy - \int_0^x F^{(n+1)*}(y)\,dy
\end{aligned}$$

Hence

$$M*\tilde{F}(x) = \frac{1}{E[X]} \sum_{n=0}^{\infty} \left(\int_0^x F^{n*}(y)\,dy - \int_0^x F^{(n+1)*}(y)\,dy\right)$$
$$= \frac{x}{E[X]} \qquad (2.19)$$

Substituting this result into (2.18) yields

$$P\{\tilde{R}(t)>x\} = \frac{1}{E[X]} \int_0^x (1-F(t-s+x)\,ds + 1-\tilde{F}(t+x)$$
$$= \frac{1}{E[X]} \int_x^{t+x} (1-F(s))\,ds + 1-\tilde{F}(t+x) \qquad (2.20)$$
$$= \tilde{F}(x)$$

Thus we have shown that for any t>0 the residual life has pdf
$\tilde{F}$(.).

A specific consequence of (2.20) is that, since

$$E[\tilde{R}(t)] = E[\tilde{N}(t)+1]E[X] - E[\tilde{X}_1]-t$$

and

$$E[\tilde{R}(t)] = E[\tilde{X}_1],$$

the following holds

$$E[\tilde{N}(t)] = \frac{t}{E[X]}, \qquad (2.21)$$

i.e. in the delayed renewal process the expected number of renewals in (0,t] is proportional to t.

This concludes this section on the basics of renewal theory. In the next section we summarize a number of limit theorems which will be applied in chapters 3 to 7. The theorems are stated without a proof. In section 2.3 we return to the analysis of renewal theory in the context of inventory management models.

2.2. Practical theorems

Define

$$\mu_k = E[X^k] \qquad k=1,2,... \tag{2.22}$$

<u>Theorem 2.1</u>

$$\lim_{t\to\infty} E[N(t)] - \left(\frac{t}{\mu_1} + \frac{\mu_2}{2\mu_1^2} - 1 \right) = 0 \tag{2.23}$$

<u>Theorem 2.2</u>

$$\lim_{t\to\infty} E[N^2(t)] - \left(\frac{t^2}{\mu_1^2} + \left(\frac{2\mu_2}{\mu_1^3} - \frac{3}{\mu_1} \right) t + \frac{3\mu_2^2}{2\mu_1^4} - \frac{2\mu_3}{3\mu_1^3} - \frac{3\mu_2}{2\mu_1^2} + 1 \right) = 0 \tag{2.24}$$

Theorem 2.3

$$\lim_{t\to\infty} E[N^3(t)] - \left(\frac{t^3}{\mu_1^3} + \left(\frac{9\mu_2}{2\mu_2^4} - \frac{6}{\mu_1^2} \right) t^2 \right.$$

$$+ \left(\frac{9\mu_2^2}{\mu_1^5} - \frac{3\mu_3}{\mu_1^4} - \frac{12\mu_1}{\mu_1^3} + \frac{7}{\mu_1} \right) t \tag{2.25}$$

$$+ \frac{3\mu_4}{4\mu_1^4} - \frac{6\mu_3\mu_3}{\mu_1^5} + \frac{15\mu_2^3}{2\mu_1^6} + \frac{4\mu_3}{\mu_1^3} - \frac{9\mu_2^2}{\mu_1^4}$$

$$\left. + \frac{7\mu_2}{2\mu_1^2} - \mu_1 \right) = 0$$

Theorem 2.4

$$\lim_{t\to\infty} \int_0^t E[N(y)]\,dy - \left(\frac{t^2}{2\mu_1} + \left(\frac{\mu_2}{2\mu_1^2} - 1 \right) t + \frac{\mu_2^2}{4\mu_1^3} - \frac{\mu_3}{6\mu_1^2} \right) = 0 \tag{2.26}$$

Theorem 2.5

$$\lim_{t\to\infty} \int_0^t E[N^2(y)]\,dy - \left(\frac{t^3}{3\mu_1^3} + \left(\frac{\mu_2}{\mu_1^3} - \frac{3}{2\mu_1} \right) t^2 \right.$$

$$+ \left(\frac{3\mu_2^2}{2\mu_1^4} - \frac{2\mu_3}{3\mu_1^3} - \frac{3\mu_2}{2\mu_1^2} + 1 \right) t \tag{2.27}$$

$$\left. + \frac{\mu_4}{6\mu_1^3} - \frac{\mu_2\mu_3}{\mu_1^4} + \frac{\mu_2^3}{\mu_1^5} + \frac{\mu_3}{2\mu_1^2} - \frac{3\mu_2^2}{4\mu_1^3} \right) = 0$$

Theorem 2.6

$$\lim_{t\to\infty} E[\tilde{N}^2(t)] - \left(\frac{t^2}{\mu_1^2} + \left(\frac{\mu_2}{\mu_1^3} - \frac{1}{\mu_1} \right) t + \frac{\mu_2^2}{2\mu_1^4} - \frac{\mu_3}{3\mu_1^3} \right) = 0 \tag{2.28}$$

Theorem 2.7

$$\begin{aligned}\lim_{t\to\infty} E[\tilde{N}^3(t)] - \Bigg(& \frac{t^3}{\mu_1^3} + \left(3\frac{\mu_2}{\mu_1^4} - \frac{3}{\mu_1^2} \right) t^2 \\ & + \left(\frac{9\mu_2^2}{2\mu_1^5} - \frac{2\mu_3}{\mu_1^4} - \frac{3\mu_2}{\mu_1^3} + \frac{1}{\mu_1} \right) t \\ & + \frac{\mu_4}{2\mu_1^4} - \frac{3\mu_2\mu_3}{\mu_1^5} + \frac{3\mu_2^3}{\mu_1^6} + \frac{\mu_3}{\mu_1^3} - \frac{3\mu_2^2}{2\mu_1^4} \Bigg) = 0\end{aligned} \tag{2.29}$$

Theorem 2.8

$$\lim_{t\to\infty} \int_0^t M(y)\,dy - \left(\frac{t^2}{2\mu_1} + \frac{\mu_2}{2\mu_1^2} t + \frac{\mu_2^2}{4\mu_1^3} - \frac{\mu_3}{6\mu_1^2} \right) = 0 \tag{2.30}$$

Theorem 2.9

$$\lim_{t\to\infty} \int_0^t yM(y)\,dy - \left(\frac{t^3}{3\mu_1} + \frac{\mu_2}{4\mu_1^2} t^2 - \frac{\mu_4}{24\mu_1^2} + \frac{\mu_2\mu_3}{6\mu_1^3} - \frac{\mu_2^3}{8\mu_1^4} \right) = 0 \tag{2.31}$$

Let Z be any random variable with pdf $F_Z(.)$. Define

$$\nu_k = E[Z^k] \tag{2.32}$$

<u>Theorem 2.10</u>

$$\lim_{t\to\infty} \int_0^t M(t-y)\,dF_Z(y) \;-\; \left(\frac{t}{\mu_1} - \frac{\nu_1}{\mu_1} + \frac{\mu_2}{2\mu_1^2} \right) = 0 \tag{2.33}$$

<u>Theorem 2.11</u>

$$\lim_{t\to\infty} \int_0^t \int_0^{t-y} (t-y-z)\,dM(z)\,dF_Z(y) - \left(\frac{t^2}{2\mu_1} + \left(\frac{\mu_2}{2\mu_1^2} - \frac{\nu_1}{\mu_1} \right) t + \frac{\nu_2}{2\mu_1} - \frac{\mu_3}{6\mu_1^2} + \frac{\mu_2^2}{4\mu_1^3} - \frac{\nu_1\mu_2}{2\mu_1^2} \right) = 0 \tag{2.34}$$

<u>Theorem 2.12</u>

$$\lim_{t\to\infty} \int_0^t \int_0^{t-y} (t-y-z)^2\,dM(z)\,dF_Z(y) - \left(\frac{t^3}{3\mu_1} + \left(\frac{\mu_2}{2\mu_1^2} - \frac{\nu_1}{\mu_1} \right) t^2 + \left(\frac{\nu_2}{\mu_1} - \frac{\mu_3}{3\mu_1^2} + \frac{\mu_2^2}{2\mu_1^3} - \frac{\nu_1\mu_2}{\mu_1^2} \right) t - \frac{\nu_3}{3\mu_1} + \frac{\mu_2}{2\mu_1^2}\nu_2 + \left(\frac{\mu_2}{3\mu_1^2} - \frac{\mu_2^2}{2\mu_1^3} \right) \nu_1 + \frac{\mu_4}{12\mu_1^2} - \frac{\mu_2\mu_3}{3\mu_1^3} + \frac{\mu_2^3}{4\mu_1^4} \right) = 0 \tag{2.35}$$

Theorem 2.13

$$\lim_{t\to\infty} \int_0^t \int_0^{t-y} (t-y-z)^3 dM(z)\, dF_Z(y)$$

$$- \left(\frac{t^4}{4\mu_1} + \left(\frac{\mu_2}{2\mu_1^2} - \frac{\nu_1}{\mu_1} \right) t^3 + \left(\frac{3\nu_2}{2\mu_1} - \frac{\mu_3}{2\mu_1^2} + \frac{3\mu_2^2}{4\mu_1^3} - \frac{3\nu_1\mu_2}{2\mu_1^2} \right) t^2 \right.$$

$$+ \left(- \frac{\nu_3}{\mu_1} + \frac{3\mu_2}{2\mu_1^2} \nu_2 + \left(\frac{\mu_3}{\mu_1^2} - \frac{3\mu_2^2}{2\mu_1^3} \right) \nu_1 + \frac{\mu_4}{4\mu_1^2} - \frac{\mu_2\mu_3}{\mu_1^3} + \frac{3\mu_2^3}{4\mu_1^4} \right) t \qquad (2.36)$$

$$+ \frac{\nu_4}{4\mu_1} - \frac{\mu_2}{2\mu_1^2} \nu_3 + \left(\frac{3\mu_2^2}{4\mu_1^3} - \frac{\mu_3}{2\mu_1^2} \right) \nu_2 + \left(\frac{\mu_3\mu_2}{\mu_1^3} - \frac{\mu_4}{4\mu_1^2} - \frac{3\mu_2^3}{4\mu_1^4} \right) \nu_1$$

$$\left. - \frac{\mu_5}{20\mu_1^2} + \frac{\mu_4\mu_2}{4\mu_1^3} + \frac{\mu_3^2}{6\mu_1^3} - \frac{3\mu_3\mu_2^2}{4\mu_1^4} + \frac{3\mu_2^4}{8\mu_1^5} \right) = 0$$

2.3. Renewal theory and inventory management models

In the preceding sections we derived a large number of asymptotic results from renewal theory. We also motivated the use of these asymptotic results for rather small values of the argument t of the functions under consideration. The fact that the asymptotic results provide excellent approximations even for relatively small values of t opens the possibility of applications of these results in the context of inventory management models.

There are a number of basic problems to be solved in the analysis of inventory management models. We state them without further explanation. The motivation for these statements follows in the chapters to follow.

Basic problems

(i) What is the distribution of demand during an interval (0,t].

(ii) What is the distribution of the undershoot of the 0-level, given that the initial inventory position equals x>0.

(iii) What is the expected time the inventory position is positive during the interval (0,t] ((0,∞)), given that the initial inventory position equals x>0.

(iv) What is the expected holding cost incurred during the interval (0,t] ((0,∞)), given that the initial inventory position equals x>0. We assume linear holding costs per item per time unit.

(v) What is the expected penalty cost incurred during the interval (o,t], given that the initial inventory equals x. We assume linear penalty costs per item short per time unit.

We subsequently deal with these problems and come up with (approximate) solutions. In the course of the analysis some auxiliary results are derived which turn out to be useful as well.

For the course of the section we assume that no replenishments to stock are made. We further assume that at time 0 the demand process has evolved over time infinitely long, such that at time 0 the demand process is stationary. The demand process is described by a compound renewal demand process, $\{(A_n,D_n)\}$,

A_n := the n^{th} interarrival time after time 0.

D_n := the demand of the n^{th} customer arriving after time 0.

The state of the system under consideration is the inventory position. Since no replenishments are bound to arrive after time 0 we may just as well say that the inventory position equals the net stock. We assume that at time 0 the inventory position equals $x \geq 0$.

We choose the time origin in two ways:

(A) At time 0 an arrival occurred after which the inventory position equals x.

(B) The time origin is some arbitrary point in time at which the demand process is stationary and the inventory position at time 0 equals x.

As might be expected the answers to questions (i) to (iv) depend on the initial conditions (A) and (B). Assumption (B) will be referred to as the APIT-assumption.

APIT := Arbitrary Point In Time.

When convenient we choose specific points in time as the time origin and in case (A) does not hold, we assume that (B) holds. In general this is not true, yet it turns out that the results derived from the APIT-assumption do provide excellent approximations.

Problem (i) The distribution of demand during an interval (0,t].

The basic problem in inventory theory is to determine the distribution of demand during some, possibly stochastic, time interval. We restrict ourselves to the demand during some fixed interval (0,t]. Define

N(t) := the number of arrivals during (0,t].

D(0,t] := the demand during (0,t].

Clearly

$$D(0,t] = \sum_{n=1}^{N(t)} D_n$$

In general the distribution of D(0,t] is intractable. To obtain tractable results we resort to two- (or three-) moment approximations. Some algebra reveals that

$$E[D(0,t]] = E[N(t)]E[D]$$

$$E[D^2(0,t]] = E[N(t)]\sigma^2(D) + E[N^2(t)]E^2[D]$$

$$E[D^3(0,t]] = E[N(t)](E[D^3] - 3E[D^2]E[D] + 2E^3[D])$$
$$+ E[N^2(t)](3E[D^2]E[D] - 3E^3[D]) + E[N^3(t)]E^3[D]$$

It suffices to have expressions for the first three moments of N(t). Note that we used the fact that $\{D_n\}$ is independent of N(t).

In section 2.1. and 2.2. we gave asymptotic expressions for the first three moments of N(t). For sake of completeness we give them again below. We distinguish between the results under assumption (A) and (B). Define

$$\nu_k = E[A_n^k]$$

Assumption (A) holds:

$$E[N(t)] \simeq \frac{t}{\nu_1} + \frac{\nu_2}{2\nu_1^2} - 1 \quad (2.37)$$

$$E[N^2(t)] \simeq \frac{t^2}{\nu_1^2} + \left(\frac{2\nu_2}{\nu_1^3} - \frac{3}{\nu_1}\right)t \quad (2.38)$$

$$+ \frac{3\nu_2^3}{2\nu_1^4} - \frac{2\nu_3}{3\nu_1^3} - \frac{3\nu_3}{2\nu_1^2} + 1 \quad (2.39)$$

$$E[N^3(t)] \simeq \frac{t^3}{\nu_1^3} + \left(\frac{9\nu_2}{2\nu_1^4} - \frac{t}{\nu_1^2}\right)t^2 + \left(\frac{9\nu_2^2}{\nu_1^5} - \frac{3\nu_3}{\nu_1^4} - \frac{12\nu_2}{\nu_1^3} + \frac{7}{\nu_1}\right)t + \frac{3\nu_4}{4\nu_1^4} - \frac{6\nu_2\nu_3}{\nu_1^5} + \frac{15\nu_2^3}{2\nu_1^6} + \frac{4\nu_3}{\nu_1^3} - \frac{9\nu_2^2}{\nu_1^4} + \frac{7\nu_2}{2\nu_1^2} - \nu_1 \tag{2.40}$$

Assumption (B) holds:

$$E[N(t)] \simeq \frac{t}{\nu_1} \tag{2.41}$$

$$E[N^2(t)] \simeq \frac{t^2}{\nu_1^2} + \left(\frac{\nu_2}{\nu_1^3} - \frac{1}{\nu_1}\right)t + \frac{\nu_2^2}{2\nu_1^4} - \frac{\nu_3}{3\nu_1^3} \tag{2.42}$$

$$E[N^3(t)] \simeq \frac{t^3}{\nu_1^3} + \left(\frac{3\nu_2}{\nu_1^4} - \frac{3}{\nu_1^2}\right)t^2 + \left(\frac{9\nu_2^2}{2\nu_1^5} - \frac{2\nu_3}{\nu_1^4} - \frac{3\nu_2}{\nu_1^3} + \frac{1}{\nu_1}\right)t + \frac{\nu_4}{2\nu_1^4} - \frac{3\nu_2\nu_3}{\nu_1^5} + \frac{3\nu_2^3}{\nu_1^6} + \frac{\nu_3}{\nu_1^3} - \frac{3\nu_2^2}{2\nu_1^4} \tag{2.43}$$

If the interval length is a random variable T then we take the expectation of the right hand side of (2.37)-(2.43) with respect to T. In order to obtain reasonable approximations we must assume

$P\{T<\Delta\} \simeq 0$

with Δ given by (2.15).

Problem (ii) The distribution of the undershoot of the 0-level, given that the initial inventory position equals x>0.

Let us define the random variable N(x) by

N(x) := the number of arrivals until the inventory position drops below 0.

Figure 2.1. Evolution of inventory position after time 0.

The random variable U(x) is defined by

$$U(x) = \sum_{n=1}^{N(x)} D_n - x \tag{2.44}$$

Then clearly U(.) is equivalent to the residual lifetime at time x of the renewal process $\{D_n\}$. Hence

$$P\{U(x) \le u\} \simeq \frac{1}{\mu_1} \int_0^u (1-F(y))\,dy, \tag{2.45}$$

where

$$\mu_k := E[D^k].$$

Note that U is independent of $\{A_n\}$.

Now let us consider the following related problem, which occurs when dealing with periodic review models. Besides the arrival times there are other important points in time, review moments. Let R denote the length of a review period. At review moments we decide about replenishments. The replenishment amounts depend on the inventory position at these review moments. In case a reorder level exists then we only order an amount at the supplier if the inventory position is below the reorder level. The problem is that the reorder level has been undershot at some point in time between the last review moment and the present one. We would like to know the distribution of the random variable T(x), the time between the moment of undershoot of the reorder level and the next review moment. In that case we must distinguish between the case of constant interarrival times and stochastic interarrival times. For both cases we define

$$T(x) = \left\lceil \frac{\sum_{n=1}^{N(x)} A_n}{R} \right\rceil R - \sum_{n=1}^{N(x)} A_n$$

where

$[x] = \min \{n | n \varepsilon N, n \geq x\}$.

We need expressions for $P\{T(x) \leq t\}$, $P\{U(x) \leq u, T(x) \leq t\}$.

Case $\sigma^2(A_n) = 0$

If $A_n \equiv A$ for all n then we assume that R is an integral multiple of A. Then T(x) is also an integral multiple of A.

$T(x) \varepsilon \{kA | k=0,1,...,R-1\}$.

Without loss of generality we assume A=1. We furthermore assume that at time 0 a customer arrived (i.e. assumption (A) holds).

$$
\begin{aligned}
P\{T(x)=k\} &= \sum_{m=1}^{\infty} P\{N(x)=mR-k\} \\
&= \sum_{m=1}^{\infty} F_D^{(mR-k-1)*}(X) - F_D^{(mR-k)*}(X), \qquad 0 \leq k \leq R
\end{aligned}
$$

where $F_D(.)$ is the pdf of D_n.

Define the Laplace-Stieltjes transform $T_k(s)$ by

$$\tilde{T}_k(s) = \int_0^\infty e^{-sx} \, d_x \, P\{T(x)=k\}$$

Then it follows after some algebra that

$$\tilde{T}_k(s) = \tilde{F}_D(s)^{R-k-1} \frac{(1-\tilde{F}_D(s))}{(1-\tilde{F}(s))R}$$

From the theory of Laplace-Stieltjes transforms we know that

$$\lim_{s \downarrow 0} \tilde{T}_k(s) = \lim_{x \to \infty} P\{T(x)=k\}$$

Applying l'Hôpital's rule we find

$$\lim_{s \downarrow 0} \tilde{T}_k(s) = \frac{1}{R}$$

and thus

$$\lim_{x \to \infty} P\{T(x)=k\} = \frac{1}{R} \qquad 0 \le k \le R-1 \qquad (2.46)$$

Hence T(x) is asymptotically uniformly distributed on 0,1,2,..., R-1.

In a similar way we can derive

$$\lim_{x \to \infty} P\{U(x) \le u, T(x)=k\} = \frac{1}{\mu_1} \int_0^u (1-F(y))\,dy \; \frac{1}{R} \qquad (2.47)$$

and thus U(.) and T(.) are asymptotically independent.

Case $\sigma^2(A)>0$

For the case of $\sigma^2(A)>0$ we only have results for a Poisson arrival process. Simulation results suggest that the following result holds true.

$$\lim_{x\to\infty} P\{T(x)\le t\} = \frac{t}{R} \qquad 0\le t\le R \tag{2.48}$$

$$\lim_{x\to\infty} P\{U(x)\le u, T(x)\le t\} = \frac{1}{\mu_1}\int_0^u (1-F(y))\,dy \; \frac{t}{R} \tag{2.49}$$

Again U(.) and T(.) are independent. We note that the above results can be proven rigourously by the theory of point processes (Nieuwenhuis [1990]).

Problem (iii) The expected time the inventory position is positive during the interval (0,t] ((0,∞)), given that the initial inventory position equals $x\ge 0$.

$T^+(x,t)$:= the time the inventory is positive during (0,t], given an initial inventory position $x\ge 0$.

It follows from the definition of N(x) and $T^+(x,t)$ that

$$T^+(x,\infty) = \sum_{n=1}^{N(x)} A_n$$

Since $\{A_n\}$ is independent of N(x) we find

$$E[T^+(x,\infty)] = \begin{cases} E[N(x)]E[A] & \textit{assumption (A) holds} \\ E[N(x)]E[A] + \dfrac{(c_A^2-1)}{2}E[A] & \textit{assumption (B) holds} \end{cases} \tag{2.50}$$

To derive an expression for $E[T^+(x,t)]$ we again distinguish between the case of $\sigma^2(A)=0$ and the case of $\sigma^2(A)>0$.

<u>Case $\sigma^2(A)=0$</u>

For the case of $\sigma^2(A)=0$ we assume that assumption (A) holds and that t is a multiple of A. We condition on the demand during (0,t] to obtain

$$E[T^+(x,\infty)] = \int_0^x E[T^+(x,\infty)|D(0,t]=y]\,dF_{D(0,t]}(y) + \int_x^\infty E[T^+(x,\infty)|D(0,t]=y]\,dF_{D(0,t]}(y)$$

$$E[T^+(x,t)] = \int_0^x E[T^+(x,t)|D(0,t]=y]\,dF_{D(0,t]}(y) + \int_x^\infty E[T^+(x,t)|D(0,t]=y]\,dF_{D(0,t]}(y)$$

Under the above assumptions we have

$$E[T^+(x,\infty)|D(0,t]=y] = t+E[T^+(x-y,\infty)] \qquad 0\le y\le x$$

$$E[T^+(x,t)|D(0,t]=y] = t \qquad 0\le y\le x$$

$$E[T^+(x,\infty)|D(0,t]=y] = E[T^+(x,t)|D(0,t]=y] \qquad y\ge x$$

Combining the above equations we find

$$E[T^+(x,t)] = E[T^+(x,\infty)] - \int_0^x E[T^+(x-y,\infty)]\,dF_{D(0,t]}(y)$$

Since

$E[N(x)] = M(x)$

with M(.) the renewal function associated with $\{D_n\}$,

$$M(x) = \sum_{n=0}^{\infty} F_D^{n*}(x)$$

we find

$$E[T^+(x,t)] = E[A](M(x) - \int_0^x M(x-y)\,dF_{D(0,t]}(y)) \tag{2.51}$$

Case $\sigma^2(A)>0$

For the case of $\sigma^2(A)>0$ we assume that t is sufficiently large to have the residual lifetime at time t associated with $\{A_n\}$ distributed according to the stationary residual life time. Equivalently we assume the APIT-assumption holds for t. Then we have to distinguish between expressions under assumption (A) and (B).

Assumption (A) holds:

We proceed along the same lines as in the case of $\sigma^2(A)=0$. However problems occur when deriving an expression for $E[T^+(x,\infty)|D(0,t]=y]$ with $0 \le y \le x$. In that case the inventory position at time t equals $x-y \ge 0$. In general no exact formula can be given for

$E[T^+(x,\infty)|D(0,t]=y]$.

However, our APIT-assumption for t yields

$$E[T^+(x,\infty)|D(0,t]=y] = t + E[N(x-y)]E[A] + \frac{(c_A^2-1)}{2}E[A], \qquad 0 \le y \le x,$$

which follows from (2.50). Thus we find

$$E[T^+(x,t)] = E[A](M(x) - \int_0^x M(x-y)\,dF_{D(0,t]}(y)) - \frac{(c_A^2-1)}{2}E[A]F_{D(0,t]}(x) \tag{2.52}$$

<u>Assumption (B) holds</u>:

Again we derive the relation between $E[T^+(x,t)]$ and $E[T^+(x,\infty)]$, yielding

$$\begin{aligned} E[T^+(x,t)] &= E[T^+(x,\infty)] - \int_0^x E[T^+(x-y,\infty)\,dF_{D(0,t]}(y) \\ &= M(x)E[A] + \frac{(c_A^2-1)}{2}E[A] - \int_0^x (M(x-y)E[A] + \frac{(c_A^2-1)}{2}E[A])\,dF_{D(0,t]}(y) \\ &= \frac{(c_A^2-1)}{2}E[A](1-F_{D(0,t]}(x)) \\ &\quad + E[A]\left(M(x) - \int_0^x M(x-y)\,dF_{D(0,t]}(y)\right) \end{aligned} \tag{2.53}$$

Note that in all cases we have consistency with

$$\lim_{t\to\infty} E[T^+(x,t)] = E[T^+(x,\infty)]$$

$$\lim_{x\to\infty} E[T^+(x,t)] = t$$

<u>Problem (iv)</u> The expected holding cost incurred during the interval (0,t] ((0,∞), given initial inventory position $x \geq 0$.

First of all we assume without loss of generality that a cost of $ 1 is incurred per item on stock per unit time. Here we assume that the inventory position equals the net stock. Define

$H(x,t)$:= holding cost incurred during the interval (0,t], given initial inventory position $x \geq 0$.

We derive a renewal equation for $E[H(x,\infty)]$ and relate $E[H(x,t)]$ with $E[H(x,\infty)]$. Again we must distinguish between the case of $\sigma^2(A)=0$ and $\sigma^2(A)>0$.

<u>Case $\sigma^2(A)=0$</u>

As before we assume that assumption (A) holds and that t is a multiple of A. Then we find the following renewal equation for E[H(x,∞)] by conditioning on the first demand,

$$E[H(x,\infty)] = x\,E[A] + \int_0^x E[H(x-y,\infty)]\,dF_D(y) \tag{2.54}$$

Then it follows from the Key Renewal Theorem that

$$E[H(x,\infty)] = \int_0^x (x-y)\,E[A]\,dM(y), \tag{2.55}$$

with M(.) the renewal function associated with $\{D_n\}$.

To derive an expression for E[H(x,t)] we note the following.

$$E[H(x,t)] = \int_0^x E[H(x,t)\,|D(0,t)=y]\,dF_{D(0,t]}(y)$$
$$+ \int_x^\infty E[H(x,t)\,|D(0,t)=y]\,dF_{D(0,t]}(y)$$

$$E[H(x,\infty)] = \int_0^x E[H(x,\infty)\,|D(0,t)=y]\,dF_{D(0,t]}(y)$$

$$+ \int_x^\infty E[H(x,\infty)\,|D(0,t)=y]\,dF_{D(0,t]}(y)$$

Furthermore the following holds

$$E[H(x,\infty)\,|D(0,t]=y] = E[H(x,t)\,|D(0,t]=y] + E[H(x-y,\infty)] \qquad 0\le y\le x$$

$$E[H(x,\infty)\,|D(0,t]=y] = E[H(x,t)\,|D(0,t]=y] \qquad y\ge x$$

Then the above equations imply that

$$E[H(x,t)] = E[H(x,\infty) - \int_0^x E[H(x-y,\infty)]\,dF_{D(0,t]}(y)$$

and thus

$$E[H(x,t)] = E[A]\left(\int_0^x (x-y)\,dM(y) - \int_0^x \int_0^{x-y} (x-y-z)\,dM(z)\,dF_{D(0,t]}(y)\right) \qquad x\ge 0 \qquad (2.56)$$

<u>Case $\sigma^2(A)>0$</u>

For the case of $\sigma^2(A)>0$ we assume that the APIT-assumption holds for t. We again distinguish between the situation for which either assumption (A) or assumption (B) holds.

<u>Assumption (A) holds</u>:

We may proceed along the same lines as in the case of $\sigma^2(A)=0$. We also find

$$H(x,\infty) = E[A] \int_0^x (x-y)\,dM(y) \tag{2.57}$$

However, we now have problems expressing E[H(x,∞)|D(0,t]=y] in terms of E[H(x,t)|D(0,t]=y] for 0≤y≤x. Here we apply the APIT-assumption. Given that D(0,t]=y the inventory position at time t equals x-y. The next customer arrives after a time which is distributed according to the stationary residual life associated with $\{A_n\}$. Hence

$$E[H(x,\infty)|D(0,t]=y] = E[H(x,t)|D(0,t]=y] + (x-y)\frac{(c_A^2+1)}{2}E[A] + \int_0^{x-y} E[H(x-y-z,\infty)]\,dF_D(z)$$

This yields

$$E[H(x,t)] = E[H(x,\infty)] - \int_0^x (x-y)\frac{(c_A^2+1)}{2}E[A]\,dF_{D(0,t]}(y) - \int_0^x \int_0^{x-y} E[H(x-y-z,\infty)]\,dF_D(z)\,dF_{D(0,t]}(y)$$

Substitution of equation (2.57) yields

$$E[H(x,t)] = E[A]\left(\int_0^x (x-y)\,dM(y) - \int_0^x \int_0^{x-y} \int_0^{x-y-z} (x-y-z-N)\,dM(N)\,dF_D(z)\,dF_{D(0,t]}(y) - \frac{(c_A^2+1)}{2} \int_0^x (x-y)\,dF_{D(0,t]}(y) \right)$$

Since $M*F_D(.) = M(.)-1$ we obtain

$$E[H(x,t)] = E[A]\left(\int_0^x (x-y)\,dM(y) - \int_0^x \int_0^{x-y} (x-y-z)\,dM(z)\,dF_{D(0,t]}(y) - \frac{(c_A^2-1)}{2} \int_0^x (x-y)\,dF_{D(0,t]}(y) \right) \tag{2.58}$$

Assumption (B) holds:

First we derive a renewal equation for $E[H(x,\infty)]$. The first arrival time is distributed according to the stationary residual life associated with $\{A_n\}$. Hence

$$E[H(x,\infty)] = \frac{(c_A^2+1)}{2} E[A]\,x + \int_0^x E[A] \int_0^{x-y} (x-y-z)\,dM(z)\,dF_D(y)$$

Here we substituted the expression for $E[H(x,\infty)]$ under assumption (A), which holds at the first arrival time after time 0. Again applying $M*F_D(.) = M(.)-1$ yields

$$E[H(x,\infty)] = \frac{(c_A^2-1)}{2} E[A]\,x + E[A] \int_0^x (x-y)\,dM(y) \tag{2.59}$$

At time t the APIT-assumption holds. Hence the relations between $E[H(x,t)|D(0,t]=y]$ and $E[H(x,\infty)|D(0,t]=y]$ are identical to the case of $\sigma^2(A)=0$. Hence

$$E[H(x,t)] = \frac{(c_A^2-1)}{2} E[A]\left(x - \int_0^x (x-y)\,dF_{D(0,t]}(y) \right) + E[A]\left(\int_0^x (x-y)\,dM(y) - \int_0^x \int_0^{x-y} (x-y-z)\,dM(z)\,dF_{D(0,t]}(y) \right) \tag{2.60}$$

For all cases of interest we derived expressions for the holding costs during an interval (0,t]. In the sequel we most often apply assumption (B) for the case of $\sigma^2(A)>0$, since the time origin usually refers to a replenishment moment and t refers to the length of a replenishment cycle. We usually assume that both lead times and replenishment cycles are long enough to warrant application of the APIT-assumption.

Equations (2.53), (2.57) and (2.59) give an expression for $E[H(x,\infty)]$ for different situations. Below we derive another expression for $E[H(x,\infty)]$, which turns out to be useful as well.

Assumption (A) holds:

Consider again figure (2.1.). We can write $E[H(x,\infty)]$ as

$$E[H(x,\infty)] = E\left[x \sum_{n=1}^{N(x)} A_n - \sum_{n=1}^{N(x)} A_n \sum_{m=1}^{n-1} D_m \right]$$

Since N(x) is independent of A_n we find

$$E[H(x,\infty)] = x\, E[A] E[N(x)] - E[A] E\left[\sum_{n=1}^{N(x)} \sum_{m=1}^{n-1} D_m \right] \tag{2.61}$$

It follows from (2.44) that

$$E[N(x)] = \frac{x}{E[D]} + E[U(x)] \tag{2.62}$$

An expression for the second term on the right hand side of (2.61) is found as follows. Equation (2.44) yields

$$E[U^2(x)] = E\left[\left(\sum_{n=1}^{N(x)} D_n\right)^2\right] - 2xE\left[\sum_{n=1}^{N(x)} D_n\right] + x^2$$

$$= E\left[\sum_{n=1}^{N(x)} D_n^2\right] + 2E\left[\sum_{n=1}^{N(x)} \sum_{m=1}^{n-1} D_n D_m\right] - 2xE\left[\sum_{n=1}^{N(x)} D_n\right] + x^2$$

Since N(x) is a stopping time for $\{D_n\}$ we find

$$E[U^2(x)] = E[N(x)]E[D^2] + 2E[D]E\left[\sum_{n=1}^{N(x)} \sum_{m=1}^{n-1} D_m\right] - 2xE[N(x)]E[D] + x^2 \tag{2.63}$$

Substitution of (2.62) and (2.63) into (2.61) yields

$$E[H(x,\infty)] = \frac{E[A]}{E[D]}\left\{\frac{x^2}{2} - \frac{E[U^2(x)]}{2} + \frac{E[D^2]}{2E[D]}(x+E[U(x)])\right\} \tag{2.64}$$

<u>Assumption (B) holds</u>:

In this case we write E[H(x,∞)] as

$$E[H(x,\infty)] = E\left[x\sum_{n=1}^{N(x)} A_n - \sum_{n=1}^{N(x)} A_n \sum_{m=1}^{n-1} D_m\right] + E[x(\tilde{A}_1 - A_1)],$$

where $\tilde{A}_1$ is the stationary residual lifetime associated with $\{A_n\}$. This leads to

$$E[H(x,\infty)] = \frac{E[A]}{E[D]} \left\{ \frac{x^2}{2} - \frac{E[U^2(x)]}{2} + \frac{E[D^2]}{2E[D]} (x+E[U(x)]) \right\} + x \left(\frac{E[A^2]}{2E[A]} - E[A] \right) \tag{2.65}$$

The complementary holding cost function

Related to the derivation of the holding cost function is the derivation of another function. To describe this function we reconsider figure 2.1. We would like to know the expectation of the area that is bounded by the x-level, the inventory position and the intersections at time 0 and t. It is easy to see that the expected value of this are does not depend on x. We denote the expected value of this area by $H_c(t)$ and we will refer to $H_c(t)$ as

$H_c(t)$:= the complementary holding cost function.

Defining

$Y(t)$:= the inventory position at time t,

we have

$$H_c(t) = E\left[\int_0^t (x-Y(t))\,dt \middle| Y(0)=x \right]$$

Case $\sigma^2(A)=0$

We assume that both the time origin 0 and t are arrival times. We derive the following renewal equations for $H_c(t)$,

$$H_c(t) = \int_0^t ((t-s)E[D] + H_c(t-s))\,dF_A(s) \tag{2.66}$$

Since A_n is degenerated in A, we have

$H_c(t) = (t-A)E[D] + H_c(t-A)$.

This difference equation has the unique solution

$$H_c(t) = \frac{1}{2} t\left(\frac{t}{A}-1\right)E[D] \qquad , \qquad t\varepsilon\{nA|n\varepsilon N\} \tag{2.67}$$

<u>Case $\sigma^2(A)>0$</u>

Proceeding along more or less the same lines we find from application of renewal theory

$$H_c(t) = \begin{cases} \left(\int_0^t (t-s)\,dM_A(s)-t\right) E[D] & \textit{assumption (A) holds} \\ \frac{1}{2}\, t^2 \frac{E[D]}{E[A]} & \textit{assumption (B) holds} \end{cases} \tag{2.68}$$

$M_A(.)$ denotes the renewal function associated with $\{A_n\}$. In case assumption (A) holds we obtain renewal equation (2.66), which has the unique solution

$$H_c(t) = \int_0^t \int_0^{t-s} (t-s-w)E[D]\,dF_A(w)\,dM_A(s)$$

This can be rewritten into (2.68).

In case assumption (B) holds, then the first arrival time is distributed according to the stationary residual life of $\{A_n\}$. Hence

$$H_c(t) = \int_0^t \{(t-s)E[D] + H_c^A(t-s)\}\,dF_{\tilde{A}}(s)$$

where A denotes the stationary residual life and $H_c^A(.)$ denotes the expression for $H_c(.)$ under assumption (A). Substitution of this expression yields

$$H_c(t) = \int_0^t (t-s)E[D]\,dF_{\tilde{A}}(s)$$

$$+ \int_0^t \int_0^{t-s} (t-s-w)\,dM_A(w)\,dF_{\tilde{A}}(s)$$

$$- \int_0^t (t-s)E[D]\,dF_{\tilde{A}}(s)$$

Next we apply

$$M_A * F_{\tilde{A}}(t) = \frac{t}{E[A]},$$

to obtain (2.68).

Problem v What is the expected penalty cost incurred during the interval (0,t], given the initial inventory equals x.

Under the assumption of linear penalty cost per unit short per unit time it is without loss of generality that we assume that the penalty cost rate is \$1. Define

B(x,t) := the penalty cost incurred in (0,t] given that at time 0 the inventory position equals x.

By definition of Y(t) we have

$$B(x,t) = \int_0^t Y^-(t)\,dt \,|\, Y(0)=x,$$

with

$x^- = \max(0,-x)$.

Similarly we have

$$H(x,t) = \int_0^t Y^+(t)\,dt | Y(0)=x$$

with

$x^+ = \max(0,x)$.

Then it is easy to see that, since $x=x^+-x^-$,

$$\int_0^t Y(t)\,dt | Y(0)x = H(x,t)-B(x,t) \tag{2.69}$$

Taking expectations on both sides of (2.69) we obtain

$$E\left[\int_0^t Y(t)\,dt | Y(0)x \right] = E[H(x,t)]-E[B(x,t)] \tag{2.70}$$

Now recall the definition of $H_c(t)$,

$$H_c(t) = E\left[\int_0^t (x-Y(t))\,dt | Y(0)=x \right] \tag{2.71}$$

Then it follows from (2.71) that

$$E\left[\int_0^t Y(t)\,dt | Y(0)=x \right] = xt-H_c(t)$$

and substituting this result into (2.70) we find

$$E[B(x,t)] = E[H(x,t)] + H_c(t) - xt \tag{2.72}$$

Hence we can apply the expressions for $E[H(x,t)]$ and $H_c(t)$ to find an expression for $E[B(x,t)]$.

This concludes this chapter on renewal theory and renewal theory applications for inventory management. The results obtained in this chapter are extensively used throughout the rest of the monograph. In section 2.6. we derive a powerful approximation to the inverse of the incomplete gamma function, which, together with the above derived results from renewal theory and the PDF-method discussed in section 2.5., forms the corner stone of the algorithms derived in chapters 3 to 7.

2.5. A class of stochastic equations from inventory theory

In this section we define a class of equations from inventory theory by giving a typical example of such an equation.

Consider the (R,S)-model, which is discussed in detail in chapter 3. Let us define

D_L := demand during lead time L.

D_{L+R} := demand during review period R plus its consecutive lead time L.

$\beta(S)$:= fraction of demand satisfied directly from stock on hand given the order-up-to-level S.

D_R := demand during review period R.

Then it is straightforward to see that (cf. chapter 3)

$$\beta(S) = 1 - \frac{E[(D_{L+R}-S^{+}] - E[(D_L-S)^{+}]}{E[D_R]},$$

where $x^+ = \max(x,0)$.

Now note that $\beta(.)$ is monotone increasing as a function of S, $\beta(0)=0$ and $\beta(\infty)=1$. Hence $\beta(.)$ is a probability distribution function associated with some (quasi) random variable X_β, i.e.

$P\{X_\beta \leq S\} = \beta(S)$.

It is well-known in literature that two-moment fits of probability distribution functions perform quite well, provided that the coefficient of variation of the associated random variable is less than 2, say, (cf. Tijms[1986], De Kok[1987]). The basic idea is to find a two-moment fit $\hat{\beta}(.)$ of $\beta(.)$. In order to obtain such a fit we have to determine the first two moments of X_β. Applying

$$E[X_\beta^k] = k\int_0^\infty y^{k-1}\,(1-\beta(y))\,dy \tag{2.73}$$

we obtain

$$E[X_\beta] = \frac{E[D_{L+R}^2 - E[D_L^2]}{2E[D_R]}$$

$$E[X_\beta^2] = \frac{E[D_{L+R}^3] - E[D_L^3]}{3E[D_R]}$$

Hence to obtain $E[X_\beta]$ and $E[X_\beta^2]$ we need the first three moments of D_L and D_{L+R}. In the literature it is shown that the distributions of both D_L and D_{L+R} can well be approximated by a gamma distribution (cf. Burgin [1975]). Our own numerical experience confirms the applicability of this assumption. Then it is easy to see that

$$E[D_{L+R}^3] = (1+c_{L+R}^2)\ (1+2c_{L+R}^2)\ E^3[D_{L+R}]$$

$$E[D_L^3] = (1+c_L^2)\ (1+2c_L^2)\ E^3[D_L],$$

where c_{L+R}^2 and c_L^2 denote the squared coefficient of variation of D_{L+R} and D_L, respectively.

Now we assume that X_β is approximately gamma distributed and define

$\hat{\beta}(S)$:= the gamma probability distribution function associated with $(E[X_\beta],E[X_\beta^2])$.

Note that $\hat{\beta}(S)$ is uniquely defined. We claim that

$\hat{\beta}(S) \approx \beta(S)$.

Let

β^* := target fraction of demand satisfied directly from stock on hand.

We would like to solve for S^* in

$\beta(S^*) = \beta^*$.

Then we claim that

$S^* \approx \hat{S}$,

with

$\hat{\beta}(\hat{S}) = \beta^*$.

Hence

$S^* \approx \hat{\beta}^{-1} (\beta^*)$.

Now note that $\hat{\beta}^{-1}$ is the inverse of the incomplete gamma function. In principle we could solve for S^* by applying a bisection method, where in each step we evaluate the incomplete gamma function using some numerical scheme (cf. Press et al [1986]). However this is quite computer time-consuming. Instead of this we derive an explicit approximation for the inverse incomplete gamma function, which is presented in the next section.

As can be seen from our analysis any easy-to-invert probability distribution function could be used for $\hat{\beta}$. Our numerical experiments reveal that the gamme fit yields excellent results. Further research is required to see whether other candidate probability functions, which are easier to invert, perform excellent as well. Some insight into this problem can be gained by studying the Laplace-Stieltjes transform of $\beta(.)$, which can be derived explicitly.

In De Kok and Van der Heijden [1990] the basic idea introduced here is exploited to obtain approximations for performance characteristics of an (R,s)-model with compound Poisson demand. Instead of computing S given a service level constraint they compute the performance characteristics for a given value of S. In their paper β(S) is a mixture of Erlang distributions. They show that the method introduced in this paper yields excellent results.

The class of equations C, to which the above idea is applicable, might be informally defined as follows. Let *F* be the class of bounded positive monotone increasing functions, i.e.

$$\mathcal{F} = \{F \mid F \text{ monotone increasing, } F(0) \geq 0), F(\infty) < \infty\}.$$

Then C consists of the following type of equations:

$$F(x^*) = \alpha, \quad F \varepsilon \mathcal{F}, \quad F(0) < \alpha < F(\infty) \tag{2.74}$$

Our scheme can be applied as follows. Define G(.) by

$$G(x) = \frac{F(x)-F(0)}{F(\infty)-F(0)}.$$

Let $\hat{G}(.)$ be the gamma fit of G. Then define $\hat{F}(.)$ by

$\hat{F}(x) = (F(\infty) - F(0))\, \hat{G}(x) + F(0).$

We want to have a solution x^* of equation (2.74). Then x^* is approximately equal to $\hat{x}$, which is defined by

$\hat{x} = \hat{F}^{-1}(\alpha),$

or equivalently

$$\hat{x} = \hat{G}^{-1}\left(\frac{\alpha-F(0)}{F(\infty)-F(0)}\right).$$

It appears that almost all practically useful equations from inventory theory can be rewritten such that a solution of (2.74) is required. Note that the main problem to solve is finding explicit expressions for the first two moments associated with the probability function G(.) from (2.73). A brute-force approach would be to use numerical integration methods to obtain these moments.

2.6. The inverse incomplete gamma function

In Van der Veen [1981] an inverse of a stochastic equation associated with a gamma probability distribution function is found by interpolation between solutions of this equation for the special cases of the exponential distribution and the normal distribution. This idea can also be exploited when we want to invert the incomplete gamma function $F_{(\alpha,\mu)}$ defined by

$$F_{(\alpha,\mu)}(x) = \int_0^x \mu^\alpha \, y^{\alpha-1} \, \frac{e^{-\mu y}}{\Gamma(\alpha)} \, dy.$$

We want to solve for x^* in

$$F_{(\alpha,\mu)}(x^*) = \beta^*, \quad 0 \leq \beta^* < 1. \tag{2.75}$$

Let X be the random variable associated with $F_{(\alpha,\mu)}$. Hence

$$E[X] = \frac{\alpha}{\mu}$$

$$c^2 = \frac{1}{\alpha}$$

where c^2 is the coefficient of variation of X.

Let us further define $k_c(.)$ by

$$k_c(x) := \frac{x - E[X]}{cE[X]}.$$

Next define G(.) by

$$G(k_c(x)) = F_{(\alpha,\mu)}(x)$$

or equivalently

$$G(k_c) = P\left\{\frac{X - E[X]}{cE[X]} \leq k_c\right\}.$$

Now equation (2.75) is equivalent to

$$x^* = (1+ck_c^*)\ E[X],$$

with k_c^* defined by

$$G(k_c^*) = \beta^*,\ 0\le\beta^*<1. \qquad (2.76)$$

It is easy to see that if c=1 then the solution to (3.2) is

$k_1^* = -1 - \ln(1 - \beta^*)$.

Also it follows from the central limit theorem that

$$\lim_{c\downarrow 0} G(k) = \Phi(k).$$

Hence if c=0 then the solution to (2.76) is

$$k_0^* = \Phi^{-1}\ (\beta^*).$$

Now we use the following scheme to solve for $F^{-1}_{(\alpha,\mu)}\ (\beta^*)$.

Inverse complete gamma function $F^{-1}_{(\alpha,\mu)}\ (\beta^*)$.

Let x^* be defined by

$$x^* = (1+ck_c^*)E[X], \qquad (2.77)$$

where

$$k_c^* = (1-c)\ K_0^* + c\ k_1^*, \tag{2.78}$$

$$k_0^* = \Phi^{-1}\ (\beta^*),\ k_1^* = 1-1n(1-\beta^*). \tag{2.79}$$

To make this scheme practically useful, we use an excellent polynomial approximation for Φ^{-1} from Abramowitz and Stegun [1965].

$$\Phi^{-1}(p) = t(p) - \frac{a_0+a_1 t(p)+a_2 t(p)^2}{1+b_0+b_1 t(p)+b_2 t(p)^2+b_3 t(p)^3} \tag{2.80}$$

with

$$t(p) = \sqrt{-2\ell n(1-p)},\ 0.5 \le p < 1 \tag{2.81}$$

$a_0 = 2.515517$ $b_1 = 1.432788$
$a_0 = 0.802853$ $b_2 = 0.189269$
$a_0 = 0.010328$ $b_3 = 0.001308$.

To show the performance of the inversion scheme we computed both the exact and approximate α-percentiles of the gamma distribution function for α between 0.1 and 0.99. For each α we determine the range of the coefficient of variation for which the relative error of the approximate α-percentile compared with the exact α-percentile is less than 5% and 10%. The results of our experiments are depicted in figure 3.1.

We may conclude that this inversion is applicable in most practically useful cases in inventory theory. This can be motivated as follows. Statistical inventory control rules are only useful if the coefficient of variation of lead time demand is less than 1.5, say. If lead time demand is highly erratic it is economically infeasible to use statistical inventory control due to excessive safety stocks, which are typically too high in periods with low demand and too small in periods with excessive demand. In practice one turns to organizational and structural changes to cope with non-erratic demand, e.g. by delivering slow moving items with erratic demand on order only.
From our numerical experience we conclude that if lead time demand is non-erratic, then also the associated quasi-random variable is non-erratic. In that case we can apply our inversion scheme.

Figure 3.1. Range of feasible coefficients of variations