THE (s,S)-MODEL

In the (b,Q)-model we fixed the reorder quantity (or more precisely the reorder quantity must be a multiple of some minimal batch size), possibly taking into account transportation and handling characteristics. This reduces the flexibility of the stock keeping facility. We look for a inventory management policy which combines the continuous review capability of the (b,Q)-policy with the lot size flexibility of the (R,S)-policy. Such a policy is the (s,S)-policy. Under the (s,S)-policy inventory is managed as follows.

As soon as the inventory position drops below s an amount is ordered such that the inventory position is raised to S.

Usually the difference between s and S depends on holding costs and fixed ordering cost. The reorder level s depends on service level constraints or penalty costs. As with the (b,Q)-model we assume compound renewal demand, i.e.

A_n := the time between the arrival of the $(n-1)^{st}$ and n^{th} customer.

D_n := the demand of the n^{th} customer.

$\{A_n\}$ and $\{D_n\}$ are mutually independent sequences of independent identically distributed random variables. Lead times $\{L_n\}$ are identically distributed and we assume that orders arrive in the order of initiation, i.e. orders do not overtake.

Among the inventory management strategies the (s,S)-policy has been shown to be optimal. Optimality refers to minimization of order costs, holding costs and penalty costs. The reason why the (s,S)-policy is less practised is, that it is somewhat more difficult to implement from an organizational point of view than the (b,Q)-policy. We discuss the differences in costs associated with the (s,S)-policy and the (b,Q)-policy in chapter 8.

It should be noted that the (s,S)-policy and the (b,Q)-policy are identical if undershoots are negligible or fixed. This is the case for constant demand per customer or incremental demand at high rate. Therefore we only discuss the case of non-negligible undershoots.

As in the previous chapters we concentrate on service measures (section 5.1.) and the mean physical stock. As a by-product we find an expression for the penalty costs assuming linear penalty cost per unit backordered per unit time. This is all discussed in section 5.2. Section 5.3. discusses a procedure that determines the (s,S)-policy that minimizes ordering, holding and penalty costs.

5.1. Service measures

Due to the fact that we have an order-up-to-level S the (s,S)-model is regenerative, whereas the (b,Q)-model was not. On the other hand the nice result that the inventory position is homogeneously distributed between the control levels no longer holds. Though there is great similarity between the (s,S)-model and the (b,Q)-model, these differences cause differences in the expressions for all performance characteristics.

At time 0 the reorder level s is undershot by an amount U_0. At time σ_1 the reorder level s is undershot again by an amount U_1. Due to the fact that the inventory position equals S after each undershoot, we have that the inventory position processes in consecutive order cycles are independent of each other: The (s,S)-model constitutes a regenerative inventory position process.

Using standard arguments we find that the net stock immediately after arrival of the order generated at time 0 equals S-D[0,L_0]. Then it is clear that the net stock immediately before arrival of the order generated at time σ_1 equals S-D(0,σ_1+L_1].

The following equation is key to the analysis.

$$\begin{aligned} D(0,\sigma_1+L_1] &= D(0,\sigma_1] + D(\sigma_1,\sigma_1+L_1] \\ &= S-s+U_1 + D(\sigma_1,\sigma_1+L_1] \end{aligned} \tag{5.1}$$

We separate D(0,σ_1+L_1] into a fixed known part S-s, an undershoot U_1 for which we can apply the approximation from renewal theory, and a lead time demand D(σ_1,σ_1+L_1], which is independent of U_1.

As before we concentrate on the P_2- and $\dot{P}_1$-service measures. Recall their definition.

P_2 := long-run fraction of demand satisfied directly from stock on hand.

$\dot{P}_1$:= long-run fraction of time the net stock is positive.

Let us first derive an expression for P_2. Based on the evolution of the net stock over time, we find

$$P_2(s,\Delta) = 1-\frac{E[(D(0,\sigma_1+L_1]-S)^+] - E[(D(0,L_0]-(s+\Delta)^+]}{E[D(L_0,\sigma_1+L_1]]},$$

where Δ = S-s. Substitution of (5.1) and use of

$$D(L_0,\sigma_1+L_1] = D(0,\sigma_1+L_1] - D(0,L_0]$$

implies

$$P_2(s,\Delta) = 1-\frac{E[(D(\sigma_1,\sigma_1+L_1]+U_1-s)^+] - E[(D(0,L_0]-(s+\Delta))^+]}{\Delta + E[U_1]} \tag{5.2}$$

It follows from the fact that both at time 0 and at time σ_1 an order is initiated that

$$D(0,L_0] \;\; d \;\; D(\sigma_1,\sigma_1+L_1]$$

We develop expressions for P_2(s,Δ) based on two-moment approximations for U_1 and D(0,L_0].

An expression for the pdf of U_1 depends on Δ.

(i) $\Delta = 0$

For the case of an (S,S)-control rule the undershoot of S is simply the demand of the arriving customer. Hence

$$P\{U_1 \leq x\} = F_D(x), \tag{5.3}$$

where D denotes the generic demand per customer.

(ii) For the case of Δ positive we approximate the pdf of U_1 by the pdf of the stationary residual lifetime distribution of the renewal process $\{D_n\}$. In order to yield a valid and accurate approximation we must assume

$$\begin{aligned} &\Delta > E[D] \qquad && c_D^2 \leq 1 \\ &\Delta > \frac{3}{2}\, c_D^2\, E[D] \qquad && c_D^2 > 1 \end{aligned} \tag{5.4}$$

The lower bounds on Δ in (5.4) are based on extensive numerical experimentation (cf. De Kok [1987]). Provided condition (5.4) holds we claim that

$$P\{U_1 \leq x\} \simeq \frac{1}{E[D]} \int_0^x (1-\sigma_D(y))\,dy \tag{5.5}$$

It is reasonable to state that cases (i) and (ii) cover all relevant cases. As soon as Δ <= E[D] the (s,S)-policy operates more or less like an (S,S)-policy to which case (i) applies.

The first two moments of U_1 are easily derived from (5.3) and (5.5).

$$E[U_1] = \begin{cases} E[D] & \Delta=0 \\ \dfrac{E[D^2]}{2E[D]} & \Delta>LB \end{cases} \tag{5.6}$$

$$E[U_1^2] = \begin{cases} E[D^2] & \Delta=0 \\ \dfrac{E[D^3]}{3E[D]} & \Delta>LB \end{cases} \tag{5.7}$$

Here LB denotes the lower bound given by (5.4).

The first two moments of $D(0,L_0]$ are given in section 4.1. For sake of completeness we restate them here

$$E[D(0,L_0]] = \left(\frac{E[L]}{E[A]} + \frac{1}{2}(c_A^2+1) \right) E[D] \tag{5.8}$$

$$\begin{aligned} \sigma^2(D(0,L_0]) &= (c_A^2+c_D^2)\frac{E[L]}{E[A]}E^2[D] + \sigma^2(L)\frac{E^2[D]}{E^2[A]} \\ &+ \frac{(c_A^2-1)}{2}\sigma^2[D] + \frac{(1-c_A^4)}{12}E^2[D] \end{aligned} \tag{5.9}$$

Knowing the first two moments of U_1 and $D(0,L_0]$ we can apply the PDF-method. Define the pdf $\gamma(.)$ by

$$\gamma(x) = P_2(x-\Delta,\Delta) \qquad x\geq 0$$

Clearly $P_2(s,\Delta)=0$ when $S<-\Delta$, since in that case the inventory position is less than zero all the time and therefore the net stock, too. Then every demand is backordered.

As before let X_γ be the random variable associated with $\gamma(.)$. Then we can calculate the first two moments of X_γ from

$$E[X_\gamma] = \int_0^\infty (1-\gamma(x))\,dx$$

$$E[X_\gamma^2] = 2\int_0^\infty x(1-\gamma(x))\,dx$$

To get a flavour of the calculations involved in the derivation of $E[X_\gamma]$ and $E[X_\gamma^2]$ we elaborate on the derivation of $E[X_\gamma]$.

$$E[X_Y] = \int_0^\infty (1-P_2(x-\Delta,\Delta))\,dx$$

$$= \int_0^\infty \frac{(E[(D(\sigma_1,\sigma_1+L_1]+U-x+\Delta)^+] - E[(D(0,L_0]-x)^+])\,dx}{\Delta+E[U_1]}$$

$$= \frac{1}{D+E[U_1]} \int_0^\infty \int_0^\infty (y-(x-\Delta))^+ dF_{D(\sigma_1,\sigma_1+L_1]+U_1}(y)\;\; dx$$

$$- \frac{1}{\Delta+E[U_1]} \int_0^\infty \int_x^\infty (y-x)\,dF_{D(0,L_0]}(y)\;\; dx$$

$$= \frac{1}{\Delta+E[U_1]} \int_{-\Delta}^\infty \int_0^\infty (y-x)^+ dF_{D(\sigma_1,\sigma_1+L_1]+U_1}(y)\;\; dx$$

$$- \frac{1}{\Delta+E[U_1]} \int_0^\infty \int_0^y (y-x)\,dx\;\; dF_{D(0,L_0]}(y)$$

$$= \frac{1}{\Delta+E[U_1]} \int_{-\Delta}^0 \int_0^\infty (y-x)\,dF_{D(\sigma_1,\sigma_1+L-1]+U_1}(y)\;\; dx$$

$$+ \frac{1}{\Delta+E[U_1]} \int_0^\infty \int_x^\infty (y-x)\,dF_{D(\sigma_1,\sigma_1+L_1]+U_1}(y)\;\; dx$$

$$- \frac{1}{\Delta+E[U_1]} \int_0^\infty \frac{1}{2}\, y^2\;\; dF_{D(0,L_0]}(y)$$

$$= \frac{1}{\Delta+E[U_1]} \{\Delta(E[D(\sigma_1,\sigma_1+L_1]+U_1]+\frac{\Delta}{2}) + \frac{1}{2}\, E[(D(\sigma_1,\sigma_1+L_1]+U_1)^2] - \frac{1}{2}\, E[D^2(0,L_0]]\}$$

Rearrangement and gathering of terms yield

$$E[X_\gamma] = (D+E[D(0,L_0]]) + \frac{(E[U^2]-\Delta^2)}{2(\Delta+E(U_1])} \tag{5.10}$$

Without going into details we claim that $E[X_\gamma^2]$ can be found along the same lines as above to yield

$$E[X_\gamma^2] = \frac{1}{\Delta+E[U_1]} \left\{ \frac{\Delta^2}{3} + (E[D(0,L_0]]+E[U]]\Delta^2 \right.$$
$$+ (E[D^2(0,L_0]]+2E[U]E(D(0,L_0]]+E[U^2](\Delta \tag{5.11}$$
$$\left. + E[D^2(0,L_0]]E[U_0]+E[D(0,L_0]]E[U^2]+\frac{E[U^3]}{3} \right\}$$

Since an (s,S)-policy operates as a (b,Q)-policy for negligible undershoots, (5.10) and (4.7) as well as (5.11) and (4.8) should coincide when assuming $U \equiv 0$. This is easy to verify.

Equation (5.11) involves the third moment of U. To compute $E[U^3]$ we assume that U is gamma distributed. Hence

$$E[U^3] = (1+c_U^2)(1+2c_U^2)E^3[U] \tag{5.12}$$

Hence c_U denoted the coefficient of variation of U.

Once we know $E[X_\gamma]$ and $E[X_\gamma^2]$ we can fit a gamma distributed $\hat{\gamma}(.)$ to these two moments. Then we claim that

$$P_2(s,\Delta) \simeq \hat{\gamma}(s+\Delta) \qquad s \geq -\Delta$$

The inversion scheme described in chapter 2 can be applied to $\hat{\gamma}(.)$ to obtain a solution of the following equation

$$P_2(s^*,\Delta) = \beta$$

We claim that

$$s^* \simeq \hat{\gamma}^{-1}(\beta)-\Delta$$

Again we found a fast and accurate algorithm to find the reorder level s, such that the P_2-service level equals some target value. The accuracy of the approximations resulting from applications of the PDF-method is ratified by the results in table 5.1.

Next we focus our attention on the $\hat{P}_1$-measure. The analysis follows the derivation of approximation (4.33) for the $\hat{P}_1$-measure in the (b,Q)-model.

We define $T^+(s,\Delta)$ analogously to $T^+(b,Q)$,

$T^+(s,\Delta)$:= time the net stock is positive during the replenishment cycle $(L_0,\sigma_1+L_1]$.

Then $\hat{P}_1(s,\Delta)$ is given by

$$\hat{P}_1(s,\Delta) = \frac{E[T^+(s,\Delta)]}{E[\sigma_1]} \tag{5.13}$$

An expression for $E[\sigma_1]$ can be obtained from renewal theoretic arguments. Let

N := the number of customers arriving in $(0,\sigma_1]$.

Then σ_1 can be written as

$$\sigma_1 = \sum_{n=1}^{N} A_n$$

Since N is independent of $\{A_n\}$ we have

$$E[\sigma_1] = E[N]E[A]$$

Furthermore we have that

$$\Delta + U_1 = \sum_{n=1}^{N} D_n,$$

since both sides of this equation describe the total demand in $(0,\sigma_1]$. Now N is a so-called stopping time for $\{D_n\}$ (cf. Çinlar [1975]), which implies

$$E\left[\sum_{n=1}^{N} D_n\right] = E[N]E[D]$$

(Note that N is <u>not</u> independent of $\{D_n\}$). Combining the above equations we find

$$E[\sigma_1] = \frac{(\Delta + E[U])}{E[D]} E[A] \tag{5.14}$$

It remains to find an expression for $E[T^+(s,\Delta)]$. As in section (4.1) we define $T^+(x,t)$ by

$T^+(x,t)$ = the time the net stock is positive during $(0,t]$, given the net stock at time 0 equals x, $x \geq 0$.

The analysis in section 2.3. yielded the basic result (2.53), which is repeated below.

$$\begin{aligned} E[T^+(x,t)] &= (E[\hat{A}] - E[A])(1 - F_{D(0,t]}(x)) \\ &\quad + E[A]\left(M(x) - \int_0^x (M(x-y)\, dF_{D(0,t]}(x) \right) \qquad x \geq 0 \end{aligned} \tag{5.15}$$

Equation (5.15) assumes that time t is an arbitrary point in time. We assume that L_0 and $\sigma_1 + L_1$ are such arbitrary points in time.

Using the definitions of $T^+(s,\Delta)$ and $T^+(x,t)$ we find

$$E[T^+(s,\Delta)] = \int_0^{\infty} \int_0^{s+\Delta} E[T^+(s+\Delta-y,t]\, dF_{D(0,L_0]|\tau_1+L_1-L_0=t}(y)\;\; dF_{\tau_1+L_1-\tau_)}(t) \tag{5.16}$$

Combination of (5.15) and (5.16) yields after tedious algebra

$$E[T^{+}(s,\Delta)] = \Big(E[\hat{A}]-E[A])(F_{D(0,L_0]}(s+\Delta)-F_{U_1+D(\tau_1,\tau_1+L_1]}(s)) + E[A]\,(\int_0^{s+\Delta} M(s+\Delta-y)\,dF_{D(0,L_0]}(y)) - \int_0^{s} M(s-y)\,dF_{U_1+D(\tau_1,\tau_1+L_1]}(y) \Big) \tag{5.17}$$

Now we distinguish between the case of Δ=0 and Δ>0.

(i) Δ=0

For this case $U_1 \stackrel{d}{=} D$. We apply the identity

$M * F\,(x) = M(x)-1 \qquad X \geq 0$

to the last integral of (5.17).

$$\int_0^{s} M(s-y)\,dF_{U_1+D(\tau_1,\tau_1+L_1]}(y) = \int_0^{s} (M(s-y)-1)\,dF_{D(\tau_1,\tau_1+L_1]}(y) \tag{5.18}$$

Substituting (5.18) into (5.17) leads us to

$$E[T^{+}(s,0)] = (E[\tilde{A}]-E[A])(F_{D(0,L_0]}(s) - F_{U_1+D(\tau_1,\tau_1+L_1]}(s)) + E[A]\ F_{D(\tau_1,\tau_1+L_1]}(s) \tag{5.19}$$

Thus we have shown for the case of Δ=0.

$$\hat{P}_1(s,0) \simeq \frac{(c_A^2-1)}{2}(F_{D(0,L_0]}(s) - F_{U_1+D(0,L_0]}(s)) + F_{D(0,L_0]}(s) \tag{5.20}$$

(ii) Δ>0

Return to (5.17). The last integral on the right hand side can be simplified through the use of (4.30),

$$\int_0^s M(s-y)\,dF_{U_1+D(\tau_1,\tau_1+L_1]}(y) = \int_0^s \frac{(s-y)}{E[D]}dF_{D(\tau_1,\tau_1+L_1]}(y)$$

This yields

$$E[T^{+}(s,\Delta)] = \Bigg(E[\tilde{A}]-E[A])(F_{D(0,L_0]}(s+\Delta) - F_{U_1+D(\tau_1,\tau_1+L_1]}(s)) + E[A](\int_0^{s+\Delta} (M(s+\Delta-y)\,dF_{D(0,L_0]}(y) - \int_0^s \frac{(s-y)}{E[D]}\ dF_{D(\tau_1,\tau_1+L_1]}(y)\Bigg) \tag{5.21}$$

Equation (5.21) leaves us with a fundamental problem not encountered in the analysis of the (R,S) and (b,Q)-model. We cannot get rid of the renewal function M(.) in an elegant way, e.g. by convolving M(.) with F_D(.) or F_U(.).

At this particular point the power of the PDF-method surfaces most. There is no way expression (5.21) can be simplified, since in general there is no explicit expression for M(.). The PDF-method, however, approaches the problem in an indirect way. Assuming a pdf associated with $\hat{P}_1(s,\Delta)$ only moments of the associated random variable are needed for the gamma fit. It happens to be that the PDF-method yields explicit expressions for the first two moments of the pdf associated with $\hat{P}_1(s,\Delta)$.

Let us first give the expression for $\hat{P}_1(s,\Delta)$ for the case of Δ>0 that follows from (5.21).

$$\hat{P}_1(s,\Delta) = \frac{(c_A^2-1)E[D]}{2(\Delta+E[U])}\left(F_{D(0,L_0]}(s+\Delta) - F_{U+D(0,L_0]}(s)\right) + \frac{E[D]}{\Delta+E[U]}\left(\int_0^{s+\Delta} M(s+\Delta-y)\,dF_{D(0,L_0]}(y) - \int_0^{s}\frac{(s-y)}{E[D]}\,dF_{D(0,L_0]}(y)\right) \tag{5.22}$$

Let $\gamma(.)$ be the pdf associated with $\hat{P}_1(s,\Delta)$,

$\gamma(x) = \hat{P}_1(x-\Delta,\Delta)$

and let X_γ denote the random variable with pdf $\gamma(.)$. For the case of $\Delta>0$ we need tedious algebra and several limit theorems from renewal theory to obtain the expressions for $E[X_\gamma]$ and $E[X_\gamma^2]$. For the case of $\Delta=0$ only routine calculations are required. We find the following,

$$E[X_\gamma] = \begin{cases} E[D(0,L_0]] - \dfrac{(c_A^2-1)}{2}E[D] & \Delta=0 \\ \dfrac{\Delta^2}{2(\Delta+E[U]} + E[D(0,L_0] + \dfrac{(E[U^2]-2E^2[U])}{2(\Delta+E[U])} - \dfrac{(c_A^2-1)}{2}E[D] & \Delta>0 \end{cases} \tag{5.23}$$

$$E[X_Y^2] \;=\; E[D^2(0,L_0)] \;-\; \frac{(c_A^2-1)}{2}\;(E[D^2]+2E[D]E[D(0,L_0)]) \quad \Delta=0$$

$$\frac{\Delta^3}{3(\Delta+E[U]} \;+\; \frac{\Delta^2}{\Delta+E[U]}\;E[D(0,L_0)] \;+\; E[D^2(0,L_0)]$$

$$+\;\frac{(E[U^2]-2E^2[U])E[D(0,L_0)]}{\Delta+E[U]} \qquad \Delta>0 \tag{5.24}$$

$$+\;\frac{(E[U^3]-3E[U]E[U^2]+3E^3[U])}{3(\Delta+E[U])}$$

$$-\;\frac{(c_A^2)}{2}\left\{E^2[D]E[D(0,L_0)] \;+\; \frac{(\Delta^2+2\Delta E[U]+2E[U^2])E[D]}{\Delta+E[U]}\right\}$$

To have some check on validity of these intricate expressions we compare $\hat{P}_1(s,\Delta)$ with $\hat{P}_1(b,Q)$ for the case of $U \equiv 0$. Then we find from (5.23) and (5.24) for the case of $\Delta>0$.

$U\equiv 0$, $\Delta>0$.

$$E[X_Y] \;=\; \frac{\Delta}{2} \;+\; E[D(0,L_0)] \;-\; \frac{(c_A^2-1)}{2}\;E[D] \tag{5.25}$$

$$E[X_Y^2] \;=\; \frac{\Delta^2}{3} \;+\; \Delta E[D(0,L_0)] \;+\; E[D^2(0,L_0)]$$

$$-\;\frac{(c_A^2-1)}{2}\;\{2E[D]E[D(0,L_0)] \;+\; \Delta E[D]\} \tag{5.26}$$

Then indeed we find that (5.25) and (5.26) are identical to (4.34) and (4.35), respectively.

Though (5.23) and (5.24) are complicated expressions, they can considerably be simplified under the assumption of gamma distributed interarrival times, demand per customer and lead times.

This concludes the section on service measures. We found that the analysis of the P_2-measure was as straightforward as with the (R,S)-model and the (b,Q)-model. The analysis of the $\hat{P}_1$-measure turned out to be quite complicated. Yet the PDF-method provided the means to obtain explicit expressions for the moments of the random variable associated with the $\hat{P}_1$-measure.

5.2. Physical stock and backlog

Section 5.1. provided us the means to compute the reorder-level that yields the required service given the minimal order size Δ. We are still interested in the amount of capital tied up in stocks. Towards this end we derive an approximation for the mean physical stock under the (s,S)-regime. It will turn out that the derivation of the results needed is far more complicated than with the (R,S)-model or (b,Q)-model. We saw the same thing happen with the $\hat{P}_1$-measure. Readers only interested in the results should skip section 5.2.1. and 5.2.2.

5.2.1. Exploring the relation between backlog and physical stock

In this section we derive an exact expression for the mean backlog, which will be convenient for further calculation. Analogously to the analysis for the (b,Q)-model we find that

$$E[X^+(s,\Delta)] = E[Y(s,\Delta)]-E[O]+E[B(s,\Delta)], \tag{5.27}$$

where

$E[Y(s,\Delta)]$:= the mean inventory position.

$E[B(s,\Delta)]$:= the mean backlog.

The cost arguments that yield (3.53) now yield

$$E[O] = E[D]\,\frac{E[L]}{E[A]} \tag{5.28}$$

as in the (b,Q)-model.

To find an expression for $E[Y(s,\Delta)]$ we proceed as follows. Assume that the stock keeping facility incurs a cost of \$1 per item on stock.

Define $k_1(.)$ as

$k_1(x)$:= total expected cost incurred until the inventory position drops below 0, given that at time 0 the inventory position equals $x \geq 0$ and no orders are initiated after time 0.

Then it is easy to see that

$$E[Y(s,\Delta)] = s+\frac{1}{E[\tau_1]}k_1(\Delta) \tag{5.29}$$

In chapter 2 (cf. also De Kok [1987]) we derived an <u>exact</u> expressions for $k_1(.)$,

$$k_1(x) = \frac{E[A]}{E[D]} \left\{ \frac{x^2}{2} - \frac{E[U^2(x)]}{2} + \frac{E[D^2]}{2E[D]} \; (x+E[U(x)] \right\} \tag{5.30}$$

where U(.) is the undershoot of 0 at the time the inventory position drops below 0. In general we do not have exact expressions for the moments of U(.), yet we have already seen that the stationary residual life time provides a good approximation if Δ is not too small.

Combination of (5.27)-(5.30) yields

$$\begin{aligned} E[X^+(s,\Delta)] &= s+\frac{1}{\Delta+E[U(\Delta)]} \left\{ \frac{\Delta^2}{2} - \frac{E[U^2(\Delta)]}{2} + \frac{E[D^2]}{2E[D]}(\Delta+E[U(\Delta)]) \right\} \\ &- E[D]\frac{E[L]}{E[A]} + E[B(s,\Delta)] \end{aligned} \tag{5.31}$$

where we substituted equation (5.14) for $E[\sigma_1]$.

Next we give an approximate expression for $E[X^+(s,\Delta)]$ which is based on the approximation for the function k(x,t) defined in chapter 2.

The analysis is analogue to the analysis of $E[X^+(b,Q)]$. Then we obtain after lengthy algebra

$$\begin{aligned} E[X^+(s,0)] &\simeq \frac{(c_A^2-1)}{2} \left(\int_0^s (s-y)\, dF_{D(0,L_0]}(y) - \int_0^s (s-y)\, dF_{D+D(0,L_0]}(y) \right) \\ &+ \int_0^s (s-y)\, dF_{D(0,L_0]}(y) \end{aligned}$$

$$\begin{aligned} E[X^+(s,\Delta)] &\simeq \frac{(c_A^2-1)}{2} \; \frac{E[D]}{\Delta+E[U]} \left\{ \Delta+E[U] + \int_{s+\Delta}^{\infty} (y-s-\Delta)\, dF_{D(0,L_0]}(y) \right. \\ &\left. - \int_s^{\infty} (y-s)\, dF_{U+D(0,L_0]}(y) \right\} \qquad \Delta>0 \\ &+ \frac{E[D]}{(\Delta+E[U])} \left(\int_0^{s+\Delta} \int_0^{s+\Delta-y} (s+\Delta-y-z)\, dM(z)\, dF_{D(0,L_0]}(y) \right. \\ &\left. - \int_0^s \frac{(s-y)^2}{2E[D]} \; dF_{D(0,L_0]}(y) \right) \end{aligned} \tag{5.32}$$

Here U is the stationary residual life time associated with the renewal process $\{D_n\}$. The expression for $E[X^+(s,\Delta)]$ for $\Delta=0$ can be routinely calculated. It also permits application of the PDF-method. The results of the application of the PDF-method are postponed until we finished the analysis of the case $\Delta>0$. So let us for the moment assume that $\Delta>0$.

Remark: Approximation (5.32) can be applied directly to yield an approximation for $E[X^+(s,\Delta)]$ when we use the following limit theorem, which holds for any random variable X.

$$\lim_{x\to\infty} \int_0^x \int_0^{x-y} (x-y-z)\, dM(z)\, dF_X(y)$$

$$- \left(\frac{x^2}{2E[D]} + \left(\frac{E[D^2]}{2E^2[D]} - \frac{E[X]}{E[D]} \right) x \right.$$

$$\left. + \frac{E[X^2]}{2E[D]} - \frac{E[D^3]}{6E^2[D]} + \frac{E^2[D^2]}{4E^3[D]} - \frac{E[X]E[D^2]}{2E^2[D]} \right) = 0$$

Assuming s sufficiently large, we obtain after some algebra

$$E[X^+(s,\Delta)] \simeq \frac{(c_A^2-1)}{2} \frac{E[D]}{\Delta+E[U]} \left\{ \Delta+E[U] + \int_{s+\Delta}^{\infty} (y-s-\Delta)\, dF_{D(0,L_0]}(y) \right.$$

$$\left. - \int_s^{\infty} (y-s)\, dF_{U+D(0,L_0]}(y) \right\} \qquad \Delta>0 \tag{5.33}$$

$$+ s-E[D(0,L_0])+E[U]$$

$$+ \frac{\Delta^2-E[U^2]}{2(\Delta+E[U])} - \int_s^{\infty} \frac{y-s^2}{2(\Delta+E[U])}\, dF_{D(0,L_0]}(y)$$

The above expression involves only one integral with which we did not deal before. One might decide to apply the PDF-method to get rid of these integrals. We have tested this approximation by fitting mixtures of Erlang distributions to $D(0,L_0]$ and then explicitly elaborating the integrals. The performance of approximation (5.33) is quite good provided s reasonably large, i.e. P_2-level associated with s $\geq$ 0.7.

Now we combine equations (5.31) and (5.32). Considerable algebra reveals that

$$E[B(s,\Delta)] \approx \frac{(c_A^2-1)}{2} \frac{E[D]}{\Delta+E[U]} \left(\int_{s+\Delta}^{\infty} (y-s-\Delta)\, dF_{D(0,L_0]}(y) - \int_{s}^{\infty} (y-s)\, dF_{U+D(0,L_0]}(y) \right)$$
$$+ \frac{E[D]}{(\Delta+E[U])} \left(\int_0^{s+\Delta} \int_0^{s+\Delta-y} (s+\Delta-y-z)\, dM(z)\, dF_{D(0,L_0]}(y) - a_2(s+\Delta)^2 - a_1(s+\Delta) - a_0 \right) \qquad (5.34)$$
$$+ \frac{1}{2(\Delta+E[U])} \int_s^{\infty} (y-s)^2 dF_{D(0,L_0]}(y)$$

The constant a_2, a_1 and a_0 are given by

$$a_2 = \frac{1}{2E[D]}$$

$$a_1 = \frac{E[D^2]}{2E^2[D]} - \frac{E[D(0,L_0]]}{E[D]}$$

$$a_0 = \frac{E[D^2(0,L_0]]}{2E[D]} - \frac{E[D^2]}{6E^2[D]} + \frac{E^2[D^2]}{4E^2[D]} - E[D(0,L_0]]\frac{E[D^2]}{2E^2[D]}$$

We seem to have made hardly any progress when comparing equation (5.32) with equation (5.34). Yet there is an essential difference. We know that

$$\lim_{s\to\infty} E[B(s,\Delta)] = 0$$

and E[B(s,Δ)] monotone decreasing in s. This suggests application of the PDF-method, which applies not only for high values of s (as with (5.33)), but for any value of s. Indeed we apply the PDF-method to E[B(s,Δ)].

5.2.2. The PDF-method applied to the mean backlog

The mean backlog is a service measure. However, unlike the P_1 and P_2-measure it does not constitute a pdf, since the mean backlog approaches infinity as s approaches minus infinity. Therefore we have to apply a normalization.

As with the P_1- and P_2-measure we only consider values of s ≥ -Δ in our PDF-analysis. Before doing so we derive an expression for B(s,Δ) when s ≤ -Δ, though this may not be practically relevant. If s ≤ -Δ then $E[X^+(s,\Delta)]=0$. Then (5.32) reduces to

$$E[B(s,\Delta)] = E[D]\frac{E[L]}{E[A]} - s - E[U] - \frac{(\Delta^2-E[U^2])}{2(\Delta+E[U])} \qquad s\le-\Delta \tag{5.35}$$

It follows from (5.35) that

$$E(B(-\Delta,\Delta)] = E[D]\frac{E[L]}{E[A]} + \Delta - E[U] - \frac{(\Delta^2-E[U^2])}{2(\Delta+E[U])} \tag{5.36}$$

We know that E[B(s,Δ)] is monotone decreasing in s and approaches zero when s tends to infinity. By normalizing E[B(s,Δ)] by dividing it by E[B(-Δ,Δ)], we can create a pdf γ(.),

$$\gamma(x) := 1-\frac{B(x-\Delta,\Delta)}{B(-\Delta,\Delta)}, \quad x\ge0 \tag{5.37}$$

As before we define X_γ the random variable associated with γ(.). Then we have that

$$E[X_\gamma] = \int_0^\infty \frac{B(x-\Delta,\Delta)}{B(-\Delta,\Delta)}\,dx \tag{5.38}$$

$$E[X_Y^2] = 2 \int_0^\infty x \frac{B(x-\Delta,\Delta)}{B(-\Delta,\Delta)} dx \tag{5.39}$$

Considering approximation (5.34) for E[B(s,Δ)] we conclude that (5.38) and (5.39) can be routinely calculated from previously obtained results once we know the following integrals.

$$I_1 := \int_0^\infty \{ \int_0^x \int_0^{x-y} (x-y-z)\, dM(z)\, dF_{D(0,L_0]}(y) - a_2 x^2 - a_1 x - a_0 \} dx$$

$$I_2 := \int_0^\infty x \{ \int_0^x \int_0^{x-y} (x-y-z)\, dM(z)\, dF_{D(0,L_0]}(y) - a_2 x^2 - a_1 x - a_0 \} dx$$

It is not at all clear that both I_1 and I_2 exist, since in both integrals we subtract two parts which diverge for x large. Luckily both integrals do exist. Computing I_1 and I_2 is not a trivial matter and requires derivation of several limit theorems from renewal theory. We only present the final results.

$$I_1 = \frac{-E[D^3(0,L_0]]}{6E[D]} + \frac{E[D^2]}{4E^2[D]} E[D^2(0,L_0]] - \left(\frac{E^2[D^2]}{4E^3[D]} - \frac{E[D^3]}{6E^2[D]}\right) E[D(0,L_0] + \frac{E[D^4]}{24D^2[D]} - \frac{E[D^2]E[D^3]}{6E^3[D]} + \frac{E^3[D^2]}{8E^4[D]} \tag{5.40}$$

$$I_2 = \frac{-E[D^4(0,L_0)]}{24E[D]} + \frac{E[D^2]}{12E^2[D]} E[D^3(0,L_0)]$$

$$- \left(\frac{E^2[D^2]}{12E^3[D^3]} - \frac{E[D^3]}{12E^2[D]}\right) E[D^2(0,L_0)]$$

$$- \left(\frac{E[D^3]E[D^2]}{12E^2[D]} - \frac{E[D^4]}{24E^2[D]} - \frac{E^3[D^2]}{8E^4[D]}\right) E[D(0,L_0)] \qquad (5.41)$$

$$+ \frac{E[D^5]}{120E[D]} - \frac{E[D^4]E[D^2]}{24E^3[D]} - \frac{E^2[D^3]}{36E^3[D]} + \frac{E[D^3]E^2[D^2]}{18E^4[D]}$$

$$- \frac{E^4[D^2]}{16E^5[D]}$$

Clearly, I_1 and I_2 are complex expressions. Yet assuming gamma distributed demand the expressions can be considerably simplified. Also I_1 and I_2 express the dependence of approximations via the PDF-method on the moments of D and $D(0,L_0]$.

Substituting I_1 and I_2 at the appropriate places in (5.38) and (5.39) we find from (5.34),

$$E[X_Y] = \frac{1}{E[B(-\Delta,\Delta)]} \left\{ \frac{(c_A^2-1)}{2} \frac{E[D]}{\Delta+E[U]} \left(\int_0^\infty \left\{ \int_x^\infty (y-x)\, dF_{D(0,L_0]}(y) - \int_{x-\Delta}^\infty (y-x+\Delta)\, dF_{U+D(0,L_0]}(y) \right\} dx \right) \right.$$

$$+ \frac{E[D]}{\Delta+E[U]} I_1$$

$$\left. + \frac{1}{2(\Delta+E[U])} \int_0^\infty \int_{x-\Delta}^\infty (y-x+\Delta)^2 dF_{D(0,L_0]}(y) \right\}$$

$$E[X_Y^2] = \frac{1}{E[B(0,L_0]]} \left\{ \frac{(c_A^2-1)}{2} \frac{E[D]}{\Delta+E[U]} \left(\int_0^\infty \left\{ \int_x^\infty (y-x)\,dF_{D(0,L_0]}(y) - \int_{x-\Delta}^\infty (y-x+\Delta)\,dF_{U+D(0,L_0]}(y) \right\} \right) + \frac{E[D]}{\Delta+E[U]} I_2 + \frac{1}{2(\Delta+E[u])} \int_0^\infty x \int_{x-\Delta}^\infty (y-x+\Delta)^2 dF_{D(0,L_0]}(y) \right\}$$

Elaborating the integrals we obtain after some algebra

$$E[X_Y] = \frac{1}{E[B(-\Delta,\Delta)]} \left\{ \frac{(c_A^2-1)}{2} \frac{E[D]}{(\Delta+E[U])} \left(\frac{E[D^2(0,L_0]]}{2} \quad 1 \quad \frac{E[U+D(0,L_0])^2]}{2} - \Delta E[U+D(0,L_0]]\right) + \frac{E[D]}{\Delta+E[U]} I_1 + \frac{1}{2(\Delta+E[U])} \left(\frac{E[D^3(0,L_0]]}{3} + \frac{\Delta^3}{3} + \Delta^2 E[D(0,L_0]] + \Delta E[D^2(0,L_0]] \right) \right\} \tag{5.42}$$

$$E[X_\gamma^2] = \frac{2}{E[B(-\Delta,\Delta)]} \Bigg\{ \frac{(c_A^2-1)}{2} \frac{E[D]}{\Delta+E[U]} \Big(\frac{E[D^3(0,L_0]]}{6}$$

$$+ \frac{\Delta^3}{6} + \frac{E[U+D(0,L_0]]}{2}\Delta^2 + \frac{E[(U+D(0,L_0])^2]}{2}\Delta$$

$$+ \frac{E[(U+D(0,L_0])^2]}{6}$$

$$+ \frac{E[D]}{\Delta+E[U]} I_2 \qquad (5.43)$$

$$+ \frac{1}{2(\Delta+E[U])} \Bigg\{ \frac{\Delta^4}{12} + \frac{E[D(0,L_0]]\Delta^3}{3}$$

$$+ \frac{E[D^2(0,L_0]]\Delta^2}{2} + \frac{E[D^3(0,L_0]]\Delta}{3}$$

$$+ \frac{E[D^4(0,L_0]]}{12} \Bigg\} \Bigg\}$$

Note that the term $E[D^3(0,L_0]]$ vanishes in $E[X_\gamma]$, since this term occurs in (5.40) and (5.42) with opposite signs. The same holds for the term $E[D^4(0,L_0]]$ in $E[X_\gamma^2]$. We emphasize that we do not use explicit expressions for higher moments of $D(0,L_0]$ than the first two, since we assume that $D(0,L_0]$ is gamma distributed. The same holds for the higher moments of D.

Though the above expressions for $E[X_\gamma]$ and $E[X_\gamma^2]$ are quite complicated, it is a routine matter to apply them in computer software. Note also that $E[X_\gamma]$ and $E[X_\gamma^2]$ only depend on Δ and not on s, which may offer computational advantages, when calculating the physical stock for several values of s with fixed Δ.

The PDF-method now prescribes to fit a gamma-distribution $\hat{\gamma}(.)$ to $E[X_\gamma]$ and $E[X_\gamma^2]$. Then we claim that

$$E[B(x,\Delta)] \simeq E[B(-\Delta,\Delta)](1-\hat{\gamma}(x+\Delta)) \qquad x \geq -\Delta \qquad (5.44)$$

This is our approximation for $B(x,\Delta)$ for $x \geq -\Delta$. Then we can find an approximation for $E[X^+(s,\Delta)]$ from (5.31).

Thus we suggest the following approximations for $E[X^+(s,\Delta)]$.

$$E[X^+(s,\Delta)] = \begin{cases} s-\dfrac{E[D]E[L]}{E[A]} + \dfrac{(c_A^2-1)}{2}\left(\displaystyle\int_s^\infty (y-s)\,dF_{D(0,L_0]}(y) \right. \\ \qquad\qquad \left. - \displaystyle\int_s^\infty (y-s)\,dF_{\Delta+D(0,L_0]}(y) \right) \\ + \displaystyle\int_s^\infty (y-s)\,dF_{D(0,L_0]}(y) \qquad \Delta=0 \\ \\ s-\dfrac{E[D]E[L]}{E[A]} + E[U] + \dfrac{\Delta^2-E[U^2]}{2(\Delta+E[U])} \\ + E[B(-\Delta,\Delta)](1-\hat{\gamma}(s+\Delta)) \qquad \Delta>0 \end{cases} \tag{5.45}$$

To complete the analysis we also give the expressions for E[B(s,Δ] for the cases of Δ>0 and Δ=0, and s ≥ -Δ.

$$E[B(s,\Delta)] \simeq \begin{cases} \dfrac{E[L]E[D]}{E[A]}(1-\hat{\zeta}(s)) & \Delta=0 \\ E[B(-\Delta,\Delta)](1-\hat{\gamma}(x+\Delta)) & \Delta>0 \end{cases} \tag{5.46}$$

Here $\hat{\zeta}(.)$ is the gamma distribution with first and second moment $E[X_\zeta]$ and $E[X_\zeta^2]$, respectively, given by

$$E[X_\zeta] = \frac{E[A]}{E[L]E[D]}\left\{ \frac{E[D^2(0,L_0]]}{2} - \frac{(c_a^2-1)}{2}\left(\frac{E[D^2]}{2} + E[D]\; E[D(0,L_0] \right] \right\} \tag{5.47}$$

$$E[X_\zeta^2] = \frac{E[A]}{E[L]E[D]}\left\{ \frac{E[D^3(0,L_0]]}{3} - \frac{(c_a^2-1)}{2}\left(\frac{E[D^3]}{3} + \frac{E[D(0,L_0]]}{2}E[D^2] + \frac{E[D^2(0,L_0]]}{2}E[D] \right) \right\} \tag{5.48}$$

Note that E[B(0,0)] equals E[0], the average amount on order. This is intuitively clear, since for the case of Δ=0 each individual demand is ordered at the supplier and because of s=0 each demand is backordered and fulfilled after the order is received. Therefore the backlog and the outstanding orders are equivalent.

In the literature usually the mean physical stock is approximated by an interpolation formula,

$$E[X^+(s,\Delta)] = \frac{1}{2}(E(\mathit{maximum\ net\ stock\ in\ a\ cycle}] + E(\mathit{minimum\ net\ stock\ in\ a\ cycle}])$$

For the (s,S)-model this approach yields

$$\begin{aligned} E[X^+(s,\Delta)] &= \frac{1}{2}(s+\Delta - E[D(0,L_0]] + s-E[U] - E[D(0,L_0]]) \\ &= s+\frac{\Delta}{2} -E[D(0,L_0]] - \frac{E[U]}{2} \end{aligned} \tag{5.49}$$

Note that we included the undershoot, which is often ignored. If we ignore the backlog we also find such a simple approximation for $E[X^+(s,\Delta)]$ from (5.45).

$$\begin{aligned} E[X^+(s,\Delta)] &\simeq s +\frac{\Delta}{2} - E[D(0,L_0] - \frac{E[U]}{2} \\ &+ E[U] - \frac{E[U^2]-E^2[U]}{2(\Delta+E[U])} \\ &+ \frac{(c_A^2-1)}{2} E[D] \end{aligned} \tag{5.50}$$

Note that (5.49) and (5.50) only coincide when U is negligible and the arrival process is a Poisson process.

Typically for Δ large (5.49) yields an underestimate of the mean physical stock.

5.3. Cost considerations

As for the (b,Q)-model we assume that the holding cost per s.k.u. per time unit equals h and the penalty cost per unit short per time unit equals p. The fixed order cost equals K. We define g(s,Δ) by

g(s,Δ) := the mean total cost per time unit associated with the (s,s+Δ)-policy

i.e.

$$g(s,\Delta) = hE[X^+(s,\Delta)] + pE[B(s,\Delta)] + K/E[\sigma_1] \tag{5.51}$$

We want to solve for (s^*,Δ^*) satisfying

$$g(s^*,\Delta^*) \le g(s,\Delta) \quad \forall(s,\Delta) \tag{5.52}$$

We can evaluate the Kuhn-Tucher conditions to find that a necessary condition for (5.51) to hold is that

$$\hat{P}_1(s^*,\Delta^*) = \frac{p}{p+h} \tag{5.53}$$

This is identical to condition (4.54). Apparently this is a structural result for inventory management models with the above cost structure.

Again the optimization procedure consists of a one-dimensional search for Δ^* given s(Δ) derived from (5.53); i.e.

$$\min_{\Delta} \; g(s(\Delta),\Delta) \tag{5.54}$$

where s(Δ) satisfies

$$\hat{P}_1 \; (s(\Delta),\Delta) = \frac{p}{p+h} \tag{5.55}$$

Equation (5.55) is routinely solved using the PDF-method as given by (5.23) and (5.24). The minimization procedure associated with (5.54) is a routine matter due to the convexity of g(s(.),.).

A short-cut method, which applied quite well in practice, is to take Δ^* equal to the Economic Order Quantity.