THE (R,s,S)-MODEL

We finalize the discussion of the basic models for the management of independent demand items with the (R,s,S)-model. The (R,s,S)-model is an extension of the (R,S)-model, where one need not reorder every review moment. As with the (R,S)-model orders are such that they raise the inventory position to an order-up-to-level S. As with the (R,b,Q)-model an order is triggered by an undershoot of the reorder level s at a review moment.

The analysis of the (R,s,S)-model is quite similar to that of the continuous review (s,S)-model. Yet the periodic review aspects cause some additional complexities and we have to resort to a more approximate analysis. The results of this analysis prove to be quite accurate for practically relevant cases.

The outline of this chapter is like the outline of the preceding chapters. First we define the model under consideration. This is done in section 7.1. In section 7.2. an expression is derived for the P_2-measure and the $\hat{P}_1$-measure. In section 7.3. we focus on the mean physical stock and the mean backlog.

7.1. The model

The management of the stock keeping facility has decided to review the inventory periodically each R^{th} time unit. The products in stock are typically rather inexpensive and therefore it is economically infeasible to order every period. Therefore a reorder level s is introduced. An order is triggered if at a review moment the inventory position, the sum of physical stock and inventory on order minus backorders, is below s. To ensure that orders are triggered only now and then the order should exceed some minimum quantity Δ. Therefore the order size is set equal to Δ plus the undershoot of s. Or equivalently, when an order is triggered an amount is ordered at the supplier, such that the inventory position is raised to an order-up-to-level S and S equals s+Δ.

The quantity Δ is typically based on some mean demand rate and cost consideration, like fixed order costs and batch stock phenomena. The determination of the reorder level s is based on customer service incentives. Therefore s depends on both market uncertainty and supplier reliability.

The supplier reliability is incorporated through the assumption that each order is delivered after some time L. L may be a random variable. We assume that consecutive orders cannot overtake.

The market uncertainty is incorporated by making assumptions concerning the demand process. First of all, we assume that demand is stationary. To be more precise, demand over time intervals of fixed length does not depend on time itself. This can be modelled in two ways. Either we assume that demand occurs at discrete equidistant points in time, or we assume that the demand is a compound renewal process.

For the case of discrete time demand, we assume that demand occurs each time unit. The demand per time unit equals D. D is a random variable. Hence we have a series of $\{D_n\}$, where D_n denotes the demand in the n^{th} time unit. Each D_n is distributed as D. Also we assume that the D_n's are mutually independent.

For the case of the compound renewal demand process we distinguish between a series of interarrival times $\{A_n\}$ and a series of demands per customer $\{D_n\}$. Both series constitute a renewal process, i.e. the series consist of independent identically distributed random variables. The series $\{A_n\}$ and $\{D_n\}$ are independent.

Note that the discrete time case is a special case of the compound renewal case. We distinguish between these two cases, because we rely on different approximations in the two cases. So in the compound renewal case we assume that $\sigma(A)>0$, where A is the generic random variable describing the demand per customer. If $\sigma(A)=0$ the discrete time results should be applied. In that case it is reasonable to assume that R is a multiple of E[A].

7.2. The service measures

We want to determine an appropriate reorder level s, since we already know Δ. For instance, Δ is equal to the Economic Order Quantity in the deterministic model. Unless stated otherwise, we assume that the reorder level s is derived from a service level constraint. As service measures we consider the P_2-measure, the fraction of demand satisfied directly from stock on hand, and the P_1-measure, the fraction of time the net stock is positive. Expressions for the P_1-measure are trivially derived from the analysis, and is left to the reader.

7.2.1. P_2-measure

To derive an expression for the P_2-measure for given values of s and Δ we consider the order cycle $(0,\sigma_1]$ and the replenishment cycle $(L_0,\sigma_1+L_1]$. The random variables of σ_1, L_0 and L_1 have been defined in section 6.2.

At time 0 the inventory position is reviewed and it is found that the inventory position is below s. Therefore an amount is ordered such that the inventory position is raised to $s+\Delta$. At review moment σ_1 the inventory position equals $s-U_{1,R}$ and therefore an amount $\Delta+U_{1,R}$ is ordered. At time σ_1-R+T_U the reorder level s is undershot by an amount U_1. The order at time 0 arrives at time L_0, the order at time σ_1 arrives at time σ_1+L_1.

We conjecture the following results.

$$X(L_0) = s+\Delta - D(0,L_0] \tag{7.1}$$

$$X(\sigma_1+L_1] = s-U_{1,R} - D(\sigma_1,\sigma_1+L_1] \tag{7.2}$$

$$P\{U_{1,R}\le x\} \simeq \frac{1}{E[D_R]}\int_0^x (1-F_D(y))\,dy \tag{7.3}$$

Equations (7.1) and (7.2) are based on the arguments applied in chapter 5 to obtain (5.1). Equation (7.3) is equivalent to (6.3).

Then it follows from (7.1) and (7.2) that

$$\begin{aligned} P_2(s,\Delta) = 1-\{&E[(D(\sigma_1,\sigma_1+L_1]+U_{1,R}-s)^+] \\ &- E[(D(0,L_0]-(s+\Delta))^+]\} \\ &/ (\Delta+E[U_{1,R}]) \end{aligned} \tag{7.4}$$

The denominator in (7.4) is the average demand per replenishment cycle, which is equal to the average demand per order cycle. At the end of the typical order cycle $(0,\sigma_1]$ an amount $\Delta+U_{1,R}$ is ordered, which is equal to the demand in $(0,\sigma_1]$.

We can apply the PDF-method to (7.4). Let us define the pdf $\gamma(.)$ by

$$\gamma(x) = P_2(x-\Delta,\Delta) \qquad x\geq 0$$

Let X_γ denote the random variable associated with $\gamma(.)$. Then

$$E[X_\gamma] = \Delta+E[D(0,L_0]] + \frac{(E[U_{1,R}^2]-\Delta^2)}{2(\Delta+E[U_{1,R}])} \tag{7.5}$$

$$\begin{aligned} E[X_\gamma^2] = (\Delta+E[U_{1,R}])^{-1} \Bigg\{ & \frac{\Delta^3}{3} + (E[D(0,L_0]+E[U_{1,R}])\Delta^2 \\ & + (E[D^2(0,L_0]] + 2E[U_{1,R}]]+E[U_{1,R}^2])\Delta \\ & + E[D^2(0,L_0]]E[U_{1,R}] + E[D(0,L_0]]E[U_{1,R}^2]+\frac{E[U_{1,R}^3]}{3} \Bigg\} \end{aligned} \tag{7.6}$$

Since (7.4) is identical to (5.2) it sufficed to copy (5.10) and (5.11) with the appropriate random variables.

Now we define $\hat{\gamma}(.)$ as the gamma distribution with its first two moments given by (7.5) and (7.6), respectively. Then we claim that

$$P_2(s,\Delta) \simeq \hat{\gamma}(s+\Delta) \tag{7.7}$$

It remains to derive expressions for the moments of $D(0,L_0]$ and $U_{1,R}$. First of all we assume that both random variables are gamma distributed. Then it suffices to determine their first two moments.

In the last chapter concerning the (R,b,Q)-model we derived expressions for the moments of $D(0,L_0]$ and $U_{1,R}$. These expressions apply here as well, since $D(0,L_0]$ is independent of the control policy applied and $U_{1,R}$ is approximated identically. Thus we obtain the appropriate expressions from (6.8)-(6.14) for the discrete time model and from (6.8), (6.9) and (6.15)-(6.18) for the compound renewal model.

7.2.2. $\acute{P}_1$-measure

The analysis of the $\hat{P}_1$-measure for the (R,s,S)-model will be a mixture of the related analysis for the (s,S)-model and the (R,b,Q)-model. As in section (6.2) we immediately distinguish between the discrete time model and the compound renewal demand model.

Case I: <u>The discrete time model</u>

We consider the replenishment cycle $(L_0,\sigma+L_1]$. It can be shown that the long-run fraction of time the net stock is positive equals the quotient of the expected time the net stock is positive during $(L_0,\sigma_1+L_1]$ and the expected length of the replenishment cycle, which is $E[\sigma_1]$. It is easily derived that

$$E[\sigma_1] \;=\; \frac{(\Delta+E[U_{1,R}])}{E[D]} \tag{7.8}$$

The expected time the net stock is positive during $(L_0,\sigma_1+L_1]$ is computed as follows. Recall from section 6.2. that

$T^+(x,t)$ = the expected time the net stock is positive during (0,t], given that the net stock is $x \geq 0$ at time 0,

is equal to

$$T^+(x,t) \;=\; M(x) \;-\; \int_0^x M(x-y)\,dF_{D(0,t]}(y)$$

For the net stock at time L_0 we have

$$X(L_0) \;=\; s+\Delta \;-\; D(0,L_0]$$

Conditioning on $X(L_0)$ we find for $E[T^+(s,\Delta)]$, the expected time the net stock is positive during $(L_0,\sigma_1+L_1]$,

$$\begin{aligned} E[T^+(s,\Delta)] \;=\; & \int_{s+\Delta}^{s+\Delta} M(s+\Delta-y)\,dF_{D(0,L_0]}(y) \\ & -\int_0^{s+\Delta} M(s+\Delta-y)\,dF_{D(0,\sigma_1+L_1]}(y) \end{aligned} \tag{7.9}$$

We can rewrite $D(0,\sigma_1+L_1]$ as

$$D(0,\sigma_1+L_1] \;=\; \Delta+U_{1,R} \;+\; D(\sigma_1,\sigma_1+L_1],$$

which implies

$$E[T^+(s,\Delta)] = \int_0^{s+\Delta} M(s+\Delta-y)\,dF_{D(0,L_0]}(y) - \int_0^{s} M(s-y)\,dF_{U_{1,R}+D(\sigma_1,\sigma_1+L_1]}(y) \tag{7.10}$$

By definition of the demand process, $U_{1,R}$ and $D(\sigma_1,\sigma_1+L_1]$ are independent.

Let us consider $U_{1,R}$. This random variable can be written as

$$U_{1,R} = U_1 + W \tag{7.11}$$

with W defined as

$$W = \sum_{n=1}^{N(R-T_U)} D_n \tag{7.12}$$

and $N(R-T_U)$ is defined as the number of customers arriving in $[\sigma_1-R+T_U,\sigma_1)$. Substituting (7.11) into (7.10) and convolving M(.) with $F_{U_1}(.)$, we find

$$E[T^+(s,\Delta)] = \int_0^{s+\Delta} M(s+\Delta-y)\,dF_{D(0,L_0]}(y) - \int_0^{s} \frac{(s-y)}{E[D]}\,dF_{W+D(0,L_0]}(y) \tag{7.13}$$

We applied the fact that $D(0,L_0]$ is identically distributed as $D(\sigma_1,\sigma_1+L_1]$. By combining (7.8) and (7.9) we obtain

$$\hat{P}_1(s,\Delta) = \frac{E[D]}{\Delta+E[U_{1,R}]}\left(\int_0^{s+\Delta} M(s+\Delta-y)\,dF_{D(0,L_0]}(y) - \int_0^{s} \frac{(s-y)}{E[D]}\,dF_{W+D(0,L_0]}(y)\right) \tag{7.14}$$

As with the continuous review (s,S)-model we cannot get rid of M(.) in (7.13), as has appeared to be possible for the (R,b,Q)-model. This complicates matters, but we can apply the results in chapter 5 for the (s,S)-model. Indeed, (7.14) is similar to the second term on the right hand side of (5.22).

To make the similarity stronger we rewrite (7.14) as

$$\hat{P}_1(s,\Delta) \;=\; \frac{E[D]}{\Delta+E[U_{1,R}]}\left\{\left(\int_0^{s+\Delta} M(s+\Delta-y)\,dF_{D(0,L_0]}(y) \;-\; \int_0^{s}\frac{(s-y)}{E[D]}\,dF_{D(0,L_0]}(y)\right)\right.$$
$$\left. -\left(\int_0^{s}\frac{(s-y)}{E[D]}\,dF_{W+D(0,L_0]}(y) \;-\; \int_0^{s}\frac{(s-y)}{E[D]}\,dF_{D(0,L_0]}(y)\right)\right\} \qquad (7.15)$$

The above expression is not tractable. Therefore we apply the PDF-method. Define the pdf $\gamma(.)$ as

$$\gamma(x) \;=\; \hat{P}_1(x-\Delta,\Delta)$$

and let X_γ be the random variable associated with $\gamma(.)$. Applying the analysis following equation (5.22) to the first term on the right hand side of (7.15) and a straightforward analysis to the second term on the right hand side of (7.15), we obtain

$$E[X_\gamma] \;=\; \frac{\Delta^2}{2(\Delta+E[U_{1,R}])} + E[D(0,L_0]] + \frac{(E[U_1^2]-2E^2[U_1])}{2(\Delta+E[U_{1,R}])}$$
$$+\frac{\Delta\;E[W]}{\Delta+E[U_{1,R}]} + \frac{E[W^2]}{2(\Delta+E[U_{1,R}])} \qquad (7.16)$$

$$E[X_\gamma^2] \;=\; \frac{\Delta^3}{3(\Delta+E[U_{1,R}])} + \frac{\Delta^2}{\Delta+E[U_{1,R}]}\,(E[W]+E[D(0,L_0]])$$
$$+\;E[D^2(0,L_0]]$$
$$+\;\frac{(E[U_1^2]-2E^2[U_1]+E[W^2])\,E[D(0,L_0]]}{\Delta+E[U_{1,R}]} \qquad (7.17)$$
$$+\;\frac{(E[U_1^3]\;-\;3E[U_1]E[U_1^2]+3E^3[U_1]+E[W^3]}{3(\Delta+E[U_{1,R}])}$$

We fit the gamma distribution $\hat{\gamma}(.)$ to $E[X_\gamma]$ and $E[X_\gamma^{\wedge}]$ to obtain

$$\hat{P}_1(s,\Delta) \simeq \hat{\gamma}(s-\Delta) \qquad s \geq -\Delta$$

The performance of this approximation is tested in figure 7.2.

Case II: <u>Compound renewal demand</u>

Consider again the replenishment cycle $(L_0,\sigma+L_1]$. We make the "Arbitrary Points In Time"-assumption (APIT), i.e.

All review and replenishment moments are arbitrary points in time from the point of view of the arrival process.

This assumption enables us to apply an approximation for $T^+(x,t)$, which is derived in chapter 2,

$$T^+(x,t) \simeq \frac{(c_A^2-1)}{2}E[A](1-F_{D(0,t]}(x)) + E[A]\left(M(x) - \int_0^x M(x-y)\,dF_{D(0,t]}(x) \right)$$

Proceeding as in the discrete time case this yields

$$E[T^+(s,\Delta)] \simeq \frac{(c_A^2-1)}{2}E[A]\left(F_{D(0,L_0]}(s+\Delta) - F_{D(0,\sigma_1+L_1]}(s+\Delta) \right) + E[A]\left(\int_0^{s+\Delta} M(s+\Delta-y)\,dF_{D(0,L_0]}(y) - \int_0^{s+\Delta} M(s+\Delta-y)\,dF_{D(0,\sigma_1+L_1]}(y) \right) \tag{7.18}$$

The second term on the right hand side with E[A]=1 is identical to the right hand side of (7.9), the expression for $E[T^+(s,\Delta)]$ in the discrete time model. Therefore we can copy the analysis for the discrete time case with respect to this part of (7.18). The first term on the right hand side of (7.18) if identical to the first term on the right hand side of (5.17) after application of the identity

$$D(0,\sigma_1+L_1] = \Delta+U_{1,R} + D(\sigma_1,\sigma_1+L_1]$$

So we can rely on previous results to obtain expressions for the first two moments of the pdf $\gamma(.)$ associated with $\dot{P}_1(s,\Delta)$. We furthermore note that

$$E[\sigma_1] = \frac{(\Delta+E[U_{1,R}])}{E[D]}E[A]$$

and

$$\hat{P}_1(s,\Delta) = \frac{E[T^+(s,\Delta)]}{E[\sigma_1]}$$

After application of the above arguments and considerable algebra we find the following expression for $E[X_\gamma]$ and $E[X_\gamma^2]$, the first two moments associated with $\gamma(.)$,

$$E[X_\gamma] = \frac{\Delta^2}{2(\Delta+E[U_{1,R}])} + E[D(0,L_0]] + \frac{E[U_1^2]-2E^2[U_1]}{2(\Delta+E[U_{1,R}])} + \frac{\Delta\ E[W]}{\Delta+E[U_{1,R}]} + \frac{E[W^2]}{2(\Delta+E[U_{1,R}])} - \frac{(c_A^2-1)}{2}E[D] \tag{7.19}$$

$$E[X_\gamma^2] = \frac{\Delta^3}{3(\Delta+E[U_{1,R}])} + \frac{\Delta^2}{\Delta+E[U_{1,R}]}(E[W]+E[D(0,L_0]])$$
$$+ E[D^2(0,L_0]] + \frac{(E[U_1^2]-2E(U_1]+E[W^2])}{\Delta+E[U_{1,R}]}E[D(0,L_0]]$$
$$+ \frac{(E[U_1^3]-3E[U_1]E[U_1^2] + 3E^3[U_1]+E[W^3])}{3(\Delta+E[U_{1,R}])} \tag{7.20}$$
$$- \frac{(c_A^2-1)}{2} \{2E[D]E[D(0,L_0]] + \frac{(\Delta^2+2\Delta\ E[U_{1,R}+E[U_{1,R}^2])}{\Delta+E[U_{1,R}]}E[D(0,L_0]]$$

Again the gamma fit $\hat{\gamma}(.)$ to $E[X_\gamma]$ and $E[X_\gamma^2]$ provides a good approximation to $\hat{P}_1(s,\Delta)$,

$$\hat{P}_1(s,\Delta) \simeq \hat{\gamma}(s-\Delta)$$

7.3. Mean physical stock and backlog

The measurement of the physical stock is highly dependent on the monitoring abilities of the inventory management system. Therefore we again do a separate analysis of the discrete time model and the compound renewal model. In both cases the approximation obtained for the mean physical stock yields an approximation for the mean backlog as a by-product through a relation between mean backlog and mean physical stock. The results obtained are quite complicated in terms of the size of the expressions. Yet, under the assumptions made throughout the text, the expressions involve only standard calculations, which can be routinely and fast executed by a computer.

Case I: Discrete time case

Suppose we incur a cost of \$ 1 for each item and for each time unit that this item is on stock. Let

$H(s,\Delta)$:= the cost incurred in the time interval $(L_0,\sigma_1+L_1]$.

Then it follows from renewal-reward arguments that

$$E[X^+(s,\Delta)] = \frac{E[H(s,\Delta)]}{E[\sigma_1]} \tag{7.21}$$

An expression for $E[K(s,\Delta)]$ is derived from a basic result stated in chapter 2. Let the function $H(x,t)$ be defined as

$H(x,t)$:= the expected cost incurred during $(0,t]$, given that the net stock at time 0 equals $x\geq 0$ and no orders arrive in $(0,t]$.

Then equation (2.56) given an expression for $H(x,t)$,

$$H(x,t) = \int_0^x (x-y)\,dM(y) - \int_0^x \int_0^{x-y} (x-y-z)\,dM(z)\,dF_{D(0,L_0]}(y)$$

The net stock at time L_0 equals S-D(0,L_0]. The interval (0,t] in the above equation coincides with the interval (L_0, σ_1 + L_1]. Then conditioning on the net stock at time L_0 and the length of the replenishment cycle, we obtain after some algebra

$$E[K(s,\Delta)] = \int_0^{s+\Delta} \int_0^{s+\Delta-y} (s+\Delta-y-z)\,dM(z)\,dF_{D(0,L_0]}(y) - \int_0^{s} \int_0^{s-y} (s-y-z)\,dM(z)\,dF_{U_{1,R}+D(0,L_0]}(y) \tag{7.22}$$

In (7.22) we used the fact that D(0,L_0] is identically distributed to D(σ_1,σ_1+L_1]. Furthermore $U_{1,R}$ and D(σ_1,σ_1-L_1] are independent.

By the definition of W we have that

$$U_{1,R} = U_1 + W$$

Applying the approximation (6.22) we find

$$\int_0^s \int_0^{s-y} (s-y-z)\, dM(z)\, dF_{U_{1,R}+D(0,L_0]}(y)$$

$$\approx \int_0^s \frac{(s-y)^2}{2E[D]} dF_{W=D(0,L_0]}(y)$$

and thus

$$E[K(s,\Delta)] = \int_0^{s+\Delta} \int_0^{s+\Delta-y} (s+\Delta-y-z)\, dM(z)\, dF_{D(0,L_0]}(y) - \int_0^s \frac{(s-y)^2}{2E[D]} dF_{W+D(0,L_0]}(y) \tag{7.23}$$

Let us rewrite (7.23) as follows

$$E[K(s,\Delta)] = \left(\int_0^{s+\Delta} \int_0^{s+\Delta-y} (s+\Delta-y-z)\, dM(z)\, dF_{D(0,L_0]}(y) - \int_0^s \frac{(s-y)^2}{2E[D]} dF_{W+D(0,L_0]}(y) \right) \tag{7.24}$$

The first term on the right hand side of (7.24) causes problems. But this term is identical to the second term on the right hand side of (5.32) in the discussion of the mean physical stock for the (s,S)-model. The analysis following (5.32) builds on a relation between the backlog and the physical stock. We proceed analogously.

By our standard cost arguments it can be seen that

$$E[X^+(s,\Delta)] = E[Y(s,\Delta)] - E[L]E[D] + E[B(s,\Delta)] \tag{7.25}$$

We need an expression for E[Y(s,Δ)]. Suppose \$ y is incurred per time unit if the inventory position equals y during that time unit. Define

C(s,Δ) = cost incurred during $(0,\sigma_1]$.

Then

$$E[Y(s,\Delta)] = \frac{E[C(s,\Delta)]}{E[\sigma_1]} \tag{7.26}$$

Also,

$$E[C(s,\Delta)] = E\left[\sigma_1 s + \int_0^{T_U} (Y(t)-s)\,dt - \int_{\hat{T}_U}^{\sigma_1} (s-Y(t))\,dt \right]$$

and thus

$$E[C(s,\Delta)] = sE[\sigma_1] + E\left[\int_0^{T_U} (Y(t)-s)\,dt \right] - E\left[\int_{\hat{T}_U}^{\sigma_1} (s-Y(t))\,dt \right] \tag{7.27}$$

Now note that the second term on the right hand side of (7.27) is the expected cost incurred during an order cycle in the (s,S)-model. The third term on the right hand side of (7.27) is equivalent to the complementary holding cost given by (2.67). Since σ_1-T_U is homogeneously distributed on 0,...,R-1 we find after some algebra and using the above arguments

$$E[\int_0^{\hat{T}_U} (Y(t)-s)\,dt] \simeq \frac{1}{E[D]} \left\{ \frac{\Delta^2}{2} - \frac{E[U_1^2]}{2} + \frac{E[D^2]}{2E[D]}(\Delta+E[U_1]) \right\}$$

$$E[\int_{\hat{T}_U}^{\sigma_1} (s-Y(t))\,dt] \simeq \frac{(R-1)}{2} \left\{ E[U_1] + \left(\frac{2}{3}R-\frac{5}{6} \right) E[D] \right\}$$

Together with (7.26) and (7.27) this yields

$$E[Y(s,\Delta)] = s+\frac{\Delta^2-E[U_1^2] + \dfrac{E[D^2]}{E[D]}(\Delta+E[U_1])}{2(\Delta+E[U_{1,R}])} + \frac{(R-1)E[D]}{2(\Delta+E[U_{1,R}])} \left\{ E[U_1] + \left(\frac{2}{3}R-\frac{5}{6} \right) E[D] \right\} \tag{7.28}$$

It is interesting to give another derivation for (7.28). Note that

$$E[C(s,\Delta)] = (s+\Delta)E[\sigma_1] - E\left(\sum_{n=1}^{\sigma_1} D_n(\sigma_1-n) \right] \tag{7.29}$$

Furthermore note that

$$E\left[\left(\sum_{n=1}^{\sigma_1} D_n \right)^2 \right] = E\left[\sum_{n=1}^{\sigma_1} D_N^2 \right] + 2E\left[\sum_{m=1}^{\sigma_1} \sum_{n=1}^{m-1} D_n D_m \right] \tag{7.30}$$

Because σ_1 is a stopping time we have

$$E\left[\sum_{m=1}^{\sigma_1} \sum_{n=1}^{m-1} D_n D_m \right] = E[D]E\left[\sum_{n=1}^{\sigma_1} \sum_{n=1}^{m-1} D_n \right] = E[D]E\left[\sum_{n=1}^{\sigma_1} D_n(\sigma_1-n) \right] \tag{7.31}$$

Together (7.29)-(7.31) yield

$$E[C(s,\Delta)] = (s+\Delta)E[\sigma_1] - \frac{1}{2E[D]} \left(E\left[\left(\sum_{n=1}^{\sigma_1} D_n \right)^2 \right] - E[\sigma_1]E[D^2] \right) \tag{7.32}$$

Another useful relation is

$$\sum_{n=1}^{\sigma_1} D_n = \Delta + U_{1,R}$$

Then

$$E\left[\left(\sum_{n=1}^{\sigma_1} D_n\right)^2\right] = \Delta^2 + 2\Delta E[U_{1,R}] + E[U_{1,R}^2] \tag{7.33}$$

Substitution of (7.33) into (7.32) yields

$$E[C(s,\Delta)] = (s+\Delta)E[\sigma_1] - \frac{1}{2E[D]}(\Delta^2+2\Delta E[U_{1,R}] + E[U_{1,R}^2] - E[\sigma_1]E[D^2])$$

Then an alternative expression for E[Y(s,Δ)] is

$$\begin{aligned} E[Y(s,\Delta)] &= s+\Delta+\frac{E[D^2]}{2E[D]} - \frac{(\Delta^2+2\Delta E[U_{1,R}]+E[U_{1,R}^2])}{2(\Delta+E[U_{1,R}])} \\ &= s+\frac{(\Delta^2-E[U_{1,R}^2])}{2(\Delta+E[U_{1,R}])} + \frac{E[D^2]}{2E[D]} \end{aligned} \tag{7.34}$$

It is easily checked that (7.34) is identical to (7.28). In the sequel we use (7.34).

From (7.25) and (7.34) we obtain

$$E[X^+(s,\Delta)] = s+\frac{(\Delta^2-E[U_{1,R}^2])}{2(\Delta+E[U_{1,R}])} + \frac{E[D^2]}{2E[D]} - E[L]E[D] + E[B(s,\Delta)] \tag{7.35}$$

From (7.35) we find for s sufficiently large

$$E[X^+(s,\Delta)] \simeq s+\frac{(\Delta^2-E[U_{1,R}^2])}{2(\Delta-E[U_{1,R}])} + \frac{E[D^2]}{2E[D]} - E[L]E[D] \tag{7.36}$$

This is the first practical approximation for $E[X^+(s,\Delta)]$. However, not in every practical case we may assume that $E[B(s,\Delta)]$ is negligible. In that case we proceed as in the analysis for the (s,S)-model, i.e. we apply the PDF-method to $E[B(s,\Delta)]$. In order to do so we derive from (7.24) and (7.35) that

$$E[B(s,\Delta)] = \frac{E[D]}{\Delta+E[U_{1,R}]}\left\{\int_0^{s+\Delta}\int_0^{s+\Delta-y}(s+\Delta-y-z)\,dM(z)\,dF_{D(0,L_0]}(y) - \int_0^{s}\frac{(s-y)^2}{2E[D]}dF_{W+D(0,L_0]}(y) + E[L]E[D] - s-\frac{(\Delta^2-E[U_{1,R}^2])}{2(\Delta+E[U_{1,R}])} - \frac{E[D^2]}{2E[D]}\right\}$$

After some algebra we can write $E[B(s,\Delta)]$ as follows

$$E[B(s,\Delta)] = \frac{E[D]}{\Delta+E[U_{1,R}]}\left\{\int_0^{s+\Delta}\int_0^{s+\Delta-y}(s+\Delta-y-z)\,dM(z)\,dF_{D(0,L_0]}(y) - a_2(s+\Delta)^2 - a_1(s+\Delta)-a_0 + \frac{1}{2(\Delta+E[U_{1,R}])}\int_s^{\infty}(y-s)^2 dF_{W+D(0,L_0]}(y)\right\}, \qquad (7.37)$$

with a_0, a_1 and a_2 given below (5.34).

Now we are in business! The expression between brackets is identical to the second term on the right hand side of (5.34). When applying the PDF-method to E[B(s,Δ)] this term gives rise to the expressions I_1 and I_2 given by (5.40) and (5.41), respectively, where we should insert the proper expressions for the moments of $D(0,L_0]$.

So we proceed as follows. Define γ(.) as

$$\gamma(.) := 1-\frac{E[B(x-\Delta,\Delta)]}{E[B(-\Delta,\Delta)]} \qquad x\geq 0$$

From (7.37) we obtain

$$E[B(-\Delta,\Delta)] = E[D(0,L_0]] + \frac{(\Delta^2+2E[W]+E[W^2]+E[U_1^2]-2E^2[U_1])}{2(\Delta+E[U_{1,R}])} \qquad (7.38)$$

Let $E[X_\gamma]$ and $E[X_\gamma^2]$ be the first and second moment, respectively, of γ(.). Then

$$E[X_\gamma^2] = \frac{1}{E[B(-\Delta,\Delta)]}\left\{\frac{E[D]}{\Delta+E[U_{1,R}]}\; I_1 + \frac{1}{2(\Delta+E[U_{1,R}])}\left(\frac{\Delta^3}{3} + E[W+D(0,L_0]]\Delta^2 + E[(W+D(0,L_0])^2]\frac{E[(W+D(0,L_0])^2]}{3}\right)\right\} \tag{7.39}$$

$$E[X_\gamma^2] = \frac{2}{E[B(-\Delta,\Delta)]}\left\{\frac{E[D]}{\Delta+E[U_{1,R}]}\; I_2 + \frac{1}{2(\Delta+E[U_{1,R}])}\left(\frac{\Delta^4}{12}+\frac{E[W+D(0,L_0]]\Delta^3}{3} + \frac{E[(W+D(0,L_0])^2]\Delta^2}{2} + \frac{E[(W+D(0,L_0])^3]\Delta}{3} + \frac{E[(W+D(0,L_0])^4]}{12}\right)\right\} \tag{7.40}$$

Assuming $W+D(0,L_0]$ is gamma distributed, it is easy to compute $E[X_\gamma]$ and $E[X_\gamma^2]$.

Now $E[B(s,\Delta)]$ is approximated by

$$E[B(s,\Delta)] \simeq E[B(-\Delta,\Delta)](1-\hat{\gamma}(s+\Delta)) \qquad s\geq-\Delta \tag{7.41}$$

where $\hat{\gamma}(.)$ is the gamma distribution with the same first two moments as $\gamma(.)$. Then it follows from (7.35) and (7.41) that

$$E[X^+(s,\Delta)] \simeq s+\frac{(\Delta^2+E[U_{1,R}^2])}{2(\Delta+E[U_{1,R}])} + \frac{E[D^2]}{2E[D]} - E[L]E[D] + E[B(s,\Delta)](1-\hat{\gamma}(s+\Delta)) \tag{7.42}$$

Substitution of $s=-\Delta$ into (7.42) yields consistency with (7.38). For sake of completeness we also give the expression for $E[B(s,\Delta)]$, when $s\leq-\Delta$.

$$E[B(s,\Delta)] = E[D(0,L_0]] + \frac{1}{2(\Delta+E[U_{1,R}])}(s^2+2(\Delta-s)E[D(0,L_0]]$$
$$- 2sE[W] + E[W^2] + E[U_1^2] - 2E^2[U_1] \qquad s\leq-\Delta \tag{7.43}$$

In the literature usually a linear interpolation formula is applied,

$$E[X^+(s,\Delta)] = s-E[U_{1,R}] + \frac{1}{2}\Delta - E[D]E[L] \tag{7.44}$$

We compare (7.36) and (7.44). Since $E[U_{1,R}^2]\geq E^2[U_{1,R}]$ we have that

$$E[X^+(s,\Delta)] \ (7.36) \leq s+\frac{\Delta}{2} - \frac{E[U_{1,R}]}{2} + E[U_1] - E[L]E[D]$$
$$\leq E[X^+(s,\Delta)] \ (7.44) + E[U_1]$$

Hence for s large and $E[U_1]$ small, we expect that $E[X^+(s,\Delta)]$ (7.44) overestimates stock. In the case of smooth demand (7.36) and (7.44) are approximately equal. For s small we expect both (7.36) and (7.44) yield poor approximations. This is confirmed by our results

Case II: <u>The compound renewal model</u>

We derive approximations for $E[X^+(s,\Delta)]$ and $E[B(s,\Delta)]$ along the lines of section 6.3. We apply the approximation derived for the function H(x,t),

H(x,t) := expected cost incurred during (0,t], assuming no orders arrive during (0,t], the net stock at time 0 equals $x\geq 0$.

The cost structure is again as follows. For each item in stock a cost of \$ 1 is paid per time unit. To obtain an expression for the expected cost incurred during the replenishment cycle $(L_0,\sigma_1+L_1]$, $E[H(s,\Delta)]$, we condition on the net stock at time L_0,

$$X(L_0) = s+\Delta - D(0,L_0]$$

This yields

$$E[H(s,\Delta)] = \int_0^\infty \int_0^{s+\Delta} H(s+\Delta-y,t)\, dF_{D(0,L_0]|\sigma_1+L_1-L_0=t}\, dF_{\sigma_1+L_1-L_0}(t)$$

Clearly we have

$$E[X^{+}(s,\Delta)] \;=\; \frac{E[H(s,\Delta)]}{E[\sigma_1]}$$

From the analysis in chapter 2 we know that

$$H(x,t) \simeq \frac{(c_A^2-1)}{2}E[A]\left(x - \int_0^x (x-y)\,dF_{D(0,t]}(y) \right)$$
$$+ E[A]\left(\int_0^x (x-y)\,dM(y) - \int_0^x \int_0^{x-y} (x-y-z)\,dM(t)\,dF_{D(0,t]}(y) \right)$$

Combination of the above results yields an expression for $E[X^{+}(s,\Delta)]$,

$$E[X^+(s,\Delta)] = \frac{(c_A^2-1)}{2}\frac{E[D]}{(\Delta+E[U_{1,R}])}\left(\int_0^{s+\Delta}(s+\Delta-y)\,dF_{D(0,L_0]}(y) - \int_0^{s+\Delta}(s+\Delta-y)\,dF_{D(0,\sigma_1+L_1]}(y)\right)$$
$$+ \frac{E[D]}{(\Delta+E[U_{1,R}])}\left(\int_0^{s+\Delta}\int_0^{s+\Delta-y}(s+\Delta-y-z)\,dM(z)\,dF_{D(0,L_0]}(y) - \int_0^{s+\Delta}\int_0^{s+\Delta-y}(s+\Delta-y-z)\,dM(z)\,dF_{D(0,\sigma_1+L_1]}(y)\right)$$

Employing the now standard arguments we can rewrite this expression as

$$E[X^+(s,\Delta)] = \frac{(c_A^2-1)}{2}\frac{E[D]}{(\Delta+E[U_{1,R}])}\left(\int_0^{s+\Delta}(s+\Delta-y)\,dF_{D(0,L_0]}(y) - \int_0^{s}(s-y)\,dF_{U_{1,R}+D(0,L_0]}(y)\right)$$
$$+ \frac{E[D]}{(\Delta+E[U_{1,R}])}\left(\int_0^{s+\Delta}\int_0^{s+\Delta-y}(s+\Delta-y-z)\,dM(z)\,dF_{D(0,L_0]}(y) - \int_0^{s}\frac{(s-y)^2}{2E[D]}\,dF_{W+D(0,L_0]}(y)\right) \tag{7.45}$$

Equation (7.45) will be applied after the derivation of an approximate relation between E[B(s,Δ)] and E[X^+(s,Δ)].

From another cost argument we can deduce that

$$E[X^+(s,\Delta)] = E[Y(s,\Delta)] - E[D]\frac{E[L]}{E[A]} + E[B(s,\Delta)] \tag{7.46}$$

We are again confronted with the problem to derive an expression for E[Y(s,Δ)], the average inventory position. In this case we follow the arguments leading to (7.28). Assume that $ y is paid per time unit when the inventory position equals y. Define

C(s,Δ) := cost incurred during (0,σ_1].

We write E[C(s,Δ)] as

$$E[C(s,\Delta)] = SE[\sigma_1] + E\left[\int_0^{T_U} (Y(t)-s]\,dt\right] - E\left[\int_{T_U}^{\sigma_1} (s-Y(t))\,dt\right] \tag{7.47}$$

Remember that T_U is the time at which the reorder level is undershot. By numerical experimentation we found that

$$P\{\sigma_1 - T_U \le t\} = \frac{t}{R} \qquad 0<t<R$$

and σ_1-T_U independent of U_1.

The expectation of the first integral in (7.47) is equal to the expected cost incurred in an (s,S)-model with s=0 and s=Δ, corrected for the fact that the first arrival is at $\tilde{A}_1$ instead of A_1, where A_1 is an ordinary interarrival time and $\tilde{A}_1$ is the stationary residual lifetime associated with A. This yields

$$E[\int_0^{T_U} (Y(t)-s)\,dt] = \frac{E[A]}{E[D]}\left\{\frac{\Delta^2}{2} - \frac{E[U_1^2]}{2} + \frac{E[D^2]}{2E[D]}(\Delta + E[U_1])\right\} + \frac{(c_A^2-1)}{2}\Delta E[A] \tag{7.48}$$

To obtain an expression for the expectation of the second integral we proceed as follows.

We define T_{N+1} by

$$T_{N+1} = \sum_{n=1}^{N+1} A_n,$$

where N is defined as

N := the number of customers arriving in $[\hat{T}_U, \sigma_1]$.

Now we assume that $T_{N+1}-\sigma_1$ is independent of N and distributed according to the stationary residual lifetime of A. This is in fact in agreement with the APIT-assumption. Hence

$$P\{T_{N+1}-\sigma_1 \le x\} \simeq \frac{1}{E[A]} \int_0^x (1-F_A(y))\,dy$$

Furthermore we assumed that U_1 is independent of $\sigma_1-\hat{T}_U$ homogeneously distributed on (0,R). Now we can derive the following

$$E[\int_{\hat{T}_U}^{\sigma_1} (s-Y(t))\,dt] = E[U_1]\frac{R}{2} + E\left[\sum_{n=1}^{N}\sum_{m=n+1}^{N+1} D_n A_m\right] - E\left[(T_{N+1}-\sigma_1)\sum_{n=1}^{N} D_n\right]$$

Using the fact that $\{D_n\}$ is independent of N and that N+1 is stopping time for $\{A_m\}$, we find

$$E\left[\int_{\hat{T}_U}^{\sigma_1} (s-Y(t))\,dt\right] = E[U_1]\frac{R}{2} + E[D]E[A]\frac{1}{2}(E[N^2]+E[N]) - \frac{(1+c_A^2)}{2}E[A]E[D]E[N] \tag{7.49}$$

In principle (7.49) yields a tractable expression for the required expectation, since we have approximations for E[N] and $E[N^2]$ given by (6.45) and (6.46), respectively. For the convenience of further analysis we rewrite (7.49) to eliminate E[N] and $E[N^2]$ and to write the expectation on the left hand side of (7.49) in terms of E[W] and $E[W^2]$. After some straightforward algebra, where we use expectations (6.47) and (6.48) for E[W] and $E[W^2]$, respectively, we find

$$E\left[\int_{\hat{T}_U}^{\sigma_1} (s-Y(t))\,dt\right] \simeq \frac{E[W^2]}{2E[D]}E[A] - \frac{(c_A^2-1)}{2}E[A](E[U_1]+E[W]) \tag{7.50}$$

Substitution of (7.50) into (7.46) yields

$$E[X^{+}(s,\Delta)] \simeq s+\frac{1}{\Delta+E[U_{1,R}]}\left(\frac{\Delta^2}{2}-\frac{E[U_1^2]}{2}+E[U_1](\Delta+E[U_1])-\frac{E[W^2]}{2}\right) + \frac{(c_A^2-1)}{2}E[D] - E[D(0,L_0)] + E[B(s,\Delta)] \tag{7.51}$$

Equation (7.51) expresses E[X^{+}(s,Δ)] in terms of E[B(s,Δ)] and vice versa. In the preceding chapters we derived an approximation for E[B(s,Δ)] by applying the PDF-method. We proceed accordingly. Yet, before doing so, we observe that for s sufficiently large,

$$E[X^{+}(s,\Delta)] \simeq s+\frac{1}{\Delta+E[U_{1,R}]}\left(\frac{\Delta^2}{2}-\frac{E[U_1^2]}{2}+E[U_1](\Delta+E[U_1])\right) + \frac{(c_A^2-1)}{2}E[D] - E[D(0,L_0)] \tag{7.52}$$

Approximation (7.52) is of use for most practical situations. Yet a more robust approximation is derived from application of the PDF-method. Towards this end we substitute (7.45) into (7.51). Some algebra leads to the following expression for E[B(s,Δ)],

$$E[B(s,\Delta)] \simeq \frac{(c_A^2-1)}{2}\frac{E[D]}{\Delta+E[U_{1,R}]}\left(\int_{s+\Delta}^{\infty}(y-(s+\Delta))\,dF_{D(0,L_0]}(y) - \int_{s}^{\infty}(y-s)\,dF_{U_{1,R}+D(0,L_0]}(y)\right)$$
$$+\frac{E[D]}{\Delta+E[U_{1,R}]}\left(\int_0^{s+\Delta}\int_0^{s+\Delta}(s+\Delta-y-z)\,dM(z)\,dF_{D(0,L_0]}(y) - (a_2(s+\Delta)^2 + a_1(s+\Delta) + a_0) + \int_s^{\infty}\frac{(y-s)^2}{2E[D]}dF_{W+D(0,L_0]}(y)\right) \tag{7.53}$$

Equation (7.53) is partly identical to the expressions for E[B(s,Δ)] in the (s,S)-model and the discrete time (R,s,S)-model. Taking the right parts from (5.34) and (7.37), we can compute the first two moments of X_γ, which has the pdf γ(.), defined by

$$\gamma(x) := 1-\frac{E[B(x-\Delta,\Delta)]}{E[B(-\Delta,\Delta)]}, \qquad x\geq 0,$$

$$E[X_\gamma] = \frac{1}{E[B(-\Delta,\Delta)]}\left\{(c_A^2-1)\frac{E[D]}{\Delta+E[U_{1,R}]}\left(\frac{E[D^2(0,L_0]]}{2} - \frac{E[(U+D(0,L_0])^2]}{2} - \Delta\, E[U+D(0,L_0]]\right) + \frac{E[D]}{\Delta+E[U_{1,R}]}I_1 + \frac{1}{2(\Delta+E[U_{1,R}])}\left(\frac{\Delta^3}{3} + E[W+D(0,L_0]]\Delta^2 + E[(W+D(0,L_0])^2]\Delta + \frac{E[(W+D(0,L_0])^3]}{3}\right)\right\} \tag{7.54}$$

$$E[X_\gamma^2] = \frac{2}{E[B(-\Delta,\Delta)]} \Bigg\{ \frac{(c_A^2-1)}{2} \frac{E[D]}{\Delta+E[U_{1,R}]} \Bigg(\frac{E[D^3(0,L_0]]}{6} - \frac{\Delta^3}{6} - \frac{E[U+D(0,L_0]]\Delta^2}{2} - \frac{E[(U+D(0,L_0]))^2]}{2}\Delta - \frac{E[(U+D(0,L_0]))^3])}{6} \Bigg) + \frac{E[D]}{\Delta+E[U_{1,R}]} I_2 + \frac{1}{2(\Delta+E[U_{1,R}])} \Bigg(\frac{\Delta^4}{12} + \frac{E[(W+D(0,L_0]))]}{3}\Delta + \frac{E[(W+D(0,L_0]))^2]}{2}\Delta^2 + \frac{E[(W+D(0,L_0]))^4]}{12}) \Bigg) \Bigg\} \tag{7.55}$$

It is easily derived from (7.51) that

$$E[B(-\Delta,\Delta)] = \Delta - \frac{1}{\Delta+E[U_{1,R}]} \Bigg(\frac{\Delta^2}{2} - \frac{E[U_1^2]}{2} - \frac{E[W^2]}{2} + E[U_1](\Delta+E[U_1]) \Bigg) - \frac{(c_A^2-1)}{2} E[D] + E[D(0,L_0]] \tag{7.56}$$

Let $\hat{\gamma}(.)$ be the gamma distribution with its first two moments equal to $E[X_\gamma]$ and $E[X_\gamma^2]$, respectively. This yields

$$E[B(s,\Delta)] \simeq E[B(-\Delta,\Delta)](1-\gamma(s+\Delta)) \qquad s \geq -\Delta \tag{7.57}$$

An expression for $E[B(s,\Delta)]$ for $s<-\Delta$ is again derived from (7.51),

$$E[B(s,\Delta)] = -s - \frac{1}{\Delta+E[U_{1,R}]}\left(\frac{\Delta^2}{2} - \frac{E[U_1^2]}{2} + E[U_1](\Delta+E[U_1]) - \frac{E[W^2]}{2}\right) - \frac{(c_A^2-1)}{2}E[D] + E[D(0,L_0)] \tag{7.58}$$

Substitution of (7.57) into (7.51) yields the following robust approximation to $E[X^+(s,\Delta)]$,

$$E[X^+(s,\Delta)] \simeq s + \frac{1}{\Delta+E[U_{1,R}]}\left(\frac{\Delta^2}{2} - \frac{E[U_1^2]}{2} + E[U_1](\Delta+E[U_1]) - \frac{E[W^2]}{2}\right) - \frac{(c_A^2-1)}{2}E[D] - E[D(0,L_0)] + E[B(-\Delta,\Delta)](1-\gamma(s+\Delta)) \tag{7.59}$$

This concludes our analysis of the (R,s,S)-model. We have expressions for the main performance characteristics. We have tested them by computer simulation and they have proven to be practically useful. It is now time to apply the results to gain insight into the mechanics of inventory management. We want to get some feeling for the benefits and drawbacks of the various models, both in terms of service performance and in operational costs. This discussion is subject of chapter 8.