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Discrete Approximations for Singularly Perturbed Boundary Value Problems with Parabolic Layers

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Abstract

Singularly perturbed boundary value problems for equations of elliptic and parabolic type are studied. For small values of the perturbation parameter, parabolic boundary layers appear in these problems. If classical discretisation methods are used, the solution of the finite difference scheme and the approximation of the diffusive flux derived from it do not converge uniformly with respect to this parameter. In particular, the relative error of the diffusive flux becomes unbounded as the perturbation parameter tends to zero. Using the method of special condensing grids, we can construct difference schemes that allow approximation of the solution and the normalised diffusive flux uniformly with respect to the small parameter.

We also consider singularly perturbed boundary value problems for convection-diffusion equations. Also for these problems we construct special finite difference schemes, the solution of which converges ε -uniformly. We study what problems appear, when classical schemes are used for the approximation of the spatial derivatives. Also for parabolic equations ε -uniformly convergent approximations for the normalised fluxes are constructed. Results of numerical experiments are discussed.

Summarising, we consider:

1. Problems for Singularly Perturbed (SP) parabolic equation with discontinuous boundary conditions.
2. Problems for SP elliptic equations with boundary conditions of Dirichlet, Neumann and Robin type.
3. Problems for SP parabolic equations, for which the solution and the normalised diffusive fluxes are required.

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Part I

Introduction

Consider a substance (or admixture) in a solution with a flux satisfying Fick's law, and with distribution given by a diffusion equation. Let the initial concentration of the admixture in the material as

well as the concentration of the admixture on the boundary of the body be known. It is required to find the distribution of admixture in the material at any given time and also the quantity of admixture (that is the diffusive flux) emitted from the boundaries into the exterior environment. Such problems are of interest in environmental sciences in determining the pollution entering the environment from manufactured sources, such as houses, factories and vehicles, and from industrial and agricultural waste disposal sites, and also in chemical kinetics where the chemical reactions are described by reaction-diffusion equations.

In considering such problems, it is important to note that the diffusion Fourier number, which is given by the diffusion coefficient of the admixture in materials, can be sufficiently small that large variations of concentration occur along the depth of the material. For small values of the Fourier number, diffusion boundary layers appear. Therefore these problems exhibit a singularly perturbed character. The mathematical formulation of such problems have a perturbation parameter which is a small coefficient (the diffusion Fourier number) multiplying the highest derivatives of the differential equation.

Even in the case where only the approximate solution of the singularly perturbed boundary value problem is required, classical numerical methods, such as finite difference schemes and finite element methods [16, 17, 18] exhibit unsatisfactory behaviour. This arises because the accuracy of the approximate solution depends inversely on the perturbation parameter value and thus it deteriorates as the parameter decreases. In [22] it was shown that the use of classical numerical methods does not give approximate solutions with acceptable accuracy even for very fine grids. Thus, even the use of computers with extremely large capacity will not guarantee acceptable accuracy in the answer. To be more precise, it can be shown that the error in the approximate solution on any arbitrarily fine grid is greater than some positive number (independent of the number of grid nodes), for a sufficiently small value of the perturbation parameter (the diffusion Fourier number). For some applied problems such solution accuracy can be satisfactory. However even in these cases dissatisfaction can be caused by the lack of a guarantee that the use of a finer grid will increase the accuracy of the approximation.

More serious problems occur when an accurate approximation of the spatial derivatives of the solution is also required. For example, in order to determine the quantity of admixture which enters the environment per unit of time, it is necessary to compute the gradient of the concentration of the substance along the normal to the surface of the material. When classical finite difference schemes are used it can be expected that errors in the computed diffusive flux will be much larger than those of the computed concentration. Such errors in evaluating fluxes can be of unacceptable magnitude.

Similar difficulties appear also in problems of heat exchange in cases where the heat Fourier number can take any arbitrary small value. One often requires an accurate approximation of the thermal flux on a boundary of the body.

This report is devoted to the construction of numerical approximations, using finite difference schemes, of singularly perturbed boundary value problems for elliptic and parabolic equations. The simplest example of problems of such type in a one-dimensional case is the problem of a stationary diffusion process with a reacting substance :

$$\varepsilon^2 \frac{d^2}{dx^2} u(x) - c(x)u(x) = f(x), \quad x \in D,$$

$$u(x) = \varphi(x), \quad x \in \Gamma,$$

for $c(x) \geq c_0 > 0$, $x \in \overline{D}$. Here $D = (0,1)$, $\Gamma = \overline{D} \setminus D$ is a boundary of the domain D and the parameter ε can take any value in the interval $(0,1]$. The parameter ε characterises the diffusion coefficient of the substance and the function $c(x)$ characterises the intensity of decay of the diffusion matter. When the parameter tends to zero, diffusion boundary layers appear in a neighbourhood of the boundary.

In the case of regular boundary value problems the error in the approximate solution produced with the use of grid methods, is a function of the smoothness of the solution and of the distribution of

the nodes of the grids used. However, the application of classical grid methods for such singularly perturbed boundary value problems leads to loss of accuracy for the approximate solution when the parameter value is small (see, for example, [12, 22] and results in II). The following question therefore arise: how to construct and to analyse special numerical methods for solving singularly perturbed boundary value problems, the approximate solution of which converges uniformly with respect to the parameter ε (or, in short, ε -uniformly). The error of the approximate solution obtained by such methods, should be independent of the parameter value and defined only by the number of nodes of the grid used.

Detailed analytic investigations of such special numerical methods dates back to the end of 1960s (see, for example, [3, 12]). These first strong results for problems with boundary layers belong to two different approaches which are used for construction of special numerical methods:

(a) fitted methods [12] on meshes with arbitrary distribution of nodes (for example, on a uniform mesh) the coefficients of difference equations (difference approximations) are chosen (fitted) to ensure parameter-uniform accuracy of the approximate solution; or

(b) methods on special condensing grids (or adaptive meshes) [3]. Those methods use the standard classical difference equations but the nodes of the mesh are redistributed (or adapted, or condensed in the boundary layer) such that parameter-uniform convergence is achieved.

Special, fitted schemes (that is the first approach) are attractive, since they allow the use of meshes with an arbitrary distribution of nodes, e.g. uniform grids (see, for example, [1, 2, 4, 6, 12]). Using the second approach, adapted meshes with classical finite difference approximations, parameter-uniformly convergent schemes were also constructed for a series of boundary value problems (see, for example, [28] and references therein). For some boundary value problems parameter-uniformly convergent schemes were constructed using either the first or the second approach for the same problem (see, for example [6, 25]), or using both approaches together for the same problem (see, for example [22, 23], where different approaches were used in different coordinate directions). In [14] both approaches were used at the same time (a fitted scheme on a grids with condensing nodes in the boundary layer). Thus there is a large variety of special approaches tailored to individual boundary value problems in the literature.

In the case of singularly perturbed boundary value problems, for which accurate estimates of the diffusive fluxes are required, methods must be evolved which approximate both the solution and the normalised fluxes accurately. Investigations of such methods have been sparse in the literature (see, for example, [22]).

In this report we consider singular perturbed elliptic and parabolic equations with parabolic boundary layers. For boundary value problems we construct special difference schemes, solutions of which converge ε -uniformly in an ℓ^∞ - norm. Also approximations of the normalised diffusive fluxes which converge ε -uniformly, are proposed.

In II it is shown that the computed solution, for a singularly perturbed ordinary differential equation, which is found using a classical scheme does not converge ε -uniformly. We then consider the construction of special schemes which are ε -uniformly convergent. Grid approximations of solutions and diffusive fluxes for singularly perturbed parabolic equations are considered in III. Approximations of elliptic equation with mixed boundary condition, which admit Dirichlet and Neumann conditions are studied in IV. To construct the special schemes in III, IV methods based on special condensed grids are used.

In V we investigate singularly perturbed boundary value problems with discontinuous boundary conditions. In this case fitted methods are used.

The improved special finite difference schemes which allow accurate approximation of both the solutions and the normalised diffusive fluxes for boundary value problems can be effectively applied for the solution and numerical analysis of applied problems with boundary and interior layers. The methods for construction of special schemes developed here can also be used to construct and investigate special schemes for more general singularly perturbed boundary value problems (see, for example, [7, 8, 28]).

Part II

The necessity to construct special schemes. A numerical example

In order to demonstrate the problems which may appear in the numerical solution process, we consider the following simple example of a singularly perturbed ordinary differential equation for a boundary value problem:

$$L_{(1.1)}u(x) \equiv \varepsilon^2 \frac{d^2}{dx^2}u(x) - u(x) = -1, \quad x \in D, \quad (1.1a)$$

$$u(0) = u(1) = 0; \quad \varepsilon \in (0, 1], \quad (1.1b)$$

where $D = (0, 1)$. For the solution of this problem we should like to use classical numerical methods, for example finite difference schemes¹.

The standard scheme for the problem (1.1) is defined as follows. In the interval \bar{D} , we introduce the grid

$$\bar{D}_h = \bar{\omega}_1, \quad (1.2)$$

where $\bar{\omega}_1$ is a uniform grid with a step-size $h = 1/N$, and $N + 1$ is the number of nodes of the grid $\bar{\omega}_1$. For the problem (1.1) we employ the classical difference scheme

$$\Lambda_{(1.3)}z(x) \equiv \{\varepsilon^2 \delta_{x\bar{x}} - 1\}z(x) = -1, \quad x \in D_h, \quad (1.3)$$

$$z(0) = z(1) = 0.$$

Here $D_h = D \cap \bar{D}_h$, $\delta_{x\bar{x}}z(x)$ is the second order central difference approximation to the second derivative

$$\begin{aligned} \delta_{x\bar{x}}z(x) &= h^{-1}(\delta_x - \delta_{\bar{x}})z(x), \\ \delta_x z(x) &= h^{-1}(z(x+h) - z(x)), \quad \delta_{\bar{x}}z(x) = h^{-1}(z(x) - z(x-h)). \end{aligned}$$

It is known (see, for example, [8, 12]) that the error of the scheme (in the ℓ^∞ -norm) depends on the value of the parameter ε and on the grid step-size

$$|u(x) - z(x)| \leq Q(\varepsilon)h^2, \quad x \in \bar{D}_h. \quad (1.4)$$

Here the constant $Q(\varepsilon)$ essentially depends on the parameter value.

Moreover, for sufficiently small values of the parameter, that is, for $\varepsilon = \varepsilon(h) = h^{-1}$, this error becomes larger than some positive constant [22]

$$\max_{\bar{D}_h} |u(x) - z(x)| \geq m_{(1.5)} > 0 \quad \text{for } h \rightarrow 0 \quad (1.5)$$

where $u(x) = u(x; \varepsilon)$, $z(x) = z(x; \varepsilon, h)$. That is, for any very small step-size of the grid and an arbitrary value of the parameter ε , $\varepsilon \in (0, 1]$, a value of ε can be found such that the error is not less than a positive constant.

It follows from the estimate (1.4) that the difference scheme (1.3), (1.2) converges as $h \rightarrow 0$ (or $N \rightarrow \infty$) for a fixed value of the parameter. However, according to the estimate (1.5) this difference scheme does not converge uniformly with respect to the small parameter ε (that is it does not converge ε -uniformly).

¹The notation $L_{(j,k)}$, $L_{(j,k)}^i$ (or $f_{(j,k)}(x)$, $f_{(j,k)}^i(x)$) means that these operators (or functions) are first introduced in formula (j.k).

It is desirable to have numerical methods, for which the error in the approximate solution tends to zero independently of the parameter ε as $N \rightarrow \infty$, that is methods in which the approximate solution converges ε -uniformly to the actual solution for $N \rightarrow \infty$.

The importance of this criterion for applications, in particular, in the case of the boundary value problem (1.1) is demonstrated by the following numerical experiments.

The solution of problem (1.1) is given by the following expression:

$$u(x) = u(x; \varepsilon) = 1 - \frac{e^{-x/\varepsilon} + e^{-(1-x)/\varepsilon}}{1 + e^{-1/\varepsilon}}, \quad x \in \overline{D},$$

Note that the function $u(x)$ satisfies the following relations

$$0 \leq u(x) < 1, \quad x \in \overline{D},$$

$$\max_{\overline{D}} u(x) = u(1/2), \quad \lim_{\varepsilon \rightarrow 0} u(1/2) = 1.$$

In Table 1 we give the results of computing the value $E(\varepsilon, N)$,

$$E(\varepsilon, N) = \max_{\overline{D}_h} e(x; \varepsilon, N), = \max_{\overline{D}_h} |u(x; \varepsilon) - z(x; \varepsilon, N)|$$

which is the maximum local error on \overline{D}_h . Here $u(x; \varepsilon)$ is the solution of problem (1.1), and $z(x; \varepsilon, N)$ the solution of problem (1.3). The values of $\overline{E}(N)$, are also given, where

$$\overline{E}(N) = \max_{\varepsilon} E(\varepsilon, N), \quad \varepsilon = 2^{-12} \dots, 1.0$$

is the largest error of the approximate solution for a fixed value of N and ε varying over the values shown in Table 1

The value $\overline{E}(N)$ defines the best guaranteed accuracy which is obtained by using the scheme (1.3), (1.2) to solve the problem (1.1) for a given N and various values of the parameter $\varepsilon = 4^{-m}$, $m = 0, 1, \dots, 6$.

TABLE 1. Table of errors $E(N, \varepsilon)$ for the classical scheme

$\varepsilon \setminus N$	4	16	64	256	1024	$u(0.5; \varepsilon)$
1	5.295e-04	3.333e-05	2.084e-06	1.303e-07	8.142e-09	1.132e-01
2^{-2}	1.991e-02	1.328e-03	8.339e-05	5.213e-06	3.258e-07	7.342e-01
2^{-4}	3.758e-02	1.409e-02	9.526e-04	5.986e-05	3.742e-06	9.993e-01
2^{-6}	3.876e-03	3.741e-02	1.409e-02	9.526e-04	5.985e-05	1.000e+00
2^{-8}	2.440e-04	3.876e-03	3.741e-02	1.409e-02	9.526e-04	1.000e+00
2^{-10}	1.526e-05	2.440e-04	3.876e-03	3.741e-02	1.409e-02	1.000e+00
2^{-12}	9.537e-07	1.526e-05	2.440e-04	3.876e-03	3.741e-02	1.000e+00
$\overline{E}(N)$	3.758e-02	3.741e-02	3.741e-02	3.741e-02	3.741e-02	

It follows from Table 1 that the solution of the difference scheme (1.3), (1.2) converges to the solution of problem (1.1) for a fixed value of the parameter. However, the error behaviour is not regular with increasing N . The error decreases with increasing N only for $N \geq 4\varepsilon^{-1}$. The approximate solution does not converge ε -uniformly. Indeed, for a fixed value of N the largest error is found for the parameter value $\varepsilon = \varepsilon(N) = 4^{-1}N^{-1}$, and this error is equal to $3.74 \cdot 10^{-2}$. For any large value of N we cannot guarantee an accuracy better than $3.74 \cdot 10^{-2}$. For the worst realisable error $\overline{E}(N)$ the lower bound $\overline{E}(N) \geq 3.74 \cdot 10^{-2}$ holds. The relative worst realisable error for a fixed N , namely, the value $\delta(N) \equiv \overline{E}(N) [\max_{\overline{D}} |u(x)|]^{-1} \geq \overline{E}(N)$, is independent of N and is equal to 3.74 %.

Although the approximate solution does not converge ε -uniformly, this level of accuracy in the computed solution can be acceptable in some cases. The computed solution gives a good qualitative representation of the exact solution behaviour for all values of the parameter ε .

However the accuracy issue appears more significant in the case where, for problem (1.1), it is required to find the gradient of the function $u(x)$ on a boundary (at the ends of the interval D). The derivative $(d/dx)u(x)$ increases unboundedly on the boundary as the parameter ε tends to zero. However, the value $P(x) \equiv \varepsilon(d/dx)u(x)$ (we call this value the normalised diffusion flux, or more briefly the normalised flux) remains bounded ε -uniformly. Therefore it is natural to consider the following problem:

$$\begin{aligned} & \text{find for boundary value problem (1.1) the solution } u(x), x \in \overline{D} \\ & \text{and the normalised diffusion flux } P(x) \text{ on the boundary } \Gamma. \end{aligned} \quad (1.6)$$

Note that for the function $P(x) = P(x; \varepsilon)$ the relations

$$\max_{\overline{D}} |P(x)| \leq 1;$$

$$P(0; \varepsilon) = -P(1; \varepsilon) > 0; \quad \lim_{\varepsilon \rightarrow 0} P(0; \varepsilon) = 1$$

hold.

To solve the problem (1.1), (1.6) we apply the difference scheme (1.3), (1.2). The value $P(0)$ is approximated by the value

$$P^{h+}(x) \equiv \varepsilon \delta_x z(x), \quad x = 0 \quad (1.7)$$

which is the computed normalised diffusive flux at the point $x = 0$.

In Table 2 we give the results of computing the value $Q(\varepsilon, N)$

$$Q(\varepsilon, N) = |P(0) - P^{h+}(0)|,$$

which is the error in the normalised flux on the boundary $x = 0$ for various values of ε and N . Values of $\overline{Q}(N)$ are also given where

$$\overline{Q}(N) = \max_{\varepsilon=4^{-m}, m=0,1,\dots,6} Q(\varepsilon, N).$$

The value $\overline{Q}(N)$, which is the best guaranteed accuracy (for varying ε) of the computed normalised flux at $x = 0$, that can be obtained when using the scheme (1.3), (1.2), (1.7) to solve the problem (1.1), (1.6) for a given N and various values of the parameter ε .

TABLE 2. Table of errors of the normalised flux $Q(N, \varepsilon)$ for the classical scheme

$\varepsilon \setminus N$	4	16	64	256	1024	$P(0; \varepsilon)$
1	1.224e-01	3.109e-02	7.802e-03	1.952e-03	4.882e-04	4.621e-01
2^{-2}	3.926e-01	1.179e-01	3.080e-02	7.785e-03	1.951e-03	9.640e-01
2^{-4}	7.640e-01	3.820e-01	1.172e-01	3.076e-02	7.782e-03	1.000e+00
2^{-6}	9.377e-01	7.639e-01	3.820e-01	1.172e-01	3.076e-02	1.000e+00
2^{-8}	9.844e-01	9.377e-01	7.639e-01	3.820e-01	1.172e-01	1.000e+00
2^{-10}	9.961e-01	9.844e-01	9.377e-01	7.639e-01	3.820e-01	1.000e+00
2^{-12}	9.990e-01	9.961e-01	9.844e-01	9.377e-01	7.639e-01	1.000e+00
$\overline{Q}(N)$	9.990e-01	9.961e-01	9.844e-01	9.377e-01	7.639e-01	

It follows from Table 2 that the value $P^{h+}(0) = P^{h+}(0; \varepsilon, N)$, the computed normalised flux at $x = 0$, converges to the value $P(0; \varepsilon)$ with increasing N , for a fixed value of the parameter ε . However, they do not converge ε -uniformly. The error $Q(\varepsilon, N)$ remains constant for a constant product εN . Moreover, the error $Q(\varepsilon, N)$ tends to the value P_0 with decreasing ε for any fixed N , where

$$P_0 \equiv \lim_{\varepsilon \rightarrow 0} P(0; \varepsilon) = 1.$$

In Table 3 the values $\lambda(\varepsilon, N)$ are given where

$$\lambda(\varepsilon, N) \equiv \frac{P(0; \varepsilon)}{P^{h+}(0; \varepsilon, N)} = \frac{d}{dx} u(0; \varepsilon) [\delta_x z(0; \varepsilon, N)]^{-1}$$

denotes the ratio of the exact and the computed flux at $x = 0$.

TABLE 3. Table of ratios of the normalised fluxes $\lambda(N, \varepsilon)$ for the classical scheme

$\varepsilon \setminus N$	4	16	64	256	1024	$P(0; \varepsilon)$
1	1.360e+00	1.072e+00	1.017e+00	1.004e+00	1.001e+00	4.621e-01
2^{-2}	1.687e+00	1.139e+00	1.033e+00	1.008e+00	1.002e+00	9.640e-01
2^{-4}	4.237e+00	1.618e+00	1.133e+00	1.032e+00	1.008e+00	1.000e+00
2^{-6}	1.606e+01	4.236e+00	1.618e+00	1.133e+00	1.032e+00	1.000e+00
2^{-8}	6.402e+01	1.606e+01	4.236e+00	1.618e+00	1.133e+00	1.000e+00
2^{-10}	2.560e+02	6.402e+01	1.606e+01	4.236e+00	1.618e+00	1.000e+00
2^{-12}	1.024e+03	2.560e+02	6.402e+01	1.606e+01	4.236e+00	1.000e+00
$\bar{\lambda}(N)$	1.024e+03	2.560e+02	6.402e+01	1.606e+01	4.236e+00	

This ratio of the exact derivative to its computed difference approximation on the boundary $x = 0$, increases unboundedly for a fixed value of N as the parameter ε tends to zero. The value also increases very sharply when $N \rightarrow \infty$ and $\varepsilon N \rightarrow 0$ (that is for $\varepsilon \ll N^{-1}$). In these cases the computed flux gives a value which significantly underestimates the actual derivative. This means that, if the classical difference scheme (1.3), (1.2), (1.7) is used, the normalised flux is not even qualitatively approximated by the computed flux in an ε -uniform sense.

In the case of singularly perturbed elliptic equations, for which reduced equations do not contain spatial derivatives, and the principal term in the singular part of the problem solution is described by an equation similar to (1.1a) (see, for example, [22, 23, 25]), so it can be expected that, when solving such singularly perturbed elliptic and parabolic equations with classical difference schemes, computational problems will arise.

Part III

Boundary value problem for parabolic equations

1. INTRODUCTION

Here we study boundary value problems for singularly perturbed parabolic equations. Using a special condensing grid we construct special difference schemes, which approximate the solution and the normalised flux ε -uniformly. Using numerical examples we compare the classical and the special schemes and we show the effectiveness of the constructed schemes.

For the open interval $D = (0, d)$, on the domain

$$G = D \times (0, T], \quad S = S(G) = \overline{G} \setminus G, \quad (1.1)$$

we consider a boundary value problem for the parabolic equation

$$L_{(1.2)}u(x, t) \equiv \left\{ \varepsilon^2 a(x, t) \frac{\partial^2}{\partial x^2} - c(x, t) - p(x, t) \frac{\partial}{\partial t} \right\} u(x, t) = f(x, t), \quad (x, t) \in G, \quad (1.2a)$$

$$u(x, t) = \varphi(x, t), \quad (x, t) \in S. \quad (1.2b)$$

Here the functions $a(x, t)$, $c(x, t)$, $p(x, t)$, $f(x, t)$, and also the function $\varphi(x, t)$ are sufficiently smooth functions on the sets \overline{G} and S respectively. Moreover

$$a_0 \leq a(x, t) \leq a^0, \quad c(x, t) \geq 0, \quad p(x, t) \geq p_0, \quad (x, t) \in \overline{G},$$

$a_0, p_0 > 0$, $\varepsilon \in (0, 1]$. Suppose that at the corner points $S^* = \{(0, 0), (d, 0)\}$ compatibility conditions are satisfied, [13], which ensure smoothness of the solution to the boundary value problem for a fixed value of the parameter. The solution of the boundary value problem is a function $u \in C^{2,1}(G) \cap C^{1,0}(\overline{G})$, which satisfies the equation on G and the boundary condition on S . We wish to find the solution and the derivative $(\partial/\partial x)u(x, t)$, $(x, t) \in \overline{G}$.

When the parameter ε tends to zero, a parabolic boundary layer appears in the neighbourhood of the set S^1 , that is the lateral boundary of the set G . Note that the derivative $(\partial/\partial x)u(x, t)$, in the neighbourhood of the boundary layer, increases unboundedly when the parameter tends to zero. It is therefore convenient to consider, instead of the gradient $(\partial/\partial x)u(x, t)$, the value $\varepsilon(\partial/\partial x)u(x, t)$, $(x, t) \in \overline{G}$ which is bounded uniformly with respect to the parameter. The value

$$P(x, t) = \varepsilon \frac{\partial}{\partial x} u(x, t), \quad (x, t) \in \overline{G}$$

is called the normalised diffusive flux. In the case of problem (1.2) it is required to find the functions $u(x, t)$, $P(x, t)$, $(x, t) \in \overline{G}$.

On the set $\overline{G}_{(1.1)}$, we shall also consider the boundary value problem for the quasi-linear parabolic equation

$$L_{(1.3)}(u(x, t)) \equiv \left\{ \varepsilon^2 a(x, t) \frac{\partial^2}{\partial x^2} - p(x, t) \frac{\partial}{\partial t} \right\} u(x, t) - \quad (1.3a)$$

$$-g(x, t, u(x, t)) = 0, \quad (x, t) \in G,$$

$$u(x, t) = \varphi(x, t), \quad (x, t) \in S.$$

The function $g(x, t, u)$ is a sufficiently smooth function on the set $\overline{G} \times R$, satisfying

$$-m_{(1.3)} \leq \frac{\partial}{\partial u} g(x, t, u) < \infty, \quad (x, t, u) \in \overline{G} \times R, \quad (1.3b)$$

and the functions $a(x, t)$, $p(x, t)$, $\varphi(x, t)$ and the parameter ε satisfy the conditions above. Again, it is required to find the functions $u(x, t)$ and $P(x, t)$, $(x, t) \in \overline{G}$, that is, the solution of problem (1.3) and normalised flux.

We arrive at problem (1.3) by considering, for example, the diffusion of a substance (e.g. pollution) in a homogeneous layer of solid material of thickness L . When the concentration of substance C depends only on y , that is the distance to the surface, then the distribution of the substance in the layer is described by the diffusion equation

$$D \frac{\partial^2}{\partial y^2} C(y, \tau) - \frac{\partial}{\partial \tau} C(y, \tau) = F(y, \tau), \quad 0 < y < L, \quad 0 < \tau < \vartheta.$$

Here D is the diffusion coefficient, and the function $F(y, \tau)$ defines the source. Using variables $x = L^{-1}y$, $t = \vartheta^{-1}\tau$ and denoting $\varepsilon^2 = D\vartheta L^{-2}$, $u(x, t) = C(y(x), \tau(t))$, $f(x, t) = \vartheta F(y(x), \tau(t))$, we obtain an equation of the form (1.2a) where $a = p \equiv 1$, $c \equiv 0$. The parameter ε^2 is the diffusion Fourier number $F_O^D = D\vartheta L^{-2}$. The diffusive flux of the substance is defined by the formula

$$\begin{aligned} D \frac{\partial}{\partial y} C(y, \tau) &= DL^{-1} \frac{\partial}{\partial x} u(x, t) = D^{1/2} \vartheta^{-1/2} \varepsilon \frac{\partial}{\partial x} u(x, t) = \\ &= D^{1/2} \vartheta^{-1/2} P(x, t), \end{aligned}$$

where $x = x(y)$, $t = t(\tau)$.

The diffusion coefficients for different media vary considerably, ranging from $10^{-5} \text{ m}^2/\text{sec}$ for gases to $10^{-14} \text{ m}^2/\text{sec}$ for solid materials. For example, the diffusion coefficient of phenol in air and water is $0.8 \cdot 10^{-5} \text{ m}^2/\text{sec}$ and $0.8 \cdot 10^{-9} \text{ m}^2/\text{sec}$ respectively [5]. The diffusion Fourier number is determined by the diffusion coefficient and also by the size of the material samples and by the time period of the diffusion process. For dwellings or local air reservoirs, L is a value between 10 and 10^3 m and ϑ is between 10 min and 1 hour, then F_O^D takes on values between $5 \cdot 10^{-9}$ and $3 \cdot 10^{-4}$ for the process of phenol diffusion.

For thermal processes the Fourier number is defined by the formula $F_O^T = a\vartheta L^{-2}$, where a is the thermal conductivity of the material. In the case of rapidly varying processes F_O^T becomes a small parameter. It is necessary not only to solve accurately for the temperature but also for the thermal gradients, since physically important variables such as thermal stresses depend directly on them.

We now describe the problems which appear when (1.2) is solved using classical finite difference schemes. For example, consider the boundary value problem

$$\begin{aligned} L_{(1.4)} u(x, t) &\equiv \left\{ \varepsilon^2 \frac{\partial^2}{\partial x^2} - 1 - \frac{\partial}{\partial t} \right\} u(x, t) = 0, \quad (x, t) \in G, \\ u(x, t) &= \varphi(x, t), \quad (x, t) \in S, \end{aligned} \tag{1.4a}$$

where $d = 1$, $T > 1$, and $\varphi(x, t)$ is a sufficiently smooth function, defined on S , satisfying

$$\left. \begin{aligned} \varphi(0, t) &= 1 \\ \varphi(1, t) &= 0 \end{aligned} \right\} 1 < t \leq T. \tag{1.4b}$$

To solve problem (1.4) we use the following classical difference scheme [17]. On the set \overline{G} introduce a rectangular grid

$$\overline{G}_h = \overline{\omega}_1 \times \overline{\omega}_0, \tag{1.5a}$$

where $\overline{\omega}_1$ and $\overline{\omega}_0$ are uniform grids, including the end-points, respectively on the intervals $[0, 1]$, $[0, T]$, with step-sizes $h = N^{-1}$, $\tau = N_0^{-1}$, where $N + 1$ and $N_0 + 1$ are the number of nodes of the grids $\overline{\omega}_1$ and $\overline{\omega}_0$ respectively. For problem (1.4) the difference scheme

$$\Lambda_{(1.5)} z(x, t) \equiv \left\{ \varepsilon^2 \delta_x \overline{z} - 1 - \delta_t \right\} z(x, t) = 0, \quad (x, t) \in G_h, \tag{1.5b}$$

$$z(x, t) = \varphi(x, t), \quad (x, t) \in S_h$$

is used. Here $\delta_{x\bar{x}} z(x, t)$, $\delta_{\bar{t}} z(x, t)$ are the second central and the first (backward) difference derivatives respectively, $G_h = G \cap \bar{G}_h$, $S_h = S \cap \bar{G}_h$. The function $P(x, t)$ is approximated

$$P^{h+}(x, t) \equiv \varepsilon \delta_x z(x, t), \quad (x, t) \in \bar{G}_h^-, \quad (1.6)$$

where that function is defined, that is on $\bar{G}_h^- = \bar{\omega}_1^- \times \bar{\omega}_0$, where $\bar{\omega}_1^- = \bar{\omega}_1 \cap [0, d)$. In the case where $z(x, t)$ is the solution of (1.5), we use the notation $P_{(1.6;1.5)}^{h+}(x, t)$ or $P_{(1.5)}^{h+}(x, t)$ if this is not ambiguous.

Choosing T sufficiently large, we have the following inequality (see [28]):

$$|u_{(1.4)}(h, T) - z_{(1.5)}(h, T)| \geq m, \quad (1.7)$$

provided

$$\varepsilon = \varepsilon(h) = h, \quad (1.8)$$

and the inequality

$$|P_{(1.4)}(0, T) - P_{(1.5)}^{h+}(0, T)| \geq m \quad (1.9)$$

provided

$$\varepsilon = o(h) \text{ for } h \rightarrow 0. \quad (1.10)$$

It follows (see [28]) that in the case when

$$T = T(\tau) = o(1) \text{ for } \tau \rightarrow 0, \quad (1.11)$$

the ratio of the real normalised diffusion flux on the boundary, namely $P(0, T)$, and the computed normalised flux $P^{h+}(0, T)$ increase unboundedly as $h, \tau \rightarrow 0$:

$$\frac{P_{(1.4)}(0, T)}{P_{(1.5)}^{h+}(0, T)} \rightarrow \infty \text{ for } h, \tau \rightarrow 0. \quad (1.12)$$

Thus, for differences of the functions $u_{(1.4)}(x, t) - z_{(1.5)}(x, t)$ and $P_{(1.4)}(x, t) - P_{(1.5)}^{h+}(x, t)$ and also for the ratios of the real flux and the computed flux the estimates (1.7), (1.9), (1.12) hold; that is the computed solution and flux do not converge ε -uniformly for $h, \tau \rightarrow 0$.

We summarise this in the following theorem.

THEOREM 1.1 The functions $z(x, t)$, $(x, t) \in \bar{G}_h$ and $P^{h+}(x, t)$, $(x, t) \in \bar{G}_h^-$ which are respectively the solution of the finite difference scheme (1.5) for (1.4), and the computed normalised diffusive flux, do not converge ε -uniformly to the functions $u(x, t)$ and $P(x, t)$, $(x, t) \in \bar{G}$, which are respectively the solution of boundary value problem (1.4) and the exact normalised diffusive flux. The ratio of the exact normalised flux and the computed flux is not bounded ε -uniformly when $h, \tau \rightarrow 0$.

Remark 1. Instead of the function $P^{h+}(x, t)$, $(x, t) \in \bar{G}_h^-$, for the approximation of the flux $P(x, t)$ one can use the backward or central approximations

$$P^{h-}(x, t) = \varepsilon \delta_{\bar{x}} z(x, t), \quad (x, t) \in \bar{G}_h^+, \quad (1.13)$$

$$P^h(x, t) = \varepsilon \delta_{\bar{x}} z(x, t), \quad (x, t) \in G_h, \quad (1.14)$$

where $\bar{G}_h^+ = \bar{\omega}_1^+ \times \bar{\omega}_0$, $\bar{\omega}_1^+ = \omega_1 \cap (0, d]$,

$$\delta_{\bar{x}} z(x, t) = \frac{z(x^{i+1}, t) - z(x^{i-1}, t)}{h^{i-1} + h^i}, \quad x = x^i \in \omega_1.$$

Also the functions $P^{h-}(x, t)$, $P^h(x, t)$ for $h, \tau \rightarrow 0$ do not converge to $P(x, t)$ ε -uniformly, for symmetry reasons.

Thus, in the case of the singularly perturbed boundary value problem (1.2) we arrive at the problem of developing special finite difference schemes which approximate ε -uniformly both the solution and the normalised diffusive flux.

2. NUMERICAL EXPERIMENTS WITH A CLASSICAL DIFFERENCE SCHEME

Firstly, let us formulate a problem suitable for numerical experiments with the classical finite difference schemes (1.4). The qualitative behaviour of the functions $z(x, t)$, $P^{h-}(x, t)$, $P^{h+}(x, t)$, $P^h(x, t)$ is described by Theorem 1.1 and Remark 1. It is interesting to analyze more precisely the errors of the approximate solution of (1.4) and the errors in the computed normalised flux. For the pointwise errors

$$e(x, t) = |u(x, t) - z(x, t)|, \quad (x, t) \in \overline{G}_h,$$

$$q^+(x, t) = |P(x, t) - P^{h+}(x, t)|, \quad (x, t) \in \overline{G}_h^-$$

the following inequalities hold

$$e(x, t) \leq e^0(x) + |u(x, t) - u^0(x)| + |z(x, t) - z^0(x)|,$$

$$q^+(x, t) \leq q^{0+}(x) + |P(x, t) - P^0(x)| + |P^{h+}(x, t) - P^{0h+}(x)|,$$

where

$$e^0(x) = e^0(x, \varepsilon, N) = |u^0(x) - z^0(x)|, \quad (2.1)$$

$$q^{0+}(x) = q^{0+}(x, \varepsilon, N) = |P^0(x) - P^{0h+}(x)|. \quad (2.2)$$

Here, the function $u^0(x)$ is the solution of the stationary problem

$$\begin{aligned} L_{(2.3)}u(x) &\equiv \left\{ \varepsilon^2 \frac{d^2}{dx^2} - 1 \right\} u(x) = 0, & x \in D, \\ u(x) &= \varphi(x), & x \in \Gamma, \end{aligned} \quad (2.3)$$

$\Gamma = \overline{D} \setminus D$, the function $P^0(x)$ is the normalised diffusive flux for stationary problem (2.3): $P^0(x) = \varepsilon(d/dx)u^0(x)$, $x \in \overline{D}$. The boundary function $\varphi(x)$ in problem (2.3) is defined as in (1.4).

The function $z^0(x)$ is the solution of the stationary discrete problem

$$\begin{aligned} \Lambda_{(2.4)}z(x) &\equiv \left\{ \varepsilon^2 \delta_{x\bar{x}} - 1 \right\} z(x) = 0, & x \in D_h, \\ z(x) &= \varphi(x), & x \in \Gamma_h, \end{aligned} \quad (2.4)$$

the function $P^{0h+}(x)$ is the normalised diffusive flux for problem (2.4): $P^{0h+}(x) = \varepsilon \delta_x z^0(x)$, $x \in \overline{D}_h^-$. The largest contribution to the functions $e(x, t)$ and $q^+(x, t)$ for $t = T$, with T is sufficiently large, and small h and τ , is caused by the terms $e^0(x)$ and $q^{0+}(x)$. Therefore the main interest here is in the numerical investigation of the influence of the parameter ε and the number N on the behaviour of values $e^0(x; \varepsilon, N)$ and $q^{0+}(x; \varepsilon, N)$. As the derivatives of the function $u^0(x)$ become large only for small values of ε , it is most interesting to investigate the behaviour of $e^0(x, \varepsilon, N)$, $q^{0+}(x, \varepsilon, N)$ for this case.

The behaviour of the general errors $e^0(x, \varepsilon, N)$, $q^{0+}(x, \varepsilon, N)$ is complex and not particularly suitable for direct analysis of the numerical results. Therefore instead of problem (2.3), (1.4) we consider a closely related problem for which analysis of the errors for the approximate solutions and fluxes is considerably simpler. Let the function

$$W(x) = \exp(-\varepsilon^{-1}x), \quad x \in \overline{D}$$

be the solution of singularly perturbed equation (2.3). Then

$$\max_{\overline{D}} |W(x)| = W(0) = 1, \quad \max_{\overline{D}} \left| \varepsilon \frac{d}{dx} W(x) \right| = -\varepsilon \frac{d}{dx} W(0) = 1. \quad (2.5)$$

Further, the following estimate holds for $u^0(x)$,

$$|u^0(x) - W(x)|, \quad \left| \varepsilon \frac{d}{dx} u^0(x) - \varepsilon \frac{d}{dx} W(x) \right| \leq M \varepsilon^n, \quad x \in \overline{D}, \quad (2.6)$$

where n is a sufficiently large number. Thus, the function $W(x)$ and $\varepsilon(d/dx)W(x)$ approximate well the solution of problem (2.3) and the normalised diffusive flux $P^0(x)$, for sufficiently small ε .

By virtue of monotonicity of the operator $\Lambda_{(2.4)}$ for the solution of the difference scheme

$$\begin{aligned}\Lambda_{(2.4)}z(x) &= 0, & x \in D_h, \\ z(x) &= W(x), & x \in \Gamma_h\end{aligned}\tag{2.7}$$

the following estimates

$$\begin{aligned}|z_{(2.7)}(x) - z_{(2.4)}(x)| &\leq M\varepsilon^n, & x \in \overline{D}_h, \\ |P_{(2.7)}^{h+}(0) - P_{(2.4)}^{h+}(0)| &\leq M\varepsilon^n,\end{aligned}\tag{2.8}$$

are valid where $P_{(2.7)}^{h+}(0)$ and $P_{(2.4)}^{h+}(0)$ are the normalised diffusive fluxes for problems (2.7) and (2.4). According to relations (2.6), (2.8), the principal parts of the errors $|u_{(1.4)}(x, T) - z_{(1.5)}(x, T)|$ and $|P(0, T) - P_{(1.5)}^{h+}(0, T)|$ for sufficiently large T and small values of ε , are the errors $|W(x) - z_{(2.7)}(x)|$ and $|\varepsilon(d/dx)W(0) - P_{(2.7)}^{h+}(0)|$.

Therefore, let us consider the difference scheme (2.7) for the boundary value problem (2.3) with boundary condition

$$\varphi(x) = W(x), \quad x \in \Gamma.\tag{2.9}$$

We wish to demonstrate the influence of the parameter ε and the number of nodes N on the error of the approximate solution and also on the error of computed normalised flux at $x = 0$.

Suppose

$$\begin{aligned}e(x) &= e(x; \varepsilon, N) = |u(x) - z(x)|, \\ q^+(x) &= q^+(x; \varepsilon, N) = |P(x) - P^{h+}(x)|,\end{aligned}$$

where $u(x)$ is the solution of problem (2.3), (2.9), $z(x) = z_{(2.7)}(x)$ is the solution of difference scheme (2.7), and $P(x)$ and $P^{h+}(x) = P_{(2.7)}^{h+}(x)$ are the normalised fluxes for problems (2.3), (2.9) and (2.7). Note that $u(x) = W(x)$, $x \in \overline{D}$. Using (2.5) it is clear that the solution of problem (2.3), (2.9) satisfies the following conditions

$$\max_{\overline{D}} |u(x)| = u(0) = 1, \quad \max_{\overline{D}} |P(x)| = -P(0) = 1.$$

In Tables 4 and 5 we can see the results of computing the errors $E(\varepsilon, N)$

$$E(\varepsilon, N) = \max_{D_h} e(x; \varepsilon, N)\tag{2.10a}$$

that are the maximal pointwise errors on the grid \overline{D}_h , and results of computing the errors $Q(\varepsilon, N)$

$$Q(\varepsilon, N) = q^+(x = 0; \varepsilon, N),\tag{2.11a}$$

that is the errors of the normalised flux on the boundary $x = 0$. These results were obtained using the difference scheme (2.7) for various values of ε and N . The values of $\overline{E}(N)$ and $\overline{Q}(N)$ are also given, where

$$\overline{E}(N) = \max_{\varepsilon=4^{-m}} E(\varepsilon, N), \quad m = 0, 1, \dots, 6,\tag{2.10b}$$

is the largest (with respect to ε) error of the approximate solution (for a fixed value of N , and

$$\overline{Q}(N) = \max_{\varepsilon=4^{-m}} Q(\varepsilon, N), \quad m = 0, 1, \dots, 6,\tag{2.11b}$$

is the largest error of the computed normalised flux for $x = 0$. The values $\overline{E}(N)$ and $\overline{Q}(N)$ define the best guaranteed accuracy which can be obtained using the scheme (2.7) to solve the problem (2.3), (2.9) for a given N and all values of ε shown.

TABLE 4. Table of errors $E(\varepsilon, N)$ for the classical scheme

$\varepsilon \setminus N$	4	16	64	256	1024
1	3.621e-04	2.296e-05	1.437e-06	8.982e-08	5.614e-09
2^{-2}	1.395e-02	9.443e-04	5.934e-05	3.710e-06	2.319e-07
2^{-4}	3.741e-02	1.409e-02	9.526e-04	5.985e-05	3.742e-06
2^{-6}	3.876e-03	3.741e-02	1.409e-02	9.526e-04	5.985e-05
2^{-8}	2.440e-04	3.876e-03	3.741e-02	1.409e-02	9.526e-04
2^{-10}	1.526e-05	2.440e-04	3.876e-03	3.741e-02	1.409e-02
2^{-12}	9.537e-07	1.526e-05	2.440e-04	3.876e-03	3.741e-02
$\overline{E}(N)$	3.741e-02	3.741e-02	3.741e-02	3.741e-02	3.741e-02

TABLE 5. Table of errors of the normalised flux $Q(\varepsilon, N)$ for the classical scheme

$\varepsilon \setminus N$	4	16	64	256	1024
1	1.164e-01	3.071e-02	7.779e-03	1.951e-03	4.881e-04
2^{-2}	3.818e-01	1.172e-01	3.076e-02	7.782e-03	1.951e-03
2^{-4}	7.639e-01	3.820e-01	1.172e-01	3.076e-02	7.782e-03
2^{-6}	9.377e-01	7.639e-01	3.820e-01	1.172e-01	3.076e-02
2^{-8}	9.844e-01	9.377e-01	7.639e-01	3.820e-01	1.172e-01
2^{-10}	9.961e-01	9.844e-01	9.377e-01	7.639e-01	3.820e-01
2^{-12}	9.990e-01	9.961e-01	9.844e-01	9.377e-01	7.639e-01
$\overline{Q}(N)$	9.990e-01	9.961e-01	9.844e-01	9.377e-01	7.639e-01

From Table 4 we see that the solution of difference scheme (2.7) converges to the solution of boundary value problem (2.3), (2.9) for a fixed value of the parameter ε . However, these approximate solutions $z(x) = z(x; \varepsilon, N)$ do not converge ε -uniformly. For the worst realisable error $\overline{E}(N)$ the lower bound $\overline{E}(N) \geq 3.74 \cdot 10^{-2}$ holds. The relative worst realisable error for a fixed N is given by formula

$$\delta(N) \equiv \frac{\overline{E}(N)}{\max_D |u(x)|},$$

where $u(x)$ is the solution of problem (2.3), (2.9). The relative error $\delta(N)$ does not depend on N and it is equal to 3.741 %.

From Table 5 it follows that $P^{h+}(0) = P^{h+}(0; \varepsilon, N)$, the computed normalised diffusive flux at the boundary $x = 0$, converges to the value $P(0)$, with increasing N for fixed ε . However, these computed fluxes $P^{h+}(x) = P^{h+}(x; \varepsilon, N)$ also do not converge ε -uniformly. The error $Q(\varepsilon, N)$ is constant for any value of N if the product εN is constant. Moreover, for any fixed N the error $Q(\varepsilon, N)$ tends to the value $|P(0)| = 1$ as ε increases.

Table 6 gives the values of

$$\lambda(\varepsilon, N) \equiv \frac{|P(0; \varepsilon)|}{|P^{h+}(0; \varepsilon, N)|} = \frac{|\frac{d}{dx}u(0; \varepsilon)|}{|\delta_x z(0; \varepsilon, N)|}$$

which is the ratio of the exact normalised flux on the boundary $x = 0$ to the computed flux (or the ratio of the first derivative of the exact solution at $x = 0$ to the computed first difference).

From Table 6 we see that the value $\lambda(\varepsilon, N)$ satisfies the relation $\lambda(\varepsilon, N) \approx \varepsilon^{-1} N^{-1}$ and increases unboundedly for any fixed N and sufficiently small values of ε satisfying $\varepsilon N \rightarrow 0$. Even for $\varepsilon \leq 16^{-1}h$ the real flux differs from the computed flux by a factor of 10.

The relative error of the flux $\eta(\varepsilon, N)$, where

$$\eta(\varepsilon, N) \equiv \frac{Q(\varepsilon, N)}{|P(0)|} = Q(\varepsilon, N),$$

TABLE 6. Table of ratios normalised fluxes $\lambda(\varepsilon, N)$ for the classical scheme

$\varepsilon \setminus N$	4	16	64	256	1024
1	1.132e+00	1.032e+00	1.008e+00	1.002e+00	1.000e+00
2^{-2}	1.618e+00	1.133e+00	1.032e+00	1.008e+00	1.002e+00
2^{-4}	4.236e+00	1.618e+00	1.133e+00	1.032e+00	1.008e+00
2^{-6}	1.606e+01	4.236e+00	1.618e+00	1.133e+00	1.032e+00
2^{-8}	6.402e+01	1.606e+01	4.236e+00	1.618e+00	1.133e+00
2^{-10}	2.560e+02	6.402e+01	1.606e+01	4.236e+00	1.618e+00
2^{-12}	1.024e+03	2.560e+02	6.402e+01	1.606e+01	4.236e+00
$\bar{\lambda}(N)$	1.024e+03	2.560e+02	6.402e+01	1.606e+01	4.236e+00

can be guaranteed to be no larger than 20 % only for $N \geq N(\varepsilon) = 4\varepsilon^{-1}$.

Thus, the results presented illustrate the statements of Theorem 1.1 and demonstrate the weaknesses of classical difference schemes for the solution of problems of the form (1.4). Since $|P(0)| = 1$, the results of table 5 show that if we use the classical difference scheme (1.5) for solving problem (1.4), in the case where $\alpha_0 \leq \varepsilon \leq \alpha_1$, $\alpha_0 = 2.4 \cdot 10^{-4}$, $\alpha_1 = 1.56 \cdot 10^{-2}$ (which corresponds to the diffusion Fourier number $F_O^D = 5.76 \cdot 10^{-8} - 2.43 \cdot 10^{-4}$ for the phenol diffusion process discussed above), we cannot guarantee an error in the computed normalised flux through the boundary that is less than 50%, even when the number of nodes is $N = 1024$.

3. GRID APPROXIMATIONS OF SOLUTIONS AND DIFFUSIVE FLUXES

In this section, we construct special finite difference schemes for problems (1.2), (1.3) and computational formulae for the approximation of the normalised diffusion flux. We suppose that $u \in C^{4,2}(\bar{G})$ for each fixed value of ε , $\varepsilon \in (0, 1]$.

On the set \bar{G} we introduce again the grid

$$\bar{G}_h = \bar{\omega}_1 \times \bar{\omega}_0, \quad (3.1)$$

where now $\bar{\omega}_1$ is a grid, generally nonuniform, on the interval $[0, d]$ and $\bar{\omega}_0$ is an uniform grid on the interval $[0, T]$. Suppose $h^i = x^{i+1} - x^i$, $x^i, x^{i+1} \in \bar{\omega}_1$, $h = \max_i h^i$. By $N + 1$ and $N_0 + 1$ we denote the number of nodes in the grids $\bar{\omega}_1$ and $\bar{\omega}_0$ respectively, $h \leq M N^{-1}$.

On the grid \bar{G}_h we define the following difference scheme for problem (1.2),

$$\begin{aligned} \Lambda_{(3.2)} z(x, t) &= f(x, t), \quad (x, t) \in G_h, \\ z(x, t) &= \varphi(x, t), \quad (x, t) \in S_h. \end{aligned} \quad (3.2)$$

Here $G_h = G \cap \bar{G}_h$, $S_h = S \cap \bar{G}_h$,

$$\Lambda_{(3.2)} z(x, t) \equiv \varepsilon^2 a(x, t) \delta_{\bar{x}x} z(x, t) - c(x, t) z(x, t) - p(x, t) \delta_{\bar{t}} z(x, t).$$

To approximate the function $P(x, t)$, that is the normalised diffusive flux, we use the grid function $P_{(1.6;3.2)}^{h+}(x, t)$.

The difference scheme (3.2), (3.1) is monotonic for any arbitrary distribution of the nodes of the grid $\bar{\omega}_1$ and hence of the grid $\bar{G}_{h(3.1)}$. Using the maximum principle we establish convergence of the difference scheme for a fixed value of the parameter

$$|u(x, t) - z(x, t)| \leq M [\varepsilon^{-1} N^{-1} + N_0^{-1}], \quad (x, t) \in \bar{G}_h. \quad (3.3)$$

In the case of the grid

$$\bar{G}_h = \{ G_{h(3.1)}, \text{ where } \bar{\omega}_1 \text{ is an uniform grid} \} \quad (3.4)$$

the estimate

$$|u(x, t) - z(x, t)| \leq M [\varepsilon^{-2} N^{-2} + N_0^{-1}], \quad (x, t) \in \overline{G}_h(3.4) \quad (3.5)$$

holds. From (3.5), the inequality

$$|P(x, t) - P^{h+}(x, t)| \leq M [\varepsilon^{-1} N^{-1} + \varepsilon N N_0^{-1}], \quad (x, t) \in \overline{G}_h^-(3.4) \quad (3.6)$$

follows. A sufficient condition for convergence of the function $P^{h+}(x, t)$ to the function $P(x, t)$ for a fixed value of ε , is that

$$N N_0^{-1} \rightarrow 0 \text{ for } N, N_0 \rightarrow \infty. \quad (3.7)$$

Thus, the difference scheme (3.2), (3.4), (3.7) allows approximation of the solution of boundary value problem (1.2) together with the normalised diffusive flux for a fixed value of ε . In particular, under the condition

$$N_0 = N_0(N) = N^2 \quad (3.8)$$

the following estimate

$$|P(x, t) - P^{h+}(x, t)| \leq \frac{M}{\varepsilon N}, \quad (x, t) \in \overline{G}_h^-(3.4). \quad (3.9)$$

is valid.

In the case of boundary value problem (1.3), we use the difference scheme

$$\begin{aligned} \Lambda_{(3.10)}(z(x, t)) &= 0, \quad (x, t) \in G_h, \\ z(x, t) &= \varphi(x, t), \quad (x, t) \in S_h. \end{aligned} \quad (3.10)$$

Here

$$\Lambda_{(3.10)}(z(x, t)) \equiv \{ \varepsilon^2 a(x, t) \delta_{\overline{x}x} - p(x, t) \delta_{\overline{t}t} \} z(x, t) - g(x, t, z(x, t)).$$

For the solution of (3.10) and for the flux $P(x, t)$ the bounds (3.3), (3.5), (3.6) also hold. When the condition (3.7) is violated then the function $P^{h+}(x, t)$, $(x, t) \in \overline{G}_h^-$ does not, in general, converge to the function $P(x, t)$ for a fixed ε . The main result is summarised in the following theorem.

THEOREM 3.1 Let the finite difference scheme (3.2), (3.1) (or (3.10), (3.1)) be used for the solution of the boundary value problem (1.2) (respectively (1.3)). Then condition (3.7) is sufficient for convergence of $P^{h+}(x, t)$ for a fixed value of the parameter, if $u \in C^{4,2}(\overline{G})$ and the grid (3.4) is used. Moreover, estimate (3.9) holds if (3.8) is satisfied.

Now we construct a special difference scheme for problem (1.2). On the grid \overline{G}_h we introduce a special grid, condensed in the boundary layer, similar to the grid constructed in [26, 27],

$$\overline{G}_h(3.11) = \overline{G}_h(3.11)(\sigma) = \overline{\omega}_1^* \times \overline{\omega}_0, \quad (3.11a)$$

where $\overline{\omega}_1^* = \overline{\omega}_1^*(\sigma)$ is a piecewise uniform grid on $[0, d]$; σ is a parameter depending on ε and N . Step-sizes of the grid $\overline{\omega}_1^*$ on the intervals $[0, \sigma]$, $[d - \sigma, d]$ and on the interval $[\sigma, d - \sigma]$ are constant and equal to $h^{(1)} = 4\sigma N^{-1}$ and $h^{(2)} = 2(d - 2\sigma)N^{-1}$ respectively, $\sigma \leq 4^{-1}d$. The value σ is chosen to satisfy the condition

$$\sigma = \sigma_{(3.11)}(\varepsilon, N) = \min[4^{-1}d, m^{-1}\varepsilon \ln N], \quad (3.11b)$$

where $m = m_{(3.11)}$ is an arbitrary number.

In a manner similar to that in [28] we establish the ε -uniform convergence of the scheme (3.2), (3.11)

$$|u(x, t) - z(x, t)| \leq M [N^{-2} \ln^2 N + N_0^{-1}], \quad (x, t) \in \overline{G}_h. \quad (3.12)$$

For the computed flux, we have

$$|P(x, t) - P^{h+}(x, t)| \leq M\varepsilon [N^{-1} \ln^2 N + N N_0^{-1}], \quad (x, t) \in \overline{G}_h^-(3.11). \quad (3.13)$$

According to the estimate (3.13), we have ε -uniform convergence of the function $P^{h+}(x, t)$, provided the condition

$$\varepsilon N N_0^{-1} \rightarrow 0 \quad \varepsilon\text{-uniformly, for } N, N_0 \rightarrow \infty \quad (3.14)$$

is fulfilled. In particular, under condition (3.8) the estimate

$$|P(x, t) - P^{h+}(x, t)| \leq M \varepsilon N^{-1} \ln^2 N \leq M N^{-1} \ln^2 N, \quad (x, t) \in \overline{G}_{h(3.11)}^-. \quad (3.15)$$

is valid.

Note that estimates (3.12), (3.13), (3.15) are also fulfilled in the case of the boundary value problem (1.3), when the scheme (3.10), (3.11) is used for solving of this problem. Thus we have the following theorem [28].

THEOREM 3.2 Let $u \in C^{4,2}(\overline{G})$ for a fixed value of the parameter ε , $\varepsilon \in (0, 1]$. Then the solution of difference scheme (3.2), (3.11) (or (3.10), (3.11)) converges ε -uniformly to the solution of the problem (1.2) (respectively (1.3)). If condition (3.14) also holds, $P^{h+}(x, t)$, $(x, t) \in \overline{G}_{h(3.11)}^-$, converges ε -uniformly to the function $P(x, t)$. For the solution of the difference scheme the estimates (3.5), (3.12) and, for computed flux $P^{h+}(x, t)$, the estimates (3.9) (if condition (3.8) is fulfilled) and (3.15) are valid. For the flux the estimates (3.6), (3.13) also hold. Estimates similar to (3.6), (3.13) and (3.9), (3.15) also hold if $P_{(1.13)}^{h-}(x, t)$ or $P_{(1.14)}^h(x, t)$ are used as approximations of $P(x, t)$.

It is also interesting investigate numerically the influence of ε and N on the behaviour of $E_{(2.10)}(\varepsilon, N)$ and $Q_{(2.11)}(\varepsilon, N)$ computed using the special difference scheme for problem (2.3), (2.9). We use the difference scheme

$$\begin{aligned} \Lambda_{(3.16)} z(x) &\equiv \{ \varepsilon^2 \delta_{\overline{x}x} - 1 \} z(x) = 0, & x \in D_h, \\ z(x) &= W(x), & x \in \Gamma_h, \end{aligned} \quad (3.16a)$$

on the grid

$$\overline{D}_h = \overline{\omega}_{1(3.11)}^*, \quad (3.16b)$$

where $m_{(3.11)} = 1/2$. In the Tables 7, 8 and 9 we give the values $E(\varepsilon, N)$, $Q(\varepsilon, N)$, $\overline{E}(N)$, $\overline{Q}(N)$, $\lambda(\varepsilon, N)$, computed with (3.16) for various values of ε and N .

TABLE 7. Table of errors $E(N, \varepsilon)$ for the special scheme

$\varepsilon \setminus N$	4	16	64	256	1024
1	3.621e-04	2.296e-05	1.437e-06	8.982e-08	5.614e-09
2^{-2}	1.395e-02	9.443e-04	5.934e-05	3.710e-06	2.319e-07
2^{-4}	1.685e-02	1.409e-02	9.526e-04	5.985e-05	3.742e-06
2^{-6}	4.050e-02	2.410e-02	4.041e-03	4.587e-04	4.492e-05
2^{-8}	5.690e-02	2.406e-02	4.041e-03	4.587e-04	4.492e-05
2^{-10}	6.109e-02	2.404e-02	4.041e-03	4.587e-04	4.492e-05
2^{-12}	6.215e-02	2.404e-02	4.041e-03	4.587e-04	4.492e-05
$\overline{E}(N)$	6.215e-02	2.410e-02	4.041e-03	4.587e-04	4.492e-05

From the Tables 7, 8 we can see that approximate solutions and computed normalised fluxes seem to converge ε -uniformly. For example, the guaranteed accuracy for the approximate solution is not worse than 1.0 %, when $N = 64$, and for the computed flux is not worse than 10% for $N = 256$. From Table 9 we see that $\lambda(\varepsilon, N)$ tends ε -uniformly to 1 with increasing N .

4. A NUMERICAL EXAMPLE FOR THE DIFFUSION EQUATION

In order to illustrate computational problems which appear with employment of classical difference schemes to solve singularly perturbed boundary value problems for a partial differential equation and

TABLE 8. Table of errors of the normalised flux $Q(N, \varepsilon)$ for the special scheme

$\varepsilon \setminus N$	4	16	64	256	1024
1	1.164e-01	3.071e-02	7.779e-03	1.951e-03	4.881e-04
2^{-2}	3.818e-01	1.172e-01	3.076e-02	7.782e-03	1.951e-03
2^{-4}	6.679e-01	3.820e-01	1.172e-01	3.076e-02	7.782e-03
2^{-6}	6.473e-01	4.764e-01	2.267e-01	8.290e-02	2.671e-02
2^{-8}	6.413e-01	4.763e-01	2.267e-01	8.290e-02	2.671e-02
2^{-10}	6.398e-01	4.763e-01	2.267e-01	8.290e-02	2.671e-02
2^{-12}	6.395e-01	4.763e-01	2.267e-01	8.290e-02	2.671e-02
$\overline{Q}(N)$	6.679e-01	4.764e-01	2.267e-01	8.290e-02	2.671e-02

TABLE 9. Table of ratios of the normalised fluxes $\lambda(N, \varepsilon)$ for the special scheme

$\varepsilon \setminus N$	4	16	64	256	1024
1	1.132e+00	1.032e+00	1.008e+00	1.002e+00	1.000e+00
2^{-2}	1.618e+00	1.133e+00	1.032e+00	1.008e+00	1.002e+00
2^{-4}	3.012e+00	1.618e+00	1.133e+00	1.032e+00	1.008e+00
2^{-6}	2.835e+00	1.910e+00	1.293e+00	1.090e+00	1.027e+00
2^{-8}	2.788e+00	1.910e+00	1.293e+00	1.090e+00	1.027e+00
2^{-10}	2.776e+00	1.910e+00	1.293e+00	1.090e+00	1.027e+00
2^{-12}	2.774e+00	1.910e+00	1.293e+00	1.090e+00	1.027e+00
$\overline{\lambda}(N)$	3.012e+00	1.910e+00	1.293e+00	1.090e+00	1.027e+00

to find normalised fluxes, and in order to show the efficiency of special difference schemes we shall consider the simplest boundary value problem for the diffusion equation. The function

$$W(x, t) = \operatorname{erfc}\left(\frac{x}{2\varepsilon\sqrt{t}}\right)\left(\frac{x^2}{2\varepsilon^2} + t\right) - \frac{1}{\sqrt{\pi}} \exp\left(-\frac{x^2}{4\varepsilon^2 t}\right) \frac{x\sqrt{t}}{\varepsilon}, \quad 0 < x < \infty, t \geq 0$$

is the solution of the singularly perturbed diffusion equation

$$L_{(4.1)}u(x, t) \equiv \varepsilon^2 \frac{\partial^2}{\partial x^2} u(x, t) - \frac{\partial}{\partial t} u(x, t) = 0, \quad 0 < x < \infty, t > 0 \quad (4.1)$$

and satisfies the boundary conditions

$$W(x, 0) = 0, \quad 0 \leq x < \infty, \quad W(0, t) = t, \quad t \geq 0.$$

For the function $W(x, t)$ the following bounds hold

$$\max_{\substack{0 < x < \infty, \\ 0 \leq t \leq t}} W(x, t) \leq t,$$

$$\max_{0 < x < \infty} \varepsilon \left| \frac{\partial}{\partial x} W(x, t) \right| \leq \varepsilon \left| \frac{\partial}{\partial x} W(0, t) \right| = 2\pi^{-1/2} t^{1/2}, \quad t \geq 0.$$

For $x \geq x_0 > m$, $0 \leq t \leq T$, the function $W(x, t)$ decays more rapidly than any power of the parameter ε that is

$$|W(x, t)| \leq M \varepsilon^n, \quad x \geq x_0 > m, \quad 0 \leq t \leq T,$$

where n is an arbitrary large number.

Let us consider the boundary value problem

$$\begin{aligned} L_{(4.1)}u(x, t) &= 0, & (x, t) \in G, \\ u(x, t) &= W(x, t), & (x, t) \in S, \end{aligned} \quad (4.2)$$

where $G = D \times (0, T]$, $D = (0, d)$, $d = 1$, $T = 1$.

The difference scheme (3.2), (3.1) for problem (4.2) is

$$\begin{aligned} \Lambda_{(4.3)} z(x, t) &\equiv \varepsilon^2 \delta_{xx} z(x, t) - \delta_t z(x, t) = 0, & (x, t) \in G_h, \\ z(x, t) &= W(x, t), & (x, t) \in S_h. \end{aligned} \quad (4.3)$$

Here \overline{G}_h is one of the grids considered previously, either the uniform grid $\overline{G}_h^{(3.4)}$ or the special grid $\overline{G}_h^{(3.11)} = \overline{G}_h^{(3.11)}$ with $m_{(3.11)} = 1/2$. Using the solutions of the difference schemes on these meshes, we calculated the values

$$E(\varepsilon, N) = \max_{\overline{G}_h} |u(x, t) - z(x, t)|,$$

which are the errors of the approximate solution (l^∞ -norm) for various values of ε and $N = N_0$, and also the values

$$Q(\varepsilon, N) = \max_{0 \leq t \leq T} |P(x=0, t) - P^{h+}(x=0, t)|,$$

which are the errors in the computed normalised flux on the boundary $x = 0$, where $P(x, t) = \varepsilon(\partial/\partial x)u(x, t)$, $P^{h+}(x, t) = \varepsilon\delta_x z(x, t)$.

In the Tables 10 and 11 we show the values of $E(\varepsilon, N)$ and $Q(\varepsilon, N)$ computed with the uniform grid $\overline{G}_h^{(3.4)}$ for various values of ε and $N = N_0$. In the Tables 12, 13 results for the special grid $\overline{G}_h^{(3.11)}$ are given.

TABLE 10. Table of errors $E(\varepsilon, N)$ for the classical scheme

$\varepsilon \setminus N$	4	16	64	256	1024
1.0	1.630e-02	6.144e-03	1.780e-03	4.651e-04	1.176e-04
2^{-2}	4.374e-02	8.624e-03	1.960e-03	4.769e-04	1.184e-04
2^{-4}	3.601e-02	2.558e-02	3.131e-03	5.507e-04	2.484e-04
2^{-6}	2.432e-03	3.095e-02	2.061e-02	1.728e-03	2.444e-04
2^{-8}	1.526e-04	2.069e-03	2.966e-02	1.934e-02	1.376e-03
2^{-10}	9.537e-06	1.297e-04	1.978e-03	2.934e-02	1.902e-02
2^{-12}	5.960e-07	8.106e-06	1.240e-04	1.956e-03	2.926e-02
$\overline{E}(N)$	4.374e-02	3.095e-02	2.966e-02	2.934e-02	2.926e-02

TABLE 11. Table of errors of the normalised flux $Q(\varepsilon, N)$, $\overline{Q}(N)$ for the classical scheme

$\varepsilon \setminus N$	4	16	64	256	1024
1.0	1.620e-01	6.123e-02	2.362e-02	9.946e-03	4.496e-03
2^{-2}	4.516e-01	1.328e-01	4.345e-02	1.535e-02	5.904e-03
2^{-4}	8.876e-01	4.332e-01	1.228e-01	3.321e-02	1.086e-02
2^{-6}	1.066e+00	8.863e-01	4.282e-01	1.211e-01	3.111e-02
2^{-8}	1.113e+00	1.066e+00	8.860e-01	4.270e-01	1.207e-01
2^{-10}	1.124e+00	1.113e+00	1.066e+00	8.859e-01	4.267e-01
2^{-12}	1.127e+00	1.124e+00	1.113e+00	1.066e+00	8.859e-01
$\overline{Q}(N)$	1.127e+00	1.124e+00	1.113e+00	1.066e+00	8.859e-01

From tables 10, 11 we can see that the solution of the difference scheme (4.3), (3.4) for $N = N_0$ and also the computed normalised flux for $x = 0$ converge for a fixed value of the parameter.

However, approximate solutions and normalised fluxes do not converge ε -uniformly. For the $\overline{E}(N) = \max_{\varepsilon=4^{-m}} E(\varepsilon, N)$, $m = 0, 1, \dots, 6$, we find

$$\overline{E}(N) \geq 2.9 \cdot 10^{-2}.$$

The ratio of the exact normalised flux on the boundary $x = 0$ for $t = T$ and the computed flux (that is $P(0, T)/P^{h^+}(0, T)$) increases unboundedly with decreasing ε , for fixed values of N .

TABLE 12. Table of errors $E(\varepsilon, N)$ for the special scheme

$\varepsilon \setminus N$	4	16	64	256	1024
1	1.630e-02	6.144e-03	1.780e-03	4.651e-04	1.176e-04
2^{-2}	4.374e-02	8.624e-03	1.960e-03	4.769e-04	1.184e-04
2^{-4}	3.976e-02	2.558e-02	3.131e-03	5.507e-04	2.484e-04
2^{-6}	4.494e-04	4.156e-02	7.214e-03	1.077e-03	2.478e-04
2^{-8}	9.440e-03	4.156e-02	7.214e-03	1.077e-03	2.478e-04
2^{-10}	1.207e-02	4.156e-02	7.214e-03	1.077e-03	2.478e-04
2^{-12}	1.273e-02	4.156e-02	7.214e-03	1.077e-03	2.478e-04
$\overline{E}(N)$	4.374e-02	4.156e-02	7.214e-03	1.077e-03	2.478e-04

From the Tables 12 and 13 we see that approximate solutions and computed normalised fluxes (for $x = 0$) seem to converge ε -uniformly.

TABLE 13. Table of errors of the normalised flux $Q(\varepsilon, N)$, $\overline{Q}(N)$ for the special scheme

$\varepsilon \setminus N$	4	16	64	256	1024
1	1.620e-01	6.123e-02	2.362e-02	9.946e-03	4.496e-03
2^{-2}	4.516e-01	1.328e-01	4.345e-02	1.535e-02	5.904e-03
2^{-4}	7.867e-01	4.332e-01	1.228e-01	3.321e-02	1.086e-02
2^{-6}	7.727e-01	5.505e-01	2.428e-01	8.507e-02	2.701e-02
2^{-8}	7.690e-01	5.505e-01	2.428e-01	8.507e-02	2.701e-02
2^{-10}	7.680e-01	5.505e-01	2.428e-01	8.507e-02	2.701e-02
2^{-12}	7.678e-01	5.505e-01	2.428e-01	8.507e-02	2.701e-02
$\overline{Q}(N)$	7.867e-01	5.505e-01	2.428e-01	8.507e-02	2.701e-02

5. BOUNDARY VALUE PROBLEM ON A RECTANGLE

In this section we consider a quasi-linear parabolic equations on a rectangle. We shall point out computational problems accompanying the flux computation, and shall construct special difference schemes whose approximate solutions and computed normalised fluxes converge ε -uniformly.

On the rectangle

$$D = \{ x : 0 < x_s < d_s, s = 1, 2 \}$$

we consider the boundary value problem for the quasi-linear equation of parabolic type

$$\begin{aligned} L_{(5.1)}(u(x, t)) &= 0, \quad (x, t) \in G, \\ u(x, t) &= \varphi(x, t), \quad (x, t) \in S. \end{aligned} \tag{5.1a}$$

Here

$$G = D \times (0, T], \quad S = \overline{G} \setminus G, \tag{5.2}$$

$$L_{(5.1)}(u(x, t)) \equiv \{ \varepsilon^2 L_{(5.1)}^2 - p(x, t) \frac{\partial}{\partial t} \} u(x, t) - g(x, t, u(x, t)),$$

$$L_{(5.1)}^2 \equiv \sum_{s=1,2} a_s(x, t) \frac{\partial^2}{\partial x_s^2} + \sum_{s=1,2} b_s(x, t) \frac{\partial}{\partial x_s} - c^0(x, t),$$

the functions $a_s(x, t)$, $b_s(x, t)$, $c^0(x, t)$, $p(x, t)$, $s = 1, 2$, and also the functions $g(x, t, u)$, $\varphi(x, t)$ are sufficiently smooth functions on the sets \overline{G} , $\overline{G} \times \mathbb{R}$ and S respectively. In addition, we shall assume that

$$\begin{aligned} a_0 \leq a_1(x, t), \quad a_2(x, t) \leq a^0, \quad c^0(x, t) \geq 0, \quad p(x, t) \geq p_0, \quad (x, t) \in \overline{G}, \quad a_0, p_0 > 0, \\ -M_{(5.1)} \leq \frac{\partial}{\partial u} g(x, t, u) < \infty, \quad (x, t, u) \in \overline{G} \times R. \end{aligned} \quad (5.1b)$$

The parameter ε takes arbitrary values in the interval $(0, 1]$. Let $\Gamma = \overline{D} \setminus D$ and Γ^* be the set of corner points of the rectangle D , then assume that on the set $S^* = S_0^* \cup S_1^*$, $S_0^* = \{(x, t) : x \in \Gamma, t = 0\}$, $S_1^* = \{(x, t) : x \in \Gamma^*, 0 < t \leq T\}$, the usual compatibility conditions are satisfied so that smoothness of the solution is ensured for each fixed value of the ε .

The solution of the boundary value problem is the function $u(x, t)$, $(x, t) \in \overline{G}$ such that $u \in C^{2,1}(G) \cap C^{1,0}(\overline{G})$, and also this function is assumed to satisfy an equation on G at $t = 0$, and a boundary condition on S . As ε tends to zero, a parabolic boundary layer appears in the neighbourhood of the set S_1

It is required to find the solution of the boundary value problem and also its normalised gradient

$$P_s(x, t) = \varepsilon \frac{\partial}{\partial x_s} u(x, t), \quad (x, t) \in \overline{G}, \quad s = 1, 2.$$

One-sided differences are used for approximation of the first order spatial derivatives. For these schemes the accuracy of the approximate solution is normally not greater than first order. In the presence of corner points (or edges) the solution smoothness is reduced, thus causing a decrease in the convergence order for the numerical methods. It can be found that second order spatial derivatives are bounded in a neighbourhood of corner points (for a fixed value of the parameter), however in this case also the order of convergence is not greater than one with respect to the spatial variables.

For difference schemes for which the convergence order is no higher than one (with respect to the spatial variables), the first order difference derivatives do not necessarily converge with increasing the number of grid nodes. Therefore the difference derivatives of the computed solution cannot be used for the approximation of fluxes. Thus, in this case, the issue of constructing acceptable difference approximations of the diffusive fluxes appears.

On the set $\overline{G}_{(5.2)}$, we introduce the grid

$$\overline{G}_h = \overline{D}_h \times \overline{w}_0 = \overline{w}_1 \times \overline{w}_2 \times \overline{w}_0, \quad (5.3)$$

where \overline{w}_s is a grid, in general nonuniform, on the interval $[0, d_s]$ on axis x_s , $s = 1, 2$, and \overline{w}_0 is an uniform grid on the interval $[0, T]$ on axis t with a step-size $\tau = TN_0^{-1}$. We denote by $h_s^i = x_s^{i+1} - x_s^i$, $x_s^i, x_s^{i+1} \in \overline{w}_s$, $h_s = \max_i h_s^i$, $h = \max h_s$, $s = 1, 2$. By $N_s + 1$ we denote the number of nodes of the grid \overline{w}_s ; $N = \min_s N_s$, $s = 1, 2$, $h \leq M N^{-1}$. For problem (5.1) we consider the difference scheme on the grid \overline{G}_h given by

$$\begin{aligned} \Lambda_{(5.4)}(z(x, t)) &= 0, & (x, t) \in G_h, \\ z(x, t) &= \varphi(x, t), & (x, t) \in S_h. \end{aligned} \quad (5.4)$$

Here $G_h = G \cap \overline{G}_h$, $S_h = S \cap \overline{G}_h$,

$$\Lambda_{(5.4)}(z(x, t)) \equiv \varepsilon^* \Lambda_{(5.4)}^2 z(x, t) - p(x, t) \delta_{\overline{t}} z(x, t) - g(x, t, z(x, t)),$$

$$\Lambda_{(5.4)}^* \equiv \sum_{s=1,2} a_s(x,t) \delta_{x_s} + \sum_{s=1,2} [b_s^+(x,t) \delta_{x_s} + b_s^-(x,t) \delta_{x_s^-}] - c^0(x,t),$$

where b_s^+ and b_s^- are respectively the positive and the negative part of b_s . The approximation of normalised diffusive fluxes is constructed below.

Considering the difference scheme (5.4), (5.3), we assume that the estimates of Theorem 5.1 are fulfilled. Using the maximum principle [17], the estimate

$$|u(x,t) - z(x,t)| \leq M [\varepsilon^{-1} N^{-1} + N_0^{-1}], \quad (x,t) \in \overline{G}_h. \quad (5.5)$$

can be proved.

To construct a ε -uniformly convergent difference scheme, we apply a special grid condensed in the boundary layer [27]. On the set \overline{G} we introduce the grid

$$\overline{G}_h(5.6) = \overline{G}_{h(5.6)}(\sigma) = \overline{D}_{h(5.6)}(\sigma) \times \overline{\omega}_0, \quad (5.6)$$

where

$$\overline{D}_h(\sigma) = \overline{\omega}_1^* \times \overline{\omega}_2^*, \quad \overline{\omega}_s^* = \overline{\omega}_s^*(\sigma) = \overline{\omega}_s^*(\sigma, d_s), \quad s = 1, 2,$$

the grid $\overline{\omega}_s^*(5.6)$ is the grid $\overline{\omega}_1^*(3.11)$ with d and N equal to d_s and N_s respectively. For this grid, the following estimate is valid

$$|u(x,t) - z(x,t)| \leq M [N^{-1} \ln N + N_0^{-1}], \quad (x,t) \in \overline{G}_{h(5.6)}. \quad (5.7)$$

We now construct the approximation of the normalised fluxes $P_1(x,t)$, $P_2(x,t)$ for the special difference scheme (5.4), (5.6). For this purpose we need to modify the standard difference derivatives with respect to variables x_1 and x_2 . Let the estimate

$$|u(x,t) - z(x,t)| \leq \beta(N, N_0), \quad (x,t) \in \overline{G}_{h(5.6)}, \quad (5.8a)$$

hold, where $\beta(N, N_0)$ tends to zero ε -uniformly for N , $N_0 \rightarrow \infty$. The computational parameter h_s^* is defined by the relation

$$h_s^* = h_s^*(\varepsilon, \beta(N, N_0)) = \min [4^{-1} d_s, M \varepsilon \beta^{1/2}(N, N_0)], \quad (5.8b)$$

where $M = M_{(5.8)}$ is an arbitrary number. We introduce grid sets \overline{G}_h^{1-} , \overline{G}_h^{2-}

$$\overline{G}_h^{s-} = \overline{G}_{h(5.6)} \cap \{ (x,t) : x_s \leq d_s - h_s^* \}, \quad s = 1, 2.$$

By using linear interpolation along x_s for the grid function $z(x,t)$ we construct functions $\tilde{z}^s(x,t)$ which are continuous functions along x_s and grid functions along variables t , x_{3-s} , $s = 1, 2$. Then we form modified difference derivatives

$$\delta_{x_1}^* z(x,t) = (h_1^*)^{-1} [\tilde{z}^1(x_1 + h_1^*, x_2, t) - z(x,t)], \quad (x,t) \in \overline{G}_h^{1-},$$

$$\delta_{x_2}^* z(x,t) = (h_2^*)^{-1} [\tilde{z}^2(x_1, x_2 + h_2^*, t) - z(x,t)], \quad (x,t) \in \overline{G}_h^{2-}.$$

We emphasise that in order to construct the modified difference derivatives we use the function $\beta(N, N_0)$, that is the right-hand side in the estimate (5.8a). In this case the function $\beta(N, N_0)$ can be taken to be the right-hand side in the inequality (5.7).

The normalised diffusive fluxes $P_1(x,t)$, $P_2(x,t)$ are approximated by the grid functions $P_1^{*h+}(x,t)$, $P_2^{*h+}(x,t)$, where

$$P_s^{*h+}(x,t) = \varepsilon \delta_{x_s}^* z(x,t), \quad (x,t) \in \overline{G}_h^{s-}, \quad s = 1, 2.$$

Using the estimate (5.7) we establish ε -uniform convergence of the functions $P_1^{*h+}(x,t)$, $P_2^{*h+}(x,t)$, that is the computed normalised fluxes,

$$|P_s(x,t) - P_s^{*h+}(x,t)| \leq M [N^{-1} \ln N + N_0]^{1/2}, \quad (x,t) \in \overline{G}_h^{s-}, \quad s = 1, 2. \quad (5.9)$$

THEOREM 5.1 Let $a_s, b_s, c^0, p \in C^{l+\alpha}(\overline{G})$, $s = 1, 2$, $g \in C^{l+\alpha}(\overline{G} \times R)$, $\varphi \in C^{l+\alpha}(S)$, $U \in C^{l-2+\alpha}(\overline{G})$ (where $U(x, t)$ is the regular part of the solution of the boundary value problem (5.1)), $l > 6$, $\alpha > 0$. Then the solution of the difference scheme (5.4), (5.6) and the computed diffusive fluxes $P_s^{*h+}(x, t)$, $(x, t) \in \overline{G}_h^{s-}$, $s = 1, 2$ converge ε -uniformly. For the solution of the difference scheme and the computed fluxes $P_1^{*h+}(x, t)$, $P_2^{*h+}(x, t)$ the estimates (5.5), (5.7), (5.9) hold.

Part IV

Boundary value problem for elliptic equation with mixed boundary condition

1. INTRODUCTION

In this part we sketch a variety of special methods which are used for constructing ε -uniformly convergent schemes. We shall demonstrate a method which achieves improved accuracy for solving singularly perturbed boundary value problem for elliptic equations with parabolic boundary layers.

In Section 4 we shall introduce a natural class, B , of finite difference schemes, in which (by the above mentioned approaches (a) and (b)) we can construct (formally) the special finite difference schemes with approximate solutions which converge parameter-uniformly to the solution of our initial boundary value problem.

In this chapter we consider a class of singularly perturbed boundary value problems which arise when diffusion processes in a moving medium are modeled. For such boundary value problems which describe transfer with diffusion, we construct a special scheme that converges parameter-uniformly. We shall show that for the construction of such schemes from class B , the use of a special condensing grid (or an adaptive mesh) is necessary. It means that the choice (to construct special parameter-uniformly convergent schemes for our class of convection diffusion problems) is quite restricted. By condensing (or adaptive) grids we can construct finite difference schemes which converge parameter-uniformly. We shall present and discuss the results of numerical computations using both the classical and the new special finite difference schemes.

2. THE CLASS OF BOUNDARY VALUE PROBLEMS

2.1. The physical problem

The diffusion of a substance in a convective flow of an incompressible fluid in a two-dimension domain gives rise to an equation of the form

$$-\varepsilon\Delta u(x) + \vec{v}(x) \cdot \vec{\nabla} u(x) = F(x), \quad x \in \Omega, \quad (2.1a)$$

where $\vec{v}(x)$ and $F(x)$ are the velocity and source, respectively; $1/\varepsilon$ is the Peclet number (Reynolds number), if the substance is heat (diffusive matter or momentum) [5]. When the substance is heat (diffusive matter or momentum) then $u(x)$ is the temperature (density or velocity) at the point x . On the boundary of domain considered (that is the wall of the container holding the fluid) we have a boundary condition that describes the exchange of the substance with the surrounding environment

$$-\alpha(u(x) - U(x)) - \frac{\partial}{\partial n} u(x) = 0, \quad x \in \partial^0 \Omega. \quad (2.1b)$$

Here $\partial^0 \Omega$ is the boundary of the domain (the wall of the container), $\frac{\partial}{\partial n}$ is the outward normal derivative at the boundary, α characterises intensity of exchange of the substance between the medium and the wall, where the value is given by $U(x)$. When α tends to infinity the condition (2.1b) becomes the Dirichlet condition

$$u(x) = U(x), \quad x \in \partial^0 \Omega.$$

The inflow boundary, that is the part of the boundary $\partial \Omega \setminus \partial^0 \Omega$ where the stream enters the domain, we denote by $\partial^+ \Omega$, and the outflow boundary by $\partial^- \Omega$

$$\vec{n}(x) \cdot \vec{v}(x) < 0, \quad x \in \partial^+ \Omega; \quad \vec{n}(x) \cdot \vec{v}(x) > 0, \quad x \in \partial^- \Omega.$$

Here $\vec{n}(x)$ is a unit vector in the direction of the external normal. On the boundary $\partial^0\Omega$ we have condition

$$\vec{n}(x) \cdot \vec{v}(x) = 0, \quad x \in \partial^0\Omega.$$

On $\partial^+\Omega$ the value of $u(x)$ is given, and on outflow boundary $\partial^-\Omega$ we assume the flux to be known

$$u(x) = U^+(x), \quad x \in \partial^+\Omega, \quad (2.1c)$$

$$\frac{\partial}{\partial n}u(x) = \Psi(x), \quad x \in \partial^-\Omega. \quad (2.1d)$$

Problem (2.1) describes a general diffusion process into moving medium. For sufficiently large Peclet number (Reynolds number), ε can be very small. As ε tends to zero a boundary layer appears in the neighbourhood of the boundary $\partial^0\Omega$.

2.2. The class of boundary value problems

Now we describe the class of two-dimensional convection-diffusion problems with mixed boundary conditions, for which we shall study the convergence behaviour. Notice that we consider here mixed boundary conditions, where usually only Dirichlet boundary conditions are studied.

On the rectangular domain $D = \{x : 0 < x_i < d_i, i = 1, 2\}$ we consider the elliptic boundary value problem

$$L_{(2.2)}u(x) \equiv \left\{ \varepsilon^2 \sum_{s=1,2} a_s(x) \frac{\partial^2}{\partial x_s^2} - b(x) \frac{\partial}{\partial x_1} - c(x) \right\} u(x) = f(x), \quad x \in D, \quad (2.2a)$$

$$u(x) = \varphi(x), \quad x \in \Gamma^+, \quad (2.2b)$$

$$l_{(2.2)}u(x) \equiv -\varepsilon\alpha \frac{\partial}{\partial n}u(x) - (1 - \alpha)u(x) = \psi(x), \quad x \in \Gamma^0, \quad (2.2c)$$

$$\frac{\partial}{\partial n}u(x) = \eta(x), \quad x \in \Gamma^-. \quad (2.2d)$$

Here $a_s, b, c, f, \varphi, \psi$, and η are sufficiently smooth functions, $\alpha \in [0, 1]$, $\varepsilon \in (0, 1]$, $a_0 \leq a_1(x)$, $a_2(x) \leq a^0$; $b(x) \geq b_0$; $c(x) \geq 0$; $x \in \bar{D}$; $a_0, b_0 > 0$, and

$$\begin{aligned} \Gamma^+ &= \Gamma \cap \{x \mid x_1 = 0\}, \\ \Gamma^- &= \Gamma \cap \{x \mid x_1 = d_1\}, \\ \Gamma^0 &= \Gamma \cap \{x \mid 0 < x_1 < d_1\}. \end{aligned}$$

This class of problems includes, for example, the following boundary value problem for a regular differential equation

$$L_{(2.3)}U(y) \equiv \left\{ \sum_{s=1,2} A_s(y) \frac{\partial^2}{\partial y_s^2} - B(y) \frac{\partial}{\partial y_1} \right\} U(y) = F(y), \quad (2.3a)$$

$y \in \tilde{D}$, with regular boundary conditions

$$\begin{aligned} U(y) &= \Phi(y), & y \in \tilde{\Gamma}^+, \\ l_{(2.3)}U(y) &\equiv -\alpha \frac{\partial}{\partial n}U(y) - (1 - \alpha)U(y) = \Psi(y), & y \in \tilde{\Gamma}^0, \\ \frac{\partial}{\partial n}U(y) &= 0, & y \in \tilde{\Gamma}^-, \end{aligned} \quad (2.3b)$$

on the rectangular domain $\tilde{D} = \{y : 0 < y_i < \tilde{d}_i, i = 1, 2\}$, $\tilde{d}_i = \varepsilon^{-1} d_i$, if the size of domain \tilde{D} is sufficiently large.

2.3. The construction of ε -uniformly convergent schemes

When ε tends to zero in the neighbourhood of Γ^0 , boundary layers appear which are described by parabolic equations. Hence these layers are known as parabolic boundary layers.

Although classical difference approximations (see, for example, [16, 17]) converge for (2.2) to the solution of the boundary value problem for each fixed value of ε (see Theorem 3.1), the accuracy of the numerical solution depends on the value of ε and decreases, sometimes to complete loss of accuracy, when ε is less or comparable with the step-size of the uniform grid. This means that classical finite difference schemes do not converge uniformly with respect to the parameter ε , (see theorem 3.2). Therefore, for the boundary value problem (2.2) it is of interest to construct special schemes the solution of which does converge ε -uniformly.

For the case of the Dirichlet problem (2.2a), an ε -uniformly convergent finite difference scheme is found in [22, 23].

3. CLASSICAL DIFFERENCE SCHEME

To solve the problem (2.2) we first use a classical finite difference method. On the set \bar{D} we introduce the rectangular grid

$$\bar{D}_h = \bar{\omega}_1 \times \bar{\omega}_2, \quad (3.1)$$

where $\bar{\omega}_s$ is a , in general non-uniform, grid on the interval $[0, d_s]$ and N_s is the number of nodes of the grid $\bar{\omega}_s$, $s = 1, 2$. Define $h_s^i = x_s^{i+1} - x_s^i$, $h_s = \max_i h_s^i$, $h \leq M N^{-1}$, where $h = \max_s h_s$, $N = \min_s N_s$, $s = 1, 2$; $D_h = D \cap \bar{D}_h$, $\Gamma_h = \Gamma \cap \bar{D}_h$. For problem (2.2) we use the difference scheme

$$\Lambda_{(3.2)} z(x) = f(x), \quad x \in D_h, \quad (3.2a)$$

$$z(x) = \varphi(x), \quad x \in \Gamma_h^+, \quad (3.2b)$$

$$\lambda_{(3.2)} z(x) = \psi(x), \quad x \in \Gamma_h^0, \quad (3.2c)$$

$$\delta_{x_1}^- z(x) = \eta(x), \quad x \in \Gamma_h^-, \quad (3.2d)$$

where

$$\Lambda_{(3.2)} z(x) \equiv \varepsilon^2 \sum_{s=1,2} a_s(x) \delta_{x_s x_s} z(x) - b(x) \delta_{x_1} z(x) - c(x) z(x),$$

$$\lambda_{(3.2)} z(x) \equiv \begin{cases} \varepsilon \alpha \delta_{x_2} z(x) - (1 - \alpha) z(x), & x_2 = 0, \\ -\varepsilon \alpha \delta_{x_2} z(x) - (1 - \alpha) z(x), & x_2 = d_2, \end{cases}$$

$\delta_{x_s x_s} z(x)$ is the second divided difference on a non-uniform grid, and $\delta_{x_s} z(x)$ and $\delta_{x_s}^- z(x)$ are the first forward and backward divided differences.

The difference scheme (3.2), (3.1) is monotone (that is the maximum principle holds) [17]. By means of the maximum principle, and using the estimates of the derivatives (see [11]), we find that the solution of the scheme (3.2)-(3.1) converges (for a fixed value of the parameter ε) as

$$|u(x) - z(x)| \leq M \varepsilon^{-4} N^{-1}, \quad x \in \bar{D}_h. \quad (3.3)$$

THEOREM 3.1 Let $u \in C^4(\bar{D})$. Then, for a fixed value of the parameter ε , the solution of the scheme (3.2)-(3.1) converges to the solution of the boundary value problem (2.2) with an error bound given by (3.3).

Clearly (3.3) does not imply ε -uniform convergence of the difference scheme. In fact it can be shown that it is impossible to obtain ε -uniform convergence for the difference scheme (3.2)-(3.1) on a fixed ε -independent mesh. The proof is found in [21]. We summarise this result in the following theorem.

THEOREM 3.2 (see [11]) On an ε -independent grid of type (3.1), the solution of the classical finite difference scheme (3.2),(4.2) does not converge ε -uniformly to the solution of the boundary value problem (2.2).

We want to make the following interesting observation. We consider problem (2.2). If we take in $l_{(2.2)}$ the parameter $\varepsilon = 1$ (leaving ε unchanged in (2.2a) and $\Lambda_{(3.2)}$, but adapting it in $\lambda_{(3.2)}$), then no singular part will appear as a first term in the expansion w.r.t. ε . Hence, the classical scheme will be ε -uniform convergent in this case.

4. THE FITTED DIFFERENCE SCHEME

For parabolic problems with parabolic layers, it was shown in [11] that there does not exist a difference scheme only based on fitting of the coefficients, for which the solution converges ε -uniformly to the solution. Here we show a similar result for the elliptic boundary value problem (2.2). Let us consider the problem

$$L_{(4.1)}u(x) \equiv \varepsilon^2 \Delta u(x) - \frac{\partial}{\partial x_1} u(x) = 0, \quad x \in D, \quad (4.1a)$$

$$u(x) = 0, \quad x \in \Gamma^+, \quad (4.1b)$$

$$l_{(2.2)}u(x) = \psi(x), \quad x \in \Gamma^0, \quad (4.1c)$$

$$\frac{\partial}{\partial n} u(x) = 0, \quad x \in \Gamma^-, \quad (4.1d)$$

where

$$\psi(x) = \begin{cases} \psi_0(x_1), & x \in \Gamma^0, \quad x_2 = 0, \\ 0, & x \in \Gamma^0, \quad x_2 \neq 0, \end{cases}$$

and the function $\psi_0(x_1)$, $x_1 \in [0, d_1]$ is sufficiently smooth. The solution of problem (4.1) is the singular solution.

Let us introduce a class, called class A, of finite difference schemes for problem (4.1), for the construction of which we use uniform meshes:

$$\overline{D}_h^u = \{\overline{D}_{h(3.1)}, \text{ where } \overline{\omega}_s = \overline{\omega}_s^u \text{ are uniform grids, } s = 1, 2\}. \quad (4.2)$$

and also (for the approximation of equation (4.1a)) a standard five-point, fitted finite difference operator

$$\Lambda_{(4.3)}z(x) \equiv \left\{ \sum_{s=1,2} (A_s \delta_{x_s \overline{x}_s} + B_s \delta_{x_s} - C) z(x) = E, \quad x \in D_h. \right. \quad (4.3)$$

Here the coefficients A_s , B_s , C , E are functionals of the coefficients of equation (4.1a) and also depend on x , h_1 , h_2 , and ε . We suppose that for $h_2 \varepsilon^{-1} \rightarrow 0$ and $h_1 \rightarrow 0$ these coefficients A_s , B_s , C , E approximate the data of equation (4.1a), in the uniform norm, in the neighbourhood of at least one point the boundary layer region.

THEOREM 4.1 In the class A of finite difference schemes there does not exist a difference scheme of which the solution converges ε -uniformly to the solution of the boundary value problem (4.1).

The proof of this theorem is rather complex. An outline of the principle steps is found in [24, 28].

Remark 1. A statement similar to Theorem 4.1 is also true in the case when the difference schemes are constructed on a more general stencil with a finite number of nodes.

The results of Theorem 4.1 and Remark 1 can be explained as follows. All solutions of problem (4.1) (defined by different functions $\psi(x)$) are singular solutions. Those solutions can not be represented as linear combinations of a finite number of fixed functions of boundary layer type (boundary layer functions).

Let us introduce class B of finite difference schemes for problem (2.2), for the construction of which we use rectangular grids $\overline{D}_h^{(3.1)}$, which are generally non-uniform and a five-point finite difference operator (in general a fitted operator) of the standard form. The coefficients of the difference operator are, as before, functionals of the coefficients of the equation (2.2a) and also depend on x , ε and on the distance between the nodes of the stencil used. Again, we suppose that for $h \rightarrow 0$ the coefficients of the difference operator approximate (in the uniform norm) the coefficients of equation (2.2a) on the set $\overline{D}_h^{(3.1)}$.

We remark that class B is a natural class for constructing finite difference schemes for problem (2.2) as it includes both fitted methods and methods with special condensing grids. This is in contrast to class A, which contains only schemes on uniform meshes. Consequences of Theorem 4.1 include results such as:

COROLLARY 4.2

In the case of boundary value problems of type (2.2), class B of finite difference schemes does not contain any difference scheme which, on grids with arbitrary distribution of nodes, can achieve ε -uniform convergence of the solution to solution of boundary value problem (2.2) by the use of a fitted method.

COROLLARY 4.3

In the case of boundary value problems of type (2.2), the use of special condensing grids (or adaptive meshes) is necessary for the construction of ε -uniformly class B finite difference schemes.

5. DIFFERENCE SCHEME OF METHOD OF SPECIAL CONDENSING MESH

We now construct an ε -uniformly convergent scheme for the boundary value problem (2.2). We use a special condensing mesh (in the neighbourhood of the boundary layers), where the distribution of the nodes is defined by a-priori estimates of the solution and its derivatives. This approach is similar to that in [24, 25, 28], where the Dirichlet problem was studied.

Consider the special grid

$$\overline{D}_h^* = \overline{\omega}_1 \times \overline{\omega}_2^* , \tag{5.1}$$

where $\overline{\omega}_2^* = \overline{\omega}_2^*(\sigma)$ is a special piecewise uniform mesh, $\overline{\omega}_1$ is a uniform mesh, σ is a parameter which depends on ε and N_2 . The mesh $\overline{\omega}_2^*(\sigma)$ is constructed as follows. The interval $[0, d_2]$ is divided into three parts $[0, \sigma]$, $[\sigma, d_2 - \sigma]$, $[d_2 - \sigma, d_2]$, $0 < \sigma \leq d_2/4$. Each subinterval $[0, \sigma]$ and $[d_2 - \sigma, d_2]$ is divided into $N_2/4$ equal cells and the subinterval $[\sigma, d_2 - \sigma]$ into $N_2/2$ equal cells. Suppose $\sigma = \sigma(\varepsilon, N_2) = \min[d_2/4, m\varepsilon \ln N_2]$ where m is arbitrary number.

The difference scheme (3.2), (5.1) belongs to class B. The scheme is constructed using an a priori adapted mesh. Distribution of the nodes on grid \overline{D}_h^* ensures ε -uniform approximation of the boundary value problem. This is formalised in the following theorem (see also [11]).

THEOREM 5.1 The solution of difference scheme (3.2), (5.1) converges ε -uniformly to the solution of boundary value problem (2.2). The following bound holds for the error

$$|u(x) - z(x)| \leq M N^{-1/3} , \quad x \in \overline{D}_h^* . \tag{5.2}$$

The proof of this theorem will appear in a future paper.

6. NUMERICAL RESULTS

Theoretically (see Theorem 3.2) it has been shown that the classical difference scheme (3.2) on the uniform grid (4.2) does not converge ε -uniformly in the l^∞ -norm to the solution of the boundary value problem (2.2). But it could be the case that the error $\max_{\overline{D}_h} |z(x) - u(x)|$ is relatively small for the classical scheme, which would reduce the need for a special scheme.

On the other hand, Theorem 5.1 shows that the special scheme (3.2),(5.1) converges ε -uniformly, but no indication is given about the value of the order constant M in (5.2) and the order of convergence is rather small. It might be that the error is relatively large for any reasonable values of N_1, N_2 . This would reduce the practical value of the special scheme. The following numerical experiments address these issues.

6.1. The model problem

To see the effect of the special scheme in practice, for the approximation of the model problem we study the singularly perturbed elliptic equation with a mixed boundary condition

$$L_{(6.1)}u(x) \equiv \varepsilon^2 \Delta u(x) - \frac{\partial}{\partial x_1} u(x) = -1, \quad x \in D, \quad (6.1)$$

$$l_{(6.1)}u(x) = \psi(x), \quad x \in \Gamma^0,$$

$$u(x) = 0, \quad x \in \Gamma^+, \quad \frac{\partial}{\partial x_2} u(x) = 0, \quad x \in \Gamma^-,$$

where

$$l_{(6.1)}u(x) \equiv \begin{cases} \alpha \varepsilon (\partial / \partial x_2) u(x) - (1 - \alpha)u(x), & x_2 = 0, \\ -\alpha \varepsilon (\partial / \partial x_2) u(x) - (1 - \alpha)u(x), & x_2 = 1. \end{cases}$$

We compare the numerical results for the classical scheme (3.2), (4.2) and the special scheme (3.2), (5.1). Here $D = \{x : 0 < x_1, x_2 < 1\}$,

$$\psi(x) = \begin{cases} x_1, & x \in \Gamma^0, x_2 = 0, \\ 0, & x \in \Gamma^0, x_2 = 1. \end{cases}$$

For the solution of problem (6.1), we have the representation

$$u(x) = U(x) + W(x), \quad x \in \overline{D},$$

where $U(x) = x_1$, $x \in \overline{D}$, is the outer solution, and $W(x)$ represents the parabolic boundary layer in the neighbourhood of the edges at $x_2 = 0$ and $x_2 = 1$. We have the following bounds on the solution

$$-1 \leq u(x) \leq 1, \quad x \in \overline{D}.$$

Due to Theorem 5.1 the solution of the discrete problem with the adapted mesh converges ε -uniformly to the solution of our model problem (6.1). The function $u_h(x)$, which is the solution of the special scheme (3.2),(5.1) is shown in Figure 1

6.2. The behaviour of the numerical solution of the classical scheme

To see the difference between the use of the uniform and the adapted grid, for the approximation of (6.1) we first use the classical scheme (3.2),(4.2). We solve the problem for different values of the mesh width $h_1 = h_2 = N^{-1}$ and for different values of the parameters ε and α . The results for a set of numerical experiments is given in Table 14. From Table 14 we can see that the solution of scheme (3.2)-(4.2) does not converge ε -uniformly. The errors, for a fixed value of N , depend on the parameters ε and α . For $\varepsilon \geq 0.1$ and $\alpha = 0.0, 0.1, 0.5, 1.0$ the error behaviour is regular: when N increases, the error decreases. For $\varepsilon = 10^{-3}$ and $\alpha = 0.0, 0.1, 0.5$ and for $\varepsilon = 10^{-2}$ and $\alpha = 0.0$, for some values of N the error increases with increasing N . For $\alpha = 0.5, 1.0$ and a fixed N the error increases with decreasing ε . In particular, for $\varepsilon = 10^{-3}$ and $\alpha = 0.5, 1.0$ the errors for $N \leq 128$ are of the same order or larger than (in ℓ^∞ -norm) the solution of the BVP. Thus, the numerical results illustrate that the lack of ε -uniform convergence leads to large errors indeed.

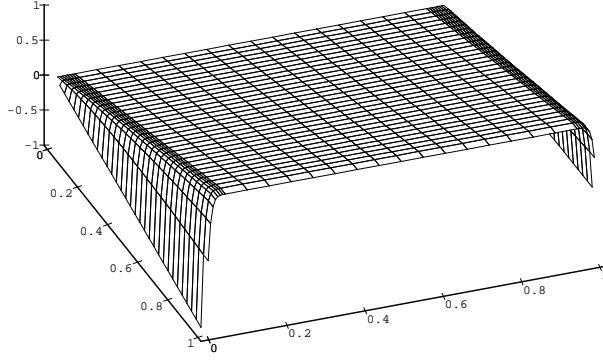


FIGURE 1. Solution computed with the adapted mesh, $\varepsilon = 0.01$, $\alpha = 0.5$, $N = 32$ and $m = 1.0$.

TABLE 14. Table of errors $E(N, \varepsilon, \alpha)$ for the classical scheme

N		4	8	16	32	64	128
ε	α						
1	0.00	0.105	0.633(-1)	0.329(-1)	0.158(-1)	0.696(-2)	0.241(-2)
10^{-1}		0.305	0.144	0.655(-1)	0.291(-1)	0.122(-1)	0.403(-2)
10^{-2}		0.247	0.127	0.822(-1)	0.107	0.882(-1)	0.246(-1)
10^{-3}		0.246	0.121	0.588(-1)	0.283(-1)	0.157(-1)	0.200(-1)
1	0.10	0.691(-1)	0.381(-1)	0.170(-1)	0.744(-2)	0.314(-2)	0.104(-2)
10^{-1}		0.312	0.149	0.701(-1)	0.310(-1)	0.129(-1)	0.423(-2)
10^{-2}		0.247	0.194	0.185	0.156	0.109	0.583(-1)
10^{-3}		0.246	0.209	0.216	0.216	0.211	0.197
1	0.5	0.852(-1)	0.401(-1)	0.183(-1)	0.842(-2)	0.359(-2)	0.119(-2)
10^{-1}		0.706	0.473	0.254	0.121	0.526(-1)	0.176(-1)
10^{-2}		1.20	1.26	1.15	0.893	0.557	0.279
10^{-3}		1.28	1.43	1.49	1.48	1.41	1.27
1	1.0	0.123	0.610(-1)	0.288(-1)	0.132(-1)	0.562(-2)	0.187(-2)
10^{-1}		1.52	0.752	0.344	0.154	0.643(-1)	0.211(-1)
10^{-2}		18.2	10.3	5.19	2.40	1.02	0.408
10^{-3}		187.	109.	57.9	29.5	14.6	7.01

In this table the error $E(N, \varepsilon, \alpha)$ is defined by

$$E(N, \varepsilon, \alpha) = \max_{x \in \mathcal{D}_h} |e(x; N, \varepsilon, \alpha)|, \quad (6.2a)$$

$$e(x; N, \varepsilon, \alpha) = z(x) - u^*(x), \quad (6.2b)$$

where $u^*(x)$ is the piecewise interpolation of $z_m^{*256}(x)$, $m = m_{(5,1)} = 1$ (see Table 15), and $z(x) \equiv z^N(x)$ is the solution of (3.2),(4.2) with $h_1 = h_2 = N^{-1}$. Notice that $u^*(x)$ is an accurate approximation of $u(x)$.

TABLE 15. Table of errors $E(N, \varepsilon, \alpha)$ for the special scheme

N		4	8	16	32	64	128
ε	α						
1	0.0	0.105	0.633(-1)	0.329(-1)	0.158(-1)	0.696(-2)	0.241(-2)
10^{-1}		0.262	0.144	0.655(-1)	0.291(-1)	0.122(-1)	0.403(-2)
10^{-2}		0.246	0.147	0.807(-1)	0.361(-1)	0.148(-1)	0.497(-2)
10^{-3}		0.246	0.147	0.807(-1)	0.361(-1)	0.148(-1)	0.497(-2)
1	0.1	0.691(-1)	0.381(-1)	0.170(-1)	0.744(-2)	0.314(-2)	0.104(-2)
10^{-1}		0.276	0.149	0.701(-1)	0.310(-1)	0.129(-1)	0.423(-2)
10^{-2}		0.246	0.170	0.887(-1)	0.461(-1)	0.240(-1)	0.924(-2)
10^{-3}		0.246	0.169	0.887(-1)	0.461(-1)	0.241(-1)	0.925(-2)
1	0.5	0.852(-1)	0.401(-1)	0.183(-1)	0.842(-2)	0.359(-2)	0.119(-2)
10^{-1}		0.611	0.473	0.254	0.121	0.526(-1)	0.176(-1)
10^{-2}		0.539	0.511	0.361	0.217	0.111	0.420(-1)
10^{-3}		0.535	0.511	0.361	0.217	0.111	0.420(-1)
1	1.0	0.123	0.610(-1)	0.288(-1)	0.132(-1)	0.562(-2)	0.187(-2)
10^{-1}		1.14	0.752	0.344	0.154	0.643(-1)	0.211(-1)
10^{-2}		0.977	0.889	0.554	0.301	0.144	0.521(-1)
10^{-3}		0.963	0.888	0.554	0.301	0.144	0.521(-1)

In this table the function $E(N, \varepsilon, \alpha)$ is defined by (6.2), but now $z(x) = z_m^{*N}(x)$ in (6.2) is the solution of (3.2),(5.1) with $m = m_{(5.1)} = 1$ and $N_1 = N_2 = N$

6.3. The behaviour of the numerical solution of the special scheme

In Table 15 we show the behaviour of (3.2),(5.1), with $m = m_{(5.1)} = 1$, applied to the model problem (6.1) From Table 15 we can see that the solution of the scheme (3.2)-(5.1) does converge ε -uniformly indeed. The errors for a fixed value of $\varepsilon = 1.0, 10^{-1}, 10^{-2}, 10^{-3}$ and $\alpha = 0.0, 0.1, 0.5, 1.0$ have all a regular behaviour and decrease for increasing N . For a fixed value of α and N the error stabilises for decreasing ε : the errors for $\varepsilon = 10^{-2}$ and $\varepsilon = 10^{-3}$ are practically the same. For $\varepsilon \leq 10^{-2}$ and a fixed value of N we find the largest error for $\alpha = 1.0$. In particular, for $\varepsilon \leq 10^{-2}$, $\alpha = 1.0$ and $N = 128$ the error is less than 6%. Also here, the numerical results illustrate the practical value of ε -convergent methods.

CONCLUSION

For the elliptic boundary value problem (2.2), where a small parameter multiplies the highest derivative, we have analysed different approaches for the construction of discrete methods. We present methods for which the accuracy of the discrete solution does not depend on the value of the small parameter, but only on the number of points in the discretisation.

We show that in a natural class of finite difference schemes, for the problem considered, no ε -uniform methods exist on a uniform grid (Theorem 4.1). As a consequence, for the construction of ε -uniform methods the use of an adapted non-uniform mesh is necessary. With a special, adapted, non-uniform mesh and a simple classical difference scheme, we are able to construct an ε -uniform approximation.

To illustrate the practical importance of our study, for a model problem we show by a numerical example that, on a uniform grid, the classical difference scheme is not ε -uniformly convergent. In our example, the error (with a Neumann boundary condition) is not less than 700% of the solution, for $N=128$ and $\varepsilon = 10^{-3}$. The same example shows that we might obtain an ε -uniformly convergent solution if we use the adapted mesh. Now the error is not larger than 6% of the solution, for any value of the parameter ε . Thus, the numerical example illustrates that the theoretical considerations have practical implications indeed.

Part V

Parabolic equations with a discontinuous boundary condition

1. INTRODUCTION

The solution of partial differential equations that are singularly perturbed and / or have discontinuous boundary conditions generally have only limited smoothness. Due to this fact difficulties appear when we solve these problems by numerical methods. For example for regular parabolic equations with discontinuous boundary conditions, classical methods (FDM or FEM) on regular rectangular grids do not converge in the ℓ^∞ -norm on a domain that includes a neighbourhood of the discontinuity [19, 20, 11].

If the parameter multiplying the highest-order derivative vanishes, boundary- and interior layers generally appear. When a discontinuity is present in the initial function (given at $t = 0$), an interior layer is generated. Outside a neighbourhood of the discontinuity classical difference schemes converge in the ℓ^∞ -norm for each fixed value of the small parameter, but they do not converge in the ℓ^∞ -norm in the neighbourhood of the discontinuity. Neither do they converge uniformly in ε in any neighbourhood of the interior layer [19, 20]. Therefore, it is interesting to construct special methods which are ℓ^∞ -convergent for parabolic PDEs with discontinuous initial functions, both in the regular and in the singularly perturbed case. In the latter case it is important to see if and when such convergence can be uniform in the small parameter on the whole domain of definition.

In [19, 20] singularly perturbed parabolic equations with discontinuous boundary conditions were studied. There, special difference schemes were constructed for these problems. In order to be able to construct a method that was uniformly convergent (in the small parameter ε), special variables were used in the neighbourhood of the discontinuity. By introducing the variables $\theta = x/(2\varepsilon\sqrt{t})$ and t , the singularity was removed from the boundary value problem and the solution became a smooth function in the new variables. This behaviour of the transformed solution allows the use of a classical scheme in the transformed variables in the neighbourhood of the singularity. Away from the singularity the classical scheme can be used with the original variables.

This transformation in the neighbourhood of the singularity implied the use of a specially condensed grid in the neighbourhood of the boundary and interior layers. So we can say that the technique is based on: (i) fitted methods in which the coefficients of the difference equations are adapted to the singularities; (ii) methods that use special, refined meshes in the neighbourhood of singularities. For these schemes ℓ^∞ -convergence on the whole domain is proved, uniformly in the small parameter, but a disadvantage of these schemes is that they are very hard to realise in practice.

Because fitting of the coefficients, combined with fitting of the mesh is generally too complex for practical application, in the present paper we propose a new method in which only the coefficients are adapted. We use a uniform rectangular grid and a special difference equation with fitted coefficients. This method is much easier to realise.

For the construction of the new scheme the coefficients are selected such that the solution of a model problem with a piecewise constant, discontinuous initial function is the exact solution of the difference equations. This difference scheme with adapted coefficients is studied in this paper and it is compared with the classical scheme.

As was shown in [19, 20], no scheme exists that converges uniformly on a uniform grid for the general problem with a parabolic layer. However, for problems with an interior layer, the present method has this favourable property, and, in addition, numerical examples show that the method has practical value for far more general equations with discontinuous boundary conditions.

2. PROBLEM FORMULATION

We consider the Dirichlet boundary value problem for the following singularly perturbed equation of parabolic type

$$\begin{aligned} L_{(2.1)}u(x, t) &= f(x, t), & (x, t) \in G, \\ u(x, t) &= \phi(x, t), & (x, t) \in S, \end{aligned} \quad (2.1a)$$

where

$$G = \{(x, t) \mid -1 < x < 1, 0 < t \leq T\}, \quad S = \overline{G} \setminus G, \quad (2.1b)$$

$$L_{(2.1)} \equiv \varepsilon^2 \frac{\partial^2}{\partial x^2} - p(t) \frac{\partial}{\partial t} - c(t). \quad (2.1c)$$

The parameter ε may take any value $\varepsilon \in (0, 1]$. The coefficients $c(t)$, $p(t)$ and the source $f(x, t)$ are sufficiently smooth functions on \overline{G} and the coefficients are positive:

$$c(t) \geq 0, \quad p(t) \geq p_0 > 0, \quad (x, t) \in \overline{G}. \quad (2.2)$$

The boundary function $\phi(x, t)$ has a discontinuity² of the first kind on the set S^* :

$$S^* = \{(x, t) \mid x = 0, t = 0\}.$$

For simplicity S^* consists of a single point only. Outside S^* the function $\phi(x, t)$ is sufficiently smooth on S .

Such boundary value problems with discontinuous boundary condition describe for example the temperature in a heat transfer problem, when two parts of a material with different temperatures are instantaneously connected [15]. Then, the small parameter ε corresponds to a small heat conduction coefficient.

The solution of the boundary value problem (2.1) is a function $u \in C(\overline{G} \setminus S^*) \cap C^{2,1}(G)$, that is on G it is C^2 in x and C^1 in t .

We say that the discrete approximation converges ε -uniformly (or uniformly in ε) on \overline{G} if the ℓ^∞ -norm of the error converges to zero on \overline{G} , uniformly in ε .

For the construction of a special difference scheme we shall use the standard function $w_0(x, t)$, which is discontinuous on S^* ,

$$w_0(x, t) = w_0(x, t; p_1) = \frac{1}{2} v\left(\frac{x}{2\varepsilon} \sqrt{\frac{p_1}{t}}\right), \quad (x, t) \in \overline{G} \setminus S^*, \quad (2.5)$$

where $p_1 = p(0)$ and $v(\xi) = \operatorname{erf}(\xi) = \frac{2}{\sqrt{\pi}} \int_0^\xi \exp(-\alpha^2) d\alpha$ is the error function. For $t = 0$, at the point $x = 0$ the function (2.5) is defined by continuous extension. The function $w_0(x, t)$ is the solution of the constant coefficient equation

$$L_{(2.6)}u(x, t) \equiv \left(\varepsilon^2 \frac{\partial^2}{\partial x^2} - p_1 \frac{\partial}{\partial t} \right) u(x, t) = 0, \quad (x, t) \in G. \quad (2.6)$$

²A piecewise continuous function $v(x, t)$, $(x, t) \in S \setminus S^*$, is redefined at the discontinuity by

$$v(x, t) = \frac{1}{2} \left\{ \lim_{s \searrow 0} v(x + s, t) + \lim_{s \nearrow 0} v(x + s, t) \right\}, \quad (x, t) \in S^*, \quad (2.3)$$

and the jump in the discontinuity is defined by

$$[v(x, t)] = \left\{ \lim_{s \searrow 0} v(x + s, t) - \lim_{s \nearrow 0} v(x + s, t) \right\}, \quad (x, t) \in S^*. \quad (2.4)$$

This function is piecewise constant on S at $t = 0$ and has a discontinuity of the first kind in S^* :

$$[w_0(0, 0)] = 1.$$

Suppose

$$W_0(x, t) = \exp\left(-\int_0^t \frac{c(\xi)}{p(\xi)} d\xi\right) w_0(x, \eta(t); p_1), \quad \text{with } \eta(t) = \int_0^t \frac{p_1}{p(\xi)} d\xi. \quad (2.7)$$

Then the function $W_0(x, t)$ is continuous on $\overline{G} \setminus S^*$, it is a solution of the homogeneous equation

$$L_{(2.1)}u(x, t) = 0, \quad (x, t) \in G. \quad (2.8)$$

On S the function $u(x, t)$ is continuous and piecewise smooth. For simplicity we suppose that $u(x, t)$ is sufficiently smooth on the boundary of G , and that a compatibility condition is satisfied at the corner points. We are interested in the solution of problem (2.1) in the neighbourhood of the point of discontinuity and in the neighbourhood of the generated interior layer. Therefore, we suppose that the boundary conditions at $x = \pm 1$ are such that no boundary layers appear.

3. AN ε -UNIFORMLY CONVERGENT SCHEME

On the set \overline{G} we introduce the rectangular grid

$$\overline{G}_h = \omega \times \omega_0. \quad (3.1)$$

Here ω and ω_0 are uniform grids on the segments $[-1, 1]$ and $[0, T]$ respectively. For some $N, N_0 > 0$ we take $x_i = ih, i \in \mathbb{Z}; -1 \leq x_i \leq 1; h = 2/N; t^j = j\tau; j = 0, 1, 2, \dots, N_0, \tau = T/N_0$; and

$$G_h = G \cap \overline{G}_h; \quad S_h = S \cap \overline{G}_h; \quad S_h^* = S^* \cap \overline{G}_h.$$

On the set S_h^* the boundary function $\phi(x, t)$ is defined by

$$\phi(x, t) = \frac{1}{2} \left\{ \lim_{s \nearrow x} \phi(s, t) + \lim_{s \searrow x} \phi(s, t) \right\}, \quad (x, t) \in S_h^*. \quad (3.2)$$

For the numerical approximation of (2.1) we may use classical difference approximations (see [16, 17]). For example, in the case of the implicit central difference scheme we have

$$\begin{aligned} \Lambda_{(3.3)}z(x, t) &= f(x, t), & (x, t) \in G_h, \\ z(x, t) &= \phi(x, t), & (x, t) \in S_h, \end{aligned} \quad (3.3a)$$

where

$$\Lambda_{(3.3)} \equiv \varepsilon^2 \delta_{x\bar{x}} - p(t) \delta_{\bar{t}} - c(t), \quad (3.3b)$$

with $\delta_{\bar{t}}z(x, t)$ and $\delta_{x\bar{x}}z(x, t)$ the usual first and second difference of $z(x, t)$ on the uniform grids ω_0 and ω respectively; the bar denotes the backward difference. It is well known that the operator $\Lambda_{(3.3)}$ is monotone [17], which implies that the maximum principle holds for (3.3).

Nevertheless, the classical difference scheme (i) does not converge on the whole domain $\overline{G}_h^* = \overline{G}_h \setminus S_h^*$ for a fixed value of ε , and (ii) outside a neighbourhood of the discontinuity it does not converge uniformly with respect to ε in the interior layer (see Section 4). To obtain uniform convergence, in the present paper we introduce a specially fitted scheme for the approximation of equation (2.1a),

$$\begin{aligned} \Lambda_{(3.4)}z(x, t) &= f(x, t), & (x, t) \in G_h, \\ z(x, t) &= \phi(x, t), & (x, t) \in S_h, \end{aligned} \quad (3.4a)$$

where

$$\Lambda_{(3.4)} \equiv \varepsilon^2 \gamma(x, t) \delta_{x\bar{x}} - p(t) \delta_{\bar{t}} - c(t). \quad (3.4b)$$

According to the principle mentioned in the introduction, here $\gamma(x, t)$ is a fitting coefficient or fitting factor, which is chosen such that the singular solution, $W_{(0)}(x, t)$, is the exact solution of the homogeneous difference equation (3.5):

$$\Lambda_{(3.4)} W_0(x, t) \equiv \{\varepsilon^2 \gamma(x, t) \delta_{x\bar{x}} - p(t) \delta_{\bar{t}} - c(t)\} W_0(x, t) = 0, \quad (x, t) \in G_h. \quad (3.5)$$

More generally we can select $\gamma(x, t)$ such that (3.5) is satisfied by $v(x, t) = W_0(x, t) + u_0(x, t)$, where W_0 is the singular solution and u_0 is some smooth solution of the homogeneous equation

$$L_{(2.1)} u(x, t) = 0, \quad (x, y) \in G. \quad (3.6)$$

This leads to the following expression for γ :

$$\gamma(x, t) = \frac{p(t) \delta_{\bar{t}} v(x, t) + c(t) v(x, t)}{\varepsilon^2 \delta_{x\bar{x}} v(x, t)}, \quad (x, t) \in G_h, \quad (3.7)$$

for any point (x, t) where $\delta_{x\bar{x}} v(x, t) \neq 0$.

We notice that for $u_0(x, t) \equiv 0$ the differences $\delta_{x\bar{x}} v(x, t)$ and $\delta_{\bar{t}} v(x, t)$ can be very small because of the exponentially small derivatives of $W_0(x, t)$ for large $x/(\varepsilon\sqrt{t})$. To improve the numerical behaviour in the computation of $\gamma(x, t)$, we choose the function u_0 such that the differences $\delta_{x\bar{x}} W_0$ and $\delta_{x\bar{x}} u_0$ have the same sign, for $(x, t) \in G_h$. In particular, in the remaining part of this paper we take, for example,

$$u_0(x, t) = - \left\{ x^3 + 6\varepsilon^2 x \int_0^t \frac{1}{p(\xi)} d\xi \right\} \exp \left(- \int_0^t \frac{c(\nu)}{p(\nu)} d\nu \right), \quad (x, t) \in \bar{G}, \quad (3.8)$$

so that, for example for $c(t) \equiv 0$ and $p(t) \equiv 1$, we obtain

$$u_0(x, t) = u_{(3.9)}(x, t) = -x^3 - 6\varepsilon^2 xt, \quad (x, t) \in \bar{G}. \quad (3.9)$$

Then, for $\gamma(x, t)$ we have the general representation

$$\gamma(x, t) = \frac{p(t)(\delta_{\bar{t}} W_0(x, t) + \delta_{\bar{t}} u_0(x, t)) + c(t)(W_0(x, t) + u_0(x, t))}{\varepsilon^2 \delta_{x\bar{x}} W_0(x, t) + \varepsilon^2 \delta_{x\bar{x}} u_0(x, t)}, \quad x \neq 0, \quad (3.10)$$

where the functions W_0 and u_0 are defined by (2.7) and (3.8) respectively. Note that $\delta_{x\bar{x}} v = \delta_{\bar{t}} v = v = 0$, at $x = 0$, $t > 0$. Therefore, for definiteness, we set $\gamma(x, t) = 1$ at $x = 0$.³ Now we define the resulting difference scheme as (3.4), where $\gamma(x, t)$ is determined by (3.10).

Under the condition that

$$\frac{\tau^{3/2}}{h} \leq \psi(h, \tau) \quad (3.11)$$

where $\psi(h, \tau) > 0$ and $\psi(h, \tau) \rightarrow 0$ for $h, \tau \rightarrow 0$, then the scheme (3.4,3.10) converges uniformly in ε :

$$\max_{\bar{G}_h} |u(x, t) - z(x, t)| \leq M \{(h + \tau)^\nu + \psi(h, \tau)\}, \quad (3.12)$$

for any $\nu \in (0, 1/3)$.

If, for instance,

$$h \geq \mathcal{O}(\tau^{\frac{3}{2(1+\nu)}}) \quad (3.13)$$

then

$$\max_{\bar{G}_h} |u(x, t) - z(x, t)| \leq M(h^{\nu_1} + \tau^{\nu_1}), \quad (3.14)$$

for any $\nu_1 \in (0, 1/3)$. Thus, we have the following theorem [10]

³According to (2.7) to compute $\delta_{x\bar{x}} W_0(x, t)$ on time layer $t = j\tau$ we use the difference derivative $\delta_{x\bar{x}} w_0(x, \eta(t))$. The difference derivatives $\delta_{\bar{t}} W_0(x, t)$, $\delta_{x\bar{x}} W_0(x, t)$, $\delta_{\bar{t}} u_0(x, t)$, $\delta_{x\bar{x}} u_0(x, t)$ can easily be found, for example when the functions $p(t)$ and $c(t)$ are analytical.

THEOREM 3.1 Under condition (3.11), the solution of the difference scheme (3.4,3.10) converges on \overline{G}_h in the discrete ℓ^∞ -norm to the solution of problem (2.1) uniformly in ε . Under the conditions (3.11) or (3.13) respectively, the estimates (3.12) or (3.14) hold for the solution of the difference problem.

4. NUMERICAL RESULTS

By theory [19, 20] and by numerical experiments [9] it is shown that, for discontinuous initial conditions, classical difference schemes do not converge in the ℓ^∞ -norm everywhere on the set $\overline{G}_h \setminus S^*$, even for a fixed value of ε . Neither do they converge uniformly in ε in the neighbourhood of the interior layer, outside a neighbourhood of S^* . However, both the true solution $u(x, t)$ and the numerical approximation $z(x, t)$ are bounded, uniformly in ε and it may be the case that the error $\max_{\overline{G}_h} |z(x, t) - u(x, t)|$ is relatively small for the classical difference scheme. That would reduce the need for a special scheme.

On the other hand, the above theorem shows that the fitted scheme converges uniformly in ε on \overline{G}_h , but no indication is given about the value of the order constant M in (3.14). Moreover, the order of convergence is rather small. It might be possible that the error is rather large for any reasonable value of h or τ . This might also reduce the practical value of our fitted scheme. To decide on the practical value of the new scheme numerical experiments are necessary to provide a more detailed comparison.

4.1. The model problem

To see the effect of the fitted scheme in practice, for the approximation of the model problem for a singularly perturbed heat equation with a discontinuous initial condition

$$\begin{aligned} L_{(4.1)}u(x, t) &\equiv \varepsilon^2 \frac{\partial^2}{\partial x^2} u(x, t) - \frac{\partial}{\partial t} u(x, t) = 0, & (x, t) \in G, \\ u(x, t) &= \phi(x, t), & (x, t) \in S, \end{aligned} \quad (4.1)$$

we compare the numerical results for the classical scheme (3.3) and the fitted scheme (3.4, 3.10). For problem (4.1) we have

$$v(x, t) = w_0(x, t; 1) + u_{(3.9)}(x, t),$$

so that the coefficient $\gamma(x, t)$ in (3.4) takes the form

$$\gamma(x, t) = \begin{cases} \frac{\delta_\tau w_0(x, t) - 6\varepsilon^2 x}{\varepsilon^2 \delta_{x\tau} w_0(x, t) - 6\varepsilon^2 x} & \text{for } (x, t) \in G_h, x \neq 0, \\ 1 & \text{for } (x, t) \in G_h, x = 0. \end{cases} \quad (4.2)$$

For $\varepsilon = 1/8$, $N = 32$, $N_0 = 40$ the solution of the model problem (4.1), with

$$\phi(x, t) = \frac{5}{2}w_0(x, t) + u_2(x, t), \quad (x, t) \in S, \quad (4.3)$$

$$u_2(x, t) = -(x + 0.5)^2 - 2\varepsilon^2 t, \quad (4.4)$$

for which we have the representation

$$u(x, t) = u_2(x, t) + \frac{5}{2}w_0(x, t), \quad (x, t) \in \overline{G} \setminus S^*, \quad (4.5)$$

is shown in Figure 1. The fitting coefficient (4.2) is shown in Figure 3. We can see that the solution has a jump at S^* for $t = 0$, and for $t > 0$ it is smooth. The space derivatives of the solution are large in the neighbourhood of the interior layer. The fitted coefficient varies strongly in the neighbourhood of the set S^* and becomes almost constant (equal to 1) away from S^* .

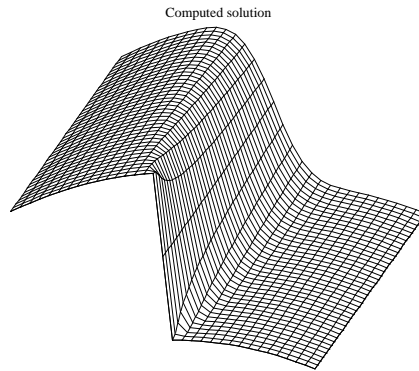


FIGURE 2. Computed solution with the fitted scheme.

The solution of problem (4.1,4.3) with $u_{(4.1)}(x, t) = \frac{5}{2}w_0(x, t) + u_2(x, t)$;
 $\varepsilon = 1/8$; $N = 32$; $N_0 = 40$.

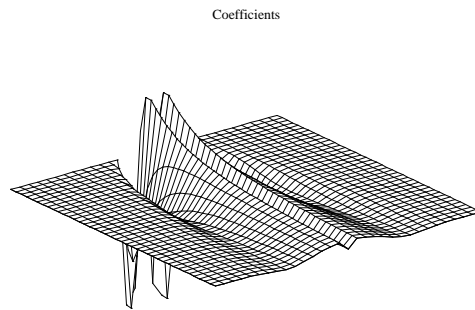


FIGURE 3. Coefficients $\gamma(x, t)$ in the fitted scheme.

Scheme (3.10), for the same problem as used in Figure 1.

4.2. Results with the classical difference approximation for the model problem

We show the behaviour of the classical difference scheme (3.3), central in x and backward in t , for the model problem (4.1,4.3). We know that this scheme converges for a fixed parameter ε on each smooth part of the solution of (4.1,4.3). Therefore we are primarily interested in the singular part of the solution for problem (4.1,4.3). Hence, we select the boundary conditions such that $u(x, t) = w_0(x, t)$,

$$\phi(x, t) = w_0(x, t), \quad (x, t) \in S. \quad (4.6)$$

For the approximation of problem (4.1,4.6) we use the classical scheme (3.3). We solve the problem for different values of the mesh, $h = 2/N$, and the time step, $\tau = 1/N_0$, and for different values of the small parameter ε . The results for a set numerical experiments are summarised in Table 16 and Table 17.

TABLE 16. Table of errors $E(N, N_0, \varepsilon)$ for the classical scheme.

N_0		N					
		8	16	32	64	128	256
10	$\varepsilon = 1$	5.76(-2)	6.08(-2)	6.16(-2)	6.25(-2)	6.26(-2)	6.26(-2)
40		2.48(-2)	5.69(-2)	6.01(-2)	6.10(-2)	6.20(-2)	6.20(-2)
160		2.93(-2)	2.47(-2)	5.69(-2)	6.01(-2)	6.10(-2)	6.20(-2)
640		3.18(-2)	2.93(-2)	2.47(-2)	5.69(-2)	6.01(-2)	6.10(-2)
10	$\varepsilon = 1/8$	3.18(-2)	2.93(-2)	2.47(-2)	5.69(-2)	6.01(-2)	6.10(-2)
40		3.27(-2)	3.18(-2)	2.93(-2)	2.47(-2)	5.69(-2)	6.01(-2)
160		3.29(-2)	3.27(-2)	3.18(-2)	2.93(-2)	2.47(-2)	5.69(-2)
640		3.29(-2)	3.29(-2)	3.27(-2)	3.18(-2)	2.93(-2)	2.47(-2)
	ε						
$N_0 = 40$	1	2.48(-2)	5.69(-2)	6.01(-2)	6.10(-2)	6.20(-2)	6.20(-2)
	0.5	2.93(-2)	2.47(-2)	5.69(-2)	6.01(-2)	6.10(-2)	6.20(-2)
	2^{-2}	3.18(-2)	2.93(-2)	2.47(-2)	5.69(-2)	6.01(-2)	6.10(-2)
	2^{-3}	3.27(-2)	3.18(-2)	2.93(-2)	2.47(-2)	5.69(-2)	6.01(-2)
	2^{-4}	2.70(-2)	3.27(-2)	3.18(-2)	2.93(-2)	2.47(-2)	5.69(-2)
	2^{-5}	7.69(-3)	2.70(-2)	3.27(-2)	3.18(-2)	2.93(-2)	2.47(-2)
	2^{-6}	1.95(-3)	7.69(-3)	2.70(-2)	3.27(-2)	3.18(-2)	2.93(-2)
	2^{-7}	4.88(-4)	1.95(-3)	7.69(-3)	2.70(-2)	3.27(-2)	3.18(-2)
	2^{-8}	1.22(-4)	4.88(-4)	1.95(-3)	7.69(-3)	2.70(-2)	3.27(-2)
	2^{-9}	3.05(-5)	1.22(-4)	4.88(-4)	1.95(-3)	7.69(-3)	2.70(-2)

In this table $E(N, N_0, \varepsilon) = \max_{(x,t) \in \overline{G}_h} |e(x, t; N, N_0, \varepsilon)|$, $e(x, t; N, N_0, \varepsilon) = z(x, t) - w_0(x, t)$ with $h = 2/N$ and $\tau = 1/N_0$.

The solution w_0 is as defined in (2.5) with $p_1 = 1$.

We notice that asymptotically for larger N or N_0 and smaller ε , the ℓ^∞ -norm of the error does not depend on ε , N and N_0 independently, but behaves as depending on a single parameter $N_0\varepsilon^{-2}$ or $N\varepsilon^{-1}$ for Table 16, and $N_0\varepsilon^{-2}$ or N for Table 17. Note that $|w_0(x, t)| \leq 0.5$. From Table 16 we see that for no value of the parameter ε we can guarantee the error on \overline{G} to be less than 12% for any sufficiently large N , N_0 :

$$\eta_1(K, \varepsilon) = \max_{N, N_0 \geq K} \{ [\max_{(x,t) \in \overline{G}} |w_0(x, t)|]^{-1} E(N, N_0, \varepsilon) \} \geq 12\%$$

when K is sufficiently large. From the results in Table 17 we see that for no values of N_0 , N we can guarantee the error on \overline{G} , $t \geq 0.2$ to be less than 6% for $\varepsilon \in (0, 1]$:

$$\eta_2(N, N_0) = \max_{\varepsilon} \{ [\max_{(x,t) \in \overline{G}, t \geq 0.2} |w_0(x, t)|]^{-1} E(N, N_0, \varepsilon) \} \geq 6\%.$$

TABLE 17. Table of errors $E_{0.2}(N, N_0, \varepsilon)$ for the classical scheme.

N_0		N					
		8	16	32	64	128	256
10	$\varepsilon = 1$	3.08(-2)	3.39(-2)	3.40(-2)	3.40(-2)	3.40(-2)	3.40(-2)
40		1.01(-2)	9.37(-3)	9.28(-3)	9.22(-3)	9.21(-3)	9.21(-3)
160		3.77(-3)	2.73(-3)	2.45(-3)	2.38(-3)	2.37(-3)	2.36(-3)
640		2.12(-3)	9.97(-4)	6.98(-4)	6.22(-4)	6.02(-4)	5.98(-4)
10	$\varepsilon = 1/8$	3.18(-2)	2.05(-2)	2.47(-2)	3.01(-2)	3.32(-2)	3.33(-2)
40		3.27(-2)	2.45(-2)	7.67(-3)	8.62(-3)	8.59(-3)	8.65(-3)
160		3.29(-2)	2.56(-2)	7.40(-3)	2.59(-3)	2.29(-3)	2.20(-3)
640		3.29(-2)	2.50(-2)	7.57(-3)	2.17(-3)	7.35(-4)	5.89(-4)
	ε						
$N_0 = 40$	1	1.01(-2)	9.37(-3)	9.28(-3)	9.22(-3)	9.21(-3)	9.21(-3)
	0.5	7.67(-3)	8.62(-3)	8.59(-3)	8.65(-3)	8.62(-3)	8.61(-3)
	2^{-2}	2.45(-2)	7.67(-3)	8.62(-3)	8.59(-3)	8.65(-3)	8.62(-3)
	2^{-3}	3.27(-2)	2.45(-2)	7.67(-3)	8.62(-3)	8.59(-3)	8.65(-3)
	2^{-4}	2.66(-2)	3.27(-2)	2.45(-2)	7.67(-3)	8.62(-3)	8.59(-3)
	2^{-5}	7.50(-3)	2.66(-2)	3.27(-2)	2.45(-2)	7.67(-3)	8.62(-3)
	2^{-6}	1.90(-3)	7.50(-3)	2.66(-2)	3.27(-2)	2.45(-2)	7.67(-3)
	2^{-7}	4.76(-4)	1.90(-3)	7.50(-3)	2.66(-2)	3.27(-2)	2.45(-2)
	2^{-8}	1.19(-4)	4.76(-4)	1.90(-3)	7.50(-3)	2.66(-2)	3.27(-2)
	2^{-9}	2.98(-5)	1.19(-4)	4.76(-4)	1.90(-3)	7.50(-3)	2.66(-2)

In this table $E_{0.2}(N, N_0, \varepsilon) = \max_{(x,t) \in \overline{G}_h, x \geq 0.2} |e(x, t; N, N_0, \varepsilon)|$, $e(x, t; N, N_0, \varepsilon) = z(x, t) - w_0(x, t)$ with $h = 2/N$ and $\tau = 1/N_0$.

The solution w_0 is as defined in (2.5) with $p_1 = 1$.

Thus, the computations also confirm that: (i) the classical scheme converges on the set \overline{G}_h with $t \geq t_0 > 0$ for a fixed value of ε ; (ii) on $\overline{G} \setminus S^*$ the classical scheme does not converge for any fixed ε ; (iii) on the set \overline{G}_h with $t \geq t_0 > 0$ the scheme does not converge uniformly in ε .

4.3. A fitted difference approximation

Let us study the behaviour of the fitted scheme applied to model problem (4.1,4.3), where the function $u(x, t)$ is the sum of a smooth and a singular part

$$u(x, t) = u_2(x, t) + \frac{5}{2}w_0(x, t), \quad (x, t) \in \overline{G} \setminus S^*. \quad (4.7)$$

Because the problem is linear, we can study both parts of the error independently. First we consider the behaviour of the fitted scheme for the singular part, that is for the model problem with

$$\phi(x, t) = w_0(x, t), \quad (x, t) \in \overline{G} \setminus S^*, \quad (4.8)$$

as we did for the classical scheme. This initial function $w_0(x, t)$ is a representative example from the class of initial functions with a discontinuity. For problem (4.1,4.8) we have the solution

$$u(x, t) = w_0(x, t), \quad (x, t) \in \overline{G} \setminus S^*. \quad (4.9)$$

Then, considering the smooth part of the solution in the expression (4.7) we study problem (4.1) with

$$\phi(x, t) = u_2(x, t) = -(x + 0.5)^2 - 2\varepsilon^2 t, \quad (x, t) \in \overline{G}. \quad (4.10)$$

For problem (4.1,4.10), we have the solution

$$u(x, t) = u_2(x, t), \quad (x, t) \in \overline{G}. \quad (4.11)$$

TABLE 18. Table of errors $E(N, N_0, \varepsilon)$ for the new scheme.

N_0	ε	N					
		8	16	32	64	128	256
10	$\varepsilon = 1$	2.26(-2)	1.96(-2)	1.89(-2)	1.87(-2)	1.87(-2)	1.87(-2)
40		1.27(-2)	1.06(-2)	1.01(-2)	1.01(-2)	1.00(-2)	1.00(-2)
160		7.74(-3)	5.30(-3)	4.30(-3)	4.16(-3)	4.08(-3)	4.07(-3)
640		6.13(-3)	3.01(-3)	1.79(-3)	1.43(-3)	1.34(-3)	1.31(-3)
10	$\varepsilon = 1/8$	5.46(-3)	3.01(-3)	1.79(-3)	1.43(-3)	1.34(-3)	1.31(-3)
40		5.56(-3)	2.30(-3)	9.47(-4)	5.28(-4)	4.17(-4)	3.88(-4)
160		5.57(-3)	2.12(-3)	7.00(-4)	2.64(-4)	1.44(-4)	1.12(-4)
640		5.58(-3)	2.07(-3)	6.36(-4)	1.92(-4)	6.90(-5)	3.66(-5)
	ε						
$N_0 = 40$	1	1.27(-2)	1.07(-2)	1.01(-2)	1.01(-2)	1.00(-2)	1.00(-2)
	0.5	7.74(-3)	5.30(-3)	4.30(-3)	4.16(-3)	4.08(-3)	4.07(-3)
	2^{-2}	6.13(-3)	3.01(-3)	1.79(-3)	1.43(-3)	1.34(-3)	1.31(-3)
	2^{-3}	5.56(-3)	2.30(-3)	9.47(-4)	5.28(-4)	4.17(-4)	3.88(-3)
	2^{-4}	4.48(-3)	1.70(-3)	6.56(-4)	2.60(-4)	1.44(-4)	1.12(-4)
	2^{-5}	1.23(-3)	6.55(-4)	3.69(-4)	1.46(-4)	5.90(-5)	3.27(-5)
	2^{-6}	3.08(-4)	1.79(-4)	8.37(-5)	5.67(-5)	2.63(-5)	1.12(-5)
	2^{-7}	7.71(-5)	4.47(-5)	2.28(-5)	1.05(-5)	7.77(-6)	4.25(-6)
	2^{-8}	1.93(-5)	1.12(-5)	5.71(-6)	2.86(-6)	1.34(-6)	1.15(-6)
	2^{-9}	4.82(-6)	2.80(-6)	1.43(-6)	7.15(-7)	3.58(-7)	2.09(-7)

In this table the scheme (3.4) is used to solve a problem (4.1,4.8) with an interior layer. $E(N, N_0, \varepsilon) = \max_{(x,t) \in \overline{G}_h} |e(x, t; N, N_0, \varepsilon)|$, $e(x, t; N, N_0, \varepsilon) = z(x, t) - w_0(x, t)$ with $h = 2/N$ and $\tau = 1/N_0$; the solution w_0 is as defined in (2.5) with $p_0 = 1$.

TABLE 19. Table of errors $E(N, N_0, \varepsilon)$.

N_0	ε	N					
		8	16	32	64	128	256
10	$\varepsilon = 1$	5.10(-2)	8.72(-2)	1.16(-1)	1.36(-1)	1.47(-1)	1.53(-1)
40		1.46(-2)	2.27(-2)	3.15(-2)	3.89(-2)	4.50(-2)	4.87(-2)
160		7.19(-3)	5.87(-3)	7.00(-3)	8.44(-3)	9.83(-3)	1.10(-2)
640		7.32(-3)	4.05(-3)	2.74(-3)	2.22(-3)	2.08(-3)	2.32(-3)
10	$\varepsilon = 1/8$	7.32(-3)	4.05(-3)	2.74(-3)	2.22(-3)	2.08(-3)	2.32(-3)
40		7.44(-3)	3.17(-3)	1.64(-3)	1.03(-3)	8.52(-4)	8.00(-4)
160		7.46(-3)	2.94(-3)	1.25(-3)	6.06(-4)	3.74(-4)	3.04(-4)
640		7.46(-3)	2.89(-3)	1.15(-3)	4.63(-4)	2.09(-4)	2.55(-4)
	ε						
$N_0 = 40$	1	1.46(-2)	2.27(-2)	3.14(-2)	3.89(-2)	4.50(-2)	4.87(-2)
	0.5	7.19(-3)	5.87(-3)	7.00(-3)	8.44(-3)	9.83(-3)	1.10(-2)
	2^{-2}	7.32(-3)	4.05(-3)	2.74(-3)	2.22(-3)	2.08(-3)	2.32(-3)
	2^{-3}	7.44(-3)	3.17(-3)	1.64(-3)	1.03(-3)	8.52(-4)	8.00(-4)
	2^{-4}	5.98(-3)	2.39(-3)	1.25(-3)	6.06(-4)	3.74(-4)	3.04(-4)
	2^{-5}	1.64(-3)	1.75(-3)	1.00(-3)	4.59(-4)	2.09(-4)	1.26(-4)
	2^{-6}	4.11(-4)	4.77(-4)	4.47(-4)	3.04(-4)	1.47(-4)	6.90(-5)
	2^{-7}	1.03(-4)	1.19(-4)	1.22(-4)	1.12(-4)	8.30(-5)	4.45(-5)
	2^{-8}	2.57(-5)	2.98(-5)	3.04(-5)	3.05(-5)	2.80(-5)	2.17(-5)
	2^{-9}	6.42(-6)	7.45(-6)	7.61(-6)	7.63(-6)	7.63(-6)	7.01(-6)

In this table the scheme (3.4) is used to solve a problem (4.1,4.10) with a smooth solution. In this table $E(N, N_0, \varepsilon) = \max_{(x,t) \in \overline{G}_h} |e(x, t; N, N_0, \varepsilon)|$, $e(x, t; N, N_0, \varepsilon) = z(x, t) - u_2(x, t)$ with $h = 2/N$ and $\tau = 1/N_0$; the solution u_1 is as defined in (4.9).

The results of these numerical experiments are given in Tables 18 and 19.

From the results in Tables 18 and 19 we see that the errors for singular and regular parts, $w_0(x, t)$ and $u_2(x, t)$ respectively, decrease for N, N_0 large enough, and a fixed value of the parameter $\varepsilon = 2^{-K}$, $K = 0, 1, \dots$. Also the errors decrease with increasing \overline{N} uniformly in ε . The relative error is less than 1% for $N \geq 8, N_0 \geq 160, \varepsilon = 2^{-K}, K \geq 0$ when $u(x, t) = w_0(x, t)$. The relative error is also less than 1% for the same parameters when $u(x, t) = u_2(x, t)$.

The functions $\frac{5}{2}w_0(x, t)$ and $u_2(x, t)$ are components of the solution of the problem (4.1), (4.3). Thus we have: (i) for the model problem (4.1), (4.3) the numerical scheme converges for a fixed ε in the discrete ℓ^∞ -norm on \overline{G}_h ; (ii) we observe ε -uniform convergence for the model problem (4.1, 4.3); (iii) the relative error for the model problem is less than 2% for N, N_0 sufficiently large.

4.4. The error analysis for the fitted difference scheme

To determine the quality of the convergence, using the data from the Tables 18 and 19 we can examine the experimental order of convergence of the fitted scheme.

When we use the classical scheme (3.3) for problem (4.1,4.11) then, according to the classical theory, we typically find an estimate of the form

$$\max_{\overline{G}_h \setminus S^*} |u_2(x, t) - z_{(3.3)}(x, t)| \leq Q(\varepsilon)(h^2 + \tau), \quad (x, t) \in \overline{G}_h. \quad (4.12)$$

This estimate means that the function $z_{(3.3)}(x, t)$ converges to the function $u_2(x, t)$ for each fixed value of ε . The constant $Q(\varepsilon)$ tends to infinity for $\varepsilon \rightarrow 0$.

From theory we know that the solution of the fitted difference scheme (3.4, 3.10) $z(x, t)$ converges ε -uniformly to the solution of problem (4.1,4.11). To investigate the ε -uniform convergence of a function $z(x, t) = z(x, t; \varepsilon, h, \tau)$, it is natural to express an error estimate in the form

$$\max_{\varepsilon} \max_{\overline{G}_h \setminus S^*} |u(x, t, \varepsilon) - z(x, t; \varepsilon, h, \tau)| \leq M(h^2 + \tau)^\nu, \quad (4.13)$$

where ν does not depend on the parameters ε, h or τ . To compute ν we shall use an inequality of the form

$$\max_{\overline{G}_h \setminus S^*} |u(x, t, \varepsilon) - z(x, t; \varepsilon, h, \tau)| \leq M(h^2 + \tau)^{\nu(\varepsilon)}. \quad (4.14)$$

We call $\nu(\varepsilon)$ in expression (4.14) the *generalised order of convergence for a fixed value of the parameter ε* , and ν in expression (4.13) the *generalised order of ε -uniform convergence*.

We determine the experimental generalised order *at the point* (N, N_0) by

$$\overline{\nu}(N, N_0, \varepsilon) = (\ln E(N, N_0, \varepsilon) - \ln E(2N, 4N_0, \varepsilon)) / \ln 4, \quad (4.15)$$

where $E(N, N_0, \varepsilon) = \max_{\overline{G}_h \setminus S^*} |u(x, t, \varepsilon) - z(x, t; \varepsilon, h, \tau)|$, $hN = 2$ and $\tau N_0 = 1$. We introduce the *experimental generalised order of convergence* for fixed ε as

$$\overline{\nu}(\varepsilon) = \min_{N, N_0} \overline{\nu}(N, N_0, \varepsilon), \quad (4.16)$$

and the *experimental generalised order of ε -uniform convergence* as

$$\overline{\nu} = \min_{\varepsilon} \overline{\nu}(\varepsilon). \quad (4.17)$$

Similarly the the *experimental ε -uniform generalised order at the point* (N, N_0) is

$$\overline{\nu}(N, N_0) = \min_{\varepsilon} \overline{\nu}(N, N_0, \varepsilon). \quad (4.18)$$

The results are given in the Tables 21 and 20.

TABLE 20. Experimental generalised order of convergence $\bar{\nu}(N, N_0, \varepsilon)$.

N_0		N				
		8	16	32	64	128
10	$\varepsilon = 1$	0.544	0.479	0.454	0.450	0.450
40		0.631	0.653	0.640	0.650	0.651
160		0.681	0.782	0.792	0.818	0.818
10	$\varepsilon = 1/8$	0.625	0.834	0.882	0.891	0.892
40		0.696	0.858	0.922	0.938	0.945
160		0.714	0.868	0.932	0.967	0.987
	ε					
$N_0 = 40$	1	0.631	0.653	0.640	0.650	0.651
	0.5	0.681	0.782	0.792	0.818	0.818
	2^{-2}	0.708	0.834	0.882	0.891	0.892
	2^{-3}	0.854	0.904	0.934	0.939	0.946
	2^{-4}	1.387	1.104	1.084	1.070	1.067
	2^{-5}	1.391	1.484	1.351	1.237	1.195
	2^{-6}	1.392	1.485	1.498	1.433	1.314
	2^{-7}	1.393	1.485	1.498	1.486	1.378
	2^{-8}	1.393	1.485	1.498	1.500	1.340

The fitted scheme (3.4,3.10) for the problem (4.1,4.8), applied to the solution $u(x, t) = w_0(x, t)$ with the interior layer. $\bar{\nu}(N, N_0, \varepsilon) = (\ln E(N, N_0, \varepsilon) - \ln E(2N, 4N_0, \varepsilon))/\ln 4$, $E(N, N_0, \varepsilon)$ from Table 18.

TABLE 21. Experimental generalised order of convergence $\bar{\nu}(N, N_0, \varepsilon)$.

N_0	ε	N				
		8	16	32	64	128
10	$\varepsilon = 1$	0.583	0.736	0.789	0.795	0.798
40		0.656	0.850	0.949	0.992	1.016
160		0.413	0.551	0.828	1.012	1.041
10	$\varepsilon = 1/8$	0.604	0.652	0.702	0.691	0.687
40		0.668	0.669	0.719	0.733	0.744
160		0.685	0.676	0.718	0.769	0.276
	ε	N				
$N_0 = 40$	1	0.656	0.850	0.949	0.992	1.016
	0.5	0.413	0.551	0.828	1.012	1.041
	0.25	0.603	0.652	0.702	0.691	0.687
	0.125	0.820	0.669	0.719	0.733	0.744
	2^{-4}	0.887	0.627	0.724	0.769	0.783
	2^{-5}	0.891	0.984	0.859	0.820	0.799
	2^{-6}	0.892	0.985	0.998	0.938	0.863
	2^{-7}	0.893	0.985	0.998	0.999	0.966
	2^{-8}	0.893	0.985	0.998	1.000	1.000

$\bar{\nu}(N, N_0, \varepsilon) = (\ln E(N, N_0, \varepsilon) - \ln E(2N, 4N_0, \varepsilon))/\ln 4$, $E(N, N_0, \varepsilon)$ from Table 19.

Computation with the new scheme (3.4,3.10) for the smooth solution $u(x, t) = u_2(x, t)$.

From the results in the Tables 20 and 21 we see: (i) for $w_0(x, t)$ and $u_2(x, t)$ the experimental generalised order of ε -uniform convergence for the fitted scheme is approximately 0.413 and 0.450 respectively; (ii) for $N \geq 16$ and $N_0 \geq 40$ the generalised orders of ε -uniform convergence for $w_0(x, t)$ and $u_2(x, t)$ are apparently not less than 0.50. This means that in practice

$$\max_{\overline{G}_h} |u(x, t) - z(x, t)| \leq M(h + \tau^{1/2})$$

for $N \geq 16$ and $N_0 \geq 40$, $0 < \varepsilon \leq 1$, for each value of the parameter ε . In accordance with the theory, for each value of ε , the experimental generalised order of convergence tends to 1 for decreasing h and τ . Thus, the experimental generalised order of convergence for the fitted scheme (3.4, 3.10) for the full model problem (4.1, 4.3) is not less than predicted by the theory. The behaviour of the errors $e(x, t; N, N_0, \varepsilon) = z(x, t) - u(x, t)$ for the fitted scheme (3.4, 3.10) and for the classical scheme (3.3) are shown in the Figures 2 and 5. We can see that the largest errors are in the neighbourhood of the set S^* and that the errors for the classical scheme are significantly larger than for the fitted scheme.

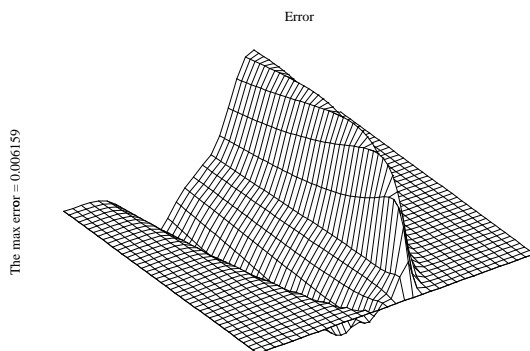


FIGURE 4. Discretisation error the fitted scheme.

Scheme (3.4,3.10) is used for the same problem as used in Figure 1.

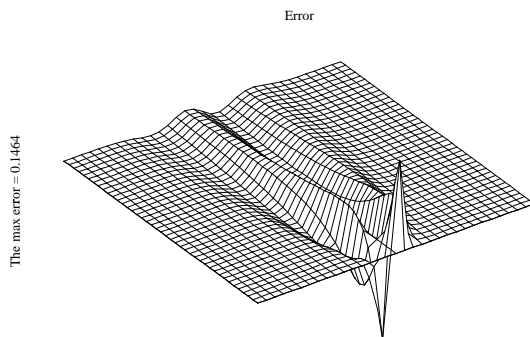


FIGURE 5. Discretisation error the classical scheme.

Scheme (3.3) is used for the same problem as used for Figure 1.

5. CONCLUSION

For a singularly perturbed boundary value problem of parabolic type with discontinuous initial condition (2.1), we have constructed a specially fitted difference scheme that converges in $\overline{G} \setminus S^*$ ε -uniformly in the ℓ^∞ -norm.

Numerical experiments for a model boundary value problem with discontinuous boundary function show that a classical difference scheme does not converge ε -uniformly. Moreover, for a fixed value of ε this scheme doesn't converge in the ℓ^∞ -norm in the neighbourhood of the discontinuity, and away from the discontinuity it does not converge ε -uniformly in the neighbourhood of the interior layer. In the case of the constant coefficient problem and a simple discontinuity, for which the error-function is the solution, an error of less than 6% on \overline{G} , $t \geq t_0 = 0.2$, and less than 12% on $\overline{G} \setminus S^*$ can not be guaranteed for arbitrarily small h or τ .

Theoretically and numerically it is also shown, that the fitted difference scheme converges ε -uniformly in the ℓ^∞ -norm on \overline{G}_h . Moreover in the case of the fitted scheme, for a model problem, an experimental generalised order of convergence of not less than 0.5 is observed if $h \leq 1/8$ and $\tau \leq 0.025$ e.g. $\nu(\varepsilon, N, N_0) \geq 0.5$ at $N \geq 16$, $N_0 \geq 40$. The experimental generalised order of convergence is substantially larger than the bound guaranteed by the theory. Both for the singular and for the regular part of the solution an error less than 1% is guaranteed for $N \geq 8$, $N_0 > 40$ and for any $\varepsilon \in (0, 1]$.

REFERENCES

- [1] M.V. Alexeevski. On difference scheme for a singularly perturbed parabolic equation. *Uchenye Zapiski (Scientific Papers), Natural Sciences*, 1:3–13, 1980. (in Russian).
- [2] D.N. Allen and R.V. Southwell. Relaxation method applied to determine the motion, in two dimensions, of viscous fluid past a fixed cylinder. *Quart. J. Mech. Appl. Math.*, 8:129–145, 1955.
- [3] N.S. Bakhvalov. On optimization of methods to solve boundary value problems with boundary layers. *Zh. Vychisl. Mat. Mat. Fiz.*, 9:841–859, 1969. (in Russian).
- [4] E.P. Doolan, J.J.H. Miller, and W.H.A. Schilders. *Uniform Numerical Methods for Problems with Initial and Boundary Layers*. Boole Press, Dublin, 1980.
- [5] E.V. Ametistov V.A. Grigor'ev B.T. Efimtsev, editor. *Heat and Mass Transfer. Heat-Technology Experiment. Handbook*. Energoizdat, Moscow, 1982. (in Russian).
- [6] K.V. Emel'yanov. Difference scheme for a three-dimensional elliptic equation with small parameters at highest-order derivatives. *Kraevye Zadachi dlya Uravn. Mat. Fiz. (Boundary Value Problems for Equations of Mathematical physics)*, pages 30–42, 1973. (in Russian).
- [7] A.F. Hegarty, J.J.H. Miller, E. O'Riordan, and G.I. Shishkin. Special meshes for two dimensional elliptic singular perturbation problems. In R.Vichnevetsky and J.J.H.Miller, editors, *IMACS'91: Proc. 13-th IMACS World Congress on Computation and Applied Mathematics*, pages 503–504, Dublin, 1991. Criterion Press.
- [8] P.W. Hemker and G.I. Shishkin. Discrete approximation of singularly perturbed parabolic PDEs with a discontinuous initial condition. In J.J.H. Miller, editor, *BAIL VI Proceedings*, pages 3–4. Boole Press, 1992.
- [9] P.W. Hemker and G.I. Shishkin. Approximation of parabolic pdes with a discontinuous initial condition. *East-West Journal of Numerical mathematics*, 1(4):287–302, 1993.
- [10] P.W. Hemker and G.I. Shishkin. Discrete approximation of singularly perturbed parabolic pdes with a discontinuous initial condition. *Computational Fluid Dynamics Journal*, 2:375–392, 1994.

- [11] P.W. Hemker and G.I. Shishkin. On a class of singularly perturbed boundary value problems for which an adaptive mesh technique is necessary. In D. Bainov and V. Covachev, editors, *Proceedings of the Second International Colloquium on Numerical Analysis*, pages 83–92. VSP, International Science Publishers, 1994. Proceedings of the “Second International Colloquium on Numerical Analysis”, August 13-17, 1993.
- [12] A.M. Il’in. Difference scheme for a differential equation with a small parameter at the highest-order. *Mat. Zametki*, 6:237–248, 1969. (in Russian).
- [13] O.A. Ladyzhenskaya and N.N. Ural’tseva. *Linear and Quasi-linear Equations of Elliptic Type*. Nauka, Moscow, 1973. (in Russian).
- [14] V.D. Liseikin. On numerical solution of two-dimensional elliptic equation with a small parameter at highest-order derivatives. *Chislennyye Metody Mekhaniki Sploshnoy Sredy (Numerical Methods of Continuum Mechanics)*, 14:110–115, 1983. (in Russian).
- [15] A. V. Lykov. *Heat and Mass Transfer*. Energia, 1978. (In Russian).
- [16] G.I. Marchuk. *Methods of Numerical Mathematics*. Springer, New York, 1982.
- [17] A.A. Samarski. *Theory of Difference Scheme*. Nauka, Moscow, 1989. (in Russian).
- [18] A.A. Samarski and E.S. Nikolaev. *Methods of solving a grids equations*. Nauka, Moscow, 1978. (in Russian).
- [19] G. I. Shishkin. A difference scheme for a singularly perturbed equation of parabolic type with discontinuous boundary conditions. *J. Vychisl. Mat. i Mat. Fiz.*, 28:1649–1662, 1988. (In Russian).
- [20] G. I. Shishkin. A difference scheme for a singularly perturbed equation of parabolic type with discontinuous initial conditions. *Dokl. Akad. Nauk.*, 300:1066–1070, 1988. (In Russian).
- [21] G.I. Shishkin. An increase in accuracy of solutions to difference scheme for parabolic equations with small parameters at highest-order derivatives. *Zh. Vychisl. Mat. Mat. Fiz.*, 24:864–875, 1984. (in Russian).
- [22] G.I. Shishkin. Approximation of solutions to singularly perturbed boundary value problems with corner boundary layers. *Doklady Akad. Nauk SSSR.*, 296:39–43, 1987. (in Russian).
- [23] G.I. Shishkin. Approximation of solutions to singularly perturbed boundary value problems with corner boundary layers. *Zh. Vychisl. Mat. Mat. Fiz.*, 27:1360–1374, 1987. (in Russian).
- [24] G.I. Shishkin. Approximation of solutions to singularly perturbed boundary value problems with parabolic boundary layers. *Zh. Vychisl. Mat. Mat. Fiz.*, 29:963–977, 1989. (in Russian).
- [25] G.I. Shishkin. Grid approximation of singularly perturbed elliptic equation in domain with characteristic bounds. *Sov. J. Numer. Anal. Math. Modelling.*, 5:327–343, 1990.
- [26] G.I. Shishkin. Grid approximation of singularly perturbed boundary value problem for quasi-linear parabolic equations in the case of complete degeneracy. *J. Vychisl. Mat. i Mat. Fiz.*, 31:1808–1826, 1991. (in Russian).
- [27] G.I. Shishkin. Grid approximation of singularly perturbed boundary value problem or quasi-linear parabolic equations in the case of complete degeneracy in spatial variables. *Sov. J. Numer. Anal. Math. Modelling.*, 6:243–261, 1991.
- [28] G.I. Shishkin. *Grid Approximation of Singularly Perturbed Elliptic and Parabolic Equations*. Ur. O. RAN, Ekaterinburg, 1992. (in Russian).

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