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REPORT*RAPPORT*

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in terms of hyperbolic functions

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Uniform Approximations of Bernoulli and Euler Polynomials in Terms of Hyperbolic Functions

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ABSTRACT

Bernoulli and Euler polynomials are considered for large values of the order. Convergent expansions are obtained for $B_n(nz + 1/2)$ and $E_n(nz + 1/2)$ in powers of n^{-1}, with coefficients being rational functions of z and hyperbolic functions of argument $1/2z$. These expansions are uniformly valid for $|z \pm i/2\pi| > 1/2\pi$ and $|z \pm i/\pi| > 1/\pi$, respectively. For real argument, accuracy of these approximations is restricted to the monotonic region. The range of validity of the uniformity parameter z is enlarged, respectively, to regions of the form $|z \pm i/2(m+1)\pi| > 1/2(m+1)\pi$ and $|z \pm i/(2m+1)\pi| > 1/(2m+1)\pi$, $m = 1, 2, 3, ..$, by adding certain combinations of incomplete gamma functions to those uniform expansions. In addition, the convergence of these improved expansions is stronger and also for real argument the accuracy of these improved approximations is better in the oscillation region.

1. Introduction

The Bernoulli and Euler polynomials of degree n and complex argument z, denoted by $B_n(z)$ and $E_n(z)$, respectively, are defined by [[1], eq. 23.1.1.],

$$\frac{we^{wz}}{e^w - 1} = \sum_{n=0}^{\infty} \frac{B_n(z)}{n!} w^n, \qquad |w| < 2\pi, \qquad (1)$$

$$\frac{2e^{wz}}{e^w + 1} = \sum_{n=0}^{\infty} \frac{E_n(z)}{n!} w^n, \qquad |w| < \pi. \qquad (2)$$

Standard texts on the classical theory of Bernoulli and Euler polynomials and numbers are, for example, chapter VI in Milne-Thomson [13] and chapters V and VI in Jordan [9]. A very complete bibliography up to 1960 concerning tables and applications of the theory of Bernoulli and Euler polynomials can be found in Fletcher et al. [6]. Fore more recent works (after 1960) the reader is referred to Weinmann [20], Todorov [19], Dilcher [3]-[4], Leeming [10], Haruki et al. [8] and Cvijovic et al. [2] and references therein for a very complete survey on formulas involving these polynomials, zeroes, asymptotic behaviour, integral representations and a number of other properties. Nevertheless, an extensive list of formulae involving Bernoulli and Euler polynomials can be found in Erdlyi et al. [[5], pp. $35 - 43$], Magnus et al. [[11], pp. $25 - 32$], Abramowitz et al. [[1], pp. $803 - 806$], Gradshteyn et al. [[7], pp. $1076 - 1080$], Prudnikov et al. [[12] vol 3, $55 - 57$] and Temme [[18], pp. $2 - 17$].

Here we are interested in approximations of these polynomials for large order n. Convergent expansions in terms of trigonometric functions for real z and $0 \leq z \leq 1$ can be found in [[1], $23.1.16 - 23.1.18$]. A more detailed study about these expansions concerning the type of convergence is investigated in [3], where similar expansions for generalized Bernoulli polynomials are obtained as well. In ref. [20], asymptotic expansions of (generalized) Bernoulli polynomials are obtained in terms of elementary functions as well as in terms of gamma functions. These expansions happen to fail when the argument z is let to grow arbitrarily. Here we are concerned with finding approximations of the Bernoulli and Euler polynomials for large order n that remain also valid for large argument z. In particular, the purpose of this paper is to obtain approximations of $B_n(nz + 1/2)$ and $E_n(nz + 1/2)$ for large n which are uniformly valid in some unbounded region of the complex variable z.

1.1. Summary of the results

Our main results are summarized as follows.

1. For $\left| z \pm \frac{i}{2\pi} \right| > \frac{1}{2\pi}$, $n \geq 1$,

$$B_n\left(nz + \frac{1}{2}\right) = \frac{n^n z^{n-1}}{2\sinh(1/2z)} \left\{ 1 + \right.$$
$$\left[1 + 4\left(z - \frac{1}{2}\coth\left(\frac{1}{2z}\right)\right)\coth\left(\frac{1}{2z}\right)\right] \frac{1}{8nz^2} + \mathcal{O}\left(\frac{1}{n^2}\right) \right\}.$$

2. For $\left| z \pm \frac{i}{\pi} \right| > \frac{1}{\pi}$, $n \geq 1$,

$$E_n\left(nz + \frac{1}{2}\right) = \frac{(nz)^n}{\cosh(1/2z)} \left[1 + \left(1 - 2\tanh^2\left(\frac{1}{2z}\right)\right)\frac{1}{8nz^2} + \mathcal{O}\left(\frac{1}{n^2}\right)\right].$$

3. For $\left| z \pm \frac{i}{4\pi} \right| > \frac{1}{4\pi}$, $n \geq 1$,

$$B_n\left(nz + \frac{1}{2}\right) = \left[\frac{e^{2\pi i n z}}{(2\pi i)^n} \Gamma(n+1, 2\pi i n z) + \frac{e^{-2\pi i n z}}{(-2\pi i)^n} \Gamma(n+1, -2\pi i n z) \right] +$$
$$(nz)^n \left\{ \left[\frac{1}{2z \sinh(1/2z)} - \frac{8\pi^2 z^2}{1 + 4\pi^2 z^2} \right] + \right.$$
$$\left[\frac{1}{16z^3 \sinh(1/2z)} \left(1 + 4\left(z - \frac{1}{2} \coth\left(\frac{1}{2z}\right) \right) \right) \coth\left(\frac{1}{2z}\right) \right) + $$
$$\left. \frac{8\pi^2 z^2 (3 - 4\pi^2 z^2)}{(1 + 4\pi^2 z^2)^3} \right] \frac{1}{n} + \mathcal{O}\left(\frac{1}{n^2}\right) \right\}.$$

4. For $\left| z \pm \frac{i}{3\pi} \right| > \frac{1}{3\pi}$, $n \geq 1$,

$$E_n\left(nz + \frac{1}{2}\right) = 2i \left[\frac{e^{\pi i n z}}{(\pi i)^{n+1}} \Gamma(n+1, \pi i n z) - \frac{e^{-\pi i n z}}{(-\pi i)^{n+1}} \Gamma(n+1, -\pi i n z) \right] +$$
$$(nz)^n \left\{ \left[\frac{1}{\cosh(1/2z)} - \frac{4\pi z^2}{1 + \pi^2 z^2} \right] + \left[\frac{4\pi z^2 (3 - \pi^2 z^2)}{(1 + \pi^2 z^2)^3} + \right. \right.$$
$$\left. \left. \frac{1}{8z^2 \cosh(1/2z)} \left(1 - 2 \tanh^2\left(\frac{1}{2z}\right) \right) \right] \frac{1}{n} + \mathcal{O}\left(\frac{1}{n^2}\right) \right\}.$$

For the given z−domains, the above approximations have the indicated asymptotic properties and, in addition, they are the first terms of convergent expansions.

2. Uniform expansions and saddle point method

From the definitions (1) and (2) we have,

$$B_n\left(z + \frac{1}{2}\right) = \frac{n!}{2\pi i} \int_C \frac{w e^{wz}}{2 \sinh(w/2)} \frac{dw}{w^{n+1}} \tag{3}$$

and

$$E_n\left(z + \frac{1}{2}\right) = \frac{n!}{2\pi i} \int_C \frac{e^{wz}}{\cosh(w/2)} \frac{dw}{w^{n+1}}, \tag{4}$$

where the contour C encircles the origin in the counterclockwise direction and contains no poles of $w \sinh^{-1}(w/2)$ or $\cosh^{-1}(w/2)$, respectively. The shift $1/2$ in the variable z is introduced in order to have reflection symmetry $z \to -z$ in these polynomials. In order to deal with both $B_n(z)$ and $E_n(z)$ simultaneously, we will start this section with generic polynomials defined by the contour integral

$$P_n(z) = \frac{n!}{2\pi i} \int_C f(w) e^{wz} \frac{dw}{w^{n+1}}, \tag{5}$$

where $f(w)$ is a meromorphic function with simple poles $w_1, w_2, \ldots$ and analytic at the origin. The contour C is a circle with center the origin and which contains no poles of

$f(w)$ inside. Bernoulli and Euler polynomials defined by (3) and (4) are nothing but particular cases.

We can write (5) in the form

$$P_n(nz) = \frac{n!}{2\pi i} \int_C f(w) e^{n(wz - \log(w))} \frac{dw}{w},$$

The key observation used for obtaining approximations of $P_n(nz)$ for large n and fixed z is the following: the main contribution of the integrand above to the integral is originated in the saddle point of the argument of the exponential [21], that is, in the point $w = z^{-1}$. Approximations of $P_n(nz)$ for large n and fixed z can be obtained by expanding $f(w)$ around the saddle point [14]-[17]. Therefore, if z^{-1} is not a pole of $f(w)$, we expand

$$f(w) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z^{-1})}{k!}(w - z^{-1})^k, \qquad |w - z^{-1}| < r, \qquad (6)$$

where r is the distance from z^{-1} to the nearest singularity of $f(w)$. The radius ϵ_1 of the contour C in the definition (5) of $P_n(z)$ can be chosen as close to zero as necessary. Then, for $w \in C$ ($|w| = \epsilon_1$), the above series is absolutely convergent if $|z^{-1}| < |z^{-1} - w_k|$ $\forall\, k = 1, 2, \ldots$ (see Figure 1 (a) where the particular case $f(w) = w/2\sinh(w/2)$ is represented).

(a)　　　　　(b)
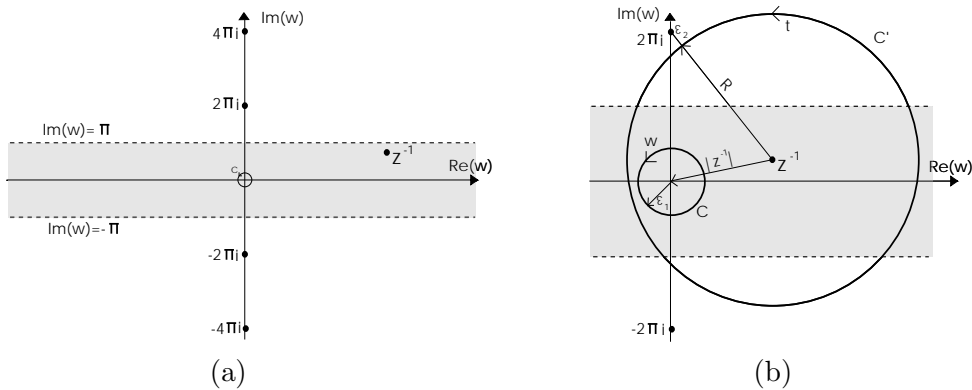
Figure 1. (a) *The meromorphic function $f(w) = w/2\sinh(w/2)$ has singularities $w_k = \pm 2k\pi i$, $k = 1, 2, \ldots$ For small ϵ_1, validity of (6) is guaranteed if the saddle point z^{-1} is closer to the origin than to any of the singularities w_k. This happens for z^{-1} inside a strip of width 2π along the real axis.* (b) *For z^{-1} inside that strip we can choose small enough ϵ_1 and $\epsilon_2 = |z^{-1} - 2\pi i| - R$ such that the circle C is inside the circle C'. Then, $a \equiv \max_{w \in C} |w - z^{-1}|/R < 1$ and (9) holds.*

Substituting expansion (6) into (5) we obtain

$$P_n(nz) = \frac{n!}{2\pi i} \int_C \sum_{k=0}^{\infty} \frac{f^{(k)}(z^{-1})}{k!} (w - z^{-1})^k e^{nwz} \frac{dw}{w^{n+1}}. \tag{7}$$

The $k-$th derivative of $f(w)$ evaluated in the point z^{-1} may be written

$$f^{(k)}(z^{-1}) = \frac{k!}{2\pi i} \int_{C'} \frac{f(t)dt}{(t - z^{-1})^{k+1}},$$

where the contour C' is a circle around the point z^{-1} with radius $R \equiv |t - z^{-1}|$ smaller than the distance from z^{-1} to any of the singularities w_k of $f(t)$. That is, $R \equiv \min_{k \in N} |w_k - z^{-1}| - \epsilon_2$ for some $\epsilon_2 > 0$ (see Figure 2 (b) where the particular case $w_k = \pm 2k\pi i$ is represented). Therefore,

$$|f^{(k)}(z^{-1})| \le K \frac{k!}{R^k}, \tag{8}$$

where K is not depending on k. Then, for $|z^{-1}| < |z^{-1} - w_k| \ \forall \ k = 1, 2, \ldots$ and small enough ϵ_1 and ϵ_2 we have $\max_{w \in C} |w - z^{-1}|/R \equiv a < 1$ for a certain $a > 0$ which only depends on z, ϵ_1 and R (see Figure 1 (b)). Taking the modulus in (7) and using (8) we obtain

$$|P_n(nz)| \le K_n \sum_{k=0}^{\infty} a^k \int_C \left| e^{nwz} \frac{dw}{w^{n+1}} \right| < \infty, \tag{9}$$

where K_n is a certain constant. Therefore, after substituting (6) into (5), we can apply Fubini's theorem and interchange sum and integral to obtain, for $|z^{-1}| < |z^{-1} - w_i| \ \forall \ i = 1, 2, 3, \ldots,$

$$P_n(nz) = (nz)^n \sum_{k=0}^{\infty} \frac{f^{(k)}(z^{-1})}{k!} \Phi_k(n, z), \tag{10}$$

where

$$\Phi_k(n, z) = \frac{1}{2\pi i} \frac{n!}{(nz)^n} \int_C (w - z^{-1})^k e^{nzw} \frac{dw}{w^{n+1}}. \tag{11}$$

The functions $\Phi_k(n, z)$ are polynomials in n divided by powers of z and constitute an asymptotic sequence for $n \to \infty$. More precisely,

Lemma 1. *The functions $\Phi_k(n, z)$ defined in (11) can be represented in the form*

$$\Phi_k(n, z) = \frac{p_k(n)}{(nz)^k}, \tag{12}$$

where

$$p_0(n) = 1, \qquad p_1(n) = 0, \qquad p_2(n) = -n, \qquad p_3(n) = 2n \tag{13}$$

and the remaining polynomials $p_k(n)$ are given by the recurrence

$$p_k(n) = (1 - k)(p_{k-1}(n) + np_{k-2}(n)). \tag{14}$$

Proof. By direct calculation from the definition (11) for $k = 0$ and $k = 1$ we obtain $\Phi_0(n, z) = 1$ and $\Phi_1(n, z) = 0$. For $k \geq 2$ we can write

$$\Phi_k(n, z) = \frac{1}{2\pi i} \frac{n!}{(nz)^{n+1}} \int_C (w - z^{-1})^{k-1} \frac{d}{dw} \left[e^{nz(w - z^{-1}\ln(w))} \right] dw.$$

Integrating by parts and after straightforward operations we obtain

$$\Phi_k(n, z) = \frac{1 - k}{nz} \left[z^{-1} \Phi_{k-2}(n, z) + \Phi_{k-1}(n, z) \right], \tag{15}$$

from which (14) follows trivially. $\diamond$

Lemma 2. *For fixed $z \neq 0$, the sequence $\Phi_k(n, z)$ is an asymptotic sequence for $n \to \infty$ that satisfies $\Phi_k(n, z) = \mathcal{O}(n^{[k/2]-k})$.*

Proof. It is trivially verified for $\Phi_0(n, z)$ and $\Phi_1(n, z)$. For $k \geq 2$ it can be easily proved by induction over k by using (12) and the recurrence (14). $\diamond$

We can summarize this discussion in the following

Theorem 1. *The polynomials $P_n(nz)$ defined in (5), where $f(w)$ is a meromorphic function with simple singularities $w_1, w_2, \ldots$ and analytic in the origin, may be expanded as the infinite sum*

$$P_n(nz) = (nz)^n \sum_{k=0}^{\infty} \frac{f^{(k)}(z^{-1})}{k!} \frac{p_k(n)}{(nz)^k}, \tag{16}$$

valid for $z \in \mathbb{C}$, $|z^{-1}| < |z^{-1} - w_i| \ \forall \ i = 1, 2, 3, \ldots$, where $p_k(n)$ are the polynomials defined in (13)-(14).

Remark 1. Observe that the polynomials $p_k(n)$ appearing in the expansion (16) of $P_n(nz)$ are the same ones for any set of polynomials of the family $P_n(z)$ defined in (5). The particular information about the particular set of polynomials defined by each $f(w)$ in (5) is all contained in the coefficients $f^{(k)}(z^{-1})$.

2.1. Uniform expansions of the Bernoulli polynomials

The Bernoulli polynomials $B_n(nz + 1/2)$ may be expanded in the form (16) with $f(w) = (w/2)/\sinh(w/2)$ and $w_k = \pm 2k\pi i$, $k = 1, 2, 3, \ldots$ In this case eq. (6) reads, for $w \in \mathbb{C}$,

$$\frac{w/2}{\sinh(w/2)} = \sum_{k=0}^{\infty} \frac{f^{(k)}(z^{-1})}{k!} (w - z^{-1})^k, \qquad |w - z^{-1}| < |\pm 2\pi i - z^{-1}|. \tag{17}$$

After straightforward operations we obtain that the derivatives $f^{(k)}(z^{-1})$ are given by the recurrence

$$f^{(k)}(z^{-1}) = - \sum_{l=1}^{\lfloor k/2 \rfloor} \binom{k}{2l} \frac{f^{k-2l}(z^{-1})}{4^l} -$$

$$\frac{1}{2} \coth\left(\frac{1}{2z}\right) \sum_{l=0}^{\lfloor (k-1)/2 \rfloor} \binom{k}{2l+1} \frac{f^{k-2l-1}(z^{-1})}{4^l}, \qquad k \geq 2,$$

where

$$f^{(0)}(z^{-1}) = \frac{1}{2z\sinh(1/2z)} \qquad \text{and} \qquad f^{(1)}(z^{-1}) = \frac{z - \frac{1}{2}\coth(1/2z)}{2z\sinh(1/2z)}. \qquad (18)$$

In the shown results we also use $f^{(2)}(z^{-1})$, which is given by

$$f^{(2)}(z^{-1}) = -\frac{1}{8z\sinh(1/2z)}\left[1 + 4\left(z - \frac{1}{2}\coth\left(\frac{1}{2z}\right)\right)\coth\left(\frac{1}{2z}\right)\right].$$

Introducing these $f^{(k)}(z^{-1})$ in (16) and retaining for convenience only the first three terms of the expansion we obtain,

Corollary 1. *For* $|\text{Im}\left(z^{-1}\right)| < \pi$ *(or* $\left|z \pm \frac{i}{2\pi}\right| > \frac{1}{2\pi}$*) and* $n \geq 1$,

$$B_n\left(nz + \frac{1}{2}\right) = \frac{n^n z^{n-1}}{2\sinh(1/2z)}\left\{1 + \right.$$
$$\left. \left[1 + 4\left(z - \frac{1}{2}\coth\left(\frac{1}{2z}\right)\right)\coth\left(\frac{1}{2z}\right)\right]\frac{1}{8nz^2} + \mathcal{O}\left(\frac{1}{n^2}\right)\right\}. \qquad (19)$$

Figure 2 shows the accuracy of approximation (19) for several values of n for real values of the uniformity parameter z.

Although expansion (16) with $f^{(k)}(z^{-1})$ given by (18) is convergent for $\left|z \pm \frac{i}{2\pi}\right| > \frac{1}{2\pi}$, convergence is slow for $|z| \leq \pi^{-1}$ (and $\left|z \pm \frac{i}{2\pi}\right| > \frac{1}{2\pi}$) and quite fast for $|z| > \pi^{-1}$. The relative error decreases for increasing $|z|$ or n. For example, for $|z| \sim \pi^{-1}$ and $n = 10$, the relative error is $\sim 10^{-2}$. For $n = 40$ and $|z| \sim 1$, the relative error is $\sim 10^{-5}$. The accuracy is even better for real argument, as shown in Figure 2. For real argument, the oscillatory region of $B_n(nx + 1/2)$ is contained in $|x| \leq \pi^{-1}$, whereas the monotonic region contains $|x| > \pi^{-1}$; therefore, for real argument, accuracy of approximation (19) is restricted to the monotonic region.

(a) $n = 3$. (b) $n = 10$.
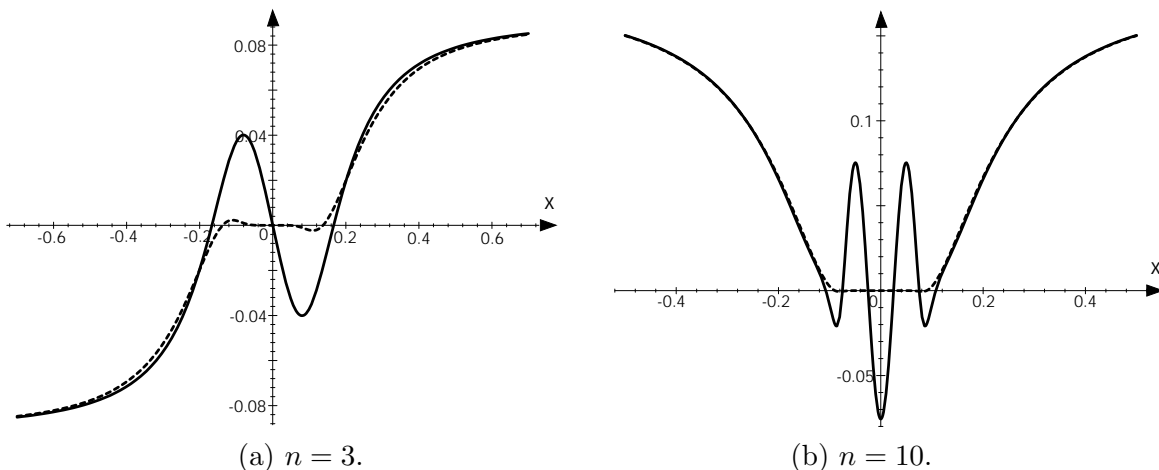
Figure 2. *Solid lines represent* $B_n(nx + 1/2)$ *for several values of* n, *whereas dashed lines represent the right-hand side of* (19) *with* $z \equiv x$, *both normalized by the factor* $(1 + |x/a|^n)^{-1}$, *where* a *is half the width of the oscillation region.*

2.2. Uniform expansions of the Euler polynomials

The Euler polynomials $E_n(nz+1/2)$ may be expanded in the form (16) with $f(w) = 1/\cosh(w/2)$ and $w_j = \pm(2j+1)\pi i$, $j = 0, 1, 2, \ldots$. Similar calculations to those of the above subsection may be performed in this case. For $w \in C$, eq. (6) reads

$$\frac{1}{\cosh(w/2)} = \sum_{k=0}^{\infty} \frac{f^{(k)}(z^{-1})}{k!}(w - z^{-1})^k, \qquad |w - z^{-1}| < |\pm \pi i - z^{-1}|. \qquad (20)$$

After straightforward operations, we obtain that the derivatives $f^{(k)}(z^{-1})$ are given by the recurrence

$$f^{(k)}(z^{-1}) = -\sum_{l=1}^{\lfloor k/2 \rfloor} \binom{k}{2l} \frac{f^{k-2l}(z^{-1})}{4^l} -$$

$$\frac{1}{2}\tanh\left(\frac{1}{2z}\right) \sum_{l=0}^{\lfloor (k-1)/2 \rfloor} \binom{k}{2l+1} \frac{f^{k-2l-1}(z^{-1})}{4^l}, \qquad k \geq 1,$$

where

$$f^{(0)}(z^{-1}) = \frac{1}{\cosh(1/2z)}. \qquad (21)$$

In the shown results we also use $f^{(2)}(z^{-1})$, which is given by

$$f^{(2)}(z^{-1}) = \frac{1}{4\cosh(1/2z)}\left(2\tanh^2\left(\frac{1}{2z}\right) - 1\right).$$

Introducing these $f^{(k)}(z^{-1})$ in (16) and retaining for convenience only the first three terms of the expansion we obtain,

Corollary 2. *For $|\mathrm{Im}\left(z^{-1}\right)| < \frac{\pi}{2}$ (or $\left|z \pm \frac{i}{\pi}\right| > \frac{1}{\pi}$) and $n \geq 1$,*

$$E_n\left(nz + \frac{1}{2}\right) = \frac{(nz)^n}{\cosh(1/2z)}\left[1 + \left(1 - 2\tanh^2\left(\frac{1}{2z}\right)\right)\frac{1}{8nz^2} + \mathcal{O}\left(\frac{1}{n^2}\right)\right]. \qquad (22)$$

Figure 3 shows the accuracy of approximation (22) for several values of n for real values of the uniformity parameter z.

As in the Bernoulli case, although expansion (16) with $f^{(k)}(z^{-1})$ given by (22) is convergent for $\left|z \pm \frac{i}{\pi}\right| > \frac{1}{\pi}$, convergence is slow for $|z| \leq 2\pi^{-1}$ (and $\left|z \pm \frac{i}{\pi}\right| > \frac{1}{\pi}$) and quite fast for $|z| > 2\pi^{-1}$. The relative error decreases for increasing $|z|$ or n. For example, for $|z| \sim 2\pi^{-1}$ and $n = 10$, the relative error is $\sim 10^{-2}$. For $n = 40$ and $|z| \sim 1$, the relative error is $\sim 10^{-4}$. The accuracy is even better for real argument, as shown in Figure 3. Also, for real argument, the oscillatory region of $E_n(nx + 1/2)$ is contained in $|x| \leq 2\pi^{-1}$, whereas the monotonic region contains $|x| > 2\pi^{-1}$; therefore, accuracy of approximation (22) is restricted to the monotonic region.

(a) $n = 3$.   (b) $n = 10$.
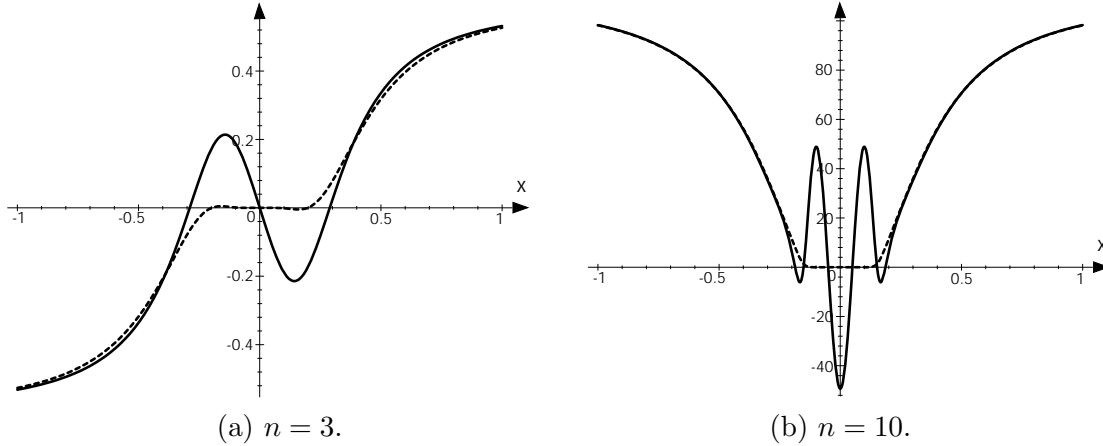
Figure 3. *Solid lines represent $E_n(nx + 1/2)$ for several values of n, whereas dashed lines represent the right-hand side of (22) with $z \equiv x$, both normalized by the same factor as in Fig. 2.*

3. Enlarging the region of validity of the uniformity parameter z

We have proved in Theorem 1 that validity of approximation (16) is restricted to the region $|z^{-1}| < |z^{-1} - w_i| \ \forall \ i = 1, 2, 3, \ldots$. (in the particular cases of the Bernoulli and Euler polynomials, the uniformity parameter z is restricted to be outside the discs $\left| z \pm \frac{i}{2\pi} \right| \leq \frac{1}{2\pi}$ and $\left| z \pm \frac{i}{\pi} \right| \leq \frac{1}{\pi}$, respectively). In these $z-$domains, the expansions are convergent series, as in (16). From computer experiments (see also Figures 2 and 3) we conclude that, in addition, in the Bernoulli and Euler cases, the uniformity parameter should satisfy $|z| \geq \pi^{-1}$ and $|z| \geq 2\pi^{-1}$, respectively, in order to obtain a good approximation by using only the first few terms of the expansion (16).

However, the region $\{z \in \mathbb{C}, |z^{-1}| < |z^{-1} - w_i| \ \forall \ i = 1, 2, 3, \ldots\}$ may be enlarged by 'isolating' the contribution of the poles $w_1, w_2, \ldots$ of $f(w)$: take the (simple) poles $\{w_i \in \mathbb{C}, i = 1, 2, 3, \ldots\}$ of $f(w)$, ordered by increasing modulus $|w_i| \leq |w_{i+1}|$ and define

$$f_m(w) = f(w) - \sum_{k=1}^{m} \frac{r_k}{w - w_k}, \tag{23}$$

where r_k are the residues of $f(w)$ in w_k. The function $f_m(w)$ has no poles inside the disc $\{w \in \mathbb{C}, |w| < |w_{m+1}|\}$. Introducing (23) in (5) we obtain

$$P_n(z) = P_n^m(z) + Q_n^m(z), \tag{24}$$

where

$$P_n^m(z) = \frac{n!}{2\pi i} \int_C f_m(w) e^{wz} \frac{dw}{w^{n+1}} \tag{25}$$

and

$$Q_n^m(z) = \frac{n!}{2\pi i} \sum_{k=1}^{m} r_k \int_C \frac{e^{wz}}{w - w_k} \frac{dw}{w^{n+1}}. \tag{26}$$

Now, the poles w_{m+1}, $w_{m+2}, \ldots$ of the function $f_m(w)$ are farther away from the origin than the poles of the function $f(w)$. Repeating the process of last section for $P_n^m(z)$ instead of $P_n(z)$ we obtain

$$P_n^m(nz) = (nz)^n \sum_{k=0}^{\infty} \frac{f^{(k)}(z^{-1}) + h_m^{(k)}(z^{-1})}{k!} \frac{p_k(n)}{(nz)^k}, \tag{27}$$

valid for $|z^{-1}| < |z^{-1} - w_i| \ \forall \ i = m+1, m+2, \ldots$ (and $z \neq 0$), where $h_m^{(k)}(w)$ is the $k-$th derivative of

$$h_m(w) = -\sum_{l=1}^{m} \frac{r_l}{w - w_l}. \tag{28}$$

Therefore, the range of validity of the expansion (27) is now larger than that of the expansion (16). On the other hand, the functions $Q_n^m(z)$ defined in (26) are just combinations of incomplete gamma functions: shifting the integration contour by $w = w_k + t$ in each integral in (26), writing

$$\frac{e^{tz}}{t} = \int_0^z e^{tx} dx + \frac{1}{t}$$

and after straightforward operations, using elementary properties of incomplete gamma functions [[18], chap. 11, sec. 2], we obtain

$$\frac{n!}{2\pi i} \int_C \frac{e^{wz}}{w - w_k} \frac{dw}{w^{n+1}} = e^{w_k z} \left[\int_0^z e^{-w_k t} t^n dt - \frac{n!}{w_k^{n+1}} \right] = -\frac{e^{w_k z}}{w_k^{n+1}} \Gamma(n+1, w_k z),$$

that is nothing but $-n!/w_k^{n+1}$ times the Taylor polynomial of degree n in $z = 0$ of $e^{w_k z}$. Therefore,

$$Q_n^m(z) = -\sum_{k=1}^{m} \frac{r_k e^{w_k z}}{w_k^{n+1}} \Gamma(n+1, w_k z),$$

and we have the following,

Theorem 2. *The polynomials $P_n(z)$ defined in (5) by means of a meromorphic function $f(w)$ analytic in the origin with simple poles w_1, $w_2, \ldots$ (and respective residues r_1, $r_2, \ldots$), can be represented, for each integer $m > 0$, as*

$$\begin{aligned} P_n(nz) = &-\sum_{k=1}^{m} \frac{r_k e^{w_k nz}}{w_k^{n+1}} \Gamma(n+1, w_k nz) \\ &+ (nz)^n \sum_{k=0}^{\infty} \frac{f^{(k)}(z^{-1}) + h_m^{(k)}(z^{-1})}{k!} \frac{p_k(n)}{(nz)^k}, \end{aligned} \tag{29}$$

valid for $z \in \mathbb{C}$, $|z^{-1}| < |z^{-1} - w_i|$, $\forall \ i = m+1, m+2, \ldots$, where the polynomials $p_k(n)$ are given in (14) and $h_m^{(k)}(z^{-1})$ is the $k-$th derivative of the function $h_m(w)$ defined in

(28). *Each term of the finite sum in the first line in the above equation equals $n!r_k/w_k^{n+1}$ times the Taylor polynomial of degree n in $z = 0$ of $e^{w_k nz}$.*

The 'isolation' technique used above for enlarging the region of validity of z, is quite similar to the one employed for obtaining uniform asymptotic expansions of contour integrals with a saddle point near a pole [[21], chap. 7, sec. 2]. There, by means of a change of variable, the contribution of the poles (eq. (26)) is expressed as an error function. Here, it is expressed just as a combination of incomplete gamma functions.

The series expansion of $P_n^m(nz)$ on the right-hand side of (27) converge in a larger region $|z^{-1}| < |z^{-1} - w_i| \; \forall \; i = m+1, m+2, \dots$. But moreover, the polynomial $Q_n^m(z)$ has been defined in (26) by using a function $-h_m(w)$ in the integral that has the same first $2m$ poles and the same residues as the function $f(w)$ defining $P_n(z)$ in (5). The contour C in (5) and in (26) may be chosen to pass near the singularities of $f(w)$ closest to the origin. If $|z|$ is small, the saddle point is far away from the path C and the greatest contribution to the integral is given by the piece of integral closest to these singularities. Therefore both integrals in (5) and in (26) are dominated by the same singularity and then, these integrals should be very similar, that is, $P_n(nz) \simeq Q_n^m(nz)$ for small $|z|$ and $m \geq 1$. Therefore, for small $|z|$, it must happen that $|P_n^m(nz)| << |Q_n^m(nz)|$. We check these facts in the Bernoulli and Euler examples.

3.1. Bernoulli polynomials

The first $2m$ poles and residues of $f(w) = (w/2)/\sinh(w/2)$ are $w_k = \pm 2k\pi i$, $r_k = (-1)^k w_k$, $k = 1, 2, \dots, m$. From (28) and after a straightforward algebra one may easily check that the derivatives $h_m^{(k)}(z^{-1})$ of $h_m(w)$ in the saddle point z^{-1} can be obtained from

$$h_m^{(k)}(z^{-1}) = \sum_{l=1}^{m} (-1)^l h_l^{(k)},$$

where $h_l^{(k)}$ are given by the recursive formula

$$h_l^{(k)} = -\frac{k}{z^{-2} + 4\pi^2 l^2} \left(2z^{-1} h_l^{(k-1)} + (k-1) h_l^{(k-2)} \right) \tag{30}$$

for $k \geq 2$, with

$$h_l^{(0)} = \frac{8\pi^2 l^2}{z^{-2} + 4\pi^2 l^2}, \qquad \text{and} \qquad h_l^{(1)} = -\frac{16\pi^2 l^2}{z(z^{-2} + 4\pi^2 l^2)^2}. \tag{31}$$

In the shown results we also use $h_l^{(2)}$, which is given by

$$h_l^{(2)} = \frac{16\pi^2 l^2 (3z^{-2} - 4l^2\pi^2)}{(z^{-2} + 4l^2\pi^2)^3}.$$

Introducing $f^{(k)}(z^{-1})$ given in (18) and $h_m^{(k)}(z^{-1})$ given in (30)-(31) in eq. (29), using $w_k = \pm 2k\pi i$, $r_k = (-1)^k w_k$, $k = 1, 2, \dots, m$ and retaining only the first three terms of the infinite sum in the second line of (29) we obtain,

Corollary 3. *For* $|\mathrm{Im}(z^{-1})| < (m+1)\pi$ *(or* $\left|z \pm \frac{i}{2(m+1)\pi}\right| > \frac{1}{2(m+1)\pi}$*) and* $n, m \geq 1$,

$$
\begin{aligned}
B_n\left(nz + \frac{1}{2}\right) = &-\sum_{k=1}^{m}(-1)^k\left[\frac{e^{2k\pi i n z}}{(2k\pi i)^n}\Gamma(n+1, 2k\pi i n z) + \right. \\
&\left.\frac{e^{-2k\pi i n z}}{(-2k\pi i)^n}\Gamma(n+1, -2k\pi i n z)\right] + \\
&(nz)^n\left\{\left[\frac{1}{2z\sinh(1/2z)} + 8\pi^2\sum_{k=1}^{m}\frac{(-1)^k k^2 z^2}{1 + 4k^2\pi^2 z^2}\right] + \right. \\
&\left[\frac{1}{16z^3\sinh(1/2z)}\left(1 + 4\left(z - \frac{1}{2}\coth\left(\frac{1}{2z}\right)\right)\coth\left(\frac{1}{2z}\right)\right) - \right. \\
&\left.\left. 8\pi^2\sum_{k=1}^{m}\frac{(-1)^k k^2 z^2(3 - 4k^2\pi^2 z^2)}{(1 + 4k^2\pi^2 z^2)^3}\right]\frac{1}{n} + \mathcal{O}\left(\frac{1}{n^2}\right)\right\}.
\end{aligned}
\tag{32}
$$

Remark 2. For $m = 1$, it follows, by using

$$
\Gamma(n + 1, z) = n!\, e^{-z}\sum_{j=0}^{n}\frac{z^j}{j!},
$$

that the first sum on the right-hand side, that is, the polynomial $Q_n^1(nz)$, reduces to

$$
Q_n^1(z) = 2n!z^n\sum_{k=0}^{\lfloor n/2 \rfloor}\frac{(-1)^k}{(4\pi^2 z^2)^k(n - 2k)!}.
$$

This is nothing but $2n!/(2\pi)^n$ times the Taylor polynomial of degree n in $z = 0$ of $\cos(2\pi n z - \pi n/2)$, which in turn is the first term of the Fourier expansion of $B_n(nz + 1/2)$ [[1], 23.1.16], which converges very fast if n is large.

Figure 4 shows the strong accuracy of approximation (32) already obtained from $n = 3$ with $m = 1$ for real values of the uniformity parameter z.

Approximation (32) is not valid for $\left|z \pm \frac{i}{2(m+1)\pi}\right| \leq \frac{1}{2(m+1)\pi}$. Convergence is slow in the vicinity of these discs and grows sharply for increasing distance from z to these discs and/or increasing n. For example, for $m = 1$ and $n = 10$, the relative error is $\sim 10^{-2}$ for $\left|z \pm \frac{i}{4\pi}\right| \sim \frac{1}{4\pi}$ and $\sim 10^{-5}$ for $|z| \sim 1$. For $m = 1$ and $n = 40$, the relative error is $\sim 10^{-3}$ for $\left|z \pm \frac{i}{4\pi}\right| \sim \frac{1}{4\pi}$ and $\sim 10^{-6}$ for $|z| \sim 1$. Accuracy is even better for real argument, as shown in Figure 4 and, as a difference with approximation (19), accuracy is also good in the oscillation region. For real argument and in the oscillation region, $Q_n^1(nx) \simeq B_n(nx + 1/2)$ and $P_n^1(nx)$ tends to zero exponentially fast. On the other hand, in the monotonic region, the approximation of $B_n(nx + 1/2)$ given by $P_n^1(nx) + Q_n^1(nx)$ is better than the approximation given by $P_n(nx)$.

(a) $n = 3$.
(b) $n = 10$.
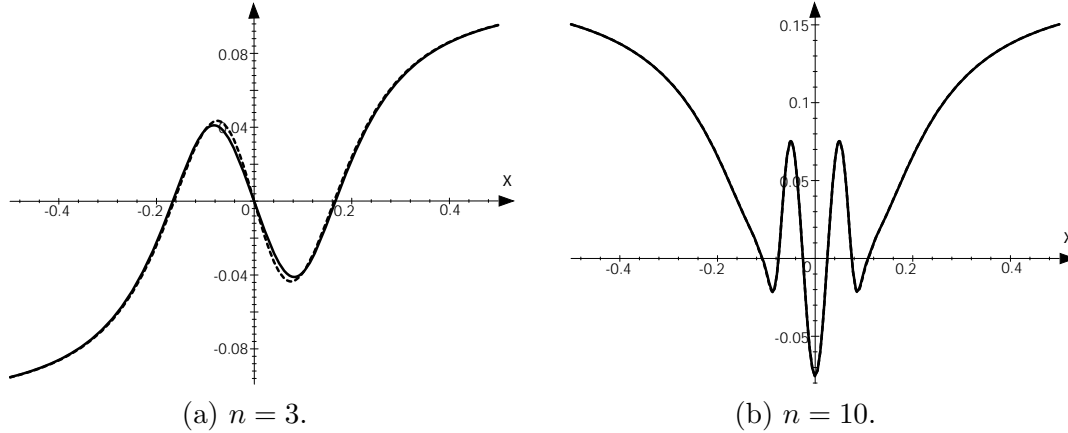
Figure 4. *Solid lines represent* $B_n(nx + 1/2)$ *for several values of* n, *whereas dashed lines represent the right-hand side of* (32) *with* $z \equiv x$, *both normalized by the same factor as in Fig. 2.*

3.2. Euler polynomials

The first $2m$ poles and residues of $f(w) = 1/\cosh(w/2)$ are, respectively, $w_k = \pm(2k + 1)\pi i$ and $r_k = \mp 2i(-1)^k$, $k = 0, 1, 2, \ldots, m - 1$. From (28) and after a straightforward algebra one may easily check that the derivatives $h_m^{(k)}(z^{-1})$ of $h_m(w)$ at the saddle point z^{-1} can be obtained from

$$h_m^{(k)}(z^{-1}) = \sum_{l=0}^{m-1}(-1)^l h_l^{(k)},$$

where $h_l^{(k)}$ are given by the recursive formula

$$h_l^{(k)} = -\frac{k}{z^{-2} + (2l + 1)^2\pi^2}\left(2z^{-1}h_l^{(k-1)} + (k - 1)h_l^{(k-2)}\right) \tag{33}$$

valid for $k \geq 2$ with

$$h_l^{(0)} = -\frac{4\pi(2l + 1)}{z^{-2} + (2l + 1)^2\pi^2}, \quad \text{and} \quad h_l^{(1)} = \frac{8\pi(2l + 1)z^{-1}}{(z^{-2} + (2l + 1)^2\pi^2)^2}. \tag{34}$$

In the shown results we also use $h_l^{(2)}$, which is given by

$$h_l^{(2)} = -8(2l + 1)\pi\frac{(3z^{-2} - (2l + 1)^2\pi^2)}{(z^{-2} + (2l + 1)^2\pi^2)^3}.$$

Introducing $f^{(k)}(z^{-1})$ given in eq. (21) and $h_m^{(k)}(z^{-1})$ given in (33)-(34) in eq. (29), using $w_k = \pm(2k + 1)\pi i$ and $r_k = \mp 2i$, $k = 0, 1, 2, \ldots, m - 1$ and retaining only the first three terms of the infinite sum in the second line of (29) we obtain,

Corollary 4. *For* $|\text{Im}\left(z^{-1}\right)| < (2m+1)\pi/2$ *(or* $\left|z \pm \frac{i}{(2m+1)\pi}\right| > \frac{1}{(2m+1)\pi}$*) and* $n, m \geq 1$,

$$
\begin{aligned}
E_n\left(nz + \frac{1}{2}\right) = & 2i \sum_{k=0}^{m-1} (-1)^k \left[\frac{e^{(2k+1)\pi inz}}{((2k+1)\pi i)^{n+1}} \Gamma(n+1, (2k+1)\pi inz) - \right. \\
& \left. \frac{e^{-(2k+1)\pi inz}}{(-(2k+1)\pi i)^{n+1}} \Gamma(n+1, -(2k+1)\pi inz) \right] + \\
& (nz)^n \left\{ \left[\frac{1}{\cosh(1/2z)} - 4\pi \sum_{k=0}^{m-1} \frac{(-1)^k (2k+1)z^2}{1 + (2k+1)^2 \pi^2 z^2} \right] + \right. \\
& \left[4\pi \sum_{k=0}^{m-1} \frac{(-1)^k (2k+1)z^2 (3 - (2k+1)^2 \pi^2 z^2)}{(1 + (2k+1)^2 \pi^2 z^2)^3} + \right. \\
& \left. \left. \frac{1}{8z^2 \cosh(1/2z)} \left(1 - 2\tanh^2\left(\frac{1}{2z}\right)\right) \right] \frac{1}{n} + \mathcal{O}\left(\frac{1}{n^2}\right) \right\}.
\end{aligned}
\tag{35}
$$

Remark 3. For $m = 1$, the first line on the right-hand side, that is, the polynomial $Q_n^1(nz)$, is nothing but $4n!/\pi^{n+1}$ times the Taylor polynomial of degree n in $z = 0$ of $\cos(\pi nz - \pi n/2)$, the first term of the Fourier expansion of $E_n(nz + 1/2)$ [[1], 23.1.16],

$$
Q_n^1(z) = \frac{4n! z^n}{\pi} \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{(-1)^k}{(\pi^2 z^2)^k (n-2k)!}.
$$

Figure 5 shows the strong accuracy of approximation (35) already obtained from $n = 3$ with $m = 1$ for real values of the uniformity parameter z.

Approximation (35) is not valid for $\left|z \pm \frac{i}{(2m+1)\pi}\right| \leq \frac{1}{(2m+1)\pi}$. Convergence is slow in the vicinity of these discs and grows sharply for increasing distance from z to these discs and/or increasing n. For example, for $m = 1$ and $n = 10$, the relative error is $\sim 10^{-3}$ for $\left|z \pm \frac{i}{3\pi}\right| \sim \frac{1}{3\pi}$ and $\sim 10^{-5}$ for $|z| \sim 1$. For $m = 1$ and $n = 40$, the relative error is $\sim 10^{-5}$ for $\left|z \pm \frac{i}{3\pi}\right| \sim \frac{1}{3\pi}$ and $\sim 10^{-6}$ for $|z| \sim 1$. Accuracy is even better for real argument, as shown in Figure 5 and, as a difference with approximation (22), accuracy is also good in the oscillation region. For real argument, and in the oscillation region, $Q_n^1(nx) \simeq E_n(nx + 1/2)$ and $P_n^1(nx)$ tends to zero exponentially fast. On the other hand, in the monotonic region, the approximation of $E_n(nx + 1/2)$ given by $P_n^1(nx) + Q_n^1(nx + 1/2)$ is better than the approximation given by $P_n(nx)$.

(a) $n = 3$.   (b) $n = 10$.
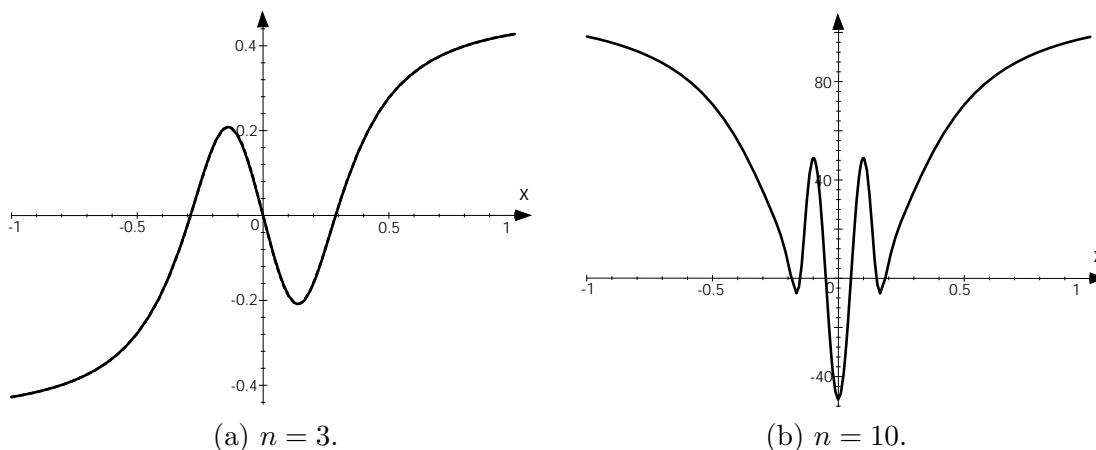
Figure 5. *Solid lines represent $E_n(nx + 1/2)$ for several values of n, whereas dashed lines represent the right-hand side of (35) with $z \equiv x$, both normalized by the same factor as in Fig. 2.*

4. Conclusions

Convergent expansions of the family of polynomials $P_n(nz)$ defined by formula (5) have been given in equation (10). For the particular cases of Bernoulli $B_n(nz+1/2)$ and Euler $E_n(nz + 1/2)$ polynomials, these expansions are given in equations (19) and (22), respectively. They are uniformly valid for $|z^{-1}| < |z^{-1} - w_i| \; \forall \; i = 1, 2, \ldots$, where w_i are the singularities of the meromorphic function $f(w)$ defining $P_n(nz)$ in (5) ordered by increasing modulus, $|w_i| \leq |w_{i+1}|$. For the particular cases of Bernoulli $B_n(nz+1/2)$ and Euler $E_n(nz + 1/2)$ polynomials, convergence is restricted to the region $\left| z \pm \frac{i}{2\pi} \right| > \frac{1}{2\pi}$ and $\left| z \pm \frac{i}{\pi} \right| > \frac{1}{\pi}$, respectively. Although the convergence is quite strong only outside the discs $|z| \leq \pi^{-1}$ and $|z| \leq 2\pi^{-1}$, respectively, and slow inside these discs. Figures 2 and 3 show the accuracy of these approximations for several values of n and real argument z. Strong convergence is restricted the monotonic region.

The convergence may be strongly accelerated by 'isolating' the poles of $f(w)$, such as it is described in Section 3. We obtain in this way that the convergence rate of expansion (10) is improved in the expansion (29). In particular, the convergence rate of the expansions (19) and (22) of Bernoulli and Euler polynomials are improved by (32) and (35), respectively. These 'improved' expansions are valid in a larger region of the uniformity parameter z. They are uniformly valid for $|z^{-1}| < |z^{-1} - w_i| \; \forall \; i = m + 1, m + 2, \ldots$ with m a positive integer. In the particular cases of the Bernoulli and Euler polynomials, expansions (32) and (35) are uniformly valid for $\left| z \pm \frac{i}{2(m+1)\pi} \right| > \frac{1}{2(m+1)\pi}$ and $\left| z \pm \frac{i}{(2m+1)\pi} \right| > \frac{1}{(2m+1)\pi}$, respectively. For real argument, the convergence is quite strong everywhere including the oscillation region. Figures 4 and 5 show the accuracy of these approximations for several values of n and real argument. We notice that the expansions in (32) and (35) are much better at the origin $z = 0$ than the simpler expansions (19) and (22). This can be explained by observing that the latter

expansions, though not being correct, are exponentially small at the origin and that the contributions coming from the incomplete gamma functions can be viewed as first terms of fast converging expansions for the Bernoulli and Euler polynomials (see Remark 2 and Remark 3).

Approximations (19) and (22) may be quite useful for practical evaluations of Bernoulli $B_n(nz + 1/2)$ and Euler $E_n(nz + 1/2)$ polynomials for large n in the appropriate region of z. When the uniformity parameter satisfies $|z| > \pi^{-1}$ or $|z| > 2\pi^{-1}$, respectively, only the three first terms of the expansion (19) or (22) are needed to approximate $B_{10}(10z + 1/2)$ or $E_{10}(10z + 1/2)$ with 2 digits. For $|z| > 1$, the first three terms approximate these polynomials with 4 digits. Because of the asymptotic character the accuracy of the approximation increases for increasing n.

Also, only the first few terms of the expansion of $P_n^1(nz)$ are needed to obtain a good approximation of Bernoulli and Euler polynomials in a larger region of the uniformity parameter z by using the improved approximations (32) and (35) containing incomplete gamma functions. Besides, the accuracy of these expansions is larger. On the other hand, the knowledge of the zeros of the incomplete gamma functions may be used for approximating the zeros of the Bernoulli and Euler polynomials by means of the approximations (32) and (35). This will be investigated in a following publication.

5. Acknowledgements

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