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ABSTRACT
The convergence behaviour is investigated of solution algorithms for the anisotropic Poisson problem on partially ordered, sparse families of regular grids in 3D. In order to study multilevel techniques on sparse families of grids, first we consider the convergence of a two-level algorithm that applies semi-coarsening successively in each of the coordinate directions. This algorithm shows good convergence, but recursive application of the successive semi-coarsening is not sufficiently efficient. Therefore we introduce another algorithm, which uses collective 3D semi-coarsened coarse grid corrections. The convergence behaviour of this collective version is worse, due to the lack of correspondence between the solutions on the different grids. By solving for the trivial solution we demonstrate that a good convergence behaviour of the collective version of the algorithm can be retained when the different solutions are sufficiently coherent. In order to solve also non-trivial problems, we develop a defect correction process. This algorithm makes use of hierarchical smoothing in order to deal with the problems related to the lack of coherence between the solutions on the different grids. Now good convergence rates are obtained also for non-trivial solutions. All convergence results are obtained for two-level processes. The results show convergence rates which are bounded, independent of the discretisation level and of the anisotropy in the problem.

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1. Introduction

Classical multigrid (MG) uses sequentially nested families of grids. In the regular, d-dimensional case every coarse grid-cell is formed by combination of 2^d finer grid cells. For more dimensions, as d becomes larger, this implies that many frequencies that can be represented on a fine grid cannot be seen on the next coarser one. Since the two basic mechanisms in a MG method are smoothing and coarse-grid-correction, this means that, for higher dimensions (e.g. $d = 3$) very strong relaxation techniques are required to obtain good convergence results. A way to simplify this heavy requirement is to use semi-coarsening, i.e., a coarser grid is created by only combining a pair of grid cells to a coarser grid cell. Obviously, this process of semi-coarsening can be applied in each of the d coordinate directions. Subsequent application this process in all coordinate directions makes a partially ordered set of grids rather than a sequentially ordered one. All grids that can be seen as the coarsenings of one particular grid are called a *full grid of grids*. Semi-coarsening is earlier described for $d = 2$ in [9, 10] and for $d = 3$ in [1, 2, 3, 6, 7, 8].

It is an additional disadvantage of regular refinement in each direction, especially for higher dimensions, that the number of the degrees of freedom increases very fast when more levels of refinement are introduced. In the battle against the increasing number of degrees of freedom, a promising development is the sparse grid approach of Zenger [12]. Sparse grids are formed by taking a subset from the partially ordered family of semi-refined grids. This subset, the *sparse grid of grids*, is formed by taking all grids for which the cells exceed a certain specified volume. For sufficiently smooth functions

it can be shown that the representation of a function on a sparse grid can be much more efficient than on a regular full grid [7, 12].

The problem with a representation on a sparse grid of grids is that there is no unique finest grid. There is a whole set of finest grids and the question arises of how to represent a final solution. Basically there are two ways to answer the question. The first is to introduce hierarchical basis functions instead of standard basis functions [1, 2, 3, 12]. In this way one has to add the contributions (the *hierarchical surplus*) from all grids in the sparse family to form the final solution. The second method is based on extrapolation. Then a linear combination of the approximations on the finest grids is formed. This is called the *combination technique* [4, 5, 6].

An advantage of the hierarchical basis technique is the existence of a straightforward and unique representation of an approximation, and the possibility to write down the FE discretisation for a differential equation. It is a disadvantage that the FE discrete operator is not sparse and that –for variable coefficient equations– no efficient method is available for its evaluation. In contrast to the hierarchical basis technique, the combination technique makes use of standard basis FE solutions on the finest grids. This has the advantage that existing techniques can be applied for the solution process on these grids.

The above arguments are the motivation to develop methods which use 3D MG semi-coarsening techniques together with a sparse grid of grids. The combination should also combine the advantages of both approaches, i.e., MG convergence rates should be obtained without the requirement of strong relaxation techniques, and the the solution of the problem can be represented with sufficient accuracy by a relative small number of degrees of freedom. An additional advantage can be that the method is not sensitive for anisotropies in the problem.

2. The equation

In this paper we study the 3D anisotropic Poisson equation with homogeneous boundary conditions:

$$-\nabla\cdot(\mathbf{a}\nabla u) = f \qquad \text{on} \qquad \Omega=[0,1]^3, \tag{2.1}$$

where $\mathbf{a} = \mathrm{diag}(a_{11}, a_{22}, a_{33})$ is a constant diagonal tensor. First, we discretise the equation by the usual finite element (FE) technique on the regular grid $\Omega_{\mathbf{k}}$, where $\mathbf{k}=(k_1,k_2,k_3)$ is a multi-integer. In $\Omega_{\mathbf{k}}$ the mesh-size in the i-th coordinate direction is $h_i = 2^{-k_i}$, $i=1,2,3$. By $B_{\mathbf{k}} = \{\varphi_{\mathbf{kj}} \mid \mathbf{o}\le\mathbf{j}\le 2^{\mathbf{k}}\}$ we denote the basis of standard piecewise tri-linear basis-functions in $\Omega_{\mathbf{k}}$. A standard FE method on grid $\Omega_{\mathbf{k}}$ is obtained by selecting trial and test functions in $B_{\mathbf{k}}$. This yields the discrete equations $\sum_{\mathbf{j}} a(\varphi_{\mathbf{kj}},\varphi_{\mathbf{ki}})u_{\mathbf{kj}} = f(\varphi_{\mathbf{kj}})$, which are also denoted in matrix form by

$$A_{\mathbf{kk}}u_{\mathbf{k}} = f_{\mathbf{k}}. \tag{2.2}$$

Let A_ℓ be the block diagonal matrix composed by all stiffness matrices on level ℓ,

$$A_\ell = \mathrm{diag}\left(A_{\mathbf{kk}} \mid \ |\mathbf{k}| = k_1+k_2+k_3 = \ell\right),$$

and let f_ℓ, u_ℓ be the concatenation of right hand side vectors $f_{\mathbf{k}}$ and solution vectors $u_{\mathbf{k}}$ respectively. Then the collection of all discretisations on the grids $\Omega_{\mathbf{k}}$ with $|\mathbf{k}|=\ell$ can be written as

$$A_\ell u_\ell = f_\ell. \tag{2.3}$$

In this paper we study iteration methods for the solution of (2.3). In (2.3) possibly the FE method can be replaced by another discretisation technique. Although we are essentially interested in MG methods, for simplicity in this paper we restrict ourselves to the corresponding two-grid methods.

3. A successive correction method

In this section we consider a method for which each cycle consists of 3 stages, one for each coordinate direction $i=1,2,3$. In each stage it uses the one-dimensional (semi-coarsened) Full Approximation

Scheme in direction i. We call the method *SCM* (Successive Correction Method). This method is similar to the one proposed by Pflaum [11] in the sense that each time the coarse-grid-correction is obtained only from coarsening in the i-th direction. For grids that have no coarser grid in the i-th direction we perform only relaxations, in order to have a constant number of relaxations for all the grids on the fine level. For relaxation on the fine level we use Jacobi-relaxation with a damping factor $\omega = \frac{1}{2}$. We applied one pre- ($\nu_1 = 1$) and one post-relaxation ($\nu_2 = 1$) for every cycle.

For convergence measurement we solve the equation (2.1) with homogeneous right hand side, $f = 0$. The trivial solution has the advantage of a possible lasting monitoring of the convergence. As an initial guess we use $u_0 = x(1-x)y(1-y)z(1-z)$. After every *SCM* cycle we calculate the residual on each grid of the fine level ℓ. In Figure 3 we see the convergence history of the *SCM* cycling. We repeat the experiment for different (plane and line) anisotropies, viz. $a_{11} = 1.0$, 0.01 and 100.0. In all cases $a_{22} = a_{33} = 1$.

We make the following observations. During convergence we can distinguish two phases: an initial and an asymptotic phase. The asymptotic convergence rate appears to be $\rho \approx 0.35$, independent of the grid and independent of the level. Also for the different anisotropies the convergence rates tend towards the same constant value for all grids, independent of the level. In contrast to the asymptotic rates, the initial convergence rates show large differences for the different grids on a level.

Despite of the good and level-independent convergence rates of the *SCM* process, this is not an efficient method for the recursive application in a multi-level algorithm, because recursive application requires a number of arithmetic operations that is more than proportional with the number of the degrees of freedom. Therefore, in the next section we study an algorithm with a better complexity.

4. Collective correction methods

4.1 The Plain Collective Correction Method *(PCCM)*

In this section we consider methods that solve only a single linear system for each coarse grid on level $\ell - 1$ (instead of three). The one solution is used to compute a correction for multiple fine grids. This is implemented in its simplest form in the Plain Collective Correction Method (*PCCM*, see Figure 1). In this algorithm we see several calls for auxiliary routines. The relaxation is damped Jacobi as in Section 3. Because of our interest in a TGM, the coarse grid equations on level $\ell - 1$ are solved by a conjugate gradient method. The prolongation of the corrections is done in the routine *P3D*. This interpolation is directly based on the hierarchical representation on the levels $\ell - 1$, $\ell - 2$ and $\ell - 3$, which representation is obtained by restriction from level $\ell - 1$ by the scheme given in Figure 2.

We again study equation (2.1) with homogeneous right hand side, $f = 0$, and initial guess $u_0 = x(1-x)y(1-y)z(1-z)$. After every cycle with *PCCM* we calculate the residual and the results are shown in Figure 4. In contrast with the *SCM* algorithm we see that the convergence behaviour of this algorithm collapses. This is due to non-coherent residuals (or solutions) on the different grids of the finest level, i.e., the residuals on the different grids do not correspond. One can even observe that residuals on different grids are equally large, but have opposite signs. Hence the effect of their restriction to the coarse grid is cancelled. This results in a bad convergence behaviour. In the initial phase some grids are converging faster, but the convergence of these particular grids is hampered as they feel the presence of the slowly converging grids. This is also the reason why the residual is not monotonically decreasing for all grids.

4.2 The Simple Collective Correction Method *(SCCM)*

In order to avoid the problem with the non-coherent solutions, as described in Section 4.1, we simply force the solution to be coherent. This is done in two steps, first changing the function from its redundant representation on level ℓ to a hierarchical representation. Second, we transfer the hierarchical representation back to the redundant representation. During the sweep from redundant representation on level ℓ to the hierarchical representation, we have to calculate for the common points on the fine grid an approximation on the coarse grid. This is done by averaging the three values of the common

routine PCCM($u_\ell, f_\ell, \nu_1, \nu_2$);
  integer $\nu_1, \nu_2, \mathbf{k}, j$;

  for $j = 1, \nu_1$ **do**
    Relax($A_\ell u_\ell = f_\ell$)
  end do;
  $r_\ell = f_\ell - A_\ell u_\ell$;
  for ($\forall |\mathbf{k}| = \ell - 1$) **do**
    $u_{\mathbf{k}} = \frac{1}{3}(R_{\mathbf{k},\mathbf{k}+\mathbf{e}_1} u_{\mathbf{k}+\mathbf{e}_1} + R_{\mathbf{k},\mathbf{k}+\mathbf{e}_2} u_{\mathbf{k}+\mathbf{e}_2} + R_{\mathbf{k},\mathbf{k}+\mathbf{e}_3} u_{\mathbf{k}+\mathbf{e}_3})$;
    $u^{old}_{\mathbf{k}} = u_{\mathbf{k}}$;
    $f_{\mathbf{k}} = A_{\mathbf{k}\mathbf{k}}(u_{\mathbf{k}}) + \frac{1}{3}(R_{\mathbf{k},\mathbf{k}+\mathbf{e}_1} r_{\mathbf{k}+\mathbf{e}_1} + R_{\mathbf{k},\mathbf{k}+\mathbf{e}_2} r_{\mathbf{k}+\mathbf{e}_2} + R_{\mathbf{k},\mathbf{k}+\mathbf{e}_3} r_{\mathbf{k}+\mathbf{e}_3})$;
  end do;
  Solve($A_{\ell-1} u_{\ell-1} = f_{\ell-1}$)
  for ($\forall |\mathbf{k}| = \ell - 1$) **do**
    $c_{\mathbf{k}} = u_{\mathbf{k}} - u^{old}_{\mathbf{k}}$
  end do;
  P3D($c_{\ell-1}$, c_ℓ);
  $u_\ell = u_\ell + c_\ell$;
  for $j = 1, \nu_2$ **do**
    Relax($A_\ell u_\ell = f_\ell$)
  end do;
end

Figure 1: The algorithm PCCM

points on the fine level. This process we call *hierarchical smoothing*, and in *SCCM* it is applied after every call for *PCCM*.

Figure 6 shows the convergence for the isotropic equation (2.1) for various levels. We see that the rate of convergence depends slightly on the level, but the convergence rate is bounded above by a reasonable (constant) value. In Figure 7 we see the the convergence results for the anisotropic equation. Again this convergence is independent of the anisotropy.

In our present case, with the trivial solution, we can simply apply hierarchical smoothing to the solution. However, in general the discrete solutions differ on the different grids, and the hierarchical smoothing is not sufficient. This is due to the fact that the solutions on the different grids all have their own discretisation error, so that the discrete solutions do not completely correspond. Hence, when we force the solutions to correspond by hierarchical smoothing we will not obtain a converging solution. Therefore, in the next section we develop a method which is able to handle also non-trivial solutions.

4.3 The Collective Correction Method *(CCM)*

In the previous section we described that we could not expect good convergence rates by *SCCM* because the solutions on all fine levels differ by their own specific truncation errors. Hence, to obtain corresponding solutions we have to adjust the right hand side of the equations (2.2) for this effect. This leads to the following defect correction equation on grid $\Omega_{\mathbf{k}}$:

$$A_{\mathbf{k}\mathbf{k}} u_{\mathbf{k}} = A_{\mathbf{k}\mathbf{k}} u_{\mathbf{k}} - R_{\mathbf{k}\ell} \left(\tilde{A}_\ell \tilde{u}_\ell - f_\ell \right) = g_{\mathbf{k}}. \tag{4.1}$$

Here $\tilde{A}_\ell$ is some (hierarchical basis) discretisation matrix, and $\tilde{u}_\ell$ is the coherent representation of the solution, obtained after a sweep of hierarchical smoothing from the redundant representation. The

algorithm *CCM* is obtained from *PCCM* by replacing the right hand side $f_{\mathbf{k}}$ in (2.2) by the adapted right hand side $g_{\mathbf{k}}$. As in *SCCM* we apply hierarchical smoothing to the solution.

As we want to obtain good convergence for arbitrary rhs, we consider equation (2.1), with $f = 100$. Figure 5 shows the convergence results for the isotropic equation, on level $\ell = 5$ and $\ell = 6$. We see that the convergence rate changes somewhat, due to the adjustment of the right hand side. Figure 8 shows the convergence of the anisotropic equation for the levels $\ell = 5, 6$ and $\ell = 7$. We see that the convergence rate hardly suffers from the anisotropy. Further we see that the convergence is not really dependent on the level of discretisation.

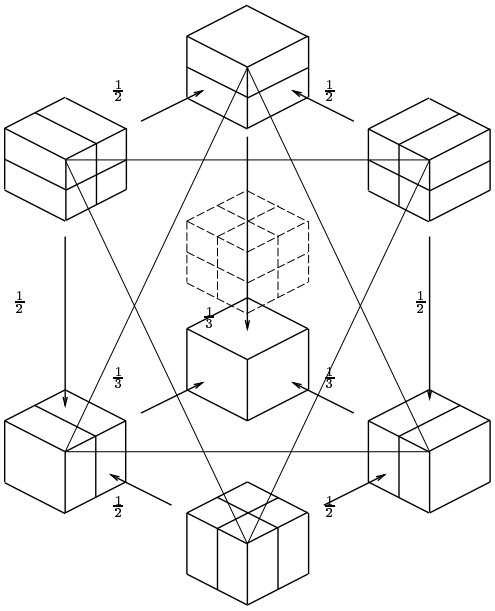

Figure 2: Restrictions of the corrections in *P3D*.

5. Conclusions

In the asymptotic phase, the convergence rates of the *SCM*-algorithm is almost completely independent on the level of discretisation and on the anisotropy of the problem. However, each cycle of this algorithm requires three coarse grid corrections. This implies that the recursive application of this algorithm requires a number of arithmetic operations that is more than proportional with the number of degrees of freedom. Collective correction algorithms overcome this difficulty.

Good convergence for the collective algorithm depends on coherence between the solutions on the different fine grids. Therefore, we apply hierarchical smoothing to obtain coherent solutions. Now the solution is solved with a defect correction process that makes use of the hierarchical basis representation. Additionally, we applied the hierarchical smoothing to speed up the convergence of the equations. With this combined process of defect correction and hierarchical smoothing we obtain good convergence results, i.e., convergence which is only slightly dependent on the discretisation level and anisotropy of the problem.

Thus far we calculated the defect correction by means of the hierarchical FE discretisation. For the general case this still can be expensive. The next challenge is to find more efficient discretisation operators to calculate the defect in (4.1).

1. R. Balder and C. Zenger. The solution of multidimensional real Helmholtz equations on sparse grids. *SIAM J. Sci. Comput.*, 17(3):631–646, May 1996.
2. H.J. Bungartz. An adaptive Poisson solver using hierarchical bases and sparse grids. In P. de Groen and R. Beauwens, editors, *Proceedings of the IMACS International Symposium on Iterative Methods in Linear Algebra*, pages 293–310. IMACS, Elsevier Science Publishers B.V. (North-Holland), 1991.
3. H.J. Bungartz. *Dünne Gitter und deren Anwendungen bei der adaptiven Lösung der dreidimensionalen Poisson-Gleichung.* PhD thesis, Technische Universität München, 1992.
4. M. Griebel. Parallel multigrid methods on sparse grids. Technical Report TUM-I-9053, SFB-Bericht Nr. 342/30/90 A, Technische Universität München, Institut für Informatik, Dezember 1990.
5. M. Griebel, W. Huber, U. Rüde, and T.Störtkuhl. The Combination Technique for Parallel Sparse-Grid-preconditioning and -Solutions of PDEs on Multiprocessor Machines and Workstation Networks. Technical Report TUM-I-9214, SFB-Bericht Nr.342/11/92 A, Technische Universität München, Institut für Informatik, Mai 1992.
6. M. Griebel, M. Schneider, and C. Zenger. A combination technique for the solution of sparse grid problems. Technical Report TUM-I-9038, SFB-Bericht Nr. 342/19/90 A, Technische Universität München, Institut für Informatik, Oktober 1990.
7. P.W. Hemker and C. Pflaum. Approximation on partially ordered sets of regular grids. *To appear in: Applied Numerical mathematics*, 1997.
8. B. Koren, P.W. Hemker, and P.M. de Zeeuw. Semi-coarsening in three directions for Euler-flow computations in three dimensions. In H. Deconinck and B. Koren, editors, *Euler and Navier-Stokes Solvers Using Multi-Dimensional Upwind Schemes and Multigrid Acceleration*, volume 57 of *European Community Research in Aeronautics,* Notes on Numerical Fluid Mechanics, pages 547–567, Braunschweig, 1997. Vieweg.
9. W.A. Mulder. A New Multigrid Approach to Convection Problems. *Journal of Computational Physics*, 83:303–323, 1989.
10. W.A. Mulder. A High-Resolution Euler Solver Based on Multigrid, Semi-coarsening, and Defect Correction. *Journal of Computational Physics*, 100:91–104, 1992.
11. C. Pflaum. A multilevel algorithm for the solution of second order elliptic differential equations on sparse grids. In N. D. Melson, T. A. Manteuffel, S. F. McCormick, and C. C. Douglas, editors, *Seventh Copper Mountain Conference on Multigrid Methods*, volume CP 3339, pages 661–672, Hampton, VA, 1996. NASA.
12. C. Zenger. Sparse grids. Technical report, Technische Universität München, Institut für Informatik, 1990.

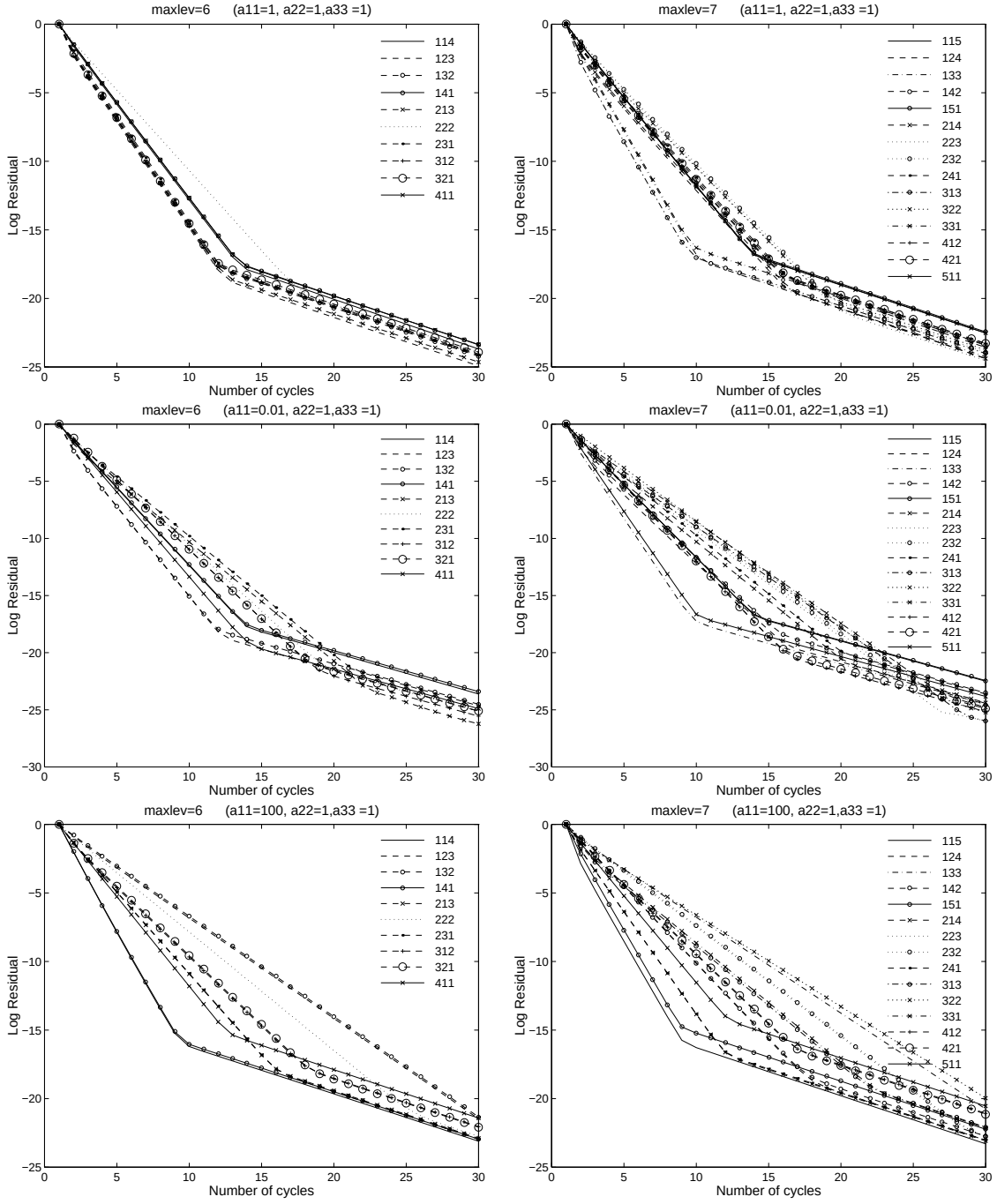

Figure 3: Convergence of *SCM* for $\ell = 6$ (left) and $\ell = 7$ (right) for different values of a_{11}.

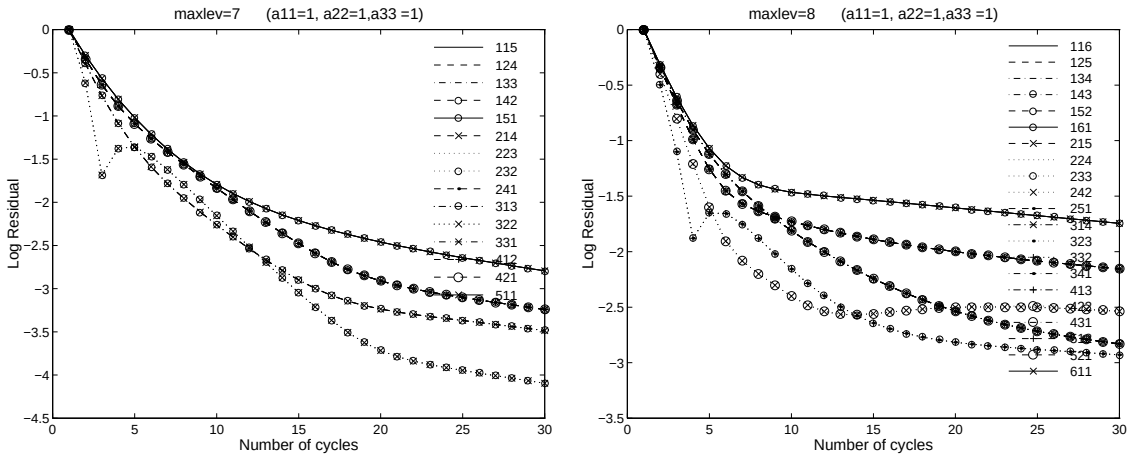

Figure 4: Convergence of $PCCM$ for the trivial solution, for level $\ell = 7$ and $\ell = 8$.

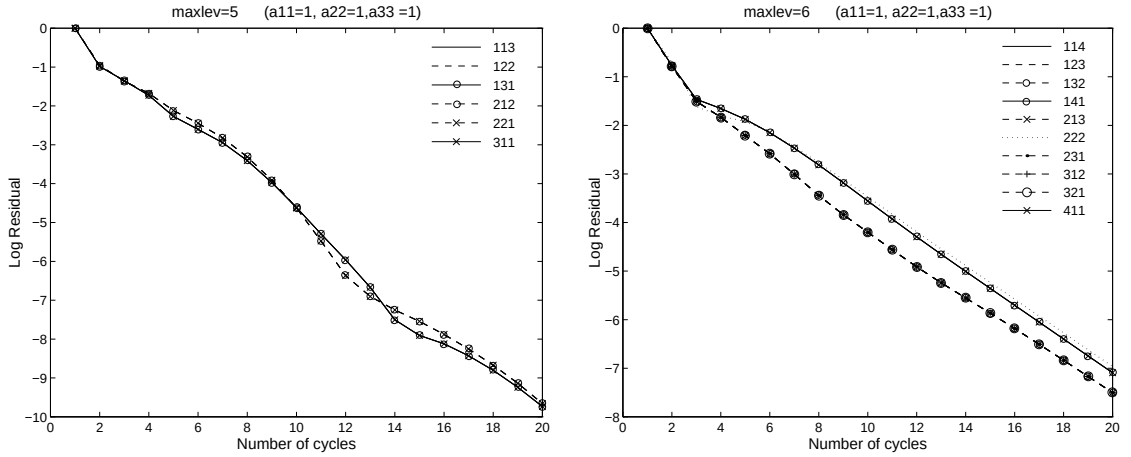

Figure 5: Convergence of CCM for the isotropic equation (2.1), with right hand side $f = 100$, for level $\ell = 5$ and $\ell = 6$.

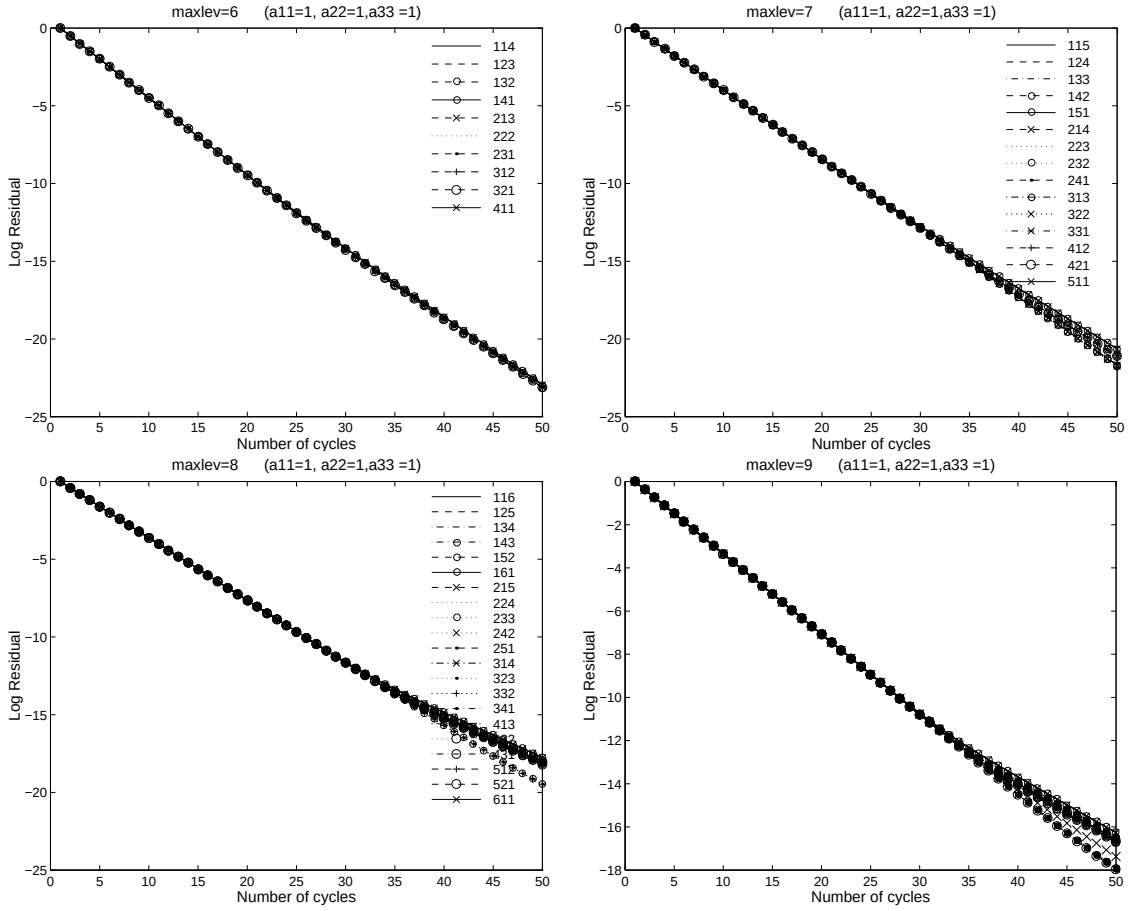

Figure 6: Convergence for the isotropic equation (2.1), for the trivial solution obtained with *SCCM*.

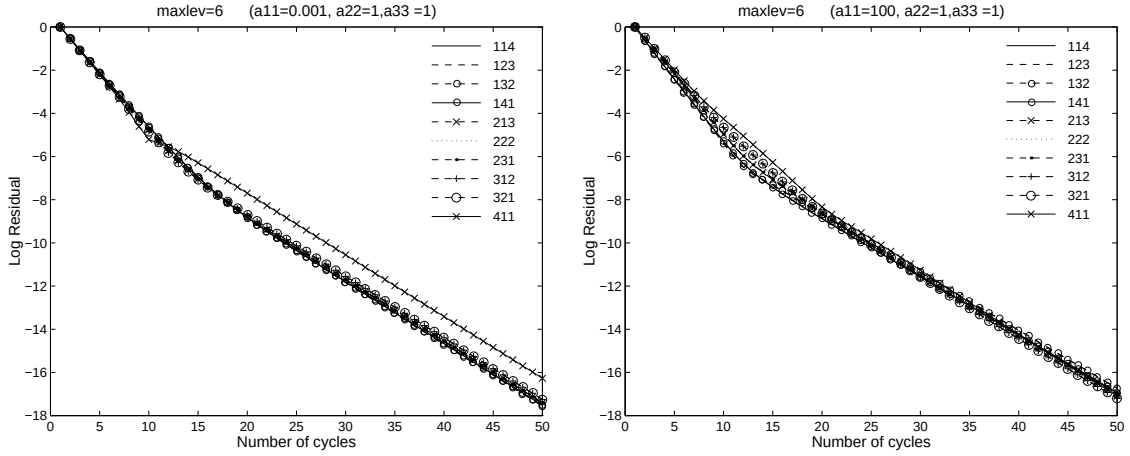

Figure 7: Convergence obtained with *SCCM*, for the anisotropic equation, $a_{11} = 0.001$ (left) and $a_{11} = 100$ (right), for the trivial solution.

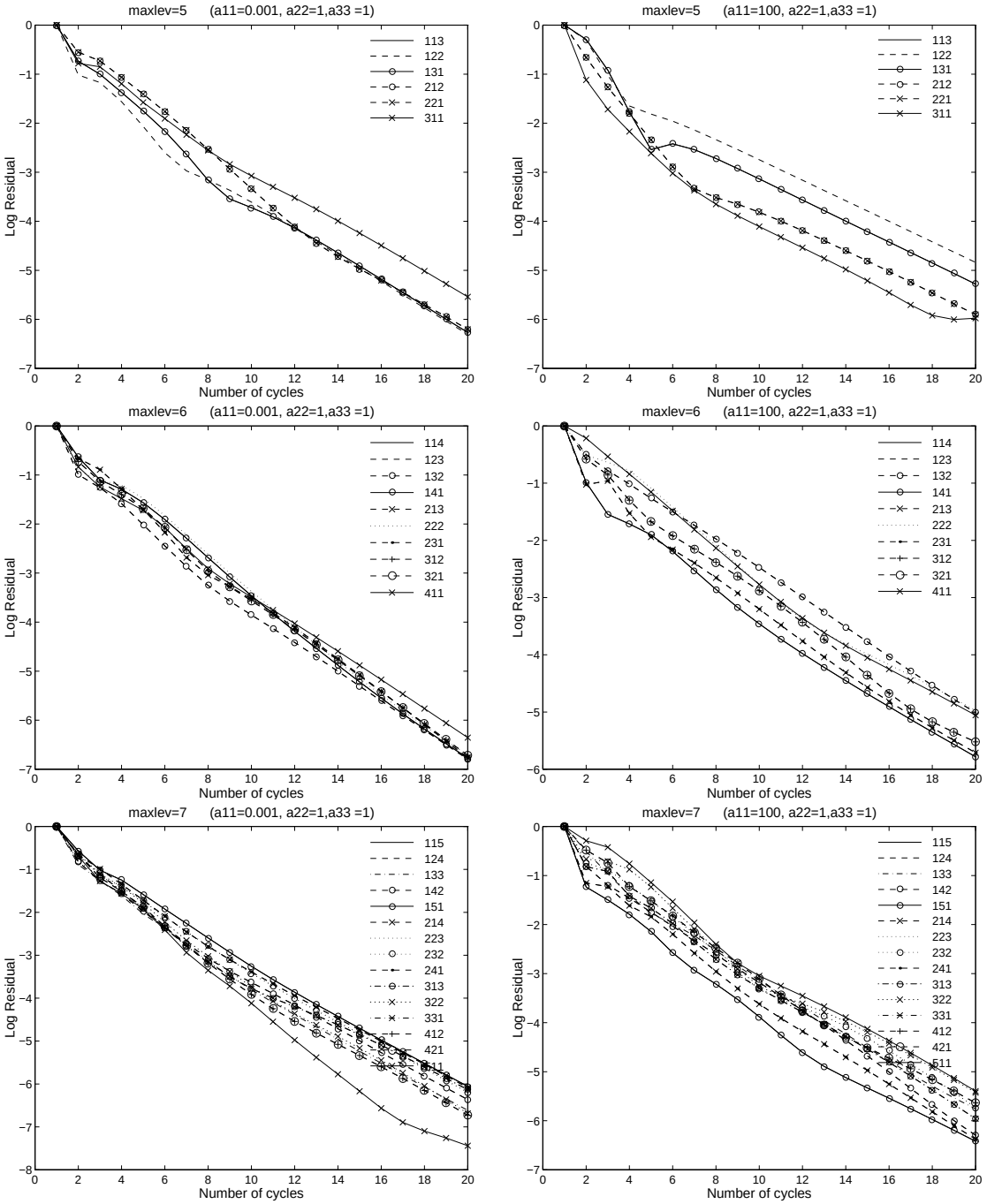

Figure 8: Convergence of *CCM* for the anisotropic equation (2.1), with $a_{11} = 0.001$ (left) and $a_{11} = 100.0$ (right) and the right hand side $f = 100$, for $\ell = 5, 6$ and $\ell = 7$.