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ABSTRACT

Detailed error analyses are given for sparse-grid function representations through the combination technique. Two- and three-dimensional, and smooth and discontinuous functions are considered, as well as piecewise-constant and piecewise-linear interpolation techniques. Where appropriate, the results of the analyses are verified in numerical experiments. Instead of the common vertex-based function representation, cell-centered function representation is considered. Explicit, pointwise error expressions for the representation error are given, rather than order estimates. The paper contributes to the theory of sparse-grid techniques.

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1. Introduction

1.1 Sparse-grid techniques

Sparse grids were introduced in 1990 by Zenger [1], in order to significantly reduce the number of degrees of freedom that describe the solution to a discretized partial differential equation (pde), while causing only a marginal increase in representation error relative to the standard discretization. Representing a solution as a piecewise-d-linear function on a conventional d-dimensional grid of mesh width h requires $\mathcal{O}(h^{-d})$ degrees of freedom, while the representation error is $\mathcal{O}(h^2)$. The piecewise-d-linear sparse-grid representation requires only $\mathcal{O}(h^{-1}(\log h^{-1})^{d-1})$ degrees of freedom. In fact, this is only a one-dimensional complexity, while the representation error is $\mathcal{O}(h^2(\log h^{-1})^{d-1})$, which is only slightly worse than for the conventional, full-grid representation. In 1992, Griebel, Schneider and Zenger [2] showed that, for two and three dimensions, the sparse-grid complexity and representation error can also be achieved by the so-called combination technique. This technique combines $\mathcal{O}((\log h^{-1})^{d-1})$ representations on conventional grids of different mesh widths in different directions, each containing $\mathcal{O}(h^{-1})$ points, into a representation on the conventional, full grid. One advantage of the combination technique relative to the sparse-grid technique, as introduced in [1], is that the former involves a straightforward discretization and solution of the pde's on the $\mathcal{O}((\log h^{-1})^{d-1})$ conventional grids while the latter requires discretization through a set of hierarchical basis functions, leading to a linear algebra problem with nearly full matrix. Since the problems to be solved on the $\mathcal{O}((\log h^{-1})^{d-1})$ conventional grids are all independent of each other, the combination technique is inherently parallelizable.

In the current work, combination techniques, for two- and three-dimensional functions, are analyzed in detail. In particular, expressions for the corresponding representation errors are derived. Within the current setup, only a single two-dimensional combination technique yields a representation error of order $\mathcal{O}(h^2 \log h^{-1})$. Likewise, only one three-dimensional combination technique yields a representation error of order $\mathcal{O}(h^2(\log h^{-1})^2)$. For these techniques, pointwise expressions for the representation

errors are obtained. The expressions are power series that describe the errors without approximation, thus allowing a derivation of leading-order terms. Furthermore, a heuristic error analysis is given for the representation of two-dimensional discontinuous functions. It is shown that for a two-dimensional step function, the L_1-norm of the representation error is $\mathcal{O}(h^{1/2})$. Contrary to [2], the present derivations do not rely on the error results for sparse grids, as given in [1]. Instead, direct analyses are given of the steps that comprise the combination technique. An important advantage of the current approach is that for smooth functions, explicit expressions for the representation error are obtained, instead of just order estimates. Numerical results that confirm the analyses are presented.

The work is directed towards the numerical solution of large-scale transport problems, governed by systems of partial differential equations of the advection-diffusion-reaction type. These equations play a prominent role in the mathematical modeling of pollution of, e.g., atmospheric air, surface water and ground water. The three-dimensional nature of these models and the necessity of modeling transport and chemical reactions between different species over long time spans, requires very efficient algorithms. When using full-grid methods, computer capacity (computing time and memory) is and will probably remain to be a severe limiting factor. Sparse-grid methods hold out the promise of alleviating these limitations.

In order to successfully implement sparse-grid methods for complex time-dependent problems, a good understanding of the interaction between sparse-grid representation errors, discretization errors and time-integration errors is crucial. The current derivations yield expressions for the sparse-grid representation error that are sufficiently detailed to be used for the study of this interaction.

1.2 The combination technique

The two-dimensional combination technique is based on a grid of grids as shown in Figure 1.

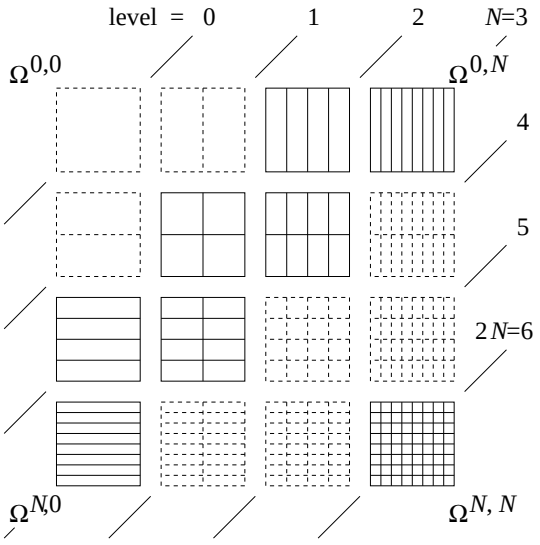

Figure 1: Grid of grids

The task at hand is to express a given function $f(x, y)$ on the grids $\Omega^{N,0}, \Omega^{N-1,1}, \ldots, \Omega^{0,N}$ and on $\Omega^{N-1,0}, \Omega^{N-2,1}, \ldots, \Omega^{0,N-1}$ and then to construct from these coarse representations a representation $\hat{f}^{N,N}$ on the grid $\Omega^{N,N}$. Throughout, upper indices label grids and lower indices label grid-point coordinates within a grid. In sparse-grid literature, it is common to use vertex-centered grids. Yet, for our future application we intend to use cell-centered grids and therefore the current work deals solely with cell-centered grids, i.e., grid-function values are located in cell centers. Furthermore, grids extend over the unit square in two dimensions and over the unit cube in three dimensions. In two dimensions,

the total number of degrees of freedom contained in the coarse representations, for two-dimensions, is given by $2^N(N-1)+1$, as can be seen by simply counting the total number of cells. The test procedure comprises the following steps:

1. The given function is restricted to the coarse grids $\Omega^{N,0},\ldots,\Omega^{0,N},\Omega^{N-1,0},\ldots,\Omega^{0,N-1}$.
2. The information on the coarse grids is used to construct a representation $\hat{f}^{N,N}$ on the finest grid.
3. The representation error is determined by comparing the representation $\hat{f}^{N,N}$ with $f^{N,N}$, i.e., with the function $f(x,y)$ directly restricted to the grid $\Omega^{N,N}$.

All restrictions are done by injection, i.e., to a cell $\Omega^{l,m}_{i,j}$, a function value

$$f^{l,m}_{i,j} \equiv f(x^l_i, y^m_j) \equiv f\left((i+\frac{1}{2})2^{-l}, (j+\frac{1}{2})2^{-m}\right)$$

is assigned. In step 2, the fine-grid representation is not found directly from the coarse-grid representations. Rather, given the representations on $\{\Omega^{l,m}, l+m = N, N-1\}$, representations on $\{\Omega^{l,m}, l+m = N+1\}$ are generated and this process is then repeated up to $l+m=2N$. Furthermore, representations are not generated from all representations on the previous levels but only from nearest neighbor representations, i.e., the representation $\hat{f}^{l,m}$ is generated only from the representations $\hat{f}^{l,m-1}$, $\hat{f}^{l-1,m}$ and $\hat{f}^{l-1,m-1}$.

2. Error accumulation

2.1 Introduction

In the following we analyze the *representation error* $E^{l,m}$, which we define as

$$E^{l,m} \equiv \hat{f}^{l,m} - f^{l,m}. \tag{2.1}$$

The quantity that we are interested in is $E^{N,N}$, the representation error on the finest grid. At this point, we introduce *prolongation operators* $P^{l,m}$ which are linear operators that map grid functions from a grid $\Omega^{l',m'}$ into grid functions on the finer grid $\Omega^{l,m}$ $(l \geq l', m \geq m')$. We consider representations that satisfy the following relation

$$\hat{f}^{l,m} = \begin{cases} f^{l,m}, & \text{for } l+m \leq N, \\ \alpha P^{l,m}\hat{f}^{l-1,m} + \beta P^{l,m}\hat{f}^{l,m-1} + \gamma P^{l,m}\hat{f}^{l-1,m-1}, & \text{for } l+m > N. \end{cases} \tag{2.2}$$

The coefficients α, β and γ, together with the choice of prolongation operator $P^{l,m}$, define the combination scheme. In Section 3, it will be shown that the choice $\alpha=\beta=1$, $\gamma=-1$ causes a number of error terms to cancel, leading to a representation error of the desired order, $E^{N,N} = \mathcal{O}(h^2 \log h^{-1})$. We denote this choice by the $[1,1,-1]$ scheme. Likewise, $[\frac{1}{2},\frac{1}{2},0]$ and $[0,0,-1]$ schemes are considered.

The *local error* $e^{l,m}$ is defined according to

$$e^{l,m} \equiv \alpha P^{l,m} f^{l-1,m} + \beta P^{l,m} f^{l,m-1} + \gamma P^{l,m} f^{l-1,m-1} - f^{l,m}, \tag{2.3}$$

in terms of which the following recursive relation for $E^{l,m}$ is obtained

$$E^{l,m} = e^{l,m} + \alpha P^{l,m} E^{l-1,m} + \beta P^{l,m} E^{l,m-1} + \gamma P^{l,m} E^{l-1,m-1}. \tag{2.4}$$

Equation (2.4) shows that to find the representation error $E^{N,N}$ we have to find an expression for the local error $e^{l,m}$ and solve $E^{N,N}$ from (2.4) such that $E^{N,N}$ is expressed solely in terms of local errors.

In the remainder of this section, we obtain expressions for the representation error $E^{N,N}$ in terms of local errors $e^{l,m}$ by solving the recursive relation (2.4) for the $[\frac{1}{2},\frac{1}{2},0]$, the $[1,1,-1]$ and the $[0,0,-1]$ schemes. Furthermore, it is shown that these schemes can also be replaced by equivalent direct schemes that directly prolongate the coarse representations on $\Omega^{N,0},\ldots,\Omega^{0,N},\Omega^{N-1,0},\ldots,\Omega^{0,N-1}$ onto the finest grid $\Omega^{N,N}$.

2.2 The $[\frac{1}{2},\frac{1}{2},0]$ combination scheme
For the $[\frac{1}{2},\frac{1}{2},0]$ *combination scheme*, the recursive relation (2.4) reduces to

$$E^{l,m} = e^{l,m} + \frac{1}{2}P^{l,m}E^{l-1,m} + \frac{1}{2}P^{l,m}E^{l,m-1}. \tag{2.5}$$

Using (2.5) and the fact that $E^{l,m} = 0$ for $l + m \leq N$, we prove the following theorem

Theorem 1 *For $\alpha = \beta = \frac{1}{2}, \gamma = 0$, the sparse-grid representation error on the finest grid is given by*

$$E^{N,N} = \sum_{l=1}^{N}\sum_{m=1}^{l} 2^{l+m-2N}\left(\begin{array}{c} 2N-l-m \\ N-l \end{array}\right) P^{N,N}e^{l,m}. \tag{2.6}$$

Proof:
Assume that

$$E^{N,N} = \sum_{n=0}^{m-1} 2^{-n}\sum_{i=0}^{n}\left(\begin{array}{c} n \\ i \end{array}\right) P^{N,N}e^{N-i,N-n+i} + 2^{-m}\sum_{i=0}^{m}\left(\begin{array}{c} m \\ i \end{array}\right) P^{N,N}E^{N-i,N-m+i} \tag{2.7}$$

holds for a certain m. (Note that it is true for $m = 1$ because then it simply reduces to (2.5).) Then, by substituting (2.5) into (2.7), we obtain

$$\begin{aligned}
E^{N,N} &- \textstyle\sum_{n=0}^{m-1} 2^{-n}\sum_{i=0}^{n}\left(\begin{array}{c} n \\ i \end{array}\right) P^{N,N}e^{N-i,N-n+i} - 2^{-m}\sum_{i=0}^{m}\left(\begin{array}{c} m \\ i \end{array}\right) P^{N,N}e^{N-i,N-m+i} \\
&= 2^{-(m+1)}\textstyle\sum_{i=0}^{m}\left(\begin{array}{c} m \\ i \end{array}\right)\left(P^{N,N}E^{N-i-1,N-m+i} + P^{N,N}E^{N-i,N-m+i-1}\right) \\
&= 2^{-(m+1)}\textstyle\sum_{i=1}^{m+1}\left(\begin{array}{c} m \\ i-1 \end{array}\right) P^{N,N}E^{N-i,N-m+i-1} \\
&\quad +2^{-(m+1)}\textstyle\sum_{i=0}^{m}\left(\begin{array}{c} m \\ i \end{array}\right) P^{N,N}E^{N-i,N-m+i-1} \\
&= 2^{-(m+1)}\textstyle\sum_{i=1}^{m}\left(\left(\begin{array}{c} m \\ i-1 \end{array}\right) + \left(\begin{array}{c} m \\ i \end{array}\right)\right) P^{N,N}E^{N-i,N-m+i-1} \\
&\quad +2^{-(m+1)}\left(P^{N,N}E^{N-m-1,N-m} + P^{N,N}E^{N,N-m-1}\right) \\
&= 2^{-(m+1)}\textstyle\sum_{i=1}^{m}\left(\begin{array}{c} m+1 \\ i \end{array}\right) P^{N,N}E^{N-i,N-m+i-1} \\
&\quad +2^{-(m+1)}\left(\left(\begin{array}{c} m+1 \\ m+1 \end{array}\right) P^{N,N}E^{N-m-1,N-m} + \left(\begin{array}{c} m+1 \\ 0 \end{array}\right) P^{N,N}E^{N,N-m-1}\right) \\
&= 2^{-(m+1)}\textstyle\sum_{i=0}^{m+1}\left(\begin{array}{c} m+1 \\ i \end{array}\right) P^{N,N}E^{N-i,N-(m+1)+i}
\end{aligned} \tag{2.8}$$

and thus

$$E^{N,N} = \sum_{n=0}^{m} 2^{-n}\sum_{i=0}^{n}\left(\begin{array}{c} n \\ i \end{array}\right) P^{N,N}e^{N-i,N-n+i} + 2^{-(m+1)}\sum_{i=0}^{m+1}\left(\begin{array}{c} m+1 \\ i \end{array}\right) P^{N,N}E^{N-i,N-(m+1)+i}. \tag{2.9}$$

Therefore, if (2.7) holds for m, then it holds for $m+1$ and since it is true for $m = 1$ it follows that (2.7) holds for all $m \geq 1$. Substituting $m = N$ into (2.7) and using the fact that $E^{l,m} = 0$ for $l + m \leq N$, yields

$$E^{N,N} = \sum_{n=0}^{N-1} 2^{-n}\sum_{i=0}^{n}\left(\begin{array}{c} n \\ i \end{array}\right) P^{N,N}e^{N-i,N-n+i}, \tag{2.10}$$

which, after substituting $l = N - i$ and $m = N - n + i$, yields (2.6). □

Theorem 2 *For $\alpha = \beta = \frac{1}{2}, \gamma = 0$, the representation on the finest grid is given by*

$$\hat{f}^{N,N} = 2^{-N} \sum_{l=0}^{N} \begin{pmatrix} N \\ N-l \end{pmatrix} P^{N,N} f^{l,N-l}. \tag{2.11}$$

Proof: Assume that

$$\hat{f}^{N,N} = 2^{-m} \sum_{i=0}^{m} \begin{pmatrix} m \\ i \end{pmatrix} P^{N,N} \hat{f}^{N-i,N-m+i} \tag{2.12}$$

holds for a certain m. (Note that it holds for $m = 1$ because then it reduces to (2.2).) Then, by substituting (2.2) into (2.12), we obtain,

$$\begin{aligned}
\hat{f}^{N,N} &= 2^{-m} \textstyle\sum_{i=0}^{m} \begin{pmatrix} m \\ i \end{pmatrix} \frac{1}{2} \left(P^{N,N} \hat{f}^{N-i-1,N-m+i} + P^{N,N} \hat{f}^{N-i,N-m+i-1} \right) \\
&= 2^{-(m+1)} \left(\textstyle\sum_{i=1}^{m+1} \begin{pmatrix} m \\ i-1 \end{pmatrix} P^{N,N} \hat{f}^{N-i,N-(m+1)+i} + \sum_{i=0}^{m} P^{N,N} \hat{f}^{N-i,N-(m+1)+i} \right) \\
&= 2^{-(m+1)} \textstyle\sum_{i=1}^{m} \begin{pmatrix} m \\ i-1 \end{pmatrix} \begin{pmatrix} m \\ i \end{pmatrix} P^{N,N} \hat{f}^{N-i,N-(m+1)+i} \\
&\quad + P^{N,N} \hat{f}^{N,N-(m+1)} + P^{N,N} \hat{f}^{N-(m+1),N} \\
&= 2^{-(m+1)} \textstyle\sum_{i=0}^{m+1} \begin{pmatrix} m+1 \\ i \end{pmatrix} P^{N,N} \hat{f}^{N-i,N-(m+1)+i}.
\end{aligned} \tag{2.13}$$

Therefore, if (2.12) holds for m, then it holds for $m+1$ and since it is true for $m = 1$ it follows that (2.12) holds for all $m \geq 1$. Substituting $m = N$ into (2.7) and using the fact that $\hat{f}^{l,m} = f^{l,m}$ for $l + m \leq N$, yields

$$\hat{f}^{N,N} = 2^{-N} \sum_{i=0}^{N} \begin{pmatrix} N \\ i \end{pmatrix} P^{N,N} f^{N-i,i}, \tag{2.14}$$

which is equivalent to (2.11). □

2.3 The $[1, 1, -1]$ combination scheme

For the $[1, 1, -1]$ *combination scheme*, the recursive relation (2.4) reads

$$E^{l,m} = e^{l,m} + P^{l,m} E^{l-1,m} + P^{l,m} E^{l,m-1} - P^{l,m} E^{l-1,m-1}. \tag{2.15}$$

Using (2.15), we proof the following theorem

Theorem 3 *For $\alpha = \beta = 1, \gamma = -1$, the representation error on the finest grid is given by*

$$E^{N,N} = \sum_{l=1}^{N} \sum_{m=1}^{l} P^{N,N} e^{l,m}. \tag{2.16}$$

Proof Assume that

$$E^{N,N} = \sum_{n=0}^{m-1} \sum_{i=0}^{n} P^{N,N} e^{N-i,N-n+i} + \sum_{i=0}^{m} P^{N,N} E^{N-i,N-m+i} - \sum_{i=1}^{m} P^{N,N} E^{N-i,N-m+i-1} \tag{2.17}$$

holds for a certain m. (Note that it is true for $m = 1$ because then it reduces to (2.15).) Then, by substituting (2.15) into (2.17), we obtain

$$\begin{aligned} E^{N,N} &= \sum_{n=0}^{m}\sum_{i=0}^{n} P^{N,N} e^{N-i,N-n+i} \\ &= \sum_{i=0}^{m}\left(P^{N,N}E^{N-i-1,N-m+i} + P^{N,N}E^{N-i,N-m+i-1} - P^{N,N}E^{N-i-1,N-m+i-1}\right) \\ &\quad - \sum_{i=1}^{m} P^{N,N}E^{N-i,N-m+i-1} \\ &= \sum_{i=0}^{m} P^{N,N}E^{N-i-1,N-m+i} - \sum_{i=0}^{m} P^{N,N}E^{N-i-1,N-m+i-1} + P^{N,N}E^{N,N-m-1} \\ &= \sum_{i=1}^{m+1} P^{N,N}E^{N-i,N-m+i-1} - \sum_{i=1}^{m+1} P^{N,N}E^{N-i,N-m+i-2} + P^{N,N}E^{N,N-m-1}, \end{aligned} \tag{2.18}$$

hence,

$$E^{N,N} = \sum_{n=0}^{m}\sum_{i=0}^{n} P^{N,N} e^{N-i,N-n+i} + \sum_{i=0}^{m+1} P^{N,N}E^{N-i,N-(m+1)+i} - \sum_{i=1}^{m+1} P^{N,N}E^{N-i,N-(m+1)+i-1}. \tag{2.19}$$

Thus, if (2.17) holds for m, then it holds for $m+1$ and since it holds for $m = 1$, it follows that (2.17) holds for all $m \geq 1$. Substituting $m = N$ into (2.17) and using the fact that $E^{l,m} = 0$ for $l + m \leq N$, yields

$$E^{N,N} = \sum_{n=0}^{N-1}\sum_{i=0}^{n} P^{N,N} e^{N-i,N-n+i}, \tag{2.20}$$

which is equivalent to (2.16). □

Theorem 4 *For $\alpha = \beta = 1, \gamma = -1$, the representation on the finest grid is given by*

$$\hat{f}^{N,N} = \sum_{l=0}^{N} P^{N,N} f^{l,N-l} - \sum_{l=0}^{N-1} P^{N,N} f^{l,N-1-l}, \tag{2.21}$$

Proof: The proof is given by induction. Assume that

$$\hat{f}^{N,N} = \sum_{i=0}^{m} P^{N,N} f^{N-i,N-m+i} - \sum_{i=0}^{m-1} P^{N,N} f^{N-1-i,N-m+i} \tag{2.22}$$

holds for a certain m. (Note that it holds for $m = 1$ because then it reduces to (2.2).) Then, by substituting (2.2) into (2.22), we obtain

$$\begin{aligned} \hat{f}^{N,N} &= \sum_{i=0}^{m}\left(P^{N,N}f^{N-i-1,N-m+i} + P^{N,N}f^{N-i,N-m+i-1} - P^{N,N}f^{N-i-1,N-m+i-1}\right) \\ &\quad - \sum_{i=0}^{m-1} P^{N,N}f^{N-1-i,N-m+i} \\ &= \sum_{i=0}^{m}\left(P^{N,N}f^{N-i,N-m+i-1} - P^{N,N}f^{N-i-1,N-m+i-1}\right) + P^{N,N}f^{N-m-1,N} \\ &= \sum_{i=0}^{m+1} P^{N,N}f^{N-i,N-(m+1)+i} - \sum_{i=0}^{m} P^{N,N}f^{N-i-1,N-(m+1)+i}. \end{aligned} \tag{2.23}$$

Therefore, if (2.22) holds for m, then it holds for $m+1$ and since it is true for $m = 1$ it follows that (2.22) holds for all $m \geq 1$. Substituting $m = N$ into (2.17) and using the fact that $\hat{f}^{l,m} = f^{l,m}$ for $l + m \leq N$, yields

$$\hat{f}^{N,N} = \sum_{i=0}^{N} P^{N,N} f^{N-i,i} - \sum_{i=0}^{N-1} P^{N,N} f^{N-1-i,i}, \tag{2.24}$$

which is equivalent to (2.21). □

2.4 The $[0,0,1]$ combination scheme
For $\alpha = \beta = 0$, $\gamma = 1$, the recursive relation (2.4) reduces to

$$E^{l,m} = e^{l,m} + P^{l,m} E^{l-1,m-1}. \tag{2.25}$$

It is straightforward to show that (2.25) leads to

$$E^{N,N} = \sum_{l=\lceil N/2 \rceil + 1}^{N} P^{N,N} e^{l,l}, \tag{2.26}$$

and to

$$\hat{f}^{N,N} = P^{N,N} f^{\lceil N/2 \rceil, \lceil N/2 \rceil}, \tag{2.27}$$

where $\lceil N/2 \rceil$ denotes the integer part of $N/2$.

2.5 Discussion
In the current section, the representation error $E^{N,N}$ was expressed in terms of the local errors $e^{l,m}$ for the $[\frac{1}{2}, \frac{1}{2}, 0]$, the $[1,1,-1]$ and the $[0,0,-1]$ schemes; see equations (2.6), (2.16) and (2.26), respectively. Furthermore, expressions (2.11), (2.21) and (2.27) were obtained. They express the representation $\hat{f}^{N,N}$ directly in terms of the coarse representations $f^{N,0}, f^{N-1,1}, \ldots, f^{0,N}$ and $f^{N-1,0}, f^{N-2,1}, \ldots, f^{0,N-1}$. Equation (2.21) corresponds to the combination technique as introduced in [2]. Inspection of (2.21) shows that the combination technique can be viewed as an extrapolation technique, see [4] and [6] for discussions of the combination technique from the extrapolation point of view. Note that for the $[1,1,-1]$ scheme, the expression for the representation error (2.16) simply states that the representation error $E^{N,N}$ is equal to the sum of the local errors on the grids $\Omega^{l,m}$ satisfying $N > l + m \leq 2N$ (the lower-right half of the grid of grids depicted in Figure 1).

3. Local errors

We now turn to analyzing the local error $e^{l,m}$ for two-dimensional functions f, i.e., we will determine the error that we make when we approximate a grid function $f^{l,m}$ by the combination

$$\alpha P^{l,m} f^{l-1,m} + \beta P^{l,m} f^{l,m-1} + \gamma P^{l,m} f^{l-1,m-1}. \tag{3.1}$$

In Figure 2, corresponding sections from the grids $\Omega^{l-1,m}$, $\Omega^{l,m-1}$, $\Omega^{l-1,m-1}$ and $\Omega^{l,m}$ are shown. The squares mark locations for which function values are defined on $\Omega^{l-1,m-1}$. Likewise, the circles and the diamonds belong to $\Omega^{l-1,m}$ and $\Omega^{l,m-1}$, respectively. The cross ($\times$) represents the location of the cell center, on $\Omega^{l,m}$, at which the combination (3.1) will be generated. For the prolongations $P^{l,m} f^{l-1,m}$, $P^{l,m} f^{l,m-1}$ and $P^{l,m} f^{l-1,m-1}$, we take linear combinations of the function values on grids $\Omega^{l-1,m}, \Omega^{l,m-1}$ and $\Omega^{l-1,m-1}$, respectively, i.e.,

$$\left(P^{l,m} f^{l',m'} \right)_{i_\times, j_\times} = \sum_{i',j'} \psi^{l'-l,m'-m}_{i',j'} f^{l',m'}_{i',j'}. \tag{3.2}$$

Note that in Figure 2 both $i_\times$ and $j_\times$ are even; the $\psi^{l'-l,m'-m}_{i',j'}$ in (3.2) also correspond to this case, the dependence of the $\psi^{l'-l,m'-m}_{i',j'}$ on $i_\times$ and $j_\times$ is suppressed in the notation. The function values $f^{l',m'}_{i',j'}$ at positions $(x^{m'}_{i'}, y^{l'}_{j'})$, corresponding to the squares, circles and diamonds, are expressed as Taylor series taken at the location of the cross ($\times$), yielding

$$f^{l',m'}_{i',j'} = \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \left(X^{l'-l,m'-m}_{i',j'} \frac{\Delta x^{l'}}{2} \right)^p \left(Y^{l'-l,m'-m}_{i',j'} \frac{\Delta y^{m'}}{2} \right)^q \frac{\partial_x^p \partial_y^q f^{l',m'}_{i_\times, j_\times}}{p! q!}, \tag{3.3}$$

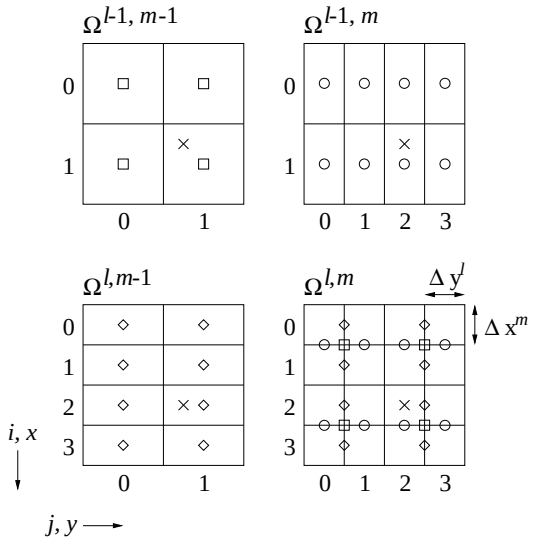

Figure 2: Sections of grids involved in combination

where

$$\left[X^{-1,-1}\right] = \left[Y^{-1,-1}\right]^T = \begin{pmatrix} -3 & -3 \\ 1 & 1 \end{pmatrix}, \quad \left[X^{-1,0}\right] = \left[Y^{0,-1}\right]^T = \begin{pmatrix} -4 & -4 \\ -2 & -2 \\ 0 & 0 \\ 2 & 2 \end{pmatrix},$$
$$\left[X^{0,-1}\right] = \left[Y^{-1,0}\right]^T = \begin{pmatrix} -3 & -3 & -3 & -3 \\ 1 & 1 & 1 & 1 \end{pmatrix}. \tag{3.4}$$

Note that $X^{l'-l,m'-m}_{i',j'}$ and $Y^{l'-l,m'-m}_{i',j'}$ are scalars; they are elements of the matrices $\left[X^{l'-l,m'-m}\right]$ and $\left[Y^{l'-l,m'-m}\right]$, respectively. The indices on the matrix elements start at zero, i.e.,

$$[A] = \begin{pmatrix} A_{0,0} & A_{0,1} & \cdots \\ A_{1,0} & A_{1,1} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}.$$

Again, the matrices $\left[X^{l'-l,m'-m}\right]$ and $\left[Y^{l'-l,m'-m}\right]$, as given by (3.4), are valid when $i_\times$ and $j_\times$ are both even, as in Figure 2. Combining equations (2.3), (3.2) and (3.3), the following expression for the local error is obtained,

$$e^{l,m}_{i_\times,j_\times} = -f^{l,m}_{i_\times,j_\times} + \sum_{p=0}^{\infty}\sum_{q=0}^{\infty} \phi_{p,q} \left(\frac{(-1)^{i_\times}\Delta x^l}{2}\right)^p \left(\frac{(-1)^{j_\times}\Delta y^m}{2}\right)^q \frac{\partial_x^p \partial_y^q f^{l,m}_{i_\times,j_\times}}{p!q!}, \tag{3.5a}$$

$$\begin{aligned} \phi_{p,q} \quad &\equiv \quad \alpha \textstyle\sum_{i=0}^{3}\sum_{j=0}^{1} \psi^{-1,0}_{i,j} \left(X^{-1,0}_{i,j}\right)^p \left(Y^{-1,0}_{i,j}\right)^q \\ &+ \quad \beta \textstyle\sum_{i=0}^{1}\sum_{j=0}^{3} \psi^{0,-1}_{i,j} \left(X^{0,-1}_{i,j}\right)^p \left(Y^{0,-1}_{i,j}\right)^q \\ &+ \quad \gamma \textstyle\sum_{i=0}^{1}\sum_{j=0}^{1} \psi^{-1,-1}_{i,j} \left(X^{-1,-1}_{i,j}\right)^p \left(Y^{-1,-1}_{i,j}\right)^q. \end{aligned} \tag{3.5b}$$

The factors $(-1)^{i_\times}$ and $(-1)^{j_\times}$ have been inserted to ensure that (3.5a) is valid for arbitrary $i_\times$ and $j_\times$ while $\left[\psi^{l'-l,m'-m}\right]$, $\left[X^{l'-l,m'-m}\right]$ and $\left[Y^{l'-l,m'-m}\right]$ are taken to correspond to even $i_\times$ and $j_\times$. We refer to (3.5a) as the *error expansion*. We will now work out the error coefficients $\phi_{p,q}$ for two specific prolongations, i.e., for specific choices of the interpolation weights $\psi^{l',m'}_{i',j'}$.

3.1 Piecewise-constant interpolation

For the prolongations, the simplest choice is piecewise-constant interpolation, which amounts to taking

$$\left[\psi^{-1,-1}\right] = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \left[\psi^{0,-1}\right] = \left[\psi^{-1,0}\right]^T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{3.6}$$

From (3.5b), we find that this leads to

$$[\phi] = \begin{pmatrix} \alpha+\beta+\gamma & \alpha+\gamma & \alpha+\gamma & \cdots \\ \beta+\gamma & \gamma & \gamma & \cdots \\ \beta+\gamma & \gamma & \gamma & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{3.7}$$

and therefore, according to the error expansion (3.5a), to

$$\begin{aligned} e^{l,m}_{i_\times,j_\times} = (\alpha+\beta+\gamma-1)\, f^{l,m}_{i_\times,j_\times} \quad & + (\beta+\gamma)\sum_{p=1}^{\infty}\left((-1)^{i_\times}\tfrac{\Delta x^l}{2}\right)^p \tfrac{\partial_x^p f^{l,m}_{i_\times,j_\times}}{p!} \\ & + (\alpha+\gamma)\sum_{q=1}^{\infty}\left((-1)^{j_\times}\tfrac{\Delta y^m}{2}\right)^q \tfrac{\partial_y^q f^{l,m}_{i_\times,j_\times}}{q!} \\ & + \gamma\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\left((-1)^{i_\times}\tfrac{\Delta x^l}{2}\right)^p\left((-1)^{j_\times}\tfrac{\Delta y^m}{2}\right)^q \tfrac{\partial_x^p\partial_y^q f^{l,m}_{i_\times,j_\times}}{p!q!}. \end{aligned} \tag{3.8}$$

From (3.8), it is apparent that, to obtain consistency, $\alpha+\beta+\gamma = 1$ must hold.

The $[\frac{1}{2},\frac{1}{2},0]$ piecewise-constant scheme. For a combination scheme that requires representation on only a single level of grids, either $\gamma = 0$ or $\alpha = \beta = 0$ must hold. In principle, the choice $\gamma = 0$ leaves us the freedom of choosing α and β, provided they satisfy $\alpha+\beta = 1$. However, we only consider the choice $\alpha = \beta = \frac{1}{2}$. This choice is not completely arbitrary; it provides a symmetric dependence of the local error $e^{l,m}$ on Δx^l and Δy^m. We thus obtain the $[\frac{1}{2},\frac{1}{2},0]$ *piecewise-constant scheme*, to which corresponds the following local error

$$e^{l,m}_{i_\times,j_\times} = \frac{1}{2}\sum_{p=1}^{\infty}\left((-1)^{i_\times}\frac{\Delta x^l}{2}\right)^p \frac{\partial_x^p f^{l,m}_{i_\times,j_\times}}{p!} + \frac{1}{2}\sum_{q=1}^{\infty}\left((-1)^{j_\times}\frac{\Delta y^m}{2}\right)^q \frac{\partial_y^q f^{l,m}_{i_\times,j_\times}}{q!}. \tag{3.9}$$

Using (3.9) and (3.2), we obtain the following for $\left\|P^{N,N}e^{l,m}\right\|_\infty$

$$\begin{aligned} \left\|P^{N,N}e^{l,m}\right\|_\infty &= \Big\| \tfrac{1}{2}\textstyle\sum_{p=1}^{\infty}\tfrac{1}{p!}\left(\tfrac{\Delta x^l}{2}\right)^p \sum_{i',j'}\psi^{l-N,m-N}_{i',j'}(-1)^{i'p}\partial_x^p f^{l,m}_{i',j'} \\ &\quad + \tfrac{1}{2}\textstyle\sum_{q=1}^{\infty}\tfrac{1}{q!}\left(\tfrac{\Delta y^m}{2}\right)^q \sum_{i',j'}\psi^{l-N,m-N}_{i',j'}(-1)^{j'q}\partial_y^q f^{l,m}_{i',j'}\Big\|_\infty \\ &\le \tfrac{1}{2}\textstyle\sum_{p=1}^{\infty}\tfrac{1}{p!}\left(\tfrac{\Delta x^l}{2}\right)^p \left\|\partial_x^p f\right\|_\infty + \tfrac{1}{2}\sum_{q=1}^{\infty}\tfrac{1}{q!}\left(\tfrac{\Delta y^m}{2}\right)^q \left\|\partial_y^q f\right\|_\infty \\ &= \tfrac{1}{2}\textstyle\sum_{p=1}^{\infty}\tfrac{1}{p!}\left\{\left(\tfrac{\Delta x^l}{2}\right)^p \left\|\partial_x^p f\right\|_\infty + \left(\tfrac{\Delta y^m}{2}\right)^p \left\|\partial_y^p f\right\|_\infty\right\}. \end{aligned} \tag{3.10}$$

To obtain the desired expression for $E^{N,N}$, equation (3.10) is now substituted into (2.6), yielding

$$\begin{aligned} \left\|E^{N,N}\right\|_\infty \le\ & \tfrac{1}{2}\textstyle\sum_{p=1}^{\infty}\tfrac{1}{p!}\sum_{l=1}^{N}\sum_{m=1}^{l} 2^{l+m-2N}\begin{pmatrix} 2N-l-m \\ N-l \end{pmatrix} \\ & \left\{\left(\tfrac{\Delta x^l}{2}\right)^p \left\|\partial_x^p f\right\|_\infty + \left(\tfrac{\Delta y^m}{2}\right)^p \left\|\partial_y^p f\right\|_\infty\right\}. \end{aligned} \tag{3.11}$$

Since the grids $\Omega^{l,m}$ extend over the unit square, we can write $\Delta x^l = 2^{-l}$ and $\Delta y^m = 2^{-m}$. Substitution of these relations into (3.11) gives

$$\begin{aligned} \left\|E^{N,N}\right\|_\infty &\leq \tfrac{1}{2}\sum_{p=1}^{\infty}\tfrac{2^{-p}}{p!}\sum_{l=1}^{N}\sum_{m=0}^{l} 2^{l+m-2N}\begin{pmatrix} 2N-l-m \\ N-l \end{pmatrix} \\ &\quad \left\{2^{-lp}\left\|\partial_x^p f\right\|_\infty + 2^{-mp}\left\|\partial_y^p f\right\|_\infty\right\}. \end{aligned} \tag{3.12}$$

Performing the summations over l and m yields

$$\left\|E^{N,N}\right\|_\infty \leq \sum_{p=1}^{\infty}\frac{2^{-(N+1)p}}{p!}\frac{1-2^{-N}\left(2^p+1\right)^N}{1-2^p}\left\{\left\|\partial_x^p f\right\|_\infty + \left\|\partial_y^p f\right\|_\infty\right\}. \tag{3.13}$$

Writing out the first few terms of this error expansion gives

$$\begin{aligned} \left\|E^{N,N}\right\|_\infty &\leq \tfrac{1}{2}\left(\left(\tfrac{3}{4}\right)^N - \left(\tfrac{1}{2}\right)^N\right)\left\{\left\|\partial_x f\right\|_\infty + \left\|\partial_y f\right\|_\infty\right\} \\ &\quad + \tfrac{1}{24}\left(\left(\tfrac{5}{8}\right)^N - \left(\tfrac{1}{4}\right)^N\right)\left\{\left\|\partial_x^2 f\right\|_\infty + \left\|\partial_y^2 f\right\|_\infty\right\} + \cdots, \end{aligned} \tag{3.14}$$

so, to leading order,

$$\left\|E^{N,N}\right\|_\infty \leq \frac{1}{2}\left(\frac{3}{4}\right)^N\left\{\left\|\partial_x f\right\|_\infty + \left\|\partial_y f\right\|_\infty\right\} + \mathcal{O}\left(\left(\frac{5}{8}\right)^N\right). \tag{3.15}$$

On the finest grid $\Omega^{N,N}$, the mesh widths in x- and y-directions are identical, $h = \Delta x^N = \Delta y^N = 2^{-N}$. Rewriting (3.15) in terms of this mesh width yields

$$\left\|E^{N,N}\right\|_\infty \leq \frac{1}{2}h^{(2-\log_2 3)}\left\{\left\|\partial_x f\right\|_\infty + \left\|\partial_y f\right\|_\infty\right\} + \mathcal{O}\left(h^{(3-\log_2 5)}\right). \tag{3.16}$$

Equation (3.16) shows that the $[\frac{1}{2}, \frac{1}{2}, 0]$ piecewise-constant scheme has a representation error of order $2 - \log_2 3 \approx 0.42$.

As a test of the above derivations, we examine the simple case $f(x,y) = x + y$. This case is particularly attractive because it allows us to obtain an explicit expression for $\left\|E^{N,N}\right\|_\infty$ (in contrast to an upper bound). For $f(x,y) = x + y$, equation (3.9) reduces to

$$e^{l,m}_{i_\times, j_\times} = \frac{1}{4}\left((-1)^{i_\times}\Delta x^l + (-1)^{j_\times}\Delta y^m\right) \tag{3.17}$$

and thus

$$E^{N,N}_{i_\times, j_\times} = \frac{1}{4}\sum_{l=1}^{N}\sum_{m=1}^{l} 2^{2N-l-m}\begin{pmatrix} 2N-l-m \\ N-l \end{pmatrix}\sum_{i',j'}\psi^{l-N,m-N}_{i',j'}\left((-1)^{i'}2^{-l} + (-1)^{j'}2^{-m}\right). \tag{3.18}$$

For piecewise-constant interpolation

$$\psi^{l-N,m-N}_{i',j'} = \delta_{\lceil i_\times 2^{l-N}\rceil - i', \lceil j_\times 2^{m-N}\rceil - j'}, \tag{3.19}$$

where δ is the Kronecker delta. Using (3.19), we transform (3.18) into

$$E^{N,N}_{i_\times, j_\times} = \frac{1}{4}\sum_{l=1}^{N}\sum_{m=0}^{l} 2^{l+m-2N}\begin{pmatrix} 2N-l-m \\ N-l \end{pmatrix}\left((-1)^{\lceil i_\times 2^{l-N}\rceil}2^{-l} + (-1)^{\lceil j_\times 2^{m-N}\rceil}2^{-m}\right). \tag{3.20}$$

This expression is maximal for $i_\times = j_\times = 0$, thus

$$\left\|E^{N,N}\right\|_\infty = \frac{1}{4}\sum_{l=1}^{N}\sum_{m=0}^{l} 2^{l+m-2N}\begin{pmatrix} 2N-l-m \\ N-l \end{pmatrix}\left(2^{-l} + 2^{-m}\right) = \left(\frac{3}{4}\right)^N - \left(\frac{1}{2}\right)^N. \tag{3.21}$$

Numerical tests show that, for $f(x,y) = x + y$, the error $\left\|E^{N,N}\right\|_\infty$ is indeed exactly given by $\left(\frac{3}{4}\right)^N - \left(\frac{1}{2}\right)^N$.

$[1,1,-1]$ *piecewise-constant scheme.* Equation (3.8) reveals that when we take $\alpha+\gamma=\beta+\gamma=0$, the error terms that depend only on Δx or only on Δy vanish. Combining these requirements with $\alpha+\beta+\gamma=1$ gives

$$\alpha=\beta=1, \quad \gamma=-1. \tag{3.22}$$

This choice of α, β and γ constitutes the $[1,1,-1]$ *combination scheme.* For the present case, $[1,1,-1]$ *combination with piecewise-constant interpolation*, equation (3.8) yields

$$e^{l,m}_{i_\times,j_\times} = -\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\left((-1)^{i_\times}\frac{\Delta x}{2}\right)^p\left((-1)^{j_\times}\frac{\Delta y}{2}\right)^q\frac{\partial_x^p\partial_y^q f^{l,m}_{i_\times,j_\times}}{p!q!} \tag{3.23}$$

and thus

$$\left\|P^{N,N}e^{l,m}\right\|_\infty \le \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{1}{p!q!}\left(\frac{\Delta x}{2}\right)^p\left(\frac{\Delta y}{2}\right)^q\left\|\partial_x^p\partial_y^q f\right\|_\infty. \tag{3.24}$$

Substitution of (3.24) into (2.16) yields

$$\left\|E^{N,N}\right\|_\infty \le \sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\frac{2^{-p-q}}{p!q!}\left\|\partial_x^p\partial_y^q f\right\|_\infty\sum_{l=1}^{N}\sum_{m=1}^{l}2^{-lp-mq}. \tag{3.25}$$

Asymptotically, this yields

$$\left\|E^{N,N}\right\|_\infty \le \frac{1}{4}\left(\frac{1}{2}\right)^N N\left\|\partial_x\partial_y f\right\|_\infty + \mathcal{O}\left(\left(\frac{1}{2}\right)^N\right), \tag{3.26}$$

in terms of the mesh width h, this becomes

$$\left\|E^{N,N}\right\|_\infty \le \frac{1}{4}h\log_2 h^{-1}\left\|\partial_x\partial_y f\right\|_\infty + \mathcal{O}(h). \tag{3.27}$$

Thus, the $[1,1,-1]$ piecewise-constant scheme has a representation error of order $h\log_2 h^{-1}$.

Again we examine a simple test case, viz. $f(x,y)=xy$, which yields

$$\left\|E^{N,N}\right\|_\infty = \frac{1}{4}\left(\left(\frac{1}{2}\right)^N N - \left(\frac{1}{2}\right)^N + \left(\frac{1}{4}\right)^N\right). \tag{3.28}$$

Numerical results confirm that representation of $f(x,y)=xy$ by the $[1,1,-1]$ piecewise-constant scheme agrees with (3.28) within machine accuracy.

$[0,0,1]$ *piecewise-constant scheme.* We now consider the choice $\alpha=\beta=0$, $\gamma=1$. This choice does not represent a real sparse-grid combination scheme because it constructs $\hat{f}^{N,N}$ from only a single coarse grid-function, e.g., from $f^{\lceil N/2\rceil,\lceil N/2\rceil}$. Yet, we do include the $[0,0,1]$ scheme for comparison, in particular with the $[\frac{1}{2},\frac{1}{2},0]$ scheme. We make this comparison because Hemker [3] pointed out that direct prolongation of $f^{\lceil N/2\rceil,\lceil N/2\rceil}$ should be superior to the $[\frac{1}{2},\frac{1}{2},0]$ scheme. It will turn out that this is indeed true. The $[0,0,1]$ piecewise-constant local error is given by

$$\begin{aligned} e^{l,m}_{i_\times,j_\times} &= \textstyle\sum_{p=1}^{\infty}\left((-1)^{i_\times}\frac{\Delta x^l}{2}\right)^p\frac{\partial_x^p f^{l,m}_{i_\times,j_\times}}{p!} + \sum_{q=1}^{\infty}\left((-1)^{j_\times}\frac{\Delta y^m}{2}\right)^q\frac{\partial_y^q f^{l,m}_{i_\times,j_\times}}{q!} \\ &\quad \textstyle-\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\left((-1)^{i_\times}\frac{\Delta x^l}{2}\right)^p\left((-1)^{j_\times}\frac{\Delta y^m}{2}\right)^q\frac{\partial_x^p\partial_y^q f^{l,m}_{i_\times,j_\times}}{p!q!}, \end{aligned} \tag{3.29}$$

therefore,

$$\begin{aligned}\left\|P^{N,N}e^{l,m}\right\|_\infty &\leq \sum_{p=1}^{\infty}\tfrac{1}{p!}\left(\tfrac{\Delta x^l}{2}\right)^p\left\|\partial_x^p f\right\|_\infty+\sum_{q=1}^{\infty}\tfrac{1}{q!}\left(\tfrac{\Delta y^m}{2}\right)^q\left\|\partial_y^q f\right\|_\infty\\ &\quad+\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\tfrac{1}{p!q!}\left(\tfrac{\Delta x^l}{2}\right)^p\left(\tfrac{\Delta y^m}{2}\right)^q\left\|\partial_x^p\partial_y^q f\right\|_\infty.\end{aligned}\tag{3.30}$$

Substitution of (3.30) into (2.26) yields

$$\begin{aligned}\left\|E^{N,N}\right\|_\infty &\leq \sum_{p=1}^{\infty}\tfrac{2^{-Np}}{p!}\tfrac{1-2^{p(N/2+1)}}{1-2^p}\left\{\left\|\partial_x^p f\right\|_\infty+\left\|\partial_y^p f\right\|_\infty\right\}\\ &\quad+\sum_{p=1}^{\infty}\sum_{q=1}^{\infty}\tfrac{2^{-N(p+q)}}{p!q!}\tfrac{1-2^{(p+q)(N/2+1)}}{1-2^{p+q}}\left\|\partial_x^p\partial_y^q f\right\|_\infty,\end{aligned}\tag{3.31}$$

or, asymptotically,

$$\left\|E^{N,N}\right\|_\infty\leq 2\left(2^{-1/2}\right)^N\left\{\|\partial_x f\|_\infty+\|\partial_y f\|_\infty\right\}+\mathcal{O}\left(\left(\frac{1}{2}\right)^N\right).\tag{3.32}$$

In terms of the mesh width h, this reads

$$\left\|E^{N,N}\right\|_\infty\leq 2h^{1/2}\left\{\|\partial_x f\|_\infty+\|\partial_y f\|_\infty\right\}+\mathcal{O}(h).\tag{3.33}$$

We see that, for piecewise-constant interpolation, the $[0,0,1]$ scheme has a representation error of order $\frac{1}{2}$, which is superior to the order $2-\log_2 3\approx 0.42$ for the $[\frac{1}{2},\frac{1}{2},0]$ piecewise-constant scheme.

3.2 Piecewise bi-linear interpolation

Next, we consider bi-linear interpolation as a means of prolongation. The prolongations are therefore described by the following interpolation weights

$$\left[\psi^{-1,-1}\right]=\begin{pmatrix}\frac{1}{16} & \frac{3}{16}\\ \frac{3}{16} & \frac{9}{16}\end{pmatrix},\quad \left[\psi^{0,-1}\right]=\left[\psi^{-1,0}\right]^T=\begin{pmatrix}0 & 0 & \frac{1}{4} & 0\\ 0 & 0 & \frac{3}{4} & 0\end{pmatrix},\tag{3.34}$$

leading to the following error coefficients

$$[\phi]=\begin{pmatrix}\alpha+\beta+\gamma & 0 & (\beta+\gamma)\lambda_{0,2} & (\beta+\gamma)\lambda_{0,3} & \cdots\\ 0 & 0 & 0 & 0 & \cdots\\ (\alpha+\gamma)\lambda_{2,0} & 0 & \gamma\lambda_{2,2} & \gamma\lambda_{2,3} & \cdots\\ (\alpha+\gamma)\lambda_{3,0} & 0 & \gamma\lambda_{3,2} & \gamma\lambda_{3,3} & \cdots\\ \vdots & \vdots & \vdots & \vdots & \ddots\end{pmatrix},\quad \lambda_{p,q}=\frac{(-3)^p+3}{4}\frac{(-3)^q+3}{4}\tag{3.35}$$

and the following local error expansion

$$\begin{aligned}e^{l,m}_{i_\times,j_\times} &= (\alpha+\beta+\gamma-1)f^{l,m}_{i_\times,j_\times}\\ &\quad+(\alpha+\gamma)\sum_{p=2}^{\infty}\lambda_{p,0}\left((-1)^{i_\times}\tfrac{\Delta x^l}{2}\right)^p\tfrac{\partial_x^p f^{l,m}_{i_\times,j_\times}}{p!}\\ &\quad+(\beta+\gamma)\sum_{q=2}^{\infty}\lambda_{0,q}\left((-1)^{j_\times}\tfrac{\Delta y^m}{2}\right)^q\tfrac{\partial_y^q f^{l,m}_{i_\times,j_\times}}{q!}\\ &\quad+\gamma\sum_{p=2}^{\infty}\sum_{q=2}^{\infty}\lambda_{p,q}\left((-1)^{i_\times}\tfrac{\Delta x^l}{2}\right)^p\left((-1)^{j_\times}\tfrac{\Delta y^m}{2}\right)^q\tfrac{\partial_x^p\partial_y^q f^{l,m}_{i_\times,j_\times}}{p!q!},\\ \lambda_{p,q} &= \tfrac{(-3)^p+3}{4}\tfrac{(-3)^q+3}{4}.\end{aligned}\tag{3.36}$$

Again, for consistency, we must have $\alpha+\beta+\gamma=1$.

$[\frac{1}{2},\frac{1}{2},0]$ *piecewise-bi-linear scheme.* For bi-linear interpolation, the choice $\alpha=\beta=\frac{1}{2},\gamma=0$ gives the following expansion for the local error

$$e^{l,m}_{i_\times,j_\times} = \frac{1}{2}\sum_{p=2}^{\infty}\frac{(-3)^p+3}{4}\left((-1)^{i_\times}\frac{\Delta x^l}{2}\right)^p\frac{\partial_x^p f^{l,m}_{i_\times,j_\times}}{p!} + \frac{1}{2}\sum_{q=2}^{\infty}\frac{(-3)^q+3}{4}\left((-1)^{j_\times}\frac{\Delta y^m}{2}\right)^q\frac{\partial_y^q f^{l,m}_{i_\times,j_\times}}{q!}. \tag{3.37}$$

We write the first terms of the summations separately, yielding

$$\begin{array}{rcl} e^{l,m}_{i_\times,j_\times} & = & \frac{3}{16}\left\{\left(\Delta x^l\right)^2\partial_x^2 f^{l,m}_{i_\times,j_\times} + \left(\Delta y^m\right)^2\partial_y^2 f^{l,m}_{i_\times,j_\times}\right\} \\ & & +\frac{1}{2}\sum_{p=3}^{\infty}\frac{(-3)^p+3}{4p!}\left\{\left((-1)^{i_\times}\frac{\Delta x^l}{2}\right)^p\partial_x^p f^{l,m}_{i_\times,j_\times} + \left((-1)^{j_\times}\frac{\Delta y^m}{2}\right)^p\partial_y^q f^{l,m}_{i_\times,j_\times}\right\}. \end{array} \tag{3.38}$$

For the prolongation of the local error we obtain

$$\begin{array}{rcl} P^{N,N}e^{l,m} & = & \frac{3}{16}\sum_{i',j'}\psi^{l-N,m-N}_{i',j'}\left\{\left(\Delta x^l\right)^2\partial_x^2 f^{l,m}_{i',j'} + \left(\Delta y^m\right)^2\partial_y^2 f^{l,m}_{i',j'}\right\} \\ & & +\frac{1}{2}\sum_{p=3}^{\infty}\frac{(-3)^p+3}{4p!}\sum_{i',j'}\psi^{l-N,m-N}_{i',j'} \\ & & \left\{\left((-1)^{i'}\frac{\Delta x^l}{2}\right)^p\partial_x^p f^{l,m}_{i',j'} + \left((-1)^{j'}\frac{\Delta y^m}{2}\right)^p\partial_y^q f^{l,m}_{i',j'}\right\} \\ & = & \frac{3}{16}\left\{\left(\Delta x^l\right)^2\partial_x^2 f^{N,N} + \left(\Delta y^m\right)^2\partial_y^2 f^{N,N}\right\} + \mathcal{O}\left(\left(\Delta x^l\right)^3 + \left(\Delta y^m\right)^3\right). \end{array} \tag{3.39}$$

In obtaining (3.39), use has been made of the following property of bi-linear interpolation

$$\sum_{i',j'}\psi^{l-N,m-N}_{i',j'}f^{l,m} = P^{N,N}f^{l,m} = f^{N,N} + \mathcal{O}\left(\left(\Delta x^l\right)^2 + \left(\Delta y^m\right)^2\right). \tag{3.40}$$

Substitution of (3.39) into (2.6) yields

$$E^{N,N} = \frac{1}{8}\left(\frac{5}{8}\right)^N\left\{\partial_x^2 f^{N,N} + \partial_y^2 f^{N,N}\right\} + \mathcal{O}\left(\left(\frac{9}{16}\right)^N\right), \tag{3.41}$$

or, in terms of the mesh width h,

$$E^{N,N} = \frac{1}{8}h^{(3-\log_2 5)}\left\{\partial_x^2 f^{N,N} + \partial_y^2 f^{N,N}\right\} + \mathcal{O}\left(h^{(4-\log_2 9)}\right). \tag{3.42}$$

Thus, the $\frac{1}{2},\frac{1}{2}$-bi-linear combination scheme has a representation error of order $3-\log_2 5 \approx 0.68$.

$[1,1,-1]$ *piecewise-bi-linear scheme.* Just as for the piecewise-constant case, taking $\alpha+\gamma=\beta+\gamma=0$ removes the error terms that depend only on Δx or only on Δy. So, again the choice $\alpha=\beta=1,\gamma=-1$ raises the order of the local error. For this choice we obtain

$$e^{l,m}_{i_\times,j_\times} = -\sum_{p=2}^{\infty}\sum_{q=2}^{\infty}\frac{(-3)^p+3}{4}\frac{(-3)^q+3}{4}\left((-1)^{i_\times}\frac{\Delta x^l}{2}\right)^p\left((-1)^{j_\times}\frac{\Delta y^m}{2}\right)^q\frac{\partial_x^p\partial_y^q f^{l,m}_{i_\times,j_\times}}{p!q!}. \tag{3.43}$$

Substitution of (3.43) into (2.16) yields

$$\begin{array}{rcl} E^{N,N} & = & \sum_{p=2}^{\infty}\sum_{q=2}^{\infty}\frac{2^{-p-q}}{p!q!}\frac{(-3)^p+3}{4}\frac{(-3)^q+3}{4} \\ & & \sum_{l=1}^{N}\sum_{m=1}^{l}2^{-lp-mq}\sum_{i_\times,j_\times}\psi^{l-N,m-N}_{i_\times,j_\times}(-1)^{i_\times p+j_\times q}\partial_x^p\partial_y^q f^{l,m}_{i_\times,j_\times}, \end{array} \tag{3.44}$$

which, in leading order, can be written as

$$E^{N,N} = -\frac{3}{64}\left(\frac{1}{4}\right)^N N\partial_x^2\partial_y^2 f^{N,N} + \mathcal{O}\left(\left(\frac{1}{4}\right)^N\right), \tag{3.45}$$

or, in terms of the mesh width h,

$$E^{N,N} = -\frac{3}{64}h^2\log_2 h^{-1}\partial_x^2\partial_y^2 f^{N,N} + \mathcal{O}\left(h^2\right). \tag{3.46}$$

So, the $[1,1,-1]$-bi-linear scheme has a representation error of order $h^2\log_2 h^{-1}$.

$[0,0,1]$ *piecewise-bi-linear scheme.* For $\alpha=\beta=0,\gamma=1$ and prolongation by bi-linear interpolation we obtain

$$P^{N,N}e^{l,m} = \frac{3}{8}\left\{\left(\Delta x^l\right)^2\partial_x^2 f^{N,N} + \left(\Delta y^m\right)^2\partial_y^2 f^{N,N}\right\} + \mathcal{O}\left(\left(\Delta x^l\right)^3 + \left(\Delta y^m\right)^3 + \left(\Delta x^l\right)^2\left(\Delta y^m\right)^2\right). \tag{3.47}$$

Substitution of (3.47) into (2.26) yields, asymptotically,

$$E^{N,N} = 2\left(\frac{1}{2}\right)^N\left\{\partial_x^2 f^{N,N} + \partial_y^2 f^{N,N}\right\} + \mathcal{O}\left(\left(\frac{1}{4}\right)^N\right). \tag{3.48}$$

In terms of the mesh width h, this reads

$$E^{N,N} = 2h\left\{\partial_x^2 f^{N,N} + \partial_y^2 f^{N,N}\right\} + \mathcal{O}\left(h^2\right). \tag{3.49}$$

We see that, for bi-linear interpolation, the $[0,0,1]$ scheme has a representation error of order 1, which is superior to the order $3-\log_2 5 \approx 0.68$ for the $[\frac{1}{2},\frac{1}{2},0]$ bi-linear scheme.

3.3 A numerical test

We now turn to analyzing the representation error, corresponding to the $[1,1,-1]$ piecewise-bi-linear scheme, for a specific example. We take

$$f(x,y) = \sin(\pi x)\sin(\pi y) \tag{3.50}$$

and compare the numerically observed error with the expression for the leading-order error term (3.45) and with the full error expansion (3.44). According to (3.45), the error corresponding to (3.50) is given by

$$E^{N,N} = -\frac{3\pi^4}{64}\left(\frac{1}{4}\right)^N N\sin(\pi x_i^N)\sin(\pi y_j^N) + \mathcal{O}\left(\left(\frac{1}{4}\right)^N\right). \tag{3.51}$$

In Figure 3, the solid line represents the analytical result (3.51) for the leading-order error term, the dotted line represents the numerically observed error. We consider the pointwise error measured at a grid point nearest to $x=y=\frac{1}{2}$ (four grid points qualify, but due to the symmetry of the function this is not a problem). From Figure 3, it appears that the experimental error is indeed converging to the analytical leading-order result as N increases. In Table 1, the ratio $(E^{N-1,N-1}_{\text{analytical}} - E^{N-1,N-1}_{\text{numerical}})/(E^{N,N}_{\text{analytical}} - E^{N,N}_{\text{numerical}})$ is listed for several values of N. Table 1 indicates that $E^{N,N}_{\text{analytical}} - E^{N,N}_{\text{numerical}} = \mathcal{O}\left((1/4)^N\right)$, as it should be according to (3.45). Figure 4 displays $E^{N,N}$ for $N=4,5,6$. In this Figure, we do indeed recognize the product of sines prescribed by (3.51).

As a test of the validity of the error expansion (3.44), the numerically observed error is also compared with higher-order approximations of the error. The expansion (3.44) is evaluated for the test case (3.50) up to $p+q \le 4,5,6,7,8$ and compared with the numerically observed error. The results are displayed in Table 2. Table 2 clearly suggests that the series (3.44) converges to the numerically observed error, as $\max(p+q)$ increases.

3.4 Discussion

In this section, the local errors $e^{l,m}$ were determined for the $[\frac{1}{2},\frac{1}{2},0]$, $[1,1,-1]$ and $[0,0,1]$ piecewise-constant and piecewise-bi-linear schemes. The local errors were inserted into the expressions for the representation error $E^{N,N}$, yielding error results for the six schemes. For the piecewise-constant schemes, upper bounds were given instead of pointwise expressions. The motivation for this is that for pointwise expressions for the piecewise-constant schemes, the summation over the grid of grids cannot be performed due to the factors $(-1)^{i\times p}$ and $(-1)^{j\times q}$ in the local error $e^{l,m}$. This complication does not appear for the bi-linear schemes since for these schemes the leading-order term corresponds to $p=q=2$, which guarantees that $(-1)^{i\times p}=(-1)^{j\times q}=1$.

The $[1,1,-1]$ piecewise-bi-linear scheme is clearly the most interesting of the schemes considered since it has the smallest approximation error. In fact, the $[\frac{1}{2},\frac{1}{2},0]$ and $[0,0,1]$ schemes were only included for comparison, they are not intended for actual use. When the leading-order error result is insufficient (on coarse grids or when higher derivatives are not small), it may be necessary to predict the error with the full error expansion. For the $[1,1,-1]$ piecewise-bi-linear scheme, the full error expansion is given by (3.44).

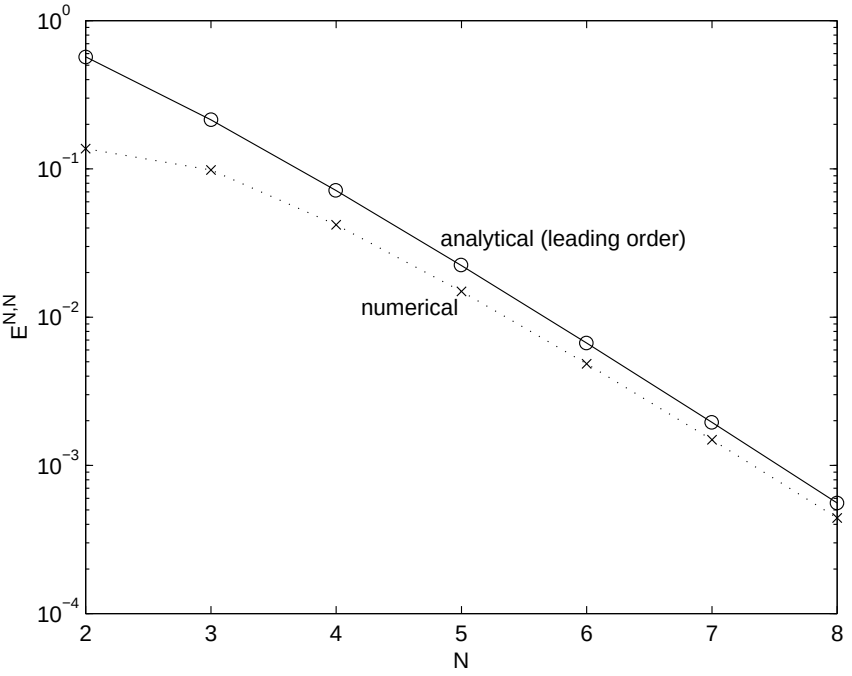

Figure 3: Numerically observed error converges to analytical leading-order result for $N\to\infty$

4. Extension to three dimensions

The current derivation for the sparse-grid representation error can easily be extended to three spatial dimensions. The given function $f(x,y,z)$ is then restricted to grids $\Omega^{l,m,n}$ satisfying $l+m+n=N-2,N-1,N$. The total number of degrees of freedom contained in these grids is given by

N	$\frac{E^{N-1,N-1}_{\text{analytical}} - E^{N-1,N-1}_{\text{numerical}}}{E^{N,N}_{\text{analytical}} - E^{N,N}_{\text{numerical}}}$
3	3.7553
4	3.9332
5	3.9817
6	3.9962
7	3.9984
8	3.9996

Table 1: Orders of convergence

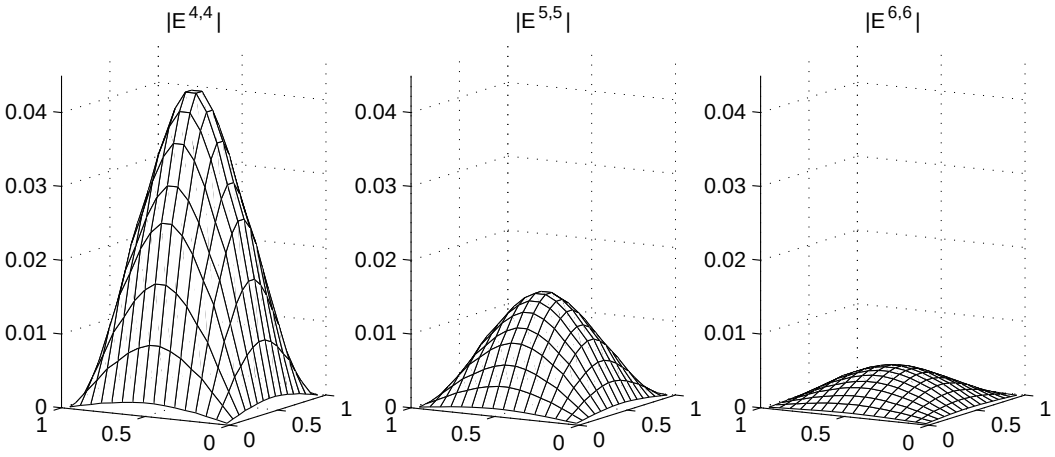

Figure 4: Spatial error distributions for $N = 4, 5, 6$

$\max(p+q)$	$\left\| E^{4,4}_{analytical} - E^{4,4}_{numerical} \right\|_\infty$
4	0.0131
5	0.0068
6	0.0036
7	0.0010
8	0.0005

Table 2: Higher-order error approximations

$2^N\left(N^2-3N+2\right)-1$. The three-dimensional representations are taken to satisfy

$$\begin{array}{lcll}\hat{f}^{l,m,n} & = & \left\{\begin{array}{l} f^{l,m,n}, \\ \sum_{l'=-1}^{0}\sum_{m'=-1}^{0}\sum_{n'=-1}^{0}\alpha^{l',m',n'}P^{l,m,n}\hat{f}^{l+l',m+m',n+n'}\end{array}\right. & \begin{array}{l}\text{for } l+m+n\leq N, \\ \text{for } l+m+n>N,\end{array} \\ \alpha^{0,0,0} & \equiv & 0, & \\ \alpha^{-1,m',n'} & = & 0 \text{ if } l=0, & \\ \alpha^{l',-1,n'} & = & 0 \text{ if } m=0, & \\ \alpha^{l',m',-1} & = & 0 \text{ if } n=0. & \end{array} \tag{4.1}$$

The last three relations ensure that no reference is made to non-existing grid-functions. Note that, due to the last three relations, the coefficients $\alpha^{l',m',n'}$ are now dependent on l, m and n. This dependence is suppressed in the notation. The local error is now given by

$$e^{l,m,n}=\sum_{l'=-1}^{0}\sum_{m'=-1}^{0}\sum_{n'=-1}^{0}\alpha^{l',m',n'}P^{l,m,n}\hat{f}^{l+l',m+m',n+n'}-f^{l,m,n}. \tag{4.2}$$

The recursive relation for $E^{l,m,n}$, for the three-dimensional case, reads

$$E^{l,m,n}=e^{l,m,n}+\sum_{l'=-1}^{0}\sum_{m'=-1}^{0}\sum_{n'=-1}^{0}\alpha^{l',m',n'}P^{l,m,n}E^{l+l',m+m',n+n'}. \tag{4.3}$$

For the two-dimensional case, the optimal combination scheme was found to be $[\alpha=\beta=1,\gamma=-1]$. For the three-dimensional case, the choice

$$\begin{array}{llllll}\alpha^{-1,0,0} & =\alpha^{0,-1,0} & =\alpha^{0,0,-1} & =\alpha^{-1,-1,-1}= & 1 \\ \alpha^{-1,-1,0} & =\alpha^{-1,0,-1} & =\alpha^{0,-1,-1} & = & -1\end{array} \tag{4.4}$$

represents the optimal combination scheme. Analogous to (2.16) for the $[\alpha=\beta=1,\gamma=-1]$ scheme, the combination scheme given by (4.4) leads to

$$E^{N,N,N}=\sum_{\substack{0\leq l,m,n\leq N \\ N<l+m+n}}P^{N,N,N}e^{l,m,n}. \tag{4.5}$$

To evaluate (4.5), we need the following equivalent form

$$\begin{array}{rcl}E^{N,N,N} & = & \sum_{l=1}^{N}\sum_{m=1}^{l}P^{N,N,N}e^{l,m,0} \\ & & +\sum_{l=1}^{N}\sum_{n=1}^{l}P^{N,N,N}e^{l,0,n} \\ & & +\sum_{l=m}^{N}\sum_{n=m}^{l}P^{N,N,N}e^{0,m,n} \\ & & +\left(\sum_{l=1}^{N}\sum_{m=1}^{N}\sum_{n=1}^{N}-\sum_{l=1}^{N-2}\sum_{m=1}^{N-1-l}\sum_{n=1}^{N-l-m}\right)P^{N,N,N}e^{l,m,n}.\end{array} \tag{4.6}$$

Just as for the two-dimensional case, the combination scheme given by (4.1) and (4.4) can be expressed in a direct form that expresses $\hat{f}^{N,N,N}$ directly in terms of the coarse representations $\{\hat{f}^{l,m,n}, l+m+n=N-2,N-1,N\}$. The direct form reads

$$\hat{f}^{N,N,N}=\left(\sum_{\substack{0\leq l,m,n\leq N \\ l+m+n=N}}-2\sum_{\substack{0\leq l,m,n\leq N \\ l+m+n=N-1}}+\sum_{\substack{0\leq l,m,n\leq N \\ l+m+n=N-2}}\right)P^{N,N,N}\hat{f}^{l,m,n}. \tag{4.7}$$

The three-dimensional local error is given by

$$\begin{aligned} e^{l,m,n}_{i_\times,j_\times,k_\times} &= -f^{l,m,n}_{i_\times,j_\times,k_\times} + \textstyle\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\sum_{r=0}^{\infty}\phi_{p,q,r} \\ &\quad \left(\frac{(-1)^{i_\times}\Delta x^l}{2}\right)^p \left(\frac{(-1)^{j_\times}\Delta y^m}{2}\right)^q \left(\frac{(-1)^{k_\times}\Delta z^n}{2}\right)^r \frac{\partial_x^p \partial_y^q \partial_z^r f^{l,m,n}_{i_\times,j_\times,k_\times}}{p!q!r!}, \end{aligned} \tag{4.8a}$$

$$\phi_{p,q,r} \equiv \sum_{l'=-1}^{0}\sum_{m'=-1}^{0}\sum_{n'=-1}^{0}\sum_{i=0}^{1-2l'}\sum_{j=0}^{1-2m'}\sum_{k=0}^{1-2n'} \alpha^{l',m',n'}\psi^{l',m',n'}_{i,j,k}\left(X^{l',m',n'}_{i,j,k}\right)^p\left(Y^{l',m',n'}_{i,j,k}\right)^q\left(Z^{l',m',n'}_{i,j,k}\right)^r, \tag{4.8b}$$

where

$$\begin{aligned} X^{l',m',n'}_{i,j,k} &= -4 - l' + 2(1-l')i, \\ Y^{l',m',n'}_{i,j,k} &= -4 - m' + 2(1-m')j, \\ Z^{l',m',n'}_{i,j,k} &= -4 - n' + 2(1-n')k. \end{aligned} \tag{4.9}$$

4.1 Piecewise-constant interpolation

For piecewise-constant interpolation, the interpolation weights $\psi^{l',m',n'}_{i,j,k}$ are given by

$$\psi^{l',m',n'}_{i,j,k} = \delta_{i-2-l'}\delta_{j-2-m'}\delta_{k-2-n'}. \tag{4.10}$$

Substitution of (4.10) into (4.8b) yields

$$\phi_{p,q,r} = \sum_{l'=-1}^{0}\sum_{m'=-1}^{0}\sum_{n'=-1}^{0}\left(\delta_{l'+1}+\delta_{l'}\delta_p\right)\left(\delta_{m'+1}+\delta_{m'}\delta_q\right)\left(\delta_{n'+1}+\delta_{n'}\delta_r\right)\alpha^{l',m',n'}. \tag{4.11}$$

Substitution of (4.11) into (4.8a) and next of (4.8a) into (4.2) yields

$$\left\|E^{N,N,N}\right\|_\infty \le \frac{1}{16}N^2\left(\frac{1}{2}\right)^N \left\|\partial_x\partial_y\partial_z f\right\|_\infty + \mathcal{O}\left(N\left(\frac{1}{2}\right)^N\right), \tag{4.12}$$

or, in terms of the mesh width,

$$\left\|E^{N,N,N}\right\| \infty \le \frac{1}{16}h\log_2^2 h^{-1}\left\|\partial_x\partial_y\partial_z f\right\|_\infty + \mathcal{O}\left(h\log_2 h^{-1}\right). \tag{4.13}$$

Thus, in three-dimensions, the piecewise-constant scheme has a representation error of order $h\log_2^2 h^{-1}$.

4.2 Piecewise tri-linear interpolation

For tri-linear interpolation, the $\psi^{l',m',n'}_{i,j,k}$ are given by

$$\begin{aligned} \psi^{l',m',n'}_{i,j,k} &= \textstyle\sum_{l'=-1}^{0}\sum_{m'=-1}^{0}\sum_{n'=-1}^{0}\chi^{l'}_i\chi^{m'}_j\chi^{n'}_k, \\ \chi^{l'}_i &\equiv \delta_{l'}\delta_{i-2} + \delta_{l'+1}\left(\tfrac{1}{4}\delta_i + \tfrac{3}{4}\delta_{i-1}\right). \end{aligned} \tag{4.14}$$

Substitution of (4.14) into (4.8b) yields

$$\begin{aligned} \phi_{p,q,r} &= \textstyle\sum_{l'=-1}^{0}\sum_{m'=-1}^{0}\sum_{n'=-1}^{0}\left(\delta_{l'+1}\frac{(-3)^p+3}{4}+\delta_{l'}\delta_p\right)\left(\delta_{m'+1}\frac{(-3)^q+3}{4}+\delta_{m'}\delta_q\right) \\ &\quad \left(\delta_{n'+1}\frac{(-3)^r+3}{4}+\delta_{n'}\delta_r\right)\alpha^{l',m',n'}. \end{aligned} \tag{4.15}$$

Substitution of (4.15) into (4.8a) and next of (4.8a) into (4.2) yields

$$E^{N,N,N} = \sum_{p=2}^{\infty}\sum_{q=2}^{\infty}(\eta_{xy})^{p,q} + \sum_{p=2}^{\infty}\sum_{r=2}^{\infty}(\eta_{xz})^{p,r} + \sum_{q=2}^{\infty}\sum_{r=2}^{\infty}(\eta_{yz})^{q,r} + \sum_{p=2}^{\infty}\sum_{q=2}^{\infty}\sum_{r=2}^{\infty}(\eta_{xyz})^{p,q,r}, \qquad (4.16)$$

where

$$\begin{aligned}
(\eta_{xy})^{p,q} &\equiv \frac{(-3)^p+3}{4}\frac{(-3)^q+3}{4}\frac{2^{-p-q}}{p!q!}\sum_{l=1}^{N}\sum_{m=1}^{l}2^{-lp-mq} \\
&\quad \sum_{i_\times,j_\times,k_\times}\psi_{i_\times,j_\times,k_\times}^{l-N,m-N,-N}(-1)^{i_\times p+j_\times q}\partial_x^p\partial_y^q f_{i_\times,j_\times,k_\times}^{l,m,0}, \\
(\eta_{xz})^{p,r} &\equiv \frac{(-3)^p+3}{4}\frac{(-3)^r+3}{4}\frac{2^{-p-r}}{p!r!}\sum_{l=1}^{N}\sum_{n=1}^{l}2^{-lp-nr} \\
&\quad \sum_{i_\times,j_\times,k_\times}\psi_{i_\times,j_\times,k_\times}^{l-N,-N,n-N}(-1)^{i_\times p+k_\times r}\partial_x^p\partial_z^r f_{i_\times,j_\times,k_\times}^{l,0,n}, \\
(\eta_{yz})^{q,r} &\equiv \frac{(-3)^r+3}{4}\frac{(-3)^r+3}{4}\frac{2^{-q-r}}{q!r!}\sum_{m=1}^{N}\sum_{n=1}^{m}2^{-mq-nr} \\
&\quad \sum_{i_\times,j_\times,k_\times}\psi_{i_\times,j_\times,k_\times}^{-N,m-N,r-N}(-1)^{j_\times q+k_\times r}\partial_y^q\partial_z^r f_{i_\times,j_\times,k_\times}^{0,m,n}, \\
(\eta_{xyz})^{p,q,r} &\equiv \frac{(-3)^p+3}{4}\frac{(-3)^q+3}{4}\frac{(-3)^r+3}{4}\frac{2^{-p-q-r}}{p!q!r!}\left(\sum_{l=1}^{N}\sum_{m=1}^{N}\sum_{n=1}^{N}-\sum_{l=1}^{N-2}\sum_{m=1}^{N-1-l}\sum_{n=1}^{N-l-m}\right) \\
&\quad 2^{-lp-mq-nr}\sum_{i_\times,j_\times,k_\times}\psi_{i_\times,j_\times,k_\times}^{l-N,m-N,n-N}(-1)^{i_\times p+j_\times q+k_\times r}\partial_x^p\partial_y^q\partial_z^r f_{i_\times,j_\times,k_\times}^{l,m,n}.
\end{aligned}$$

The corresponding leading-order term is

$$E_{i,j,k}^{N,N,N} = \frac{9}{1024}N^2\left(\frac{1}{4}\right)^N\partial_x^2\partial_y^2\partial_z^2 f_{i,j,k}^{N,N,N} + \mathcal{O}\left(N\left(\frac{1}{4}\right)^N\right), \qquad (4.17)$$

or, in terms of the mesh width,

$$E_{i,j,k}^{N,N,N} = \frac{9}{1024}h^2\log_2^2 h^{-1}\partial_x^2\partial_y^2\partial_z^2 f_{i,j,k}^{N,N,N} + \mathcal{O}\left(h^2\log_2 h^{-1}\right). \qquad (4.18)$$

Thus, the three-dimensional piecewise-tri-linear scheme has a representation error of order $h^2\log_2^2 h^{-1}$.

4.3 The semi-sparse grid

The combination procedure in the current section started with restricting $f(x,y,z)$ to grids $\Omega^{l,m,n}$ satisfying $l+m+n = N-2, N-1, N$. As an alternative, we now consider the semi-sparse approach as introduced in [5], which amounts to restricting the function to the grids $\Omega^{l,m,n}$ satisfying $l+m+n = 2N-2, 2N-1, 2N$, causing the number of degrees of freedom to increase to $2^{2N-3}(7N^2+29N+1)$. This is an asymptotically two-dimensional complexity, as opposed to the one-dimensional complexity of the sparse-grid approach. Of course, the semi-sparse approach is expected to have a smaller representation error.

For the semi-sparse, three-dimensional combination technique, the representations are taken to satisfy

$$\hat{f}^{l,m,n} = \begin{cases} f^{l,m,n}, & \text{for } l+m+n \le 2N, \\ \sum_{l'=-1}^{0}\sum_{m'=-1}^{0}\sum_{n'=-1}^{0}\alpha^{l',m',n'}P^{l,m,n}\hat{f}^{l+l',m+m',n+n'} & \text{for } l+m+n > 2N. \end{cases} \qquad (4.19)$$

The coefficients $\alpha^{l',m',n'}$ are again taken to be given by (4.8b), yielding the following direct form of the semi-sparse combination technique

$$\begin{aligned}\hat{f}^{N,N,N} &= \left(\sum_{\substack{0\le l,m,n\le N \\ l+m+n=2N}} -2 \sum_{\substack{0\le l,m,n< N \\ l+m+n=2N-1}} + \sum_{\substack{0\le l,m,n< N \\ l+m+n=2N-2}} \right) P^{N,N,N} \hat{f}^{l,m,n} \\ &- \left(\sum_{\substack{l=N \\ 0\le m,n< N \\ l+m+n=2N-1}} + \sum_{\substack{m=N \\ 0\le l,n< N \\ l+m+n=2N-1}} + \sum_{\substack{n=N \\ 0\le l,m< N \\ l+m+n=2N-1}} \right) P^{N,N,N} \hat{f}^{l,m,n}.\end{aligned} \tag{4.20}$$

The error coefficients $\phi^{p,q,r}$ are the same as for the truely-sparse approach, e.g., for piecewise-constant prolongation they are given by (4.11) and for piecewise-tri-linear prolongation they are given by (4.15). The representation error is now given by

$$E^{N,N,N} = \sum_{\substack{0<l,m,n\le N \\ 2N<l+m+n}} e^{l,m,n}. \tag{4.21}$$

For piecewise-constant prolongation, the error expansion reads

$$E^{N,N,N} = \sum_{p=1}^{\infty} \sum_{q=1}^{\infty} \sum_{r=1}^{\infty} \left(\eta^{\text{const}}_{xyz}\right)^{p,q,r},$$

where

$$\begin{aligned}\left(\eta^{\text{const}}_{xyz}\right)^{p,q,r} \equiv\ & \frac{2^{-p-q-r}}{p!q!r!} \sum_{l=1}^{N} \sum_{m=N+1-l}^{N} \sum_{n=2N+1-l-m}^{N} 2^{-lp-mq-nr} \\ & \sum_{i_\times,j_\times,k_\times} \psi^{l-N,m-N,n-N}_{i_\times,j_\times,k_\times} (-1)^{i_\times p + j_\times q + k_\times r} \partial_x^p \partial_y^q \partial_z^r f^{l,m,n}_{i_\times,j_\times,k_\times}.\end{aligned}$$

The corresponding leading-order result is

$$\left\| E^{N,N,N}_{i,j,k} \right\|_\infty \le \frac{1}{16} N^2 \left(\frac{1}{4}\right)^N \partial_x \partial_y \partial_z f^{N,N,N}_{i,j,k} + \mathcal{O}\left(N \left(\frac{1}{4}\right)^N \right), \tag{4.22}$$

or, in terms of the mesh width,

$$\left\| E^{N,N,N}_{i,j,k} \right\|_\infty \le \frac{1}{16} h^2 \log_2^2 h^{-1} \partial_x \partial_y \partial_z f^{N,N,N}_{i,j,k} + \mathcal{O}\left(h^2 \log_2 h^{-1} \right). \tag{4.23}$$

For piecewise-tri-linear prolongation, the error expansion reads

$$E^{N,N,N} = \sum_{p=2}^{\infty} \sum_{q=2}^{\infty} \sum_{r=2}^{\infty} \left(\eta^{\text{lin}}_{xyz}\right)^{p,q,r}, \tag{4.24}$$

where

$$\begin{aligned}\left(\eta^{\text{lin}}_{xyz}\right)^{p,q,r} \equiv\ & \frac{(-3)^p+3}{4} \frac{(-3)^q+3}{4} \frac{(-3)^r+3}{4} \frac{2^{-p-q-r}}{p!q!r!} \sum_{l=1}^{N} \sum_{m=N+1-l}^{N} \sum_{n=2N+1-l-m}^{N} \\ & 2^{-lp-mq-nr} \sum_{i_\times,j_\times,k_\times} \psi^{l-N,m-N,n-N}_{i_\times,j_\times,k_\times} (-1)^{i_\times p + j_\times q + k_\times r} \partial_x^p \partial_y^q \partial_z^r f^{l,m,n}_{i_\times,j_\times,k_\times}.\end{aligned}$$

The corresponding leading-order term is

$$E_{i,j,k}^{N,N,N} = \frac{9}{1024} N^2 \left(\frac{1}{16}\right)^N \partial_x^2 \partial_y^2 \partial_z^2 f_{i,j,k}^{N,N,N} + \mathcal{O}\left(N \left(\frac{1}{16}\right)^N\right), \tag{4.25}$$

or, in terms of the mesh width,

$$E_{i,j,k}^{N,N,N} = \frac{9}{1024} h^4 \log_2^2 h^{-1} \partial_x^2 \partial_y^2 \partial_z^2 f_{i,j,k}^{N,N,N} + \mathcal{O}\left(h^4 \log_2 h^{-1}\right). \tag{4.26}$$

4.4 A numerical test

As a test of the derivations in the current section, consider the following test case

$$f(x, y, z) = \sin(\pi x) \sin(\pi y) \sin(\pi z).$$

In Figures 5 and 6, the first-order error expressions (4.17) and (4.25) for the sparse and the semi-sparse schemes, respectively, are compared with corresponding numerical results. The errors are evaluated at a grid point nearest to $x = y = z = \frac{1}{2}$ (eight grid points qualify but due to the symmetry of the function this is not a problem). From Figures 5 and 6, it appears that the asymptotic expressions (4.17) and (4.25) indeed describe the numerical error of the sparse and the semi-sparse schemes, respectively, for $N \to \infty$. Convergence of the error expansions (4.16) and (4.24) for the sparse and the semi-sparse schemes to the corresponding numerical results as $\max(p + q + r) \to \infty$ is shown in Figures 7 and 8, respectively.

4.5 Discussion

In the current section, the error analysis introduced in Sections 2 and 3 was extended to three dimensions. Besides the sparse grid, also a so-called semi-sparse grid was considered. The semi-sparse grid was shown to have a representation error of $\mathcal{O}\left(h^4 \log_2^2 h^{-1}\right)$. If the (semi-) sparse-grid representation error would be the only error to deal with, then the semi-sparse-grid approach would be superior to the sparse-grid approach. This is illustrated in Figure 9, in which the numerically observed error at a grid point nearest to $x = y = z = \frac{1}{2}$ is plotted versus the number of degrees of freedom for the tri-linear sparse and semi-sparse schemes, for the test function $f(x, y, z) = \sin(\pi x) \sin(\pi y) \sin(\pi z)$. Figure 9 suggests that the semi-sparse-grid approach yields a smaller error for the same number of degrees of freedom than the sparse-grid approach. However, this suggestion is misleading since the sparse-grid representation error is not the only relevant error.

In the current setup, the sparse and semi-sparse representations $\hat{f}^{N,N,N}$ are piecewise-constant or piecewise-d-linear and hence contain an additional error of $\mathcal{O}(h)$ or $\mathcal{O}\left(h^2\right)$, respectively, when evaluated outside grid points. A sensible comparison of the sparse and semi-sparse approaches includes this error. In Figure 10, the sparse and semi-sparse approaches are again compared, now with inclusion of the $\mathcal{O}\left(h^2\right)$ tri-linear representation error. This error is included by comparing the average of grid-function-values nearest to $x = y = z = \frac{1}{2}$ with the exact value at $x = y = z = \frac{1}{2}$. From Figure 10, it is apparent that when the tri-linear representation error on the finest grid is included, the sparse-grid approach yields a smaller error than the semi-sparse-grid approach for the same number of degrees of freedom, as was expected. In Figure 10, we also plotted the conventional tri-linear representation error at $x = y = z = \frac{1}{2}$ versus the complexity of the conventional grid, 2^{3N}. Figure 10 clearly indicates that for the current test function, the sparse-grid representation is more efficient than the conventional representation and, for more than 10^5 degrees of freedom, the semi-sparse representation is also more efficient than conventional representation, but less efficient than a truely-sparse representation.

If we would only be interested in the solution at grid points of the finest grid $\Omega^{N,N,N}$, then we might argue that there is no reduction in representation error for the semi-sparse approach and hence that the semi-sparse approach is more efficient than the truely-sparse approach. However, so far, we have assumed that the function f is known exactly at the points contained in the coarse grids. Of course,

when solving a differential equation this is not true. Then, the coarse-grid functions are subject to a discretization error. In general, a discretization error of order $\mathcal{O}\left(h_{\text{coarse}}^n\right)$ on the coarse grids leads to an error of order $\mathcal{O}\left(h^n\right)$ on the finest grid $\Omega^{N,N,N}$. Therefore, a very common discretization error of $\mathcal{O}\left(h_{\text{coarse}}^2\right)$ also reduces the representation error of the semi-sparse approach to $\mathcal{O}\left(h^2\right)$. To exploit the $\mathcal{O}\left(h^4 \log_2^2 h^{-1}\right)$ representation error of the semi-sparse approach, a discretization of $\mathcal{O}\left(h_{\text{coarse}}^4\right)$ would be required. However, if such a discretization were feasible, then it would be wiser to stick to the conventional full grid, since this would be more efficient then.

5. Discontinuous functions

In this section, we do not require f to be a smooth function. In particular, we examine the behavior of the error in the case that f is a two-dimensional step function of the type

$$f(x,y) = \begin{cases} -1, & (1+\lambda)x + (1-\lambda)y < 1. \\ 0, & (1+\lambda)x + (1-\lambda)y = 1. \\ +1, & (1+\lambda)x + (1-\lambda)y > 1. \end{cases} \tag{5.1}$$

We will obtain expressions for the local error $e^{l,m}$ directly from its defining equation (2.3) by substitution of values for α, β, γ and $\psi_{i,j}^{l,m}$. In general, we have

$$\begin{aligned} e_{i_\times,j_\times}^{l,m} = -f_{i_\times,j_\times}^{l,m} \quad &+ \quad \alpha \textstyle\sum_{i=0}^{3} \sum_{j=0}^{1} \psi_{i,j}^{-1,0} f\left(x_{i_\times}^l + X_{i,j}^{-1,0}(-1)^{i_\times} \frac{\Delta x^l}{2}, y_{j_\times}^m\right) \\ &+ \quad \beta \textstyle\sum_{i=0}^{1} \sum_{j=0}^{3} \psi_{i,j}^{0,-1} f\left(x_{i_\times}^l, y_{j_\times}^m + Y_{i,j}^{0,-1}(-1)^{j_\times} \frac{\Delta y^m}{2}\right) \\ &+ \quad \gamma \textstyle\sum_{i=0}^{1} \sum_{j=0}^{1} \psi_{i,j}^{-1,-1} f\left(x_{i_\times}^l + X_{i,j}^{0,-1}(-1)^{i_\times} \frac{\Delta x^l}{2}, y_{j_\times}^m + Y_{i,j}^{-1,0}(-1)^{j_\times} \frac{\Delta y^m}{2}\right), \end{aligned} \tag{5.2}$$

where $X_{i,j}^{l',m'}$ and $Y_{i,j}^{l',m'}$ are given by (3.4) and where the coefficients $\psi_{i,j}^{l',m'}$ determine the prolongation. Since now $f(x,y)$ is a step function, we assume that prolongation by bi-linear interpolation will not be superior to piecewise-constant interpolation. Hence, we will only consider piecewise-constant interpolation. For piecewise-constant interpolation, $\psi_{i,j}^{l',m'}$ is given by (3.6). Substitution of (3.6) into (5.2) yields

$$\begin{aligned} e_{i_\times,j_\times}^{l,m} = -f_{i_\times,j_\times}^{l,m} \quad &+ \quad \alpha f\left(x_{i_\times}^l + (-1)^{i_\times} \tfrac{\Delta x^l}{2}, y_{j_\times}^m\right) + \beta f\left(x_{i_\times}^l, y_{j_\times}^m + (-1)^{j_\times} \tfrac{\Delta y^m}{2}\right) \\ &+ \quad \gamma f\left(x_{i_\times}^l + (-1)^{i_\times} \tfrac{\Delta x^l}{2}, y_{j_\times}^m + (-1)^{j_\times} \tfrac{\Delta y^m}{2}\right). \end{aligned} \tag{5.3}$$

5.1 The $[\frac{1}{2}, \frac{1}{2}, 0]$ piecewise-constant scheme.

For $\alpha = \beta = 1, \gamma = -1$, the local error $e^{l,m}$ takes the form

$$e_{i_\times,j_\times}^{l,m} = -f_{i_\times,j_\times}^{l,m} + \frac{1}{2} f\left(x_{i_\times}^l + (-1)^{i_\times} \frac{\Delta x^l}{2}, y_{j_\times}^m\right) + \frac{1}{2} f\left(x_{i_\times}^l, y_{j_\times}^m + (-1)^{j_\times} \frac{\Delta y^m}{2}\right). \tag{5.4}$$

This expression is only non-zero for a limited number of points, determined by the line $(1+\lambda)x + (1-\lambda)y = 1$. In Figure 11, black triangles have been drawn that correspond to equation (5.4). Triangles that are intersected by the line $(1+\lambda)x + (1-\lambda)y = 1$ correspond to points for which $e^{l,m}$ is of order 1 (proportional to the step). The number of triangles that are intersected is assumed to be proportional to $\max\left(2^l, \frac{1-\lambda}{1+\lambda} 2^m\right)$ if $\lambda \geq 0$ and $\lambda \neq 1$, and to $\max\left(\frac{1+\lambda}{1-\lambda} 2^l, 2^m\right)$ if $\lambda \leq 0$ and $\lambda \neq -1$. Thus, for $\lambda \geq 0$ and $\lambda \neq 1$, there is a $\kappa \in \mathbb{R}$ such that for all $l \geq 0$ and $m \geq 0$

$$\left\| e^{l,m} \right\|_1 \leq 2^{-l-m} \kappa \max\left(2^l, \frac{1-\lambda}{1+\lambda} 2^m\right). \tag{5.5}$$

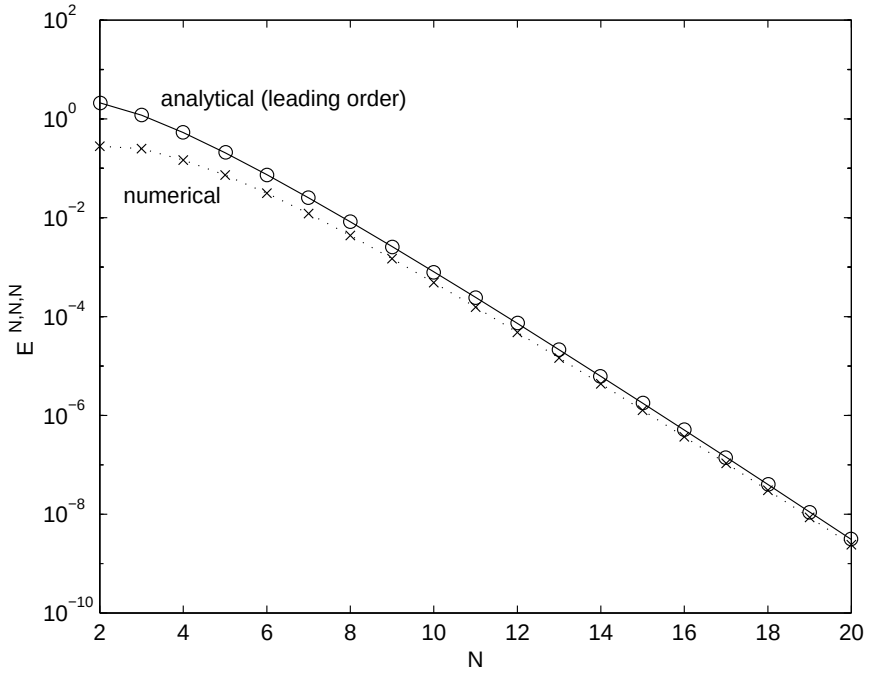

Figure 5: Convergence of the sparse-grid representation error to the analytical result

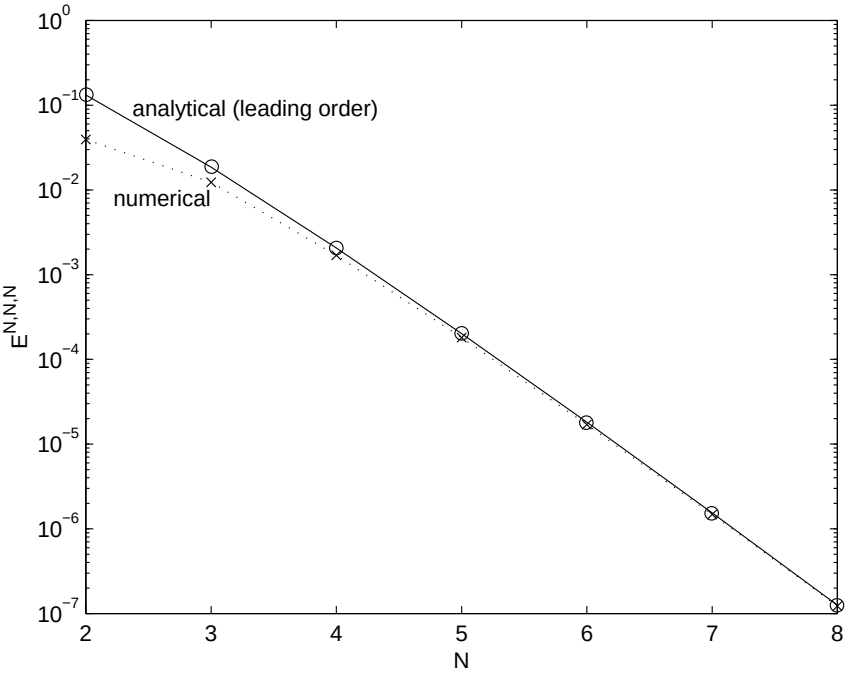

Figure 6: Convergence of the semi-sparse-grid representation error to the analytical result

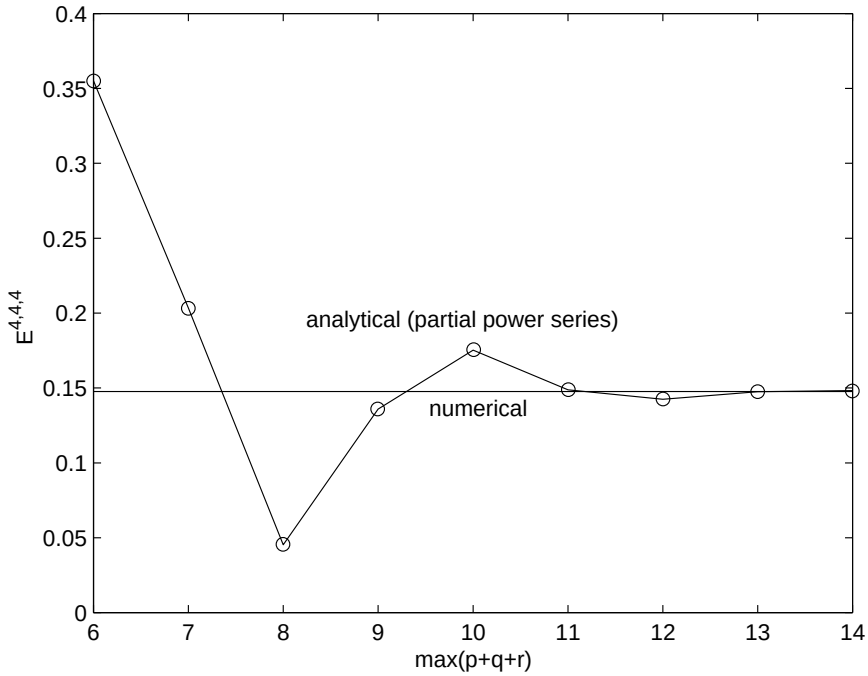

Figure 7: Convergence of the power series for the sparse-grid representation error

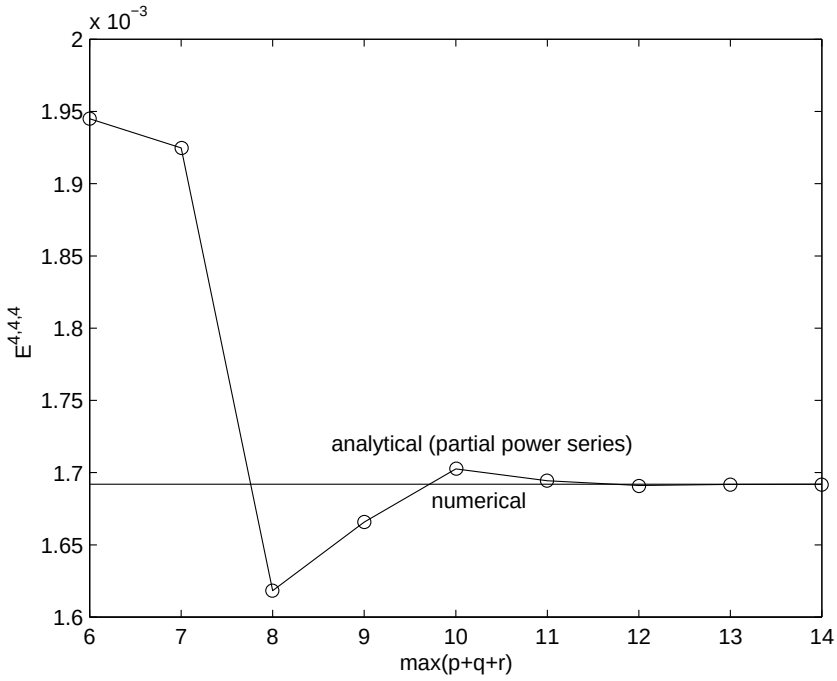

Figure 8: Convergence of the power series for the semi-sparse-grid representation error

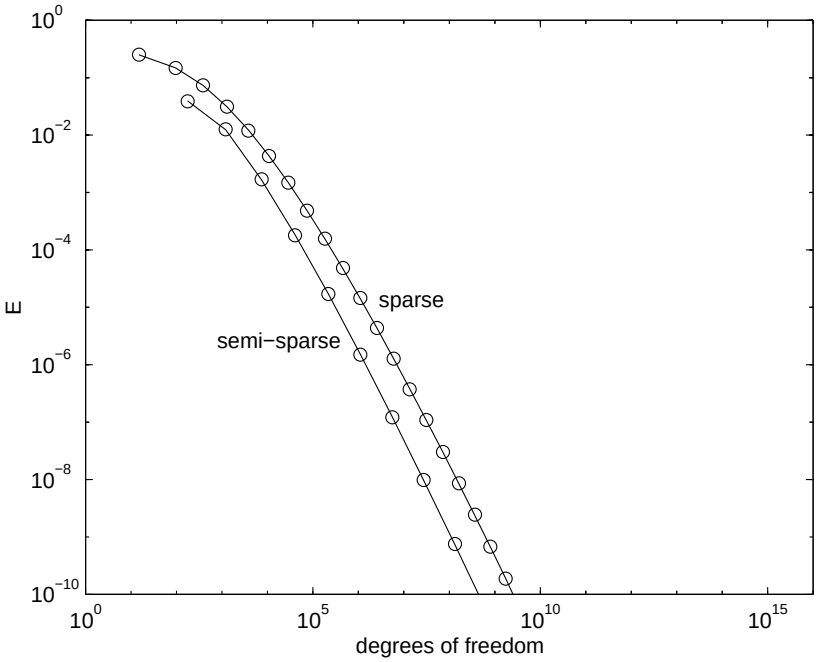

Figure 9: Sparse and semi-sparse representation errors (numerical), the conventional representation error is neglected

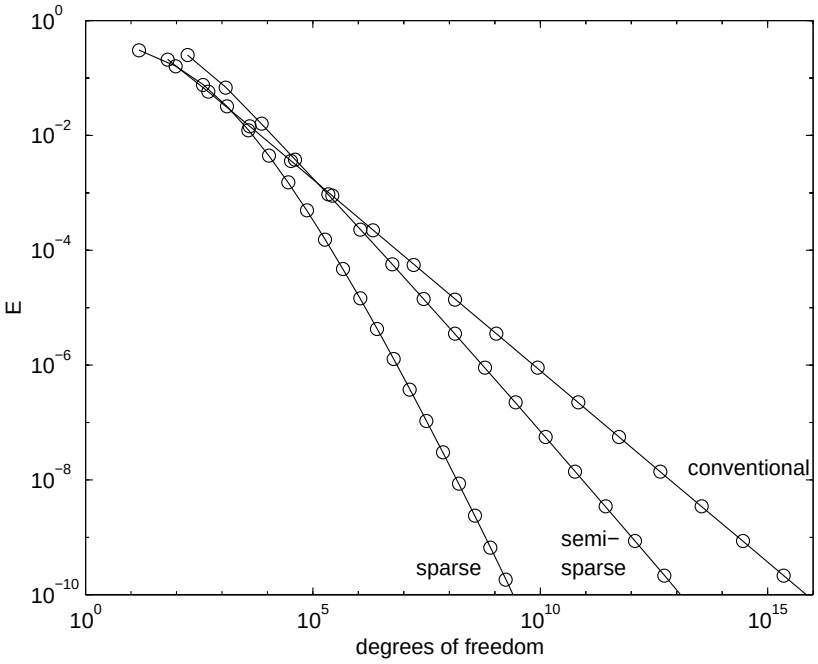

Figure 10: Conventional, sparse and semi-sparse representation errors (numerical), sparse and semi-sparse errors include the conventional representation error

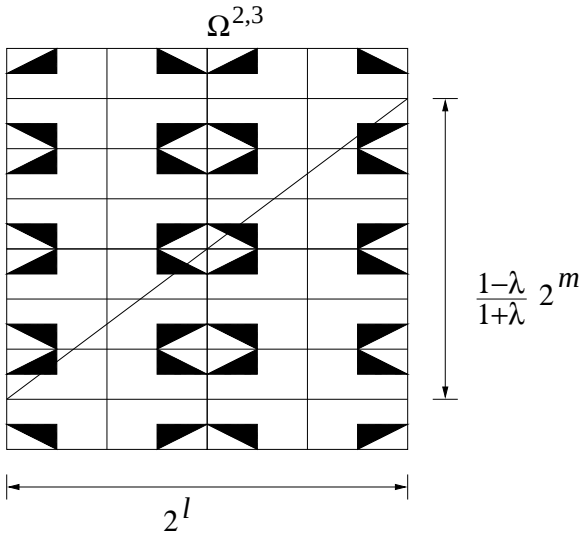

Figure 11: Counting errors, $\frac{1}{2}, \frac{1}{2}$ combination

For the $[\frac{1}{2}, \frac{1}{2}, 0]$ scheme, the representation error $E^{N,N}$ is given by (2.6), which we use to obtain the following expression for $\left\|E^{N,N}\right\|_1$

$$\begin{aligned}
\left\|E^{N,N}\right\|_1 &= \left\| \sum_{n=0}^{N-1} 2^{-n} \sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} P^{N,N} e^{N-i,N-n+i} \right\|_1 \\
&\leq \sum_{n=0}^{N-1} 2^{-n} \sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} \left\| P^{N,N} e^{N-i,N-n+i} \right\|_1 \\
&= \sum_{n=0}^{N-1} 2^{-n} \sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} \left\| e^{N-i,N-n+i} \right\|_1 .
\end{aligned} \tag{5.6}$$

Substitution of (5.5) into (5.6) gives

$$\begin{aligned}
\left\|E^{N,N}\right\|_1 &\leq \kappa \sum_{n=0}^{N-1} 2^{-n} \sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} 2^{n-2N} \max\left(\tfrac{1-\lambda}{1+\lambda} 2^{N-i}, 2^{N-n+i}\right) \\
&\leq \kappa \sum_{n=0}^{N-1} 2^{-n} \sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} 2^{n-2N} \max\left(2^{N-i}, 2^{N-n+i}\right) \\
&= \kappa \sum_{n=0}^{N-1} 2^{-n} \sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} 2^{-N} \max\left(2^{n-i}, 2^{i}\right) \\
&\leq 2^{1-N} \kappa \sum_{n=0}^{N-1} \sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} 2^{-i} \\
&= 4\kappa \left(\left(\tfrac{3}{4}\right)^N - \left(\tfrac{1}{2}\right)^N \right).
\end{aligned} \tag{5.7}$$

Thus,

$$\left\|E^{N,N}\right\|_1 = \mathcal{O}\left(\left(\frac{3}{4}\right)^N \right). \tag{5.8}$$

Rewriting the last equation in terms of the mesh width yields

$$\left\|E^{N,N}\right\|_1 = \mathcal{O}\left(h^{2-\log_2 3}\right). \tag{5.9}$$

Note that we have taken $\lambda \geq 0$, $\lambda \neq 1$. It is obvious that $\lambda < 0$, $\lambda \neq -1$ gives the same result. Thus, for a step function described by (5.1), the $[\frac{1}{2}, \frac{1}{2}, 0]$ piecewise-constant scheme has a representation error of order $2 - \log_2 3 \approx 0.42$.

5.2 The $[1,1,-1]$ *piecewise-constant scheme.*
For $\alpha=\beta=1,\gamma=-1$, the local error $e^{l,m}$ takes the form

$$\begin{aligned} e^{l,m}_{i_\times,j_\times} = -f^{l,m}_{i_\times,j_\times} \quad &+ \quad f\left(x^l_{i_\times}+(-1)^{i_\times}\tfrac{\Delta x^l}{2}, y^m_{j_\times}\right)+f\left(x^l_{i_\times}, y^m_{j_\times}+(-1)^{j_\times}\tfrac{\Delta y^m}{2}\right) \\ &- \quad f\left(x^l_{i_\times}+(-1)^{i_\times}\tfrac{\Delta x^l}{2}, y^m_{j_\times}+(-1)^{j_\times}\tfrac{\Delta y^m}{2}\right). \end{aligned} \tag{5.10}$$

This expression is also only non-zero for a limited number of points, determined by the line $(1+\lambda)x+(1-\lambda)y=1$. In Figure 12, rectangles have been drawn that correspond to equation (5.4). Squares

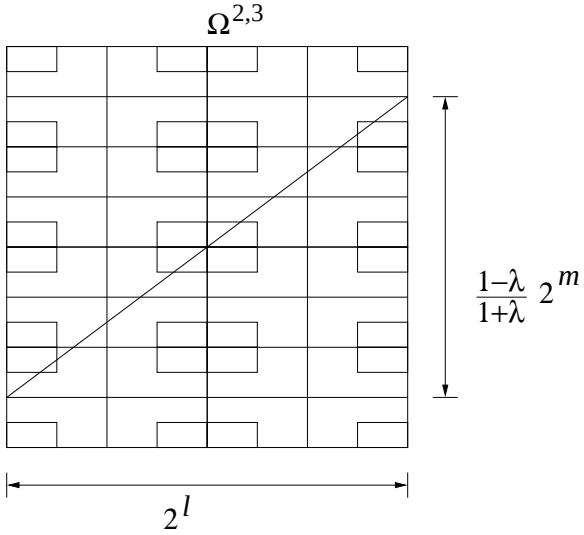

Figure 12: Counting errors, $1,1,-1$ combination

that are cut, through a horizontal and a vertical side, by the line $(1+\lambda)x+(1-\lambda)y=1$ correspond to points for which $e^{l,m}$ is of order 1 (proportional to the step). The number of rectangles that are cut, through a horizontal and a vertical side, is assumed to be proportional to $\min\left(2^l, \frac{1-\lambda}{1+\lambda}2^m\right)$ if $\lambda\geq 0$ and $\lambda\neq 1$, and to $\min\left(\frac{1+\lambda}{1-\lambda}2^l, 2^m\right)$ if $\lambda\leq 0$ and $\lambda\neq -1$. Thus, for $\lambda\geq 0$ and $\lambda\neq 1$, there is a $\kappa\in\mathbb{R}$ such that for all $l\geq 0$ and $m\geq 0$

$$\left\|e^{l,m}\right\|_1 \leq 2^{-l-m}\kappa\min\left(2^l, \frac{1-\lambda}{1+\lambda}2^m\right). \tag{5.11}$$

For the $[1,1,-1]$ scheme, the representation error $E^{N,N}$ is given by (2.16), from which we obtain the following relation for $\left\|E^{N,N}\right\|_1$

$$\left\|E^{N,N}\right\|_1 \leq \sum_{n=0}^{N-1}\sum_{i=0}^{n}\left\|e^{N-i,N-n+i}\right\|_1. \tag{5.12}$$

Substitution of (5.11) into (5.12) gives

$$\begin{aligned} \left\|E^{N,N}\right\|_1 &\leq \kappa\textstyle\sum_{n=0}^{N-1}\sum_{i=0}^{n}2^{n-2N}\min\left(\frac{1-\lambda}{1+\lambda}2^{N-i}, 2^{N-n+i}\right) \\ &\leq \kappa\textstyle\sum_{n=0}^{N-1}\sum_{i=0}^{n}2^{n-2N}\min\left(2^{N-i}, 2^{N-n+i}\right) \\ &= 2^{-N}\kappa\textstyle\sum_{n=0}^{N-1}\sum_{i=0}^{n}\min\left(2^{n-i}, 2^i\right) \\ &= 2^{1-N}\kappa\textstyle\sum_{n=0}^{N-1}\sum_{i=0}^{n/2}2^i \\ &= \frac{4}{\sqrt{2}-1}\kappa\left(\left(\frac{1}{\sqrt{2}}\right)^N-\left(\frac{1}{2}\right)^N\right)-2\left(\frac{1}{2}\right)^N N. \end{aligned} \tag{5.13}$$

Rewriting in terms of the mesh width yields

$$\left\|E^{N,N}\right\|_1 = \mathcal{O}\left(h^{1/2}\right). \tag{5.14}$$

Thus, for a step function described by (5.1), the $[1,1,-1]$ piecewise-constant scheme has a representation error of order $\frac{1}{2}$.

5.3 The $[0,0,1]$ *piecewise-constant scheme*
For $\alpha = \beta = 0, \gamma = 1$, the local error $e^{l,m}$ takes the form

$$e^{l,m}_{i_\times,j_\times} = -f^{l,m}_{i_\times,j_\times} + f\left(x^l_{i_\times} + (-1)^{i_\times}\frac{\Delta x^l}{2}, y^m_{j_\times} + (-1)^{j_\times}\frac{\Delta y^m}{2}\right). \tag{5.15}$$

This expression is again only non-zero for a limited number of points, determined by the line $(1+\lambda)x + (1-\lambda)y = 1$. In Figure 13, diagonal lines have been drawn that correspond to equation (5.15). Diagonal lines that are cut by $(1+\lambda)x + (1-\lambda)y = 1$ correspond to points for which $e^{l,m}$ is of order

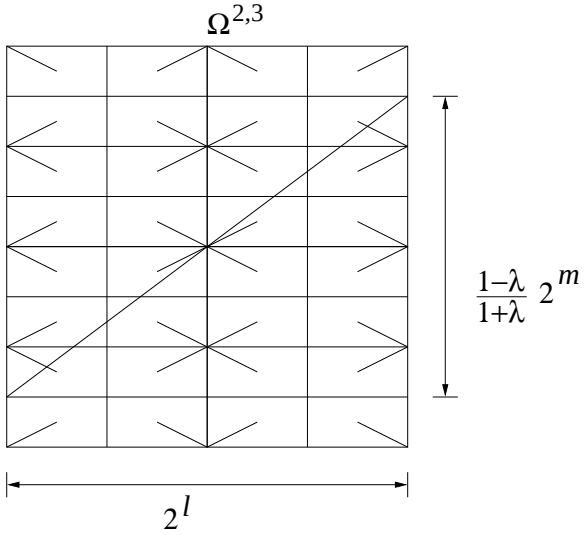

Figure 13: Counting errors, $\gamma = 1$ combination

1 (proportional to the step). The number of diagonal lines that are cut is assumed to be proportional to $\max\left(2^l, \frac{1-\lambda}{1+\lambda}2^m\right)$ if $\lambda \geq 0$ and $\lambda \neq 1$, and to $\max\left(\frac{1+\lambda}{1-\lambda}2^l, 2^m\right)$ if $\lambda \leq 0$ and $\lambda \neq -1$. Thus, for $\lambda \geq 0$ and $\lambda \neq 1$, there is a $\kappa \in \mathbb{R}$ such that for all $l \geq 0$ and $m \geq 0$

$$\left\|e^{l,m}\right\|_1 \leq 2^{-l-m}\kappa \max\left(2^l, \frac{1-\lambda}{1+\lambda}2^m\right). \tag{5.16}$$

For the $\gamma = 1$ scheme, the representation error $E^{N,N}$ is given by (2.26), from which we obtain the following relation for $\left\|E^{N,N}\right\|_1$

$$\left\|E^{N,N}\right\|_1 \leq \sum_{l=N/2}^{N-1}\left\|e^{l,l}\right\|_1. \tag{5.17}$$

Substitution of (5.16) into (5.17) gives

$$\begin{aligned} \left\| E^{N,N} \right\|_1 &\leq \kappa \sum_{l=N/2}^{N-1} 2^{-l} \\ &= 2\kappa \left(\left(\tfrac{1}{\sqrt{2}} \right)^N - \left(\tfrac{1}{2} \right)^N \right). \end{aligned} \tag{5.18}$$

Rewriting in terms of the mesh width yields

$$\left\| E^{N,N} \right\|_1 = \mathcal{O}\left(h^{1/2} \right). \tag{5.19}$$

Thus, for a step function described by (5.1), the $[0,0,1]$ piecewise-constant scheme has a representation error of order $\frac{1}{2}$.

5.4 A numerical test

To test the validity of the conclusion that the $[1,1,-1]$ scheme has a representation error of order $\mathcal{O}\left(h^{1/2}\right)$ for the representation of a discontinuous function of the type (5.1), we now represent (5.1), with the $[1,1,-1]$ combination scheme, for $\lambda = 0$ (a diagonal line through the domain). In Table 3, the representation error in the L_1-norm is listed for $N = 2, 3, \ldots, 12$, together with convergence ratios. Table 3 shows that the L_1-norm of the representation error on sparse grids with N even is twice as small as on $N-1$, while going from even N to (odd) $N+1$ actually leads to a small rise in error. The explanation that the diagonal step function is better represented for N even than for N odd is that for N even there is a grid $\Omega^{N/2,N/2}$ within the set of coarse grids $\{\Omega^{l,m}, l+m = N-1, N\}$ on which the diagonal step function can be reasonably described. A more important observation is that the average convergence ratio (rightmost column) seems to tend to $\sqrt{2}$, as it should according to (5.14).

N	$\left\| E^{N,N} \right\|_1$	$\frac{\left\| E^{N-1,N-1} \right\|_1}{\left\| E^{N,N} \right\|_1}$	$\left(\frac{\left\| E^{N-2,N-2} \right\|_1}{\left\| E^{N,N} \right\|_1} \right)^{\frac{1}{2}}$
2	0.125000		
3	0.187500	0.666667	
4	0.093750	2.000000	1.154701
5	0.109375	0.857143	1.309307
6	0.054688	2.000000	1.309307
7	0.058594	0.933333	1.366260
8	0.029297	2.000000	1.366260
9	0.030273	0.967742	1.391217
10	0.015137	2.000000	1.391217
11	0.015381	0.984127	1.402945
12	0.007690	2.000000	1.402945

Table 3: Orders of convergence

5.5 Discussion

In the current section, it was shown that the combination technique has a representation error of order $\mathcal{O}\left(h^{1/2}\right)$ when a step function is represented. This accuracy can also be obtained by interpolating solely from the grid $\Omega^{N/2,N/2}$, e.g., by conventional representation on the grid $\Omega^{N/2,N/2}$ which contains less degrees of freedom than the set of coarse grids $\{\Omega^{l,m}, l+m = N-1, N\}$ comprising the sparse grid. For the representation of genuinely discontinuous functions, the combination technique is not superior to conventional representation.

6. Conclusions

The sparse-grid combination technique is an attractive alternative to the conventional representation of a function on a full grid. The reason for this is that, for the same number of degrees of freedom, the sparse-grid combination technique yields a significantly smaller representation error than conventional representation; see for instance Figure 10.

By analyzing the steps that make up the combination technique, explicit expressions for the representation error were obtained. The leading-order error terms contain cross derivatives of the function to be represented, instead of single-variable derivatives like the conventional representation error. The deficiency of the combination technique is that it will be less effective for functions that have large cross derivatives. This problem may be alleviated by adapting the grids to the geometry of the problem at hand.

For comparison, an alternative to the combination technique introduced in [2] was considered. This alternative technique, the $[\frac{1}{2}, \frac{1}{2}, 0]$ technique, appeared to perform less well than the technique in [2], the $[1, 1, -1]$ technique. In fact, the alternative technique even appeared to be inferior to conventional representation, such as the $[0, 0, 1]$ technique.

It was shown that for a step-function, which is not aligned with the grid, the combination technique performs less well than the standard representation. For such a non-aligned step-function, the order of the representation error was found to be $\mathcal{O}\left(h^{1/2}\right)$. (The explicit error expression derived may be useful for a combination technique that relies on grid refinement.)

The representation for the 3D semi-sparse combination technique, as proposed in [5], was analyzed. The representation error was found to be $\mathcal{O}\left(h^4(logh^{-1})^2\right)$. At first sight, this result implies that the 3D semi-sparse combination technique is to be preferred above the 3D truely-sparse combination technique. However, due to additional representation errors or discretization errors of $\mathcal{O}\left(h^2\right)$, the 3D semi-sparse representation error reduces to $\mathcal{O}\left(h^2\right)$, which makes it less attractive than the 3D truely-sparse combination technique.

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