Centrum voor Wiskunde en Informatica
REPORT RAPPORT

Interpolation on Sparse Gauss-Chebyshev Grids in Higher Dimensions

F. Sprengel

Modelling, Analysis and Simulation (MAS)

MAS-R9824 November 1998

Report MAS-R9824
ISSN 1386-3703

CWI
P.O. Box 94079
1090 GB Amsterdam
The Netherlands

Interpolation on Sparse Gauß–Chebyshev Grids in Higher Dimensions

Frauke Sprengel
CWI
P.O. Box 94079, 1090 GB Amsterdam, The Netherlands

ABSTRACT

In this paper, we give a unified approach to error estimates for interpolation on sparse Gauß–Chebyshev grids for multivariate functions from Besov–type spaces with dominating mixed smoothness properties. The error bounds obtained for this method are almost optimal for the considered scale of function spaces.

1991 Mathematics Subject Classification: 41A05, 41A63, 65D05, 46E35
Keywords and Phrases: Sparse Grids, Besov–type Spaces of Dominating Mixed Smoothness, Multivariate Interpolation
Note: This work was carried out under CWI project MAS2.1 "Computational Fluid Dynamics"

1. Introduction

The Gauß–Chebyshev knots are quite often used as interpolation points for functions given on the interval $[-1, 1]$. Beside the well–studied polynomial interpolation, one can investigate interpolation by splines adapted to these special nodes.

Using generalized translates, the so–called Chebyshev–shifts, the interpolation of univariate functions on Gauß–Chebyshev knots can be seen as interpolation by translates. Shift–invariant spaces corresponding to such translates and their wavelet analysis are described in detail in [11]. Forming tensor products yields interpolation on full Gauß–Chebyshev grids. This is a reasonable choice for fairly small dimensions only.

For higher dimension, interpolation on sparse grids, using essentially less points, is much more appropriate. It can be realized by j–th order blending. Some recent papers use this fact for determining the quadrature error for smooth functions [6, 9, 10] or for investigating interpolatory wavelets for sparse Gauß–Chebyshev grids [15]. Interpolation and approximation on sparse grids has been fairly well investigated for periodic functions [5, 12, 14]. It is closely related to hyperbolic approximation (see e.g. [17]).

Error estimates for interpolation on sparse Gauß–Chebyshev grids were given up to now only for functions from Sobolev–type spaces [16] and for polynomial interpolation of functions with bounded mixed derivatives measured in the maximum norm [1].

In the present paper, we give a unified approach to error estimates for interpolation on sparse Gauß–Chebyshev grids for functions from certain Besov–type spaces to be introduced in Section 3.

Therefore, we can adapt the concept of Strang–Fix conditions for this special situation, see Section 4. Furthermore, we use essentially the properties of the multivariate function spaces which can be represented as tensor products of the univariate function spaces provided with uniform crossnorms. In this way, we propose interpolation methods which are almost optimal for a wide range of Besov–type spaces.

2. Notation

We denote our reference interval by $I := [-1, 1]$ and the Chebyshev weight by $w(x) := (1 - x^2)^{-1/2}$ $(x \in (-1, 1))$. Let $L^2_w(I)$ be the weighted Hilbert space of all measurable functions $f : I \longrightarrow \mathbb{R}$, with

$$\int\limits_I f(x)^2 w(x)\, dx < \infty.$$

For $f, g \in L^2_w(I)$, the corresponding inner product is given by

$$\langle f, g\rangle := \frac{2}{\pi} \int\limits_I f(x) g(x) w(x)\, dx.$$

By Π_n, we denote the set of all real valued polynomials of degree at most n restricted on I. Furthermore, let $T_n \in \Pi_n$ be the Chebyshev polynomials

$$T_n(x) := \cos(n \arccos x).$$

They form a complete orthogonal basis

$$\langle T_k, T_\ell\rangle = \begin{cases} 2 & \text{for } k = \ell = 0, \\ 1 & \text{for } k = \ell \neq 0, \\ 0 & \text{otherwise,} \end{cases}$$

of $L^2_w(I)$. With the help of the Chebyshev coefficients

$$a_k[f] := \langle f, T_k\rangle\,, \qquad f \in L^2_w(I),\ k \in \mathbb{N}_0,$$

we characterize the Wiener algebra

$$A(I) := \Big\{ f \in L^2_w(I)\ ;\ \sum_{k \in \mathbb{N}_0}{}' |a_k[f]| < \infty \Big\}$$

of functions with an absolutely summable Fourier–Chebyshev series. Here and in the sequel, we use the notation $\sum'$ if we halve the first term and $\sum''$ where both the first and the last term in the sum are halved. By $\mathcal{G}_N$, we denote the grid of the Gauß–Chebyshev nodes

$$\mathcal{G}_N := \left\{ g_k := \cos k\tfrac{\pi}{N}\,;\ k = 0, \ldots, N \right\}.$$

The discrete Chebyshev coefficients are given as

$$a_k^N[f] := \frac{2}{N} \sum_{\ell=0}^{N}{}'' f(g_\ell)\, T_k(g_\ell)\,, \qquad k = 0, \ldots, N.$$

For functions $f \in A(I)$, one can prove the aliasing formula

$$a_k^N[f] = \sum_{\ell \in \mathbb{N}_0} a_{2\ell N+k}[f] + a_{2(\ell+1)N-k}[f]\,, \qquad k = 0, \ldots, N.$$

3. Function Spaces

We want to interpolate multivariate functions. Their smoothness will be measured in the scale of Besov–type spaces. In this section, we give a definition of these function spaces via the summability of their Fourier–Chebyshev series. The Chebyshev coefficient of an n–variate function is given in the usual way as

$$a_{\boldsymbol{k}}[f] = \langle f, T_{k_1} \otimes T_{k_2} \otimes \cdots \otimes T_{k_n} \rangle, \qquad \text{for } \boldsymbol{k} \in \mathbb{N}_0^n.$$

Furthermore, we need the index sets

$$\begin{aligned} Q_0^n &= \{0\}, \\ Q_j^n &= \{\boldsymbol{k} \in \mathbb{N}_0^n\,;\ k_r < 2^j, r = 1, \ldots, n\} \\ &\qquad\qquad \setminus \{\boldsymbol{k} \in \mathbb{N}_0^n\,;\ k_r < 2^{j-1}, r = 1, \ldots, n\}. \end{aligned}$$

Definition 1 *Let $1 \le q \le \infty$ and $s \ge 0$. Then we define the isotropic Besov–type space $B^s_{2,q,w}(I^n)$ as*

$$\begin{aligned} B^s_{2,q,w}(I^n) &:= \Big\{ f \in L^2_w(I^n)\,;\ \|f \mid B^s_{2,q,w}(I^n)\| \\ &\qquad = \Big(\sum_{j=0}^{\infty} 2^{jsq} \Big(\sum_{\boldsymbol{k} \in Q_j^n} a^2_{\boldsymbol{k}}[f] \Big)^{q/2} \Big)^{1/q} < \infty \Big\} \end{aligned}$$

for $q < \infty$ and

$$\begin{aligned} B^s_{2,\infty,w}(I^n) &:= \Big\{ f \in L^2_w(I^n)\,;\ \|f \mid B^s_{2,\infty,w}(I^n)\| \\ &\qquad = \sup_{j \in \mathbb{N}_0} 2^{js} \Big(\sum_{\boldsymbol{k} \in Q_j^n} a^2_{\boldsymbol{k}}[f] \Big)^{1/2} < \infty \Big\}, \end{aligned}$$

respectively.

For the definition of the spaces of functions with dominating mixed smoothness properties, we put the index sets

$$P_{\boldsymbol{j}}^n = Q_{j_1}^1 \times Q_{j_2}^1 \times \cdots \times Q_{j_n}^1, \quad \boldsymbol{j} \in \mathbb{N}_0^n,$$

and denote the inner product in $\mathbb{N}_0^n$ by $\boldsymbol{j} \cdot \boldsymbol{r} := j_1 r_1 + j_2 r_2 + \cdots + j_n r_n$.

Definition 2 *Let $1 \leq q \leq \infty$ and $\boldsymbol{r} \in \mathbb{R}_+^n$. Then the Besov–type space $S_{2,q,w}^{\boldsymbol{r}} B(I^n)$ of n–variate functions with dominating mixed smoothness properties is defined as*

$$\begin{aligned} S_{2,q,w}^{\boldsymbol{r}} B(I^n) \;\; := \;\; & \Big\{ f \in L_w^2(I^n) \; ; \; \| f \mid S_{2,q,w}^{\boldsymbol{r}} B(I^n) \| \\ & = \Big(\sum_{\boldsymbol{j} \in \mathbb{N}_0^n}^{\infty} 2^{(\boldsymbol{j} \cdot \boldsymbol{r}) q} \Big(\sum_{\boldsymbol{k} \in P_{\boldsymbol{j}}^n} a_{\boldsymbol{k}}^2 [f] \Big)^{q/2} \Big)^{1/q} < \infty \Big\} \end{aligned}$$

for $q < \infty$ and

$$\begin{aligned} S_{2,\infty,w}^{\boldsymbol{r}} B(I^n) \;\; := \;\; & \Big\{ f \in L_w^2(I^n) \; ; \; \| f \mid S_{2,\infty,w}^{\boldsymbol{r}} B(I^n) \| \\ & = \sup_{\boldsymbol{j} \in \mathbb{N}_0^n} 2^{(\boldsymbol{j} \cdot \boldsymbol{r})} \Big(\sum_{\boldsymbol{k} \in P_{\boldsymbol{j}}^n} a_{\boldsymbol{k}}^2 [f] \Big)^{1/2} < \infty \Big\}, \end{aligned}$$

respectively.

By construction, it holds that

$$B_{2,2,w}^0(I^n) = L_w^2(I^n) \qquad \text{and} \qquad S_{2,2,w}^{\boldsymbol{0}} B(I^n) = L_w^2(I^n). \tag{3.1}$$

Furthermore, we have the imbeddings

$$S_{2,q,w}^{\boldsymbol{s}_i} B(I^n) \hookrightarrow B_{2,q,w}^s(I^n) \hookrightarrow S_{2,q,w}^{\boldsymbol{s}_o} B(I^n) \tag{3.2}$$

between isotropic and dominating mixed smoothness spaces with $\boldsymbol{s}_i = (s, s, \ldots, s)$ and $\boldsymbol{s}_o = (s/n, s/n, \ldots, s/n)$.

The Besov spaces of n–variate functions with dominating mixed smoothness properties can be characterized as tensor products

$$B_{2,q,w}^{s_1}(I) \otimes_\lambda B_{2,q,w}^{s_2}(I) \otimes_\lambda \cdots \otimes_\lambda B_{2,q,w}^{s_n}(I) = S_{2,q,w}^{\boldsymbol{s}} B(I^n) \tag{3.3}$$

of the corresponding univariate Besov spaces (equivalent norms). Here, the norm λ which was used for the completion of the algebraic tensor product is the injective tensor norm for $1 \leq q < \infty$ and a certain modification thereof for $q = \infty$, which can be proved in the same manner as in the periodic case cf. [13]. These norms have the main advantage to be uniform crossnorms, cf. [8, 13]. In particular, this (together with (3.1)) means that for n operators $P_r \in \mathcal{L}(B^{s_r}_{2,q,w}(I), L^2_w(I))$, their tensor product operator given by

$$\begin{aligned} P(f_1 \otimes f_2 \otimes \cdots \otimes f_n) &:= (P_1 \otimes P_2 \otimes \cdots \otimes P_n)(f_1 \otimes f_2 \otimes \cdots \otimes f_n) \\ &:= P_1(f_1) \otimes P_2(f_2) \otimes \cdots \otimes P_n(f_n) \end{aligned}$$

is bounded, i.e., $P \in \mathcal{L}(S^{\boldsymbol{s}}_{2,q,w}B(I^n), L^2_w(I^n))$, and its norm can be estimated as

$$\|P \mid \mathcal{L}(S^{\boldsymbol{s}}_{2,q,w}B(I^n), L^2_w(I^n))\| \le C \prod_{r=1}^{n} \|P_r \mid \mathcal{L}(B^{s_r}_{2,q,w}(I), L^2_w(I))\|, \tag{3.4}$$

with some constant C independent of P. Note that for Hilbert spaces the tensor norms are uniform crossnorms by construction. In case of tensor products of Banach spaces, this has to be proved for each example separately, see [8, 13].

Because of the imbeddings

$$\begin{aligned} B^s_{2,q,w}(I^n) &\hookrightarrow B^s_{2,\infty,w}(I^n), \\ S^{\boldsymbol{s}}_{2,q,w}B(I^n) &\hookrightarrow S^{\boldsymbol{s}}_{2,\infty,w}B(I^n), \end{aligned}$$

for $q < \infty$, in the following sections, we give the error estimates for the most interesting case $q = \infty$ only.

4. Univariate Interpolation by Generalized Translates

As a preparation for n–variate interpolation on sparse grids, we need to describe the interpolation method for univariate functions first. To this end, we concentrate our investigations on interpolation by generalized translates of a single function.

The Chebyshev shift (cf. [4, 11]) $s_h f$ of a function f is defined by

$$(s_h f)(x) := \tfrac{1}{2}\, f\big(xh - \tfrac{1}{w(x)w(h)}\big) + \tfrac{1}{2}\, f\big(xh + \tfrac{1}{w(x)w(h)}\big), \qquad x \in I.$$

For interpolation, we use the special shifts $\sigma_k := s_{g_k}$, $k = 0, \dots, N$, into the Gauß–Chebyshev nodes. The Chebyshev shift effects the Chebyshev coefficients $a_k[\sigma_n f] = T_k(g_n) a_k[f]$ in the same multiplicative way as the usual shift effects the Fourier coefficients of periodic functions [11]. We assume to know a modified Lagrange function $\Lambda_N \in A(I)$ satisfying

$$\sigma_k \Lambda_N(g_\ell) = \frac{1}{\varepsilon_k}\, \delta_{k,\ell}\,, \qquad k = 0, \dots, N,$$

with

$$\varepsilon_k := \begin{cases} \frac{1}{2} & \text{for } k = 0, N, \\ 1 & \text{for } k = 1, \dots, N-1. \end{cases}$$

For the construction of such a Lagrange function, we refer to [16]. Then, we can write the corresponding interpolation operator interpolating in the Gauß–Chebyshev nodes as

$$L_N f = \sum_{k=0}^{N}{}'' f(g_k)\, \sigma_k \Lambda_N.$$

In order to characterize the approximation properties of the modified Lagrange function, we impose conditions on the decay of the Chebyshev coefficients of Λ_N, see [16]. They are the pendant for the interval of the strong cardinal Strang–Fix conditions [7] and the periodic Strang–Fix conditions [3].

Definition 3 *The Lagrange function $\Lambda_N \in A(I)$ satisfies the* Strang–Fix conditions *(for Gauß–Chebyshev grids) of order $m > 0$ if for all $k = 0, \dots, N$ the inequalities*

$$\begin{aligned} \left|1 - \frac{N}{2}\, a_k[\Lambda_N]\right| &\le b_0\, k^m\, N^{-m}, \\ \left|\frac{N}{2}\, a_{2\ell N+k}[\Lambda_N]\right| &\le b_{2\ell}\, k^m\, N^{-m}, \qquad \ell \in \mathbb{N}, \\ \left|\frac{N}{2}\, a_{2(\ell+1)N-k}[\Lambda_N]\right| &\le b_{2\ell+1}\, k^m\, N^{-m}, \qquad \ell \in \mathbb{N}_0, \end{aligned} \tag{4.1}$$

hold for some sequence $\{b_\ell\}_{\ell\in\mathbb{N}_0} \in \ell_2(\mathbb{N}_0)$.

With the help of these quite general conditions we obtain the following estimate for the univariate interpolation.

Theorem 4 *Let $m > s > 1/2$. Let the Lagrange function $\Lambda_N \in A(I)$ satisfy the Strang–Fix conditions* (4.1) *of order m. Then there exists a constant C (independent of N) such that*

$$\|f - L_N f \mid L^2_w(I)\| \;\le\; C\; N^{-s}\; \|f \mid B^s_{2,\infty,w}(I)\|,$$

for all $f \in B^s_{2,\infty,w}(I)$.

Proof: Throughout this proof, we denote by C a constant independent of N. The value of C may differ even within the same equation.

The Chebyshev coefficients of the interpolant to the function $f \in A(I)$ can be computed by using aliasing as

$$\begin{aligned} a_{2\ell N+k}[L_N f] &= \sum_{r\in\mathbb{N}_0} \big(a_{2rN+k}[f] + a_{2(r+1)N-k}[f]\big)\,\frac{N}{2} a_{2\ell N+k}[\Lambda_N], \\ a_{(2\ell+1)N+k}[L_N f] &= \sum_{r\in\mathbb{N}_0} \big(a_{(2r+1)N+k}[f] + a_{(2r+1)N-k}[f]\big)\,\frac{N}{2} a_{(2\ell+1)N+k}[\Lambda_N], \end{aligned}$$

for $k = 0, \ldots, N-1$, $\ell \in \mathbb{N}_0$.

First we prove the assertion for polynomials $f \in \Pi_{N-1}$. Using the Strang–Fix conditions, we obtain

$$\begin{aligned} &\|f - L_N f \mid L^2_w(I)\|^2 \\ &\quad\le \sum_{k\in\mathbb{N}_0} (a_k[f] - a_k[L_N f])^2 \\ &\quad= \sum_{k=0}^{N} \sum_{\ell\in\mathbb{N}_0} \Big(a_{2\ell N+k}[f] - \sum_{r\in\mathbb{N}_0} \big(a_{2rN+k}[f] + a_{2(r+1)N-k}[f]\big)\,\frac{N}{2} a_{2\ell N+k}[\Lambda_N]\Big)^2 \\ &\qquad + \Big(a_{(2\ell+1)N+k}[f] - \sum_{r\in\mathbb{N}_0} \big(a_{(2r+1)N+k}[f] + a_{(2r+1)N-k}[f]\big)\,\frac{N}{2} a_{(2\ell+1)N+k}[\Lambda_N]\Big)^2 \\ &\quad= \sum_{k=0}^{N-1} a_k^2[f] \Big(\Big(1 - \frac{N}{2} a_k[\Lambda_N]\Big)^2 + \sum_{\ell\in\mathbb{N}} \Big(\frac{N}{2} a_{2\ell N+k}[\Lambda_N]\Big)^2 + \Big(\frac{N}{2} a_{2\ell N-k}[\Lambda_N]\Big)^2\Big) \\ &\quad\le \sum_{k=0}^{N-1} a_k^2[f] k^{2m} N^{-2m} \sum_{\ell\in\mathbb{N}_0} b_\ell^2. \end{aligned}$$

Let $2^{r-1} \le N < 2^r$. Then

$$\begin{aligned} \sum_{k=0}^{N-1} a_k^2[f] k^{2m} &= \sum_{\ell=0}^{r} \sum_{k\in Q_\ell^1} k^{2m} 2^{-2\ell s} 2^{2\ell s} a_k^2[f] \\ &\le C \sum_{\ell=0}^{r} 2^{2(m-s)\ell} 2^{2\ell s} \sum_{k\in Q_\ell^1} a_k^2[f] \\ &\le C\ N^{2(m-s)} \|f \mid B^s_{2,\infty,w}(I)\|^2. \end{aligned}$$

This means that, for polynomials $f \in \Pi_{N-1}$, we proved

$$\|f - L_N f \mid L^2_w(I)\| \ \le\ C\ N^{-s}\ \|f \mid B^s_{2,\infty,w}(I)\|. \tag{4.2}$$

Now, we consider the general case $f \in B^s_{2,\infty,w}$. Because of $s > 1/2$ it holds that $f \in A(I)$. Therefore, interpolation is well–defined and aliasing is allowed. Let S_{N-1}

denote the $(N-1)$–st Fourier–Chebyshev partial sum. The Strang–Fix conditions applied with $1/2 < s' < s$ and the Cauchy–Schwarz inequality yield

$$\|L_N(f - S_{N-1}f) \mid L^2_w(I)\|^2 \le \sum_{k\in\mathbb{N}_0} a_k^2[L_N(f - S_{N-1}f)]$$

$$\begin{aligned}
\le\ & CN^{-2s}\|\{b_\ell\} \mid \ell_2(\mathbb{N}_0)\| \\
& \times \sum_{k=0}^{N} k^{2s'}\Big[\Big(\sum_{r\in\mathbb{N}_0}(2rN+k)^{-s'}(2rN+k)^{s'}a_{2rN+k}[f - S_{N-1}f]\Big)^2 \\
& \quad + \Big(\sum_{r\in\mathbb{N}_0}(2(r+1)N-k)^{-s'}(2(r+1)N-k)^{s'}a_{2(r+1)N-k}[f - S_{N-1}f]\Big)^2 \\
& \quad + \Big(\sum_{r\in\mathbb{N}_0}((2r+1)N+k)^{-s'}((2r+1)N+k)^{s'}a_{(2r+1)N+k}[f - S_{N-1}f]\Big)^2 \\
& \quad + \Big(\sum_{r\in\mathbb{N}_0}((2r+1)N-k)^{-s'}((2r+1)N-k)^{s'}a_{(2r+1)N-k}[f - S_{N-1}f]\Big)^2 \\
\le\ & CN^{-2s}\|f - S_{N-1}f \mid H^{s'}_w(I)\|^2 \sup_{k=0,\dots,N} k^{2s'}\sum_{r\in\mathbb{N}_0}(2rN+k)^{-2s'} \\
& \quad + (2(r+1)N-k)^{-2s'} + ((2r+1)N+k)^{-2s'} + ((2r+1)N-k)^{-2s'},
\end{aligned}$$

where $H^{s'}_w(I)$ denotes the Sobolev–type space with the norm

$$\|f \mid H^{s'}_w(I)\|^2 := {\sum_{k\in\mathbb{N}_0}}'(1+k^2)^{s'}\,a_k^2[f],$$

as in [2]. Since $s' > 1/2$, we have

$$\begin{aligned}
\sup_{k=0,\dots,N}\Big(\frac{k}{N}\Big)^{2s'} & \sum_{r\in\mathbb{N}_0}\Big(2r+\frac{k}{N}\Big)^{-2s'} + \Big(2(r+1)-\frac{k}{N}\Big)^{-2s'} \\
& + \Big(2r+1+\frac{k}{N}\Big)^{-2s'} + \Big(2r+1-\frac{k}{N}\Big)^{-2s'} \le C < \infty
\end{aligned}$$

and hence

$$\|L_N(f - S_{N-1}f) \mid L^2_w(I)\| \le CN^{-s'}\|f - S_{N-1}f \mid H^{s'}_w(I)\|.$$

One proves easily that for $s > s'$ it holds that

$$\|f - S_{N-1}f \mid H^{s'}_w(I)\| \le CN^{s'-s}\|f \mid B^s_{2,\infty,w}(I)\|. \tag{4.3}$$

With this, it follows

$$\|L_N(f - S_{N-1}f) \mid L^2_w(I)\| \le CN^{-s}\|f \mid B^s_{2,\infty,w}(I)\|. \tag{4.4}$$

Now, our results (4.2), (4.4) and (4.3) applied with $s' = 0$ imply

$$\begin{aligned} &\|f - L_N f \mid L^2_w(I)\| \\ &\quad \le \ \|f - S_{N-1} f \mid L^2_w(I)\| + \|S_{N-1} f - L_N(S_{N-1} f) \mid L^2_w(I)\| \\ &\qquad\qquad + \|L_N(f - S_{N-1} f) \mid L^2_w(I)\| \\ &\quad \le \ C N^{-s} \|f \mid B^s_{2,\infty,w}(I)\|. \end{aligned}$$

This proves the theorem. ■

5. Examples

The interpolation on Gauß–Chebyshev grids is closely related to periodic interpolation on equidistant grids. The function $\mathcal{M} := \Lambda_N(\cos\cdot)/2$ is an even periodic fundamental interpolant on the grid $\{\frac{2\pi k}{2N}\,;\ k = -N, \dots, N-1\}$. If $\mathcal{M}$ satisfies the periodic Strang–Fix conditions of order m (see [3, 12]) with the constants $\{d_\ell\}_{\ell\in\mathbb{Z}}$ then Λ_N satisfies the Strang–Fix conditions for Gauß–Chebyshev grids of order m with the constants $b_{2\ell} = \pi^m d_\ell$ and $b_{2\ell+1} = \pi^m d_{\ell+1}$, $\ell \in \mathbb{N}_0$.

In this way, one obtains that the interpolatory scaling functions of the multiresolution analysis for a bounded interval described in [11] fulfil Strang–Fix conditions of certain order.

So the fundamental interpolant of the transformed B–spline of even order r satisfies Strang–Fix conditions of order r. The de la Vallée Poussin means of Chebyshev polynomials also described in [11] are fundamental interpolants and satisfy Strang–Fix conditions of arbitrary order m.

The constants $\{d_\ell\}_{\ell\in\mathbb{Z}}$ for the corresponding periodic functions can be found in [3, 12].

6. n–variate Interpolation on Sparse Grids

We now want to consider interpolation on sparse grids. Therefore, we choose $N_0 \in \mathbb{N}$ and set $N_j := N_0\, 2^j$ and $L_0 = 0$, $L_j := L_{N_{j-1}}, j \in \mathbb{N}$. Furthermore, we assume the imbeddings $\mathrm{Im}\ L_j \subset \mathrm{Im}\ L_{j+1}$, $j \in \mathbb{N}$. Therefore, the interpolation operators form a chain, i.e., $L_j L_{j+1} = L_{j+1} L_j = L_j$. The corresponding Lagrange functions Λ_{N_j} have to satisfy the Strang–Fix conditions with the same sequence of constants $\{b_\ell\}_{\ell\in\mathbb{N}_0}$. For our examples in Section 5, both assumptions are fulfilled.

Then, the interpolation operator on a sparse grid is the j–th order Boolean sum (j–th order blending operator), cf. [5], Chap. 1,

$$B^n_j := \bigoplus_{|\boldsymbol{r}|=j} L_{r_1} \otimes L_{r_2} \otimes \cdots \otimes L_{r_n},$$

with $j \geq n$, $|\boldsymbol{r}| = r_1 + r_2 + \cdots + r_n$ and $A \oplus B := A + B - AB$. The Boolean sum can be rewritten in terms of ordinary sums (see [18])

$$B_j^n = \sum_{j-n+1 \leq |\boldsymbol{r}| \leq j} (-1)^{j-|\boldsymbol{r}|} \binom{n-1}{j-|\boldsymbol{r}|} L_{r_1} \otimes L_{r_2} \otimes \cdots \otimes L_{r_n}.$$

It interpolates on the sparse grid

$$\bigcup_{|\boldsymbol{r}|=j} \mathcal{G}_{r_1} \times \mathcal{G}_{r_2} \times \cdots \times \mathcal{G}_{r_n}.$$

These grids are also nested. The number $N_G = N_G(j,n)$ of interpolation nodes in the sparse grid belonging to B_j^n can be estimated as (see [9]) as

$$N_G \leq C_n \; j^{n-1} \; 2^j$$

The Boolean sum operators form a chain, i.e., $B_j^n B_{j+1}^n = B_{j+1}^n B_j^n = B_j^n$.

Theorem 5 *Let $m > s > 1/2$, $\boldsymbol{s} = (s, s, \ldots, s)$. Let Λ_{N_j} satisfy the Strang–Fix conditions of order m with the same sequence of constants $\{b_\ell\}_{\ell \in \mathbb{N}_0}$. The corresponding interpolation operators form a chain. Then there exists a constant C (independent of j) such that*

$$\|f - B_j^n f \mid L_w^2(I^n)\| \leq C \; j^{n-1} \; N_j^{-s} \; \|f \mid S_{2,\infty,w}^{\boldsymbol{s}} B(I^n)\|$$

for all $f \in S_{2,\infty,w}^{\boldsymbol{s}} B(I^n)$.

Proof: Let I_n denote the natural imbedding $S_{2,\infty,w}^{\boldsymbol{s}} B(I^n) \hookrightarrow L_w^2(I^n)$. From Theorem 4, we obtain the estimates

$$\begin{aligned} \|I_1 \mid \mathcal{L}(B_{2,\infty,w}^s(I), L_w^2(I))\| &= 1, \\ \|I_1 - L_r \mid \mathcal{L}(B_{2,\infty,w}^s(I), L_w^2(I))\| &\leq C\, 2^{-sr}, \\ \|L_r - L_{r-1} \mid \mathcal{L}(B_{2,\infty,w}^s(I), L_w^2(I))\| &\leq D\, 2^{-sr}. \end{aligned}$$

With $E := C\,(\max\{2^s, D\})^{n-1}$ and the uniform crossnorm property (3.4) of the underlying Besov–type spaces of functions with dominating mixed smoothness properties, one proves easily (see [18]) that

$$\|I_n - B_j^n \mid \mathcal{L}(S_{2,\infty,w}^{\boldsymbol{s}} B(I^n), L_w^2(I^n))\| \leq E \binom{j}{n-1} 2^{-sj}.$$

This proves the theorem. ■

Corollary 6 *Under the assumptions of Theorem 5, the error of interpolation on the sparse grid can be estimated in terms of the number of grid points as*

$$\|f - B_j^n f \mid L_w^2(I^n)\| \le C\ N_G^{-s}\ (\log N_G)^{(s+1)(n-1)}\ \|f \mid S^{\boldsymbol{s}}_{2,\infty,w}B(I^n)\|.$$

Remark 1. The interpolation on sparse grids is almost optimal (up to logarithmic factors).

One can do better by substituting the interpolation operators in the definition of B_j^n by the Fourier–Chebyshev partial sum operators. This would yield the hyperbolic Fourier–Chebyshev sum $S_{U_j^n}$ with all Chebyshev polynomials which have their degree contained in the hyperbolic cross

$$U_j^n := \bigcup_{|\boldsymbol{r}|\le j} P_{\boldsymbol{r}}^n,$$

with the index sets $P_{\boldsymbol{r}}^n$ as used in Definition 2. As usual (see [17]), one can estimate

$$\|f - S_{U_j^n} f \mid L_w^2(I^n)\| \le C\ 2^{-2js}\ \|f \mid S^{\boldsymbol{s}}_{2,\infty,w}B(I^n)\|.$$

But, of course, here one has to use general linear information instead of function values only.

Remark 2. Under the assumptions of Theorem 5 one can easily prove the estimate

$$\|f - (L_j \otimes L_j \otimes \cdots \otimes L_j)f \mid L_w^2(I^n)\| \le C\ N_j^{-s}\ \|f \mid S^{\boldsymbol{s}}_{2,\infty,w}B(I^n)\|$$

for all $f \in S^{\boldsymbol{s}}_{2,\infty,w}B(I^n)$ using the tensor product interpolation on $N_T = N_j^n$ grid points. One gets the same order of approximation already for the functions $f \in B^s_{2,\infty,w}(I^n)$, $m > s > n/2$ from the corresponding isotropic Besov–type space. In terms of grid points, we have an error of $\mathcal{O}(N_T^{-s/n})$ in both cases.

This means that for functions with dominating mixed smoothness properties, the interpolation on sparse grids is essentially better suited than the tensor product construction.

For functions from isotropic spaces, we would lose some approximation order in sparse grid interpolation and obtain with (3.2) an error of order $\mathcal{O}(j^{n-1}N_j^{-s/n}) = \mathcal{O}((\log N_G)^{(s/n+1)(n-1)}N_G^{-s/n})$ only. So, for functions not providing dominating mixed smoothness, the interpolation on sparse grids is less suited. But still it is only by a logarithmic term worse than the full grid.

Remark 3. With the help of the Strang–Fix conditions, we obtain error estimates for a wide variety of possible sparse grid constructions, including for instance splines or polynomials, which are almost optimal for functions from all the Besov–type spaces $S^{\boldsymbol{s}}_{2,\infty,w}B(I^n)$ up to the order of the Strang–Fix conditions (which can be seen to be infinity for polynomials).

References

1. V. Barthelmann, E. Novak, and K. Ritter. High dimensional polynomial interpolation on sparse grids. Preprint, 1998.
2. D. Berthold, W. Hoppe, and B. Silbermann. A fast algorithm for solving the generalized airfoil equation. *J. Comput. Appl. Math.*, 43, 1992.
3. G. Brumme. Error estimates for periodic interpolation by translates. In P. J. Laurent, A. Le Méhauté, and L. L. Schumaker, editors, *Wavelets, Images, and Surface Fitting*, pages 75–82. AK Peters, Boston, 1994.
4. P. L. Butzer and R. L. Stens. The operational properties of the Chebyshev transform. *Funct. Approx. Comment. Math.*, 5:129–160, 1977.
5. F.-J. Delvos and W. Schempp. *Boolean Methods in Interpolation and Approximation*. Pitman Research Notes in Mathematics Series 230. Longman Scientific & Technical, Harlow, 1989.
6. T. Gerstner and M. Griebel. Numerical intergration using sparse grids. Preprint, 1998.
7. K. Jetter. Multivariate approximation from the cardinal point of view. In E. W. Cheney, C. K. Chui, and L. L. Schumaker, editors, *Approximation Theory VII*, pages 131–161. Academic Press, New York, 1992.
8. W. A. Light and E. W. Cheney. *Approximation Theory in Tensor Product Spaces*. Lecture Notes in Mathematics 1169. Springer, Berlin, 1985.
9. E. Novak and K. Ritter. High dimensional integration of smooth functions over cubes. *Numer. Math.*, 75:79–98, 1996.
10. E. Novak and K. Ritter. The curse of dimension and a universal method for numerical integration. In G. Nürnberger, J. W. Schmidt, and G. Walz, editors,

Multivariate Approximation and Splines, ISNM, pages 177–188. Birkhäuser, Basel, 1997.

11. G. Plonka, K. Selig, and M. Tasche. On the construction of wavelets on a bounded interval. *Adv. in Comput. Math.*, 4:357–388, 1995.
12. G. Pöplau and F. Sprengel. Some error estimates for periodic interpolation on full and sparse grids. In A. Le Méhauté, C. Rabut, and L. L. Schumaker, editors, *Curves and Surfaces with Applications in CAGD*, pages 355–362. Vanderbilt University Press, Nashville, 1997.
13. W. Sickel. Tensor products of Besov spaces. *J. Comput. Anal. Appl.*, 1, 1999. to appear.
14. W. Sickel and F. Sprengel. Interpolation on sparse grids and Nikol'skij–Besov spaces of dominating mixed smoothness. *J. Comput. Anal. Appl.*, 1, 1999. to appear.
15. F. Sprengel. Interpolation and wavelets on sparse Gauß–Chebyshev grids. In W. Haußmann, K. Jetter, and M. Reimer, editors, *Multivariate Approximation, Recent Trends and Results*, Mathematical Research, Vo. 101, pages 269–286. Akademie–Verlag, Berlin, 1997.
16. F. Sprengel. A unified approach to error estimates for interpolation on full and sparse Gauß–Chebyshev grids. *Rostock. Math. Kolloq.*, 51:51–64, 1997.
17. V. N. Temlyakov. *Approximation of Periodic Functions.* Nova Science, New York, 1993.
18. G. W. Wasilkowski and H. Woźniakowski. Explicit cost bounds of algorithms for multivariate tensor product problems. *J. Complexity*, 11:1–56, 1995.