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An A Posteriori Adaptive Mesh Technique for Singularly Perturbed Convection-Diffusion Problems with a Moving Interior Layer

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ABSTRACT

We study numerical approximations for a class of singularly perturbed problems of convection-diffusion type with a moving interior layer. In a domain (a segment) with a moving interface between two subdomains, we consider an initial boundary value problem for a singularly perturbed parabolic convection-diffusion equation. Convection fluxes on the subdomains are directed towards the interface. The solution of this problem has a moving transition layer in the neighbourhood of the interface. Unlike problems with a stationary layer, the solution exhibits singular behaviour also with respect to the time variable. Well-known upwind finite difference schemes for such problems do not converge ε-uniformly in the uniform norm, even under the condition $N^{-1} + N_0^{-1} \approx \varepsilon$, where ε is the perturbation parameter and N and N_0 denote the number of mesh points with respect to x and t. In the case of rectangular meshes which are ($a\ priori$ or $a\ posteriori$) locally refined in the transition layer, there are no schemes that convergence uniformly in ε even under the $very\ restrictive$ condition $N^{-2} + N_0^{-2} \approx \varepsilon$. However, the condition for convergence can be $essentially\ weakened$ if we take the geometry of the layer into account, i.e., if we introduce a new coordinate system which captures the interface. For the problem in such a coordinate system, one can use either an $a\ priori$, or an $a\ posteriori$ adaptive mesh technique. Here we construct a scheme on $a\ posteriori$ adaptive meshes (based on the gradient of the solution), whose solution converges 'almost ε-uniformly', viz., under the condition $N^{-1} = o(\varepsilon^{\nu})$, where $\nu > 0$ is an arbitrary number from the half-open interval $(0, 1]$.

2000 Mathematics Subject Classification: 65M06; 65M12; 65M15

Keywords and Phrases: singular perturbation problem, convection-diffusion equation, moving interior/transition layer, a $posteriori$ adaptive mesh, almost ε-uniform convergence

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An A Posteriori Adaptive Mesh Technique for Singularly Perturbed Convection-Diffusion Problems with a Moving Interior Layer

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We study numerical approximations for a class of singularly perturbed problems of convection-diffusion type with a moving interior layer. In a domain (a segment) with a moving interface between two subdomains, we consider an initial boundary value problem for a singularly perturbed parabolic convection-diffusion equation. Convection fluxes on the subdomains are directed towards the interface. The solution of this problem has a moving transition layer in the neighbourhood of the interface. Unlike problems with a stationary layer, the solution exhibits singular behaviour also with respect to the time variable. Well-known upwind finite difference schemes for such problems do not converge ε-uniformly in the uniform norm, even under the condition $N^{-1} + N_0^{-1} \approx \varepsilon$, where ε is the perturbation parameter and N and N_0 denote the number of mesh points with respect to x and t. In the case of rectangular meshes which are ($a\ priori$ or $a\ posteriori$) locally refined in the transition layer, there are no schemes that convergence uniformly in ε even under the $very\ restrictive$ condition $N^{-2} + N_0^{-2} \approx \varepsilon$. However, the condition for convergence can be $essentially\ weakened$ if we take the geometry of the layer into account, i.e., if we introduce a new coordinate system which captures the interface. For the problem in such a coordinate system, one can use either an $a\ priori$, or an $a\ posteriori$ adaptive mesh technique. Here we construct a scheme on $a\ posteriori$ adaptive meshes (based on the gradient of the solution), whose solution converges 'almost ε-uniformly', viz., under the condition $N^{-1} = o(\varepsilon^{\nu})$, where $\nu > 0$ is an arbitrary number from the half-open interval $(0, 1]$.

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1. Introduction

In this paper we study a problem with discontinuous coefficients. In particular, we consider the case of a parabolic problem where the convection coefficient is discontinuous and has opposite signs at both sides of an interface. Such boundary value problems for singularly perturbed equations with partial derivatives arise (see, e.g., [9, 10] and the bibliography therein), when heat/mass transfer processes in composite materials with small heatconduction/diffusion in the case of stationary interfaces are studied. The terms with the highest derivatives in those equations are multiplied by a small parameter ε that gives rise to boundary and transition layers in the solution. Because coefficients in the equations (and source terms) are discontinuous, the derivatives of the solution have discontinuities at the interface even for fixed values of the parameter ε. Note that the errors of standard numerical methods, developed for regular boundary value problems, essentially depend on the value of the parameter ε. For example, the solution of a finite difference scheme, in the case of a problem with a stationary interface, converges when the stepsize in the space mesh is much smaller than ε (see remarks to theorems 3.1, 3.2 in section 3). Because of the restrictive convergence condition for the classical finite difference schemes, the interest arises to construct special schemes for which the errors in the solution depend weakly on the parameter ε. In particular, we are interested in methods where the error behaviour is independent of ε (we say, that the numerical methods converge ε-uniformly).

In the case of nonstationary interfaces, moving transition layers appear. Unlike problems with stationary singularities, in the case of moving transition layers, the solution exhibits singular behaviour also with respect to the time variable (see, e.g., estimates (9.5) in section 9). The considerably complicated character of arising singularities (as compared to problems with stationary singularities; for which numerical methods are treated, e.g., in [2, 6, 7, 11]) forces us to use a more complicated discrete construction.

The use of a technique developed to improve the accuracy of the solutions in the case of regular boundary value problems (e.g., the technique of *a priori*/*a posteriori* adaptive meshes; see [1, 4, 7] and the bibliography therein), turns out to be ineffective in the case of singularly perturbed problems (see [12] and the statement of theorem 4.1 in section 4). Therefore, the quest for conditions which are necessary (and for specific discrete methods also sufficient) for the ε-uniform convergence of numerical methods, for problems with moving transition layers is relevant.

In this paper we consider discrete approximations of an initial boundary value problem for the singularly perturbed parabolic convection-diffusion equation in a domain with a moving interface between two subdomains when convective fluxes on the subdomains are directed towards the interface. The solution of this problem has a singularity near the transition layer that moves in time.

We study finite difference schemes based on classical discrete approximations of the problem. When rectangular uniform meshes are used, the order of magnitude of the error is not smaller than the exact solution when $\varepsilon = \mathcal{O}(N^{-1} + N_0^{-1})$, where N, N_0 denote the number of nodes in the space and time variables respectively. Finite difference schemes are considered on meshes that are locally refined in a neighbourhood of the moving interface. It turns out that in the class of finite difference schemes on rectangular meshes locally condensing in x and t in a neighbourhood of the trajectory of the moving interface, there do not exist ε-uniformly convergent schemes even under the condition $\varepsilon \approx N^{-2} + N_0^{-2}$. However, if a new coordinate system is introduced, for which the interface becomes a meshline, then one can construct a scheme, whose solution converges 'almost' ε-uniformly, i.e., it converges under the condition $N^{-1} = \mathcal{O}(\varepsilon^{\nu})$, where $\nu > 0$ is an arbitrary small number. For the problem in the new coordinate system, it is possible to use either *a priori* or *a posteriori* mesh refinement techniques to obtain almost ε-uniformly accurate result. As *a posteriori* adaptive technique we use adaptive mesh refinement on the base of the gradient of the solution.

The approach can be used to construct effective numerical methods for representative classes of boundary value problems with the dominant convection with known, moving transition layers.

About the contents. Problem formulation, and the aim of the research are given in Section 2. Classical schemes and auxiliary problems related to the construction of schemes convergent ε-uniformly and almost ε-uniformly are considered in the Sections 3 and 4. Schemes on *a posteriori* adaptive meshes are constructed and studied in the Sections 5–8 (in Sections 5–7 for a problems with stationary and in Section 8 with moving interface boundaries). *A priori* estimates are given in Section 9.

The technique for the construction of the *a posteriori* adaptive meshes, based on the solution gradient, is used in [3] to construct almost ε-uniformly convergent schemes for an initial boundary value problem for a parabolic convection-diffusion equation with "standard" singularity (a "stationary" boundary layer). In [12] for an initial value problem for a parabolic reaction-diffusion equation, a special scheme is considered on meshes that are *a priori* refined in the transition layer caused by a moving point source. For the construction of the scheme, special coordinates are used, for which the location of the point source is fixed.

2. Problem formulation. Aim of research

1. In a bounded domain with a moving interface between two subdomains we consider an initial boundary value problem for a singularly perturbed parabolic convection-diffusion equation.

Let the domain $\overline{G}$ with the boundary $S = \overline{G} \setminus G$, where $G = D \times (0, T]$, $D = (-d, d)$, be decomposed into non-overlapping subdomains

$$\overline{G} = \overline{G}^1 \cup \overline{G}^2, \qquad G^1 \cap G^2 = \emptyset, \tag{2.1}$$

in each of which we consider an equation

$$L^r u(x,t) \equiv \left\{ \varepsilon a^r(x,t) \frac{\partial^2}{\partial x^2} + (-1)^r b^r(x,t) \frac{\partial}{\partial x} - c^r(x,t) - p^r(x,t) \frac{\partial}{\partial t} \right\} u(x,t) = f^r(x,t),$$
$$(x,t) \in G^r, \quad r = 1,2, \tag{2.2a}$$

where

$$G^1 = \{(x,t):\ x < \beta(t),\ \ t \in (0,T]\}, \ \ G^2 = \{(x,t):\ x > \beta(t),\ \ t \in (0,T]\}. \tag{2.3}$$

The curve $\gamma = \{(x,t):\ x = \beta(t),\ \ |\beta(t)| < d,\ \ t \in (0,T]\}$, i.e. the interface between the subdomains, is given by a sufficiently smooth function. On the set S the function $u(x,t)$ takes the prescribed values

$$u(x,t) = \varphi(x,t), \ \ (x,t) \in S, \tag{2.2b}$$

and, on the interface γ, it obeys the conjugation condition, i.e. the solution and the diffusion flux be continuous

$$[u(x,t)] = 0, \ \ lu(x,t) \equiv \varepsilon \left[a(x,t) \frac{\partial}{\partial x} u(x,t) \right] = 0, \ \ (x,t) \in \gamma. \tag{2.2c}$$

In (2.2a) $a^r(x,t), \dots, f^r(x,t)$, $(x,t) \in \overline{G}^r$, $r = 1,2$ and $\varphi(x,t)$, $(x,t) \in S$ are sufficiently smooth functions on $\overline{G}^r$ and S_0, $\overline{S}^L$, respectively, $\varphi(x,t) \in C(\overline{S})$, and also [1]

$$a_0 \le a^r(x,t) \le a^0, \ \ b_0 \le b^r(x,t) \le b^0, \ \ p_0 \le p^r(x,t) \le p^0, \tag{2.4}$$
$$0 \le c^r(x,t) \le c^0, \ \ (x,t) \in \overline{G}^r, \quad a_0,\, b_0,\, p_0 > 0;$$
$$-\beta_0 \le \beta(t) \le \beta^0, \ \ |\beta^{'}(t)| \le \beta^1, \ \ t \in [0,T], \ \ \beta_0,\, \beta^0 < d, \ \ \beta^1 < b_0\,(p^0)^{-1};$$
$$|f^r(x,t)| \le M, \ \ (x,t) \in \overline{G}^r, \ \ |\varphi(x,t)| \le M, \ \ (x,t) \in S; \ \ r = 1,2;$$

$S = S_0 \cup S^L$, S^L and S_0 are the lateral and lower parts of the boundary S, $S_0 = \overline{S}_0$; let $\beta(0) = 0$; ε is a parameter taking arbitrary values from the half-open interval $(0,1]$. The symbol $[v(x,t)]$ denotes the jump of the function $v(x,t)$ when passing through γ from the set G^1 into the set G^2:

$$[v(x^*,t)] = \lim_{x \to x^*+0} v(x,t) - \lim_{x \to x^*-0} v(x,t),$$
$$\left[a(x^*,t) \frac{\partial}{\partial x} v(x^*,t) \right] = \lim_{x \to x^*+0} a^2(x,t) \frac{\partial}{\partial x} v(x,t) - \lim_{x \to x^*-0} a^1(x,t) \frac{\partial}{\partial x} v(x,t), \ \ (x^*,t) \in \gamma.$$

For the equation (2.2a) we also use the following notation [2]

$$L_{(2.2a)} u(x,t) \equiv \left\{ \varepsilon a(x,t) \frac{\partial^2}{\partial x^2} + b(x,t) \frac{\partial}{\partial x} - c(x,t) - p(x,t) \frac{\partial}{\partial t} \right\} u(x,t) = f(x,t), \quad (x,t) \in G^{(*)},$$

where $G^{(*)} = G \setminus \gamma$, $b(x,t) = b^2(x,t)$, $(x,t) \in \overline{G}^2$, $b(x,t) = -b^1(x,t)$, $(x,t) \in \overline{G}^1$, the functions $a(x,t)$, $c(x,t)$, $p(x,t)$, $f(x,t)$ are defined by the relation $v(x,t) = v^r(x,t)$, $(x,t) \in \overline{G}^r$, $r = 1,2$.

For simplicity, we assume that the compatibility conditions are fulfilled on the sets $S^c = S_0 \cap \overline{S}^L$ and $\gamma^0 = \{(\beta(0),0)\}$ to ensure sufficient smoothness of the solution of the problem on each of the subsets $\overline{G}^r$ (for fixed values of the parameter ε); we suppose $S^r = \overline{G}^r \setminus G^r$, $r = 1,2$.

[1] Here and below M, M_i (or m) denote sufficiently large (small) positive constants which do not depend on ε and on the discretization parameters.

[2] Throughout the paper, the notation $L_{(\mathrm{j.k})}$ ($M_{(\mathrm{j.k})}$, $G_{h(\mathrm{j.k})}$) means that these operators (constants, grids) are introduced in equation (j.k).

As $\varepsilon \to 0$, in a neighbourhood of the set γ there appears a transition layer decreasing exponentially away (in the x-direction) from the set γ (the typical "width" of the layer is of order ε). Hence, in the case of the moving interface, the transition layer decreases exponentially away from γ for a fixed value of x. Under the condition

$$\beta(t) = const\ , \quad t \in [0, T] \tag{2.5}$$

(the steady interface) the t-derivatives of the singular part of the solution to the problem are bounded on $\overline{G}$ ε-uniformly (see estimates (9.4), (9.5) from Section 9).

The solution of the reduced problem is a function which is sufficiently smooth on each of the sets $\overline{G}^{\,r}$ and has a discontinuity of the first kind on $\overline{\gamma}$.

Note that a boundary layer does not appear in the neighbourhood of the lateral boundary S^L (because of the condition imposed on the coefficient $b(x,t)$ so that the convection flow is directed to the interface γ). In the case of the condition $\beta^1 < b_0(p^0)^{-1}$ (see (2.4)) the interface γ between the subdomains G^r is noncharacteristic (the characteristics of the reduced equation are not tangent to the set γ).

For definiteness, we consider that the functions $a(x,t), \ldots, f(x,t)$ on the set $\overline{\gamma}$ are equal to the half-sum of the values (mean value) of their limits from the sets G^1 and G^2.

2. The errors in the solutions of finite difference schemes based on classical difference approximations to problem (2.2), (2.1) depend on the parameter ε and become small only for those values of ε that essentially exceed the "effective" mesh steps with respect to x and t. So, by virtue of estimates (3.7), (3.13) (see Section 3), classical difference schemes (3.4), (3.3) and (3.11), (3.10), (3.3) converge under the condition

$$\varepsilon >> N^{-1} + N_0^{-1}, \tag{2.6}$$

where N, N_0 denote the number of mesh points with respect to x and t, respectively. If this condition fails, the solutions of the difference schemes do not generally converge to the solution of problem (2.2), (2.1).

By this argument, we are interested in constructing special difference schemes whose errors do not depend on the value of the parameter ε. In particular, it is of interest to develop such schemes that converge under a weaker condition than condition (2.6), which is the convergence condition for the discrete problems (3.4), (3.3) and (3.11), (3.10), (3.3).

Definition. Let $z(x,t)$, $(x,t) \in \overline{G}_h$ be a solution of some difference scheme. This scheme converges uniformly with respect to the parameter ε (or ε-uniformly), if the function $z(x,t)$ satisfies the estimate:

$$|u(x,t) - z(x,t)| \le M\, \mu\, (N^{-1}, N_0^{-1}), \quad (x,t) \in \overline{G}_h,$$

where $\mu(N^{-1},\, N_0^{-1})$ tends to zero for $N,\, N_0 \to \infty$ uniformly with respect to the parameter. We say that the solution of the scheme converges *almost ε-uniformly*, if for any arbitrarily small number $\nu > 0$ there exists a function $\mu(\varepsilon^{-\nu} N^{-1}, \varepsilon^{-\nu} N_0^{-1})$ such that the following estimate holds for the mesh function $z(x,t)$:

$$|u(x,t) - z(x,t)| \le M\, \mu\, (\varepsilon^{-\nu} N^{-1}, \varepsilon^{-\nu} N_0^{-1}), \quad (x,t) \in \overline{G}_h. \tag{2.7}$$

where $\mu(N^{-1}, N_0^{-1}) \to 0$ for $N, N_0 \to \infty$ uniformly with respect to the parameter ε. If estimate (2.7) is satisfied, then we say that the scheme converges almost ε-uniformly with the defect ν. (If estimate (2.7) is satisfied for $\nu = 0$, then the convergence is ε-uniform.)

The aim of the research. The defect of schemes (3.4), (3.3) and (3.11), (3.10), (3.3) is equal to 1. Thus, in the case of problem (2.2), (2.1) we arrive at the following question of theoretical (and practical) interest: how to construct schemes whose defect is less than 1, and in particular, how to construct almost ε-uniformly convergent schemes.

3. Classical difference schemes

Let us give classical difference schemes for problem (2.2), (2.1) and show estimates for their solutions.

1. We give a difference scheme for problem (2.2), (2.1) considering the problem in a week formulation and not focusing the conjugation condition (2.2c) in the approximation of the problem.

On the set $\overline{G}$ we introduce the rectangular mesh

$$\overline{G}_h = \overline{\omega}_1 \times \overline{\omega}_0, \tag{3.1}$$

where $\overline{\omega}_1$ and $\overline{\omega}_0$ are meshes on the segments $\overline{D}$ and $[0,T]$, respectively; $\overline{\omega}_1$ and $\overline{\omega}_0$ are meshes with any distribution of the nodes satisfying only the condition $h \le MN^{-1}$, $h_t \le MN_0^{-1}$, where $h = \max_i h^i$, $h^i = x^{i+1} - x^i$, x^i, $x^{i+1} \in \overline{\omega}_1$, $h_t = \max_j h_t^j$, $h_t^j = t^{j+1} - t^j$, t^j, $t^{j+1} \in \overline{\omega}_0$. Here $N+1$ and N_0+1 are the number of nodes in the meshes $\overline{\omega}_1$ and $\overline{\omega}_0$, respectively. It is of great interest to consider meshes that are uniform with respect to x

$$\overline{G}_h = \overline{G}_{h(3.1)}, \tag{3.2}$$

where $\overline{\omega}_1$ is a uniform mesh; such meshes for $b(x,t) \equiv 0$ allow us to obtain the second order of the approximation with respect to x for sufficiently smooth solutions. Also it is interesting to consider difference schemes on the simplest meshes which are uniform with respect to both x and t:

$$\overline{G}_h = \overline{G}_{h(3.1)}, \tag{3.3}$$

where both $\overline{\omega}_1$ and $\overline{\omega}_0$ are uniform meshes.

We approximate problem (2.2), (2.1) by the implicit finite difference scheme [8]

$$\begin{aligned} \Lambda z(x,t) \equiv \big\{ & \varepsilon a(x,t)\delta_{\overline{x}\widehat{x}} + b^+(x,t)\delta_x + b^-(x,t)\delta_{\overline{x}} - c(x,t) - \\ & - p(x,t)\delta_{\overline{t}}\big\} z(x,t) = f(x,t), \quad (x,t) \in G_h, \\ z(x,t) = {} & \varphi(x,t), \qquad (x,t) \in S_h. \end{aligned} \tag{3.4a}$$

Here $G_h = G \cap \overline{G}_h$, $S_h = S \cap \overline{G}_h$; $\delta_{\overline{x}\widehat{x}} z(x,t)$, $\delta_x z(x,t)$, $\delta_{\overline{x}} z(x,t)$, $\delta_{\overline{t}} z(x,t)$ are the second and first difference derivatives; $\delta_{\overline{x}\widehat{x}} z(x,t) = 2(h^i + h^{i-1})^{-1}\{\delta_x - \delta_{\overline{x}}\} z(x,t)$, $x = x^i$, h^{i-1} and h^i are the left and right "arms" of the three-point stencil (for the operator $\delta_{\overline{x}\widehat{x}}$) on G_h with center at the node $(x^i,t) \in G_h$;

$$b^+(x,t) = 2^{-1}(b(x,t) + |b(x,t)|), \quad b^-(x,t) = 2^{-1}(b(x,t) - |b(x,t)|).$$

For the difference scheme (3.4), (3.1) the maximum principle is valid.

By using the majorant function technique, we find the estimate

$$|z(x,t)| \le M, \quad (x,t) \in \overline{G}_h. \tag{3.5}$$

Taking into account the *a priori* estimates for the solution of the initial boundary value problem, we find the following estimate in the case of mesh (3.1):

$$|u(x,t) - z(x,t)| \le M\varepsilon^{-2}[N^{-1} + N_0^{-1}], \quad (x,t) \in \overline{G}_h. \tag{3.6}$$

For the mesh (3.3) we have

$$|u(x,t) - z(x,t)| \le M\varepsilon^{-1}[N^{-1} + N_0^{-1}], \quad (x,t) \in \overline{G}_h. \tag{3.7}$$

We summarize this in the following.

Theorem 3.1 *Let the components of the solution of initial boundary value problem* (2.2), (2.1) *in the representation* (9.1) *satisfy the a priori estimates* (9.4), (9.5). *Then the solution of the difference scheme* (3.4), (3.1) *converges for fixed values of the parameter* ε. *For the discrete solutions the estimates* (3.5), (3.6), (3.7) *are valid.*

Remark 3.1 In the case of condition (2.5) for scheme (3.4), (3.3) we have the estimate

$$|u(x,t) - z(x,t)| \le M[\varepsilon^{-1}N^{-1} + N_0^{-1}], \ \ (x,t) \in \overline{G}_h.$$

The condition

$$\varepsilon^{-1} = o(N) \tag{3.8}$$

is necessary and sufficient for the convergence of the scheme.

2. We now consider a difference scheme for problem (2.2), (2.1) based on "direct" approximation of the conjugation condition (2.2c), i.e. the interface condition. For this we need meshes which contain nodes on the set γ at each time level $t = t^j$ of this difference scheme which is an alternative to scheme (3.4), (3.1). Let us construct such meshes.

For the construction of these meshes we use the regular mesh (3.1) (or (3.2), (3.3)) as the basic mesh; we construct the mesh $\overline{G}_h^* = \overline{G}_h^*(\overline{G}_{h(3.1)})$. Let

$$G_h^* = \{(x^i,\, t^j) :\, (x^i,\, t^j) \in G_h, \ \ x^i \ne \beta(t), \ t \in [t^{j-1},\, t^j], \ t^{j-1},\, t^j \in \overline{\omega}_0\}. \tag{3.9}$$

On the time level $t = t^n$ we construct the sets G_h^{1n}, G_h^{2n}: $G_h^{rn} = G_h^* \cap \{t = t^n\}$, $t^n \in \overline{\omega}_0$, $r = 1, 2$. The corresponding nodes $(x^i,\, t^{j-1})$ generate the sets S_{0h}^{1n} and S_{0h}^{2n}, which are the lower mesh boundaries for sets G_h^{1n}, G_h^{2n}. The nodes (x^i, t^j) from S^L form the set S_h^{nL}. We assume $S_h^{1n} = \gamma_h^n \cup S_{0h}^{1n} \cup \{S_h^{nL} \cap \overline{G}^1\}$, $S_h^{2n} = \gamma_h^n \cup S_{0h}^{2n} \cup \{S_h^{nL} \cap \overline{G}^2\}$, $\overline{G}_h^{1n} = G_h^{1n} \cup S_h^{1n}$, $\overline{G}_h^{2n} = G_h^{2n} \cup S_h^{2n}$, where $\gamma_h^n = \{(\beta(t^n), t^n)\}$. We introduce the sets $\overline{G}_h^{*n} = \overline{G}_h^{1n} \cup \overline{G}_h^{2n}$, $S_h^{*n} = \overline{G}_h^{*n} \setminus G_h^{*n}$, $G_h^{*n} = G_h^{(*)n} \cup \gamma_h^n$, $G_h^{(*)n} = G_h^{1n} \cup G_h^{2n}$; $\overline{G}_h^{*n} = G_h^{*n} \cup S_h^{*n}$. The mesh $\overline{G}_h^*$ is defined by the relation

$$\overline{G}_h^* = \bigcup_{n=1}^{N_0} \overline{G}_h^{*n}. \tag{3.10}$$

We approximate problem (2.2), (2.1) by the difference scheme

$$\Lambda z(x,t) \ = \ f(x,t), \ \ (x,t) \in G_h^{(*)n}, \tag{3.11a}$$

$$l^h z(x,t) \ \equiv \ \varepsilon\{a^2(x,t)\delta_x z(x,t) - a^1(x,t)\delta_{\overline{x}} z(x,t)\} = 0, \ \ (x,t) \in \gamma_h^n, \tag{3.11b}$$

$$z(x,t) \ = \ \left\{ \begin{array}{ll} \widehat{z}^{n-1}(x,t), & (x,t) \in S_h^{*n} \setminus S \\ \varphi(x,t), & (x,t) \in S_h^{*n} \cap S \end{array} \right\}; \ \ (x,t) \in \overline{G}_h^{*n}, \ \ n = 1, \ldots, N_0. \tag{3.11c}$$

Here $z^n(x,t) = z(x,t)$ for $(x,t) \in \overline{G}_h^{*n}$, $t = t^n$, $\widehat{z}^n(x,t)$, $(x,t) \in \overline{G}$, $t = t^n \in \overline{\omega}_0$ is the linear interpolant constructed from the values of $z^n(x,t)$, $(x,t) \in \overline{G}_h^{*n}$, $t = t^n$. The function

$$z(x,t) = \left\{ \begin{array}{ll} z^n(x,t), & (x,t) \in \overline{G}_h^{*n}, \ \ t = t^n, \\ \widehat{z}^{n-1}(x,t), & (x,t) \in \overline{G}_h^{*n}, \ \ t = t^{n-1}; \end{array} \right.$$

$$(x,t) \in \overline{G}_h^{*n}, \ \ n = 1, \ldots, N_0, \ \ (x,t) \in \overline{G}_h^*$$

will be called the solution of difference scheme (3.11), (3.10).

For the difference scheme (3.11), (3.10) the maximum principle is valid.

Taking into account the *a priori* estimates for the solutions of the differential problem we establish the estimates similar to (3.6), (3.7)

$$|u(x,t) - z(x,t)| \le M\varepsilon^{-2}\left[N^{-1} + N_0^{-1}\right], \ \ (x,t) \in \overline{G}_h^*, \ \ \overline{G}_h = \overline{G}_{h(3.1)}; \tag{3.12}$$

$$|u(x,t) - z(x,t)| \le M\varepsilon^{-1}\left[N^{-1} + N_0^{-1}\right], \ \ (x,t) \in \overline{G}_h^*, \ \ \overline{G}_h = \overline{G}_{h(3.3)}. \tag{3.13}$$

Theorem 3.2 *Let the condition of theorem* 3.1 *be fulfilled. Then the solution of the difference scheme* (3.11), (3.10) *converges to the solution of the problem* (2.2), (2.1) *for fixed values of the parameter* ε. *For the discrete solutions the estimates* (3.12), (3.13) *are valid.*

Definition. Let the function $z(x,t)$, $(x,t) \in \overline{G}_h$ be the solution of some difference scheme. An estimate of the following form

$$|u(x,t) - z(x,t)| \le M \left[\varepsilon^{-\nu_1^1} N^{-\nu_1} + \varepsilon^{-\nu_0^1} N_0^{-\nu_0}\right], \quad (x,t) \in \overline{G}_h,$$

where ν_i, $\nu_i^1 \ge 0$, is said to be *unimprovable* with respect to the values of N, N_0, ε if the estimate

$$|u(x,t) - z(x,t)| \le M \left[\varepsilon^{-\alpha_1^1} N^{-\alpha_1} + \varepsilon^{-\alpha_0^1} N_0^{-\alpha_0}\right], \quad (x,t) \in \overline{G}_h,$$

in general, fails under the conditions $\alpha_i \ge \nu_i$, $\alpha_i^1 \le \nu_i^1$ and also $\alpha_1 + \alpha_0 - \alpha_1^1 - \alpha_0^1 > \nu_1 + \nu_0 - \nu_1^1 - \nu_0^1$.

Remark 3.2 In the case of condition (2.5) for the solutions of difference scheme (3.11), (3.10), (3.3) we have the unimprovable estimate

$$|u(x,t) - z(x,t)| \le M \left[\varepsilon^{-1} N^{-1} + N_0^{-1}\right], \quad (x,t) \in \overline{G}_h;$$

the scheme converges under the unimprovable condition (3.8).

3. For problem (2.2), (2.1) we discuss the conditions under which the solution of difference scheme (3.4) converges for $N, N_0 \to \infty$ and $\varepsilon \to 0$, where $\varepsilon \to 0$ for $N, N_0 \to \infty$.

It follows from estimates (3.6) and (3.7) that scheme (3.4), (3.1) converges under the condition $N^{-1}, N_0^{-1} << \varepsilon^2$, and scheme (3.4), (3.3) converges under the condition N^{-1}, $N_0^{-1} << \varepsilon$, i.e. for

$$\varepsilon^{-1} = o(N), \qquad \varepsilon^{-1} = o(N_0). \tag{3.14}$$

The estimate (3.7) (as well as estimate (3.13)) is unimprovable with respect to the values of N, N_0, ε. The defect of schemes (3.4), (3.1) and (3.4), (3.3) (schemes (3.11), (3.10), (3.1) and (3.11), (3.10), (3.3)), according to estimates (3.6) and (3.7) (to estimates (3.12) and (3.13)), is not less than 2 and 1, respectively. It follows from the unimprovability of estimate (3.7) (estimate (3.13)) that the unimprovable defect of scheme (3.4), (3.3) (scheme (3.11), (3.10), (3.3)) is equal to 1, moreover, the unimprovable defect of scheme (3.4), (3.1) (scheme (3.11), (3.10), (3.1)) is not less than 1.

Thus, the estimates of the discrete solutions essentially depend on the value of the parameter ε; if condition (3.14) fails then schemes (3.4) and (3.11), (3.10), generally speaking, do not converge.

Theorem 3.3 *Let the hypothesis of theorem* 3.1 *be fulfilled. In the case of schemes* (3.4), (3.3) *and* (3.11), (3.10), (3.3) *(schemes* (3.4), (3.1) *and* (3.11), (3.10), (3.1)*) the condition* (3.14) *is necessary and sufficient (is necessary) for the convergence of the discrete solutions to the solution of problem* (2.2), (2.1) *for* $N, N_0 \to \infty$ *and* $\varepsilon \to 0$*; the defect of schemes* (3.4), (3.3) *and* (3.11), (3.10), (3.3) *(schemes* (3.4), (3.1) *and* (3.11), (3.10), (3.1)*) is equal to* 1 *(not less than* 1*). The estimates* (3.7) *and* (3.13) *are unimprovable with respect to the values of* N, N_0, ε.

Remark 3.3 Taking into account the above considerations of classical difference schemes, in the case of problem (2.2), (2.1) one comes to the problem of construction of special schemes which converge under a weaker condition than the condition (3.14) (i.e. schemes whose defect is less than 1), in particular, ε-uniformly convergent schemes.

Remark 3.4 We construct a triangulation of the domain $\overline{G}$ on the basis of the mesh $\overline{G}_{h(3.1)}$; triangular elements obtained by dividing elementary quadrangles in halves by a diagonal have vertices at the nodes from $\overline{G}_h$; see, e.g., [5]. In the case of difference scheme (3.4), (3.3) the function $\overline{z}(x,t)$, $(x,t) \in \overline{G}$, i.e. the linear interpolant of $z(x,t)$ on triangular elements, satisfies the error estimate

$$|u(x,t) - \overline{z}(x,t)| \le M\varepsilon^{-1} \left[N^{-1} + N_0^{-1}\right], \quad (x,t) \in \overline{G}, \tag{3.15}$$

which is unimprovable with respect to the values of N, N_0, ε. A similar estimate is valid for scheme (3.11), (3.10), (3.3); the triangulation of the domain $\overline{G}$ is performed on the basis of the mesh $\overline{G}_h^*$.

4. On the construction of ε-uniformly convergent schemes on locally condensing meshes

Note that the singularity in the solution of problem (2.2), (2.1) exponentially decreases away from the set γ (see estimates (9.4), (9.5)). The singular component for $|x-\beta(t)| \geq \sigma$ does not exceed $M\delta$, where δ is a sufficiently small number, when $\sigma = m_1^{-1}\varepsilon \ln \delta^{-1}$, $m_1 = m_{1(9.4)}$. The local truncation error of a classical scheme on the solution of the problem is great, but only in this neighbourhood, which is sufficiently narrow for small values of the parameter ε.

1. Bearing in mind the possible use of schemes on sufficiently arbitrary locally condensing meshes for solving the problem (2.2), (2.1), it would be convenient to measure an "amount" of computational work in order to evaluate the efficiency of difference schemes. The amount of computational work (denoted by P) is defined by the number of the mesh points at which it is necessary to find the solution of the discrete problem. The "quality" of the solution to the difference scheme is defined by the distance, in the maximum norm, between the solution of the problem and the interpolant which is constructed from the solution of the difference scheme. In the case of schemes (3.4), (3.3) and (3.11), (3.10), (3.3) and optimal meshes (with respect to the order of convergence of the scheme), we have the following relation for such meshes

$$N \sim N_0,$$

and the unimprovable error estimate (with respect to P and ε)

$$|u(x,t) - \overline{z}(x,t)| \leq M\varepsilon^{-1}P^{-1/2}, \quad (x,t) \in \overline{G}. \tag{4.1}$$

Definition. We say that the scheme converges with *defect* ν for $P \to \infty$, if there exists a function $\mu(P^{-1})$, $\mu(P^{-1}) \to 0$ for $P \to \infty$ ε-uniformly such that the following estimate holds for the interpolated mesh function $z(x,t)$:

$$|u(x,t) - \overline{z}(x,t)| \leq M\mu(\varepsilon^{-\nu}P^{-1/2}), \quad (x,t) \in \overline{G}; \quad \nu > 0.$$

Thus, for difference schemes (3.4), (3.3) and (3.11), (3.10), (3.3) the unimprovable convergence defect (for $P \to \infty$) is equal to 1.

2. To construct schemes with improved convergence defect, it is very attractively to use a technique based on locally condensing rectangular meshes. So, in the case of regular boundary value problems whose solutions have singularities, the improvement of accuracy of a discrete solution can be achieved by means of *a priori* and/or *a posteriori* local refinement of a rectangular mesh in those subdomains where errors of the discrete solution are larger (see, e.g., [1, 4, 15]).

For problem (2.2), (2.1) it is required to redistribute the given number of nodes P in the domain so that to weaken the condition

$$P^{-1/2} = o(\varepsilon), \tag{4.2}$$

i.e. the condition of convergence of schemes (3.4), (3.3) and (3.11), (3.10), (3.3).

Note that the derivatives $(\partial^{k_1+k_0}/\partial x^{k_1}\partial t^{k_0})u(x,t)$ in an $M\varepsilon$-neighbourhood of the set γ are of the order $\varepsilon^{-(k_1+k_0)}$ if the interface boundary γ between subdomains is moving at a rate distinct from zero in some time interval (let $\beta'(t) = m$, $t \in [0,t_0]$). The derivatives are ε-uniformly bounded outside a sufficiently large (compared to ε) neighbourhood of the set γ.

The similar behaviour of derivatives of the solution is observed in the case of a problem with the moving concentrated source [12]. In papers [12, 13], it is shown that, in a class of schemes on adaptive meshes based on meshes which are rectangular in an $M\varepsilon$-neighbourhood of the moving source, there are no schemes with the convergence defect (for $P \to \infty$) less than 2^{-1}.

In the case of problem (2.2), (2.1), by considering lower errors for solutions of equations (3.4a), (3.11a) on "piecewise-uniform" meshes (i.e., meshes which are uniform in the nearest $M\varepsilon$-neighbourhood of the boundary γ for $t \in [0,t_0]$, as well as outside some larger neighbourhood), we verify, similarly to the constructions from [12], that there are no meshes on which the solution of the discrete problem converges under the condition

$$P^{-1} \geq \varepsilon. \tag{4.3}$$

Thus, in the case of schemes (3.4) and (3.11), (3.10) there are no meshes in the given class of meshes on which the schemes converge with defect not exceeding 2^{-1}.

Theorem 4.1 *For the problem* (2.2), (2.1), *in a class of schemes constructed on the basis of approximations* (3.4*a*) *and* (3.11*a*) *(the classical difference approximations of the initial boundary value problem) on locally condensing meshes that are uniform in an $M\varepsilon$-neighbourhood of the set γ, there are no schemes convergent under the condition* (4.3).

Remark 4.1 The direct use of adaptive mesh refinement techniques with no taking account of orientation of the transition layer is not sufficiently effective to solve numerically problems from this class of singularly perturbed problems with a moving transition layer. In order to construct schemes on adaptive meshes with the convergence defect less than 2^{-1}, it is necessary to use meshes condensing (in the transition layer) along a normal to the interface boundary γ.

Remark 4.2 To construct special schemes for problem (2.2), (2.1) we introduce new variables (connected with the moving interface boundary γ) in which the interface boundary is already stationary. For the problem in these new variables it is possible to construct a difference scheme on rectangular meshes (in particular, a scheme on adaptive meshes) and then to return to the old variables. It is convenient to use the variables ξ, t, $\xi = \xi_{(9.2)}(x,t)$ as new variables.

5. Grid approximations on locally refined meshes. Problem (2.2), (2.1), (2.5)

We now give an algorithm for constructing a locally refined (in the transition layer) mesh. On domains which are subjected to the refinement, this algorithm uses uniform meshes in space and time (the time mesh is not refined).

1. At first, we describe a formal iterative algorithm which we use for construction of difference schemes in order to find a numerical solution of problem (2.2), (2.1), (2.5). Suppose that a function $u_1(x,t)$ is founded on the set $\overline{G}$, i.e. the approximation to the solution of the boundary value problem, moreover,

$$|u(x,t) - u_1(x,t)| \le M\delta, \quad (x,t) \in \overline{G}, \ \ x \notin (d_1, d^1). \tag{5.1a}$$

where $\delta > 0$ is an arbitrary small number, the constant M does not depend on δ; d_1, $d^1 \in D$. By $u_{(2)}(x,t)$, $(x,t) \in \overline{G}_{(2)}$, where $G_{(2)} = D_{(2)} \times (0,T]$, $D_{(2)} = (d_1, d^1)$, we denote the solution of the problem

$$\begin{aligned} L_{(2.2)}\, u_{(2)}(x,t) &= f(x,t), \qquad (x,t) \in G_{(2)}, \\ u_{(2)}(x,t) &= \begin{cases} u_1(x,t), & (x,t) \in S_{(2)} \setminus S, \\ \varphi(x,t), & (x,t) \in S_{(2)} \bigcap S. \end{cases} \end{aligned} \tag{5.1b}$$

Here $S_{(2)} = \overline{G}_{(2)} \setminus G_{(2)}$. Let $\widetilde{u}_2^i(x,t)$, $(x,t) \in \overline{G}_{(2)}$ be an approximation of the solution $u_{(2)}(x,t)$, moreover,

$$\big|\, u_{(2)}(x,t) - \widetilde{u}_2^i(x,t) \,\big| \le M\,\delta, \quad (x,t) \in \overline{G}_{(2)}, \quad x \notin (d_2, d^2).$$

Assume

$$u_2(x,t) = \begin{cases} \widetilde{u}_2^i(x,t), & (x,t) \in \overline{G}_{(2)}, \\ u_1(x,t), & (x,t) \in \overline{G} \setminus \overline{G}_{(2)}. \end{cases}$$

Then we have: $\quad |u(x,t) - u_2(x,t)| \le M\delta, \quad (x,t) \in \overline{G}, \quad x \notin (d_2, d^2)$.

Let the function $u_{k-1}(x,t)$, $(x,t) \in \overline{G}$ have been constructed for $k \ge 3$, and this function has a convenient representation in order to compute it for $x \notin (d_{k-1}, d^{k-1})$, d_{k-1}, $d^{k-1} \in D$, and also

$$|u(x,t) - u_{k-1}(x,t)| \le M\delta, \quad (x,t) \in \overline{G}, \ \ x \notin (d_{k-1}, d^{k-1}). \tag{5.1c}$$

Here $M = M(k)$. The function $u_{(k)}(x,t)$, $(x,t) \in \overline{G}_{(k)}$, where $G_{(k)} = D_{(k)} \times (0,T]$, $D_{(k)} = (d_{k-1}, d^{k-1})$, denotes the solution of the problem

$$\begin{aligned} L_{(2.2)}\, u_{(k)}(x,t) &= f(x,t), \qquad (x,t) \in G_{(k)}, \\ u_{(k)}(x,t) &= \begin{cases} u_{k-1}(x,t), & (x,t) \in S_{(k)} \setminus S, \\ \varphi(x,t), & (x,t) \in S_{(k)} \bigcap S, \end{cases} \end{aligned} \tag{5.1d}$$

where $S_{(k)} = \overline{G}_{(k)} \setminus G_{(k)}$. Let $\widetilde{u}^i_k(x,t)$, $(x,t) \in \overline{G}_{(k)}$ be the approximation of the function $u_{(k)}(x,t)$, and

$$\big|\, u_{(k)}(x,t) - \widetilde{u}^i_k(x,t) \,\big| \le M\delta, \quad (x,t) \in \overline{G}_{(k)}, \quad x \notin (d_k, d^k).$$

Assume

$$u_k(x,t) = \begin{cases} \widetilde{u}^i_k(x,t), & (x,t) \in \overline{G}_{(k)}, \\ u_{k-1}(x,t), & (x,t) \in \overline{G} \setminus \overline{G}_{(k)}. \end{cases}$$

The function $u_k(x,t)$, $(x,t) \in \overline{G}$ satisfies the estimate:

$$|u(x,t) - u_k(x,t)| \le M\delta, \quad (x,t) \in \overline{G}, \quad x \notin (d_k, d^k).$$

If for some value of $k = K_0$ it occurs that $|u_{(k)}(x,t) - \widetilde{u}^i_k(x,t)| \le M\delta$, $(x,t) \in \overline{G}_{(k)}$ for all $x \in \overline{D}_{(k)}$, then for $k \ge K_0 + 1$ we consider that the sets $\overline{G}_{(k)}$ are empty, and further the functions $u_{(k)}(x,t)$ are not computed. For example, for $k \ge K_0$ we have $u_k(x,t) = u_{K_0}(x,t)$, $(x,t) \in \overline{G}$.

For $k = K$, where K is a given fixed number, $K \ge 1$, we assume

$$u^K(x,t) = u_K(x,t), \quad (x,t) \in \overline{G}. \tag{5.1e}$$

The functions $u^K(x,t)$ and $u_k(x,t)$, $k=1,...,K$, $(x,t) \in \overline{G}$ denote the solution and the components of the solution to iterative process (5.1).

The functions $u^K(x,t)$ have suitable representations for computing them on the subdomains, which are extending as K grows. The functions $u_k(x,t)$ and $u^K(x,t)$ satisfy the estimates

$$\begin{aligned} &|u(x,t) - u^K(x,t)| \le M\,\delta, \quad (x,t) \in \overline{G}, \quad x \notin (d_K, d^K), \\ &|u(x,t) - u_k(x,t)| \le M\,\delta, \quad (x,t) \in \overline{G}, \quad x \notin (d_k, d^k), \quad k = 1, ..., K. \end{aligned} \tag{5.2}$$

Lemma 5.1 *The functions $u^K(x,t)$ and $u_k(x,t)$, $(x,t) \in \overline{G}$, $k = 1,...,K$, i.e. the solution of the iterative process* (5.1) *and its components, satisfy the estimate* (5.2).

2. We now give a grid construction which approximate the iterative process (5.1). On the set $\overline{G}$ we introduce the coarse (primary) mesh

$$\overline{G}_{1h} = \overline{\omega}_1 \times \overline{\omega}_0, \tag{5.3a}$$

where $\overline{\omega}_1$ and $\overline{\omega}_0$ are uniform meshes, $\overline{\omega}_0 = \overline{\omega}_{0(3.3)}$; the step-size in the mesh $\overline{\omega}_1$ is equal to $h_1 = 2dN^{-1}$. We denote by $z_1(x,t)$, $(x,t) \in \overline{G}_{1h}$, where $\overline{G}_{1h} = \overline{G}_{1h(5.3)} = \overline{G}_{h(3.3)}$, the solution of problem (3.4), (5.3a).

Let the values $d_1, d^1 \in \overline{\omega}_1$ be founded in some a way so that for $x \notin (d_1, d^1)$ the discrete solution $z_1(x,t)$, $(x,t) \in \overline{G}_{1h}$ well approximates the solution of problem (2.2), (2.1), (2.5). If it occurs that $d^1 - d_1 > 0$, then we define the subdomain on which the mesh will be refined:

$$G_{(2)} = G_{(2)}(d_1, d^1), \quad G_{(2)} = D_{(2)} \times (0,T], \quad D_{(2)} = (d_1, d^1). \tag{5.3b}$$

On the subdomain $\overline{G}_{(2)}$ we introduce the mesh $\overline{G}_{(2)h} = \overline{\omega}_{(2)} \times \overline{\omega}_0$, where $\overline{\omega}_{(2)}$ is a uniform mesh with the number of nodes $N+1$. On the set $\overline{G}_{(2)h}$ we find the solution $z_{(2)}(x,t)$ of the discrete problem

$$\begin{aligned} \Lambda_{(3.4)} z_{(2)}(x,t) &= f(x,t), \qquad (x,t) \in G_{(2)h}, \\ z_{(2)}(x,t) &= \begin{cases} z_1(x,t), & (x,t) \in S_{(2)h} \setminus S, \\ \varphi(x,t), & (x,t) \in S_{(2)h} \bigcap S, \end{cases} \end{aligned}$$

where $G_{(2)h} = G_{(2)} \bigcap \overline{G}_{(2)h}$, $S_{(2)h} = S_{(2)} \bigcap \overline{G}_{(2)h}$, $S_{(2)} = \overline{G}_{(2)} \setminus G_{(2)}$. The mesh set $\overline{G}_{2h}$ and the function $z_2(x,t)$, $(x,t) \in \overline{G}_{2h}$ are defined by the relations:

$$\overline{G}_{2h} = \overline{G}_{(2)h} \bigcup \left\{ \overline{G}_{1h} \setminus \overline{G}_{(2)} \right\}, \quad z_2(x,t) = \begin{cases} z_{(2)}(x,t), & (x,t) \in \overline{G}_{(2)h}, \\ z_1(x,t), & (x,t) \in \overline{G}_{1h} \setminus \overline{G}_{(2)}. \end{cases}$$

Let the mesh set $\overline{G}_{k-1,h}$ and the mesh function $z_{k-1}(x,t)$ on this set have been constructed for $k \ge 3$. Let also d_{k-1}, $d^{k-1} \in \overline{\omega}_{k-1}$ be founded so that for $x \notin (d_{k-1}, d^{k-1})$ the discrete solution $z_{k-1}(x,t)$, $(x,t) \in \overline{G}_{k-1,h}$ well approximates the solution of problem (2.2), (2.1), (2.5). Here $\overline{\omega}_{k-1}$ is a mesh which generates the mesh $\overline{G}_{k-1,h}$: $\overline{G}_{k-1,h} = \overline{\omega}_{k-1} \times \overline{\omega}_0$; $N_k + 1$ is the number of nodes in the mesh $\overline{\omega}_k$, $k \ge 1$; $N_1 = N$. If it occurs that $d^{k-1} - d_{k-1} > 0$, then we define the domain

$$G_{(k)} = G_{(k)}(d_{k-1}, d^{k-1}), \quad G_{(k)} = D_{(k)} \times (0,T], \quad D_{(k)} = (d_{k-1}, d^{k-1}). \tag{5.3c}$$

On the set $\overline{G}_{(k)}$ we introduce the mesh

$$\overline{G}_{(k)h} = \overline{\omega}_{(k)} \times \overline{\omega}_0, \tag{5.3d}$$

where $\overline{\omega}_{(k)}$ is a uniform mesh with the number of nodes $N+1$; $h_{(k)}$ is the step-size of the mesh $\overline{\omega}_{(k)}$. Let $z_{(k)}(x,t)$, $(x,t) \in \overline{G}_{(k)h}$ be the solution of the discrete problem

$$\begin{aligned} \Lambda_{(3.4)} z_{(k)}(x,t) &= f(x,t), \qquad (x,t) \in G_{(k)h}, \\ z_{(k)}(x,t) &= \begin{cases} z_{k-1}(x,t), & (x,t) \in S_{(k)h} \setminus S, \\ \varphi(x,t), & (x,t) \in S_{(k)h} \bigcap S. \end{cases} \end{aligned} \tag{5.3e}$$

Assume $\overline{G}_{kh} = \overline{G}_{(k)h} \bigcup \{ \overline{G}_{k-1,h} \setminus \overline{G}_{(k)} \}$,

$$z_k(x,t) = \begin{cases} z_{(k)}(x,t), & (x,t) \in \overline{G}_{(k)h}, \\ z_{k-1}(x,t), & (x,t) \in \overline{G}_{k-1,h} \setminus \overline{G}_{(k)}. \end{cases}$$

If for some value $k = K_0$ it occurs that the discrete solution $z_k(x,t)$, $(x,t) \in \overline{G}_{kh}$ well approximates on $\overline{G}_{kh}$ the solution of the differential problem, then for $k \ge K_0 + 1$ we consider that the sets $\overline{G}_{(k)}$ are empty and further the functions $z_{(k)}(x,t)$ are not computed.

For example, for $k \ge K_0$ we have $z_k(x,t) = z_{K_0}(x,t)$, $\overline{G}_{kh} = \overline{G}_{K_0 h}$.

The computations are stopped also in the case when for some value $k = K_0$ the condition $d^k - d_k \ge d^{k-1} - d_{k-1}$ holds, which means that the solution cannot be further improved. For $k \ge K_0 + 1$ the function $z_{(k)}(x,t)$ is not computed; for $k \ge K_0$ we assume $z_k(x,t) = z_{K_0}(x,t)$, $\overline{G}_{kh} = \overline{G}_{K_0 h}$.

For $k = K$, where K is a given fixed number, $K \ge 1$, we suppose that

$$\overline{G}_h^K = \overline{G}_{Kh} \equiv \overline{G}_h, \quad z^K(x,t) = z_K(x,t) \equiv z(x,t). \tag{5.3f}$$

We call the function $z_{(5.3f)}(x,t)$, $(x,t) \in \overline{G}_{h(5.3f)}$, the solution of scheme (3.4), (5.3), and the functions $z_k(x,t)$, $(x,t) \in \overline{G}_{kh}$, $k = 1, ..., K$, the components of the solution to the difference scheme.

The above algorithm (we call it $A_{(5.3)}$) allows us to construct meshes condensing in transition layers. The value $N_K + 1$, i.e. the number of nodes in the mesh $\overline{\omega}^K = \overline{\omega}_K$ used for the construction of the function $z^K(x,t)$, does not exceed the value $N(K) = K(N+1)$.

In schemes (3.4), (5.3), when solving the intermediate problems (5.3e), it does not require the interpolation in order to define values of the functions $z_{(k)}(x,t)$ on the boundary $S_{(k)h}$.

3. The meshes $\overline{G}_{kh}$, $k = 1, ..., K$, generated by the algorithm $A_{(5.3)}$, are defined by the rule to choice the values d_k, d^k, $k = 1, 2, ..., K$, and also by the values of K and N, N_0.

Thus, the algorithm $A_{(5.3)}$ defines a class of difference schemes, i.e. the class of schemes (3.4), (5.3). In this class of schemes, the boundary of the subdomain, in which the refinement of the mesh is derived, pass through nodes of more rough refined mesh. Note that the smallest stepsize in the mesh $\overline{\omega}^K = \overline{\omega}_K$ is not less than the value dN^{-K}. In the meshes generated by the algorithm $A_{(5.3)}$, the values d_k are defined on the base of intermediate results obtained in the computing process; such meshes $\overline{G}_{kh}$ are *a posteriori* condensing meshes.

For schemes from the class (3.4), (5.3) the maximum principle is valid. Note that in this class there are no schemes whose solutions converge ε-uniformly to the solution of (2.2), (2.1), (2.5).

6. Adaptive scheme based on the estimate of the solution gradient

To construct *a posteriori* condensing meshes we use indicators (auxiliary functionals from solutions of intermediate problems), which help us to define boundaries of the mesh domain which is subjected to a refinement. We show a construction of the indicator based on an estimate for the gradient of the solution.

1. We define a width of the boundary layer for problem (2.2), (2.1). Let the following estimate holds for the component $U(x,t)$ from the representation (9.1):

$$\left|\frac{\partial}{\partial x} U(x,t)\right| \le M_1, \qquad (x,t) \in \overline{G}. \tag{6.1a}$$

Suppose that the values of the parameter ε are sufficiently small, $\varepsilon \le \varepsilon_0$. We say that $\sigma_0^L = \sigma_0^L(\varepsilon, M_0)$ and $\sigma_0^R = \sigma_0^R(\varepsilon, M_0)$, where M_0 is an arbitrary large number, $M_0 > M_1$, are the left and right boundaries of the transition layer in a neighbourhood of the interface boundary γ, if σ_0^L and σ_0^R are respectively the maximum and minimum values of σ^L and σ^R for which we have the estimate

$$\left|\frac{\partial}{\partial x} u(x,t)\right| \le M_0, \quad (x,t) \in \overline{G}, \quad x \notin (\sigma^L, \sigma^R); \tag{6.1b}$$

we call the value $\sigma_0 = \sigma_0^R - \sigma_0^L$ the width of the layer.

We consider such a boundary value problem as a model example:

$$\varepsilon\, u''(x) + b(x) u'(x) = 1, \quad x \in \Omega = (-1,1), \;\; u(-1) = 1, \;\; u(1) = 0, \tag{6.2}$$

where $b(x) = 1,\ x \in \overline{\Omega}^2,\ b(x) = -1,\ x \in \overline{\Omega}^1$. The solution of problem (6.2) can be decomposed into its regular $U(x)$ and singular $V(x)$ components: $u(x) = U(x) + V(x),\ x \in \overline{\Omega}^r$, $r = 1, 2$; the transition layer appears in a neighbourhood of the point $x = 0$.

In the case of problem (6.2) the width of the layer $\sigma_{0(6.1)}$ has an asymptotic behavior

$$\sigma_0 \approx \varepsilon \ln \varepsilon^{-1} \quad \text{for} \quad \varepsilon = o(1).$$

The following estimate also holds:

$$\sigma_0 \le M\varepsilon \ln(\varepsilon^{-1} M_0^{-1}), \quad \varepsilon \in (0, \varepsilon_0], \quad \varepsilon_0 = \varepsilon_0(M_0), \quad \varepsilon_0 \le m M_0^{-1},$$

where $M_0,\ m$ are any constants satisfying the conditions $M_0 > 1,\ m < 1,\ M > 1$.

2. We define the width of the transition layer for difference scheme (3.4), (3.1). We denote by $z_v(x,t)$, $(x,t) \in \overline{G}_h^r$ the solution of the difference problem

$$\begin{aligned} \Lambda_{(3.4)} z(x,t) &= L_{(2.2)}\, v(x,t), \quad (x,t) \in G_h^r, \\ z(x,t) &= \phantom{L_{(2.2)}}\, v(x,t), \quad (x,t) \in S_h^r, \quad r = 1, 2, \end{aligned}$$

where $v(x,t)$ is any sufficiently smooth function, $v \in C^{2,1}(G^r) \cap C(\overline{G})$. The solution of problem (3.4), (3.1) can be represented as a sum of functions

$$z(x,t) = z_U(x,t) + V^h(x,t), \quad (x,t) \in \overline{G}_h^r, \quad r = 1, 2, \tag{6.3}$$

where $z_U(x,t)$, $(x,t) \in \overline{G}_h^{\,r}$ is a discrete function which approximates the component $U(x,t)$ from the representation (9.1). Let the component $z_U(x,t)$ satisfies the following estimate

$$|\, \delta_x\, z_U(x,t)\,| \le M_1, \qquad (x,t) \in \overline{G}_h^{\,r}, \;\; r = 1,2; \;\; x \ne 0, d. \tag{6.4a}$$

We say that $\sigma_0^L = \sigma_0^L(M_0) = \sigma_0^L(M_0; D, \varepsilon, \overline{G}_h)$, $\sigma_0^R = \sigma_0^R(M_0) = \sigma_0^R(M_0; D, \varepsilon, \overline{G}_h)$ are the left and right boundaries of the discrete transition layer in a neighbourhood of the interface boundary γ, if σ_0^L and σ_0^R are respectively the maximum and minimum values of σ^L and σ^R, for which we have the estimate

$$|\delta_x z(x,t)|, \; |\delta_{\overline{x}} z(x,t)| \le M_0, \quad (x,t) \in \overline{G}_h, \quad x \ne (\sigma^L, \sigma^R); \tag{6.4b}$$

here $\varepsilon \in (0, \varepsilon_0]$, M_0 and ε_0 are sufficiently large and small constants, $M_0 > M_1$, $\varepsilon_0 = \varepsilon_0(M_0)$, we call the value $\sigma_0 = \sigma_0^R - \sigma_0^L$ the width of the layer. Thus, the functions $\sigma_0(M_0)$, $\sigma_0^L(M_0)$, $\sigma_0^R(M_0)$ are constructed.

In the case of the difference scheme

$$\varepsilon\, \delta_{\overline{x}\widehat{x}}\, z(x) + b^+(x)\delta_x z(x) + b^-(x)\delta_{\overline{x}} z(x) = 1, \quad x \in \Omega_h, \quad z(-1) = 1, \quad z(1) = 0,$$

which approximates problem (6.2), for the width of the boundary layer on uniform (with the step-size h) meshes we have the asymptotic

$$\sigma_0 \approx \begin{cases} \varepsilon \ln \varepsilon^{-1}, & h \le M\varepsilon, \\ h \ln^{-1}(1 + \varepsilon^{-1} h) \ln h^{-1}, & h \ge m\varepsilon; \quad \varepsilon,\, h = o(1). \end{cases}$$

The following estimate is valid:

$$\sigma_0 \;\le\; M\, \left[\, \varepsilon \ln \varepsilon^{-1} + h \ln h^{-1} \,\right], \quad \varepsilon \in (0, \varepsilon_0], \quad h \le h_0,$$

where ε_0, h_0 are sufficiently small values, $\varepsilon_0 = \varepsilon_0(M_0)$, $h_0 = h_0(M_0)$.

3. In order that the formal mesh construction (3.4), (5.3) is constructive, it is required to give values K and d_k, d^k, $k = 1, 2, \ldots, K$.

Let $K \ge 1$. We define the values $d_{k(5.3)}$, $d^k_{(5.3)}$. Assume

$$d_1 = \sigma_1^L, \;\; d^1 = \sigma_1^R, \tag{6.5a}$$

where $\sigma_1^{(L,R)} = \sigma_{0(6.4)}^{(L,R)}(M_0; D, \varepsilon, \overline{G}_h)$, $D = D_{(2.1)}$, $\overline{G}_h = \overline{G}_{h(3.3)}$, M_0 is a sufficiently large number. Let the values of d_{k-1}, d^{k-1} have been founded. Further we find the values of $\sigma_k^{(L,R)}$:

$$\sigma_k^{(L,R)} = \sigma_0^{(L,R)}\Big(kM_0;\; D_{(k)}, \varepsilon, \overline{G}_{(k)h}\Big), \quad k \ge 2, \tag{6.5b}$$

where $\sigma_{0(6.5)}^{(L,R)}(M; D, \varepsilon, \overline{G}_h) = \sigma_{0(6.4)}^{(L,R)}(M; D, \varepsilon, \overline{G}_h)$, $M = k\, M_0$, $D_{(k)} = D_{(k)(5.3)}$, $\overline{G}_{(k)h} = \overline{G}_{(k)h(5.3)}$. If the relation $\sigma_k \le m_0\, \sigma_{k-1}$ is valid, where $\sigma_k = \sigma_k^R - \sigma_k^L$, then we suppose

$$d_k = \sigma_k^L, \;\; d^k = \sigma_k^R; \tag{6.5c}$$

here m_0 is a sufficiently small number. If for some value of $k = k_0$ it occurs that $\sigma_{k_0} > m_0\, \sigma_{k_0 - 1}$, then we assume $d_k = d_{k_0}$ $d^k = d^{k_0}$ for $k \ge k_0$.

The difference scheme (3.4), (5.3), (6.5) is the scheme on the adaptive meshes which are constructed on the basis of the estimate for the gradient of the discrete solutions obtained in the process of intermediate computations. The mesh refinement is realized only in a neighbourhood of the transition layer; the diameter of such a neighbourhood (the width of the transition layer) becomes narrow when the value of k grows.

7. Analysis of scheme (3.4), (5.3), (6.5)

1. We now give some estimates for the solution of difference scheme (3.4), (3.3). We denote by $W(x)$ and $z_W(x)$ the solutions of the problems

$$\Lambda W(x) \equiv \left\{\varepsilon\, a_{(1)}\frac{d^2}{dx^2} + b_{(1)}\mathrm{sign}\, x\frac{d}{dx}\right\} u(x) = 0, \quad x \in D, \quad x \neq 0,$$
$$W(0) = 1, \quad W(x) = 0, \quad x \in \Gamma;$$
$$\Lambda\; z(x) \equiv \left\{\varepsilon a_{(1)}\delta_{\overline{x}\widehat{x}} + (b_{(1)}\mathrm{sign}\, x)^{+}\delta_x + (b_{(1)}\mathrm{sign}\, x)^{-}\delta_{\overline{x}}\right\} z(x) = 0, \quad x \in D_h, \quad x \neq 0,$$
$$z(0) = 1, \quad z(x) = 0, \quad x \in \Gamma_h,$$

where $\overline{D}_h = \overline{\omega}_{1(3.3)}$, $a_{(1)} = \max_{\overline{G}}\, a(x,t)$, $b_{(1)} = \min_{\overline{G}}\, |b(x,t)|$.

1.1. The function $z_W(x)$ satisfies the estimate

$$z_W(x) \;\leq\; q^{-r_1 h^{-1}}, \quad x \in \overline{D}_h, \tag{7.1a}$$

where $q = 1 + a_{(1)}^{-1}\, b_{(1)}\, \varepsilon^{-1}\, h$, $r_1 = r(x, \beta(t=0)) = |x|$, $r(x, x^*)$ is the distance between the points x and x^*. Thus, we have the estimate

$$z_W(x) \leq \left\{\begin{array}{ll} M\, \exp(-m\,\varepsilon^{-1} r_1), & h \leq M\,\varepsilon \\ M\,(\varepsilon h^{-1})^{m h^{-1} r_1}, & h > M\,\varepsilon \end{array}\right\}, \quad x \in \overline{D}_h. \tag{7.1b}$$

The function $z_W(x)$ is the majorant for the component $z_V(x,t)$, which corresponds to $V(x,t)$ from the representation (9.1)

$$|z_V(x,t)| \;\leq\; M\, z_W(x), \quad (x,t) \in \overline{G}{}^{r}_{h}, \quad r = 1, 2. \tag{7.2}$$

1.2. The solution of difference scheme (3.4), (3.3) satisfies the estimate

$$|u(x,t) - z(x,t)| \;\leq\; M\left[\,(\varepsilon + N^{-1})^{-1}\, N^{-1} + N_0^{-1}\,\right], \tag{7.3a}$$
$$|u(x,t) - z(x,t)| \;\leq\; M\left[\, z_W(x) + N^{-1} + N_0^{-1}\,\right], \quad (x,t) \in \overline{G}_h. \tag{7.3b}$$

It follows from estimates (7.2), (7.3) that, under condition (3.8), the scheme converges on $\overline{G}_h$, and also the scheme converges ε-uniformly outside the σ_0-neighbourhood of the set γ:

$$|u(x,t) - z(x,t)| \;\leq\; M\left[\, N^{-1/2} + N_0^{-1}\,\right], \quad (x,t) \in \overline{G}_h, \tag{7.4a}$$
$$\text{for} \quad r(x,\gamma) \geq \sigma_0, \quad \text{where} \quad \sigma_0 = \sigma_{0(6.4)}(M_0; D, \varepsilon, \overline{G}_h)$$
$$\sigma_0 \;\leq\; M\left[\,\varepsilon \ln \varepsilon^{-1} + N^{-1}\ln N\,\right]. \tag{7.4b}$$

The neighbourhood, out of which estimate (7.4a) holds, becomes narrow for $\varepsilon \to 0$, $N \to \infty$.

Theorem 7.1 *Let the solution of the boundary value problem* (2.2), (2.1) *satisfies the condition* (2.5) *and the estimates of theorem* 9.1. *Then the solution of the difference scheme* (3.4), (3.3) *converges on* $\overline{G}$ *to the solution of the boundary value problem under the condition* (3.8)*, and also ε-uniformly (with the rate* $\mathcal{O}(N^{-1/2} + N_0^{-1})$ *) outside the σ_0-neighbourhood of the set γ. The discrete solution satisfies the estimates* (7.2)–(7.4).

2. Let us consider the difference scheme (3.4), (5.3), (6.5).

For the component $z_1(x,t) = z(x,t)$ of the solution to this scheme estimate (7.3) is valid. Taking into account estimate (7.1), for the function $z_W(x)$ we find the following estimate for the value of σ_1, i.e. the width of the transition layer:

$$\sigma_1 \;\leq\; M\left[\,\varepsilon \ln \varepsilon^{-1} + N^{-1}\ln N\,\right], \quad \varepsilon \in (0, \varepsilon_0], \quad h \leq h_0.$$

The value h_2, i.e. the step-size of the mesh $\overline{\omega}_{(2)}$, satisfies the estimate

$$h_2 \le M\,N^{-1}\left[\,\varepsilon\,\ln\,\varepsilon^{-1} + N^{-1}\,\ln N\,\right].$$

Taking into consideration estimate (7.3b), we estimate $u(x,t) - z_1(x,t)$ on the boundary on the set $\overline{G}_{(2)h}$, and also $u(x,t) - z_{(2)}(x,t)$ on the whole set $\overline{G}_{(2)h}$. For $u(x,t) - z_2(x,t)$ we have the estimate

$$|u(x,t) - z_2(x,t)| \le M\left[N^{-1/2} + \varepsilon^{-1}N^{-2}\ln N + N_0^{-1}\right], \quad (x,t) \in \overline{G}_{2h},$$

and outside the σ_2-neighbourhood of the set γ we have

$$|u(x,t) - z_2(x,t)| \le M\left[N^{-1/2} + N_0^{-1}\right], \quad (x,t) \in \overline{G}_{2h}, \quad r(x,\gamma) \ge \sigma_2.$$

The value σ_2 satisfies the estimate

$$\sigma_2 \le M\,N^{-1}\,\ln N.$$

In a similar way we find the estimates

$$|u(x,t) - z_k(x,t)| \le M\left[N^{-1/2} + \varepsilon^{-1}N^{-k}\ln^{k-1}N + N_0^{-1}\right], \quad (x,t) \in \overline{G}_{kh};$$

$$|u(x,t) - z_k(x,t)| \le M\left[N^{-1/2} + N_0^{-1}\right], \quad (x,t) \in \overline{G}_{kh}, \quad r(x,\gamma) \ge \sigma_k;$$

$$\sigma_k \le \left\{\begin{array}{ll} M[\varepsilon\ln\varepsilon^{-1} + N^{-1}\ln N], & k = 1 \\ MN^{-k+1}\ln^{k-1}N, & k \ge 2 \end{array}\right\}, \quad k = 1, 2, \ldots, K; \tag{7.5}$$

$$|u(x,t) - z(x,t)| \le M\left[N^{-1/2} + \varepsilon^{-1}N^{-K}\ln^{K-1}N + N_0^{-1}\right], \quad (x,t) \in \overline{G}_h;$$

$$|u(x,t) - z(x,t)| \le M\left[N^{-1/2} + N_0^{-1}\right], \quad (x,t) \in \overline{G}_h, \quad r(x,\gamma) \ge \sigma_K;$$

$$\sigma_K \le \left\{\begin{array}{ll} M[\varepsilon\ln\varepsilon^{-1} + N^{-1}\ln N], & K = 1 \\ MN^{-K+1}\ln^{K-1}N, & K \ge 2 \end{array}\right\}; \tag{7.6}$$

where $z(x,t) = z_{(5.3e)}(x,t)$, $\overline{G}_h = \overline{G}_{h(5.3e)}$.

The functions $z(x,t)$ and $z_k(x,t)$ for N, $N_0 \to \infty$ converge (to the solution of boundary value problem (2.2), (2.1)) ε-uniformly outside the σ_K- and σ_k-neighbourhoods of the set γ, and also on the sets $\overline{G}_{kh}$ and $\overline{G}_h$ for sufficiently small (but not too small) values of the parameter ε, namely, under the condition

$$\begin{aligned} &\varepsilon \ge \varepsilon_0(N), \quad \varepsilon_0^{-1}(N) = o(N^K\ln^{-K+1}N); \\ &\varepsilon \ge \varepsilon_k(N), \quad \varepsilon_k^{-1}(N) = o(N^k\ln^{-k+1}N), \quad k = 1, 2, \ldots, K. \end{aligned} \tag{7.7}$$

Thus, difference scheme (3.4), (5.3), (6.5), i.e. the scheme on the adaptive meshes, converges almost ε-uniformly. In order to ensure the convergence defect for the function $z(x,t)$ not exceeding the values of $\nu_{(2.7)}$, it is required to choose the value K satisfying the condition

$$K > K(\nu), \quad K(\nu) = \nu^{-1}. \tag{7.8}$$

Theorem 7.2 *Let the hypothesis of theorem* 7.1 *be fulfilled. Then the functions* $z(x,t)$, $(x,t) \in \overline{G}_h$ *and* $z_k(x,t)$, $(x,t) \in \overline{G}_{kh}$, $k = 1, \ldots, K$, *i.e. the solution of the difference scheme* (3.4), (5.3), (6.5) *and its components, converge on* $\overline{G}$ *to the solution of the boundary value problem* (2.2), (2.1) *under the condition* (7.7), *and also* ε*-uniformly (with the rate* $\mathcal{O}(N^{-1/2} + N_0^{-1})$*) outside the* σ_K*- and* σ_k*-neighbourhoods of the set* γ*; the solution of scheme* (3.4), (5.3), (6.5), (7.8) *converges to the solution of the boundary value problem almost* ε*-uniformly with the defect* ν*. For the discrete solutions the estimates* (7.5), (7.6) *are valid.*

Remark 7.1 For the interpolant $\overline{z}(x,t)$, $(x,t)\in\overline{G}$ (linear on triangular elements) constructed from the function $z(x,t)$, $(x,t)\in\overline{G}_h$, we have the estimate similar to estimate (7.6):

$$|u(x,t)-\overline{z}(x,t)|\le M\left[N^{-1/2}+\varepsilon^{-1}N^{-K}\ln^{K-1}N+N_0^{-1}\right],\quad (x,t)\in\overline{G};$$

$$|u(x,t)-\overline{z}(x,t)|\le M\left[N^{-1/2}+N_0^{-1}\right],\quad (x,t)\in\overline{G},\quad r(x,\gamma)\ge\sigma_K;$$

$$\sigma_K\le\left\{\begin{array}{ll} M[\varepsilon\ln\varepsilon^{-1}+N^{-1}\ln N], & K=1\\ MN^{-K+1}\ln^{K-1}N, & K\ge 2\end{array}\right\}. \tag{7.9}$$

8. Special scheme for problem (2.2), (2.1)

In the case of problem (2.2), (2.1) with the moving interface boundary γ we pass to the system of new coordinates ξ,t. Further, for problem (9.2), (9.3) we construct a classical scheme, not focusing the conjugation condition in the approximation.

1. On the set $\overline{\widetilde{G}}$ we construct meshes. At first, we introduce the basic mesh

$$\overline{\widetilde{G}}{}_h^B=\widetilde{\omega}_1\times\overline{\omega}_0, \tag{8.1}$$

where $\widetilde{\omega}_1$ is a mesh on the axis ξ, $\overline{\omega}_0=\overline{\omega}_{0(3.1)}$; the mesh $\widetilde{\omega}_1$ is a mesh with any distribution of the nodes satisfying only the condition

$$h_\xi\le MN^{-1},$$

where $h_\xi=\max_i h_\xi^i$, $h_\xi^i=\xi^{i+1}-\xi^i$, ξ^i, $\xi^{i+1}\in\widetilde{\omega}_1$, $N+1$ is the maximal number of nodes on an interval of unit length on the axis ξ. As a basic grid, we use the following mesh which is uniform with respect to ξ, t:

$$\overline{\widetilde{G}}{}_h^B,\quad\text{where }\ \widetilde{\omega}_1,\overline{\omega}_0\ \text{ are uniform meshes.} \tag{8.2}$$

The interior nodes are defined by the relation $\widetilde{G}_h=\widetilde{G}\cap\overline{\widetilde{G}}{}_h^B$; the boundary nodes are generated by an intersection of the lines $t=t^j\in\overline{\omega}_0$ with the lateral boundary $\widetilde{S}^L$ and the lines $\xi=\xi^i\in\widetilde{\omega}$ with the lower part of the boundary $\widetilde{S}_0$; $\widetilde{S}_h=\widetilde{S}_{0h}\cup\widetilde{S}_h^L$. On the set $\overline{\widetilde{G}}$ we introduce the mesh

$$\overline{\widetilde{G}}_h=\widetilde{G}_h\cup\widetilde{S}_h;\ \ \overline{\widetilde{G}}_h=\overline{\widetilde{G}}_h(\overline{\widetilde{G}}{}_h^B). \tag{8.3}$$

Problem (9.2), (9.3) is approximated by the implicit difference scheme

$$\widetilde{\Lambda}Z(\xi,t)\equiv\left\{\varepsilon\widetilde{a}(\xi,t)\delta_{\overline{\xi}\widehat{\xi}}+B^+(\xi,t)\delta_\xi+B^-(\xi,t)\delta_{\overline{\xi}}-\widetilde{c}(\xi,t)-\widetilde{p}(\xi,t)\delta_{\overline{t}}\right\}\widetilde{z}(\xi,t)=\widetilde{f}(\xi,t),$$
$$(\xi,t)\in\widetilde{G}_h, \tag{8.4}$$
$$Z(\xi,t)=\widetilde{\varphi}(\xi,t),\quad(\xi,t)\in\widetilde{S}_h.$$

The difference scheme (8.4), (8.3), (8.1) is monotone.

Using the algorithm $A_{(5.3)}$ for scheme (8.4), we construct the meshes

$$\overline{\widetilde{G}}_{kh},\ \ k=1,2,\ldots,K,\ \ \overline{\widetilde{G}}_h, \tag{8.5a}$$

where $\overline{\widetilde{G}}_{1h}=\overline{\widetilde{G}}_{h(8.3)}(\overline{\widetilde{G}}{}^B_{h(8.2)})$, $\overline{\widetilde{G}}_h=\overline{\widetilde{G}}_{Kh}$, and then we find the functions

$$Z_k(\xi,t),\ \ (\xi,t)\in\overline{\widetilde{G}}_{kh},\ \ Z(x,t),\ \ (\xi,t)\in\overline{\widetilde{G}}_h, \tag{8.5b}$$

where $Z(\xi,t)=Z_K(\xi,t)$. The meshes are defined by the law of choice of the values

$$d_k,d^k,\ \ k=1,2,\ldots,K, \tag{8.5c}$$

and also by the values of K and N, N_0.

In the class of difference schemes (8.4), (8.5) the more precise discrete solution is produced on simple domains, i.e. domains with stationary boundaries; the boundary of the domain, in which the mesh refinement is realized, pass through nodes of the refine mesh.

For the schemes from class (8.4), (8.5) the maximum principle is valid.

The values

$$\sigma_k^L,\ \sigma_k^R,\ \sigma_k,\ \ k=1,2,\ldots,K, \tag{8.6a}$$

which define the left and right boundaries of the transition layer and its width, are constructed similarly to the values $\sigma^L_{k(6.5)}, \sigma^R_{k(6.5)}, \sigma_{k(6.5)}$. The parameters d_k, d^k are defined similarly to the parameters $d_{k(6.5)}$, $d^k_{(6.5)}$ as

$$d_k=\sigma_k^L,\ \ d^k=\sigma_k^R,\ \ k=1,2,\ldots,K, \tag{8.6b}$$

The difference scheme (8.4), (8.5), (8.6) is the scheme on the *a posteriori* adaptive meshes, which are constructed on the basis of the gradient of the intermediate discrete solutions.

2. For the function $Z_1(\xi,t)=Z_{(8.4;8.3;8.2)}$, i.e. the solution of difference scheme (8.4), (8.3), (8.2), we have the estimates

$$|\widetilde{u}(\xi,t)-Z_1(\xi,t)|\le M\left[(\varepsilon+N^{-1})^{-1}N^{-1}+N_0^{-1}\right],$$

$$|\widetilde{u}(\xi,t)-Z_1(\xi,t)|\le M\left[(Z_W(\xi)+N^{-1}+N_0^{-1}\right],\ \ (\xi,t)\in\overline{\widetilde{G}}_h.$$

Here $Z_W(\xi)$ is the solution of the problem

$$\Lambda Z(\xi)\equiv\left\{\varepsilon a_{(1)}\,\delta_{\overline{\xi}\widehat{\xi}}+(B_{(1)}\mathrm{sign}\,\xi)^+\delta_\xi+(\mathrm{B}_{(1)}\mathrm{sign}\,\xi)^-\delta_{\overline{\xi}}\right\}Z(\xi)=0,\ \ \xi\in\widetilde{D}_h^B,\ \ \xi\ne0,$$

$$Z(0)=1,\ \ Z(\xi)\to0\quad\text{for}\ \ |\xi|\to\infty,$$

$a_{(1)}=\max_{\overline{G}}a(x,t)$, $B_{(1)}=\min_{\overline{\widetilde{G}}}|B_{(9.2)}(\xi,t)|$, $\widetilde{D}_h^B$ is a uniform mesh on the axis ξ with the step-size $h_\xi=N^{-1}$. The function $Z_W(\xi)$ satisfies the estimate

$$Z_W(\xi)\le\left\{\begin{array}{ll}M\exp(-m\varepsilon^{-1}r_2), & h_\xi\le M\varepsilon\\ M(\varepsilon\,h_{(\xi)}^{-1})^{mh_\xi^{-1}r_2}, & h_\xi>M\varepsilon\end{array}\right\},\ \ \xi\in\widetilde{D}_h^B,$$

where $r_2=r(\xi,\xi(\beta(0),0))=|\xi|$.

With regard to the *a priori* estimates for the solution of problem (9.2), (9.3), for the solutions of difference scheme (8.4), (8.5), (8.6) we establish the estimates

$$|\widetilde{u}(\xi,t)-Z_k(\xi,t)|\le M\left[N^{-1/2}+\varepsilon^{-1}N^{-k}\ln^{k-1}N+N_0^{-1}\right],\ \ (\xi,t)\in\overline{\widetilde{G}}_{kh}; \tag{8.7}$$

$$|\widetilde{u}(\xi,t)-Z_k(\xi,t)|\le M\left[N^{-1/2}+N_0^{-1}\right],\ \ (\xi,t)\in\overline{\widetilde{G}}_{kh},\ \ r(\xi,\widetilde{\gamma})\ge\sigma_k;$$

$$\sigma_k\le\left\{\begin{array}{ll}M\left[\varepsilon\,\ln\varepsilon^{-1}+N^{-1}\ln N\right], & k=1\\ MN^{-k+1}\ln^{k-1}N, & k\ge2\end{array}\right\},\ \ k=1,\ldots,K;$$

$$|\widetilde{u}(\xi,t)-Z(\xi,t)|\le M\left[N^{-1/2}+\varepsilon^{-1}N^{-K}\ln^{K-1}N+N_0^{-1}\right],\ \ (\xi,t)\in\overline{\widetilde{G}}_h; \tag{8.8}$$

$$|\widetilde{u}(\xi,t)-Z(\xi,t)|\le M\left[N^{-1/2}+N_0^{-1}\right],\ \ (\xi,t)\in\overline{\widetilde{G}}_h;\ \ r(\xi,\widetilde{\gamma})\ge\sigma_K;$$

$$\sigma_K\le\left\{\begin{array}{ll}M\left[\varepsilon\,\ln\varepsilon^{-1}+N^{-1}\ln N\right], & K=1\\ MN^{-K+1}\ln^{K-1}N, & K\ge2\end{array}\right\},$$

where $Z(\xi,t) = Z_{(8.5)}(\xi,t)$, $\overline{\overline{G}}_h = \overline{\overline{G}}_{h(8.5)}$.

The function $Z(x,t)$ for $N, N_0 \to \infty$ converges (to the solution of problem (9.2), (9.3)) ε-uniformly outside the σ_K-neighbourhood of the set $\widetilde{\gamma}$, and also on $\overline{\overline{G}}$ for the values of the parameter satisfying the condition ($N^{-K} \ln^{K-1} N << \varepsilon$)

$$\varepsilon \geq \varepsilon_0(N), \quad \varepsilon_0^{-1}(N) = o(N^K \ln^{-K+1} N). \tag{8.9}$$

Thus, difference scheme (8.4), (8.5), (8.6) converges almost ε-uniformly for large K; in order to ensure the convergence defect of the function $Z(\xi,t)$ not exceeding the value of $\nu_{(2.7)}$, it is required to choose the value K satisfying the condition

$$K > K(\nu), \quad K(\nu) = \nu^{-1}. \tag{8.10}$$

Theorem 8.1 *Let the hypothesis of theorem* 7.1 *be fulfilled. Then the solution of the difference scheme* (8.4), (8.5), (8.6) *converges to the solution of problem* (9.2), (9.3) *under the condition* (8.9)*; the solution of this scheme under the condition* (8.10) *converges almost ε-uniformly. For the discrete solutions the estimates* (8.7), (8.8) *are valid.*

Remark 8.1 For the interpolant $\overline{Z}(\xi,t)$, $(\xi,t) \in \overline{\overline{G}}$ (linear on triangular elements) the following estimate holds:

$$\left|\widetilde{u}(\xi,t) - \overline{Z}(\xi,t)\right| \leq M \left[N^{-1/2} + \varepsilon^{-1} N^{-K} \ln^{K-1} N + N_0^{-1}\right], \quad (\xi,t) \in \overline{\overline{G}}.$$

In the variables x, t we have

$$\begin{aligned} &|u(x,t) - Z_x(x,t)| \leq M \left[N^{-1/2} + \varepsilon^{-1} N^{-K} \ln^{K-1} N + N_0^{-1}\right], \qquad (x,t) \in \left\{\overline{\overline{G}}_h\right\}_x; \\ &\left|u(x,t) - (\overline{Z})_x(x,t)\right| \leq M \left[N^{-1/2} + \varepsilon^{-1} N^{-K} \ln^{K-1} N + N_0^{-1}\right], \quad (x,t) \in \overline{G}, \end{aligned}$$

where $Z_x(x,t) = Z(\xi(x,t),t)$, $\{\overline{\overline{G}}_h\}_x$ is the mesh on $\overline{G}$, which corresponds to the mesh $\overline{\overline{G}}_h$ on $\overline{\overline{G}}$.

Remark 8.2 For problem (9.2), (9.3), when we have almost ε-uniform convergent scheme on the rectangular (in the variables ξ, t) *a posteriori* adaptive meshes, it is possible to rewrite the given constructions in the variables x, t. In this case we pass to a difference scheme which approximates problem (2.2), (2.1) on meshes generated by a family of (sufficiently smooth) curves adapted to the interface boundary γ. Almost ε-uniform convergence of the constructed in this way schemes for the approximation of problem (2.2), (2.1) is ensured by *a posteriori* condensation of the mesh in a neighbourhood of the set γ.

9. *A priori* estimates

In this section we consider *a priori* estimates for the solution of problem (2.2), (2.1) used in our constructions (see also [2, 6, 7, 11]).

On the set $\overline{G}^r$ the solution can be decomposed into its regular and singular components

$$u(x,t) = U(x,t) + V(x,t), \quad (x,t) \in \overline{G}^r, \quad r = 1, 2, \tag{9.1a}$$

which are defined below.

It is convenient to transform problem (2.2), (2.1) to the variables $\xi = \xi(x,t) = x - \beta(t)$, t as follows

$$\widetilde{L}\,\widetilde{u}(\xi,t) = \widetilde{f}(\xi,t), \quad (x,t) \in \widetilde{G}^{(*)}, \tag{9.2}$$

$$[\widetilde{u}(\xi,t)] = 0, \quad \widetilde{l}\,\widetilde{u}(\xi,t) = 0, \quad (\xi,t) \in \widetilde{\gamma}, \quad \widetilde{u}(\xi,t) = \widetilde{\varphi}(\xi,t), \quad (\xi,t) \in \widetilde{S}.$$

Here $\widetilde{G}_0 = \{G_0\}_\xi = G_0$ is the image of the set $G_0 \subseteq \overline{G}$ in the variables ξ, t, $\xi = \xi(x,t)$;

$$\begin{aligned}
\widetilde{\gamma} &= \{(\xi,t):\ \xi = 0,\ t \in (0,T]\}, \quad \widetilde{v}(\xi,t) = v(x(\xi,t),t) = v(\xi+\beta(t),t);\\
\widetilde{L}_{(9.2)} &\equiv \varepsilon\widetilde{a}(\xi,t)\frac{\partial^2}{\partial\xi^2} + B(\xi,t)\frac{\partial}{\partial\xi} - \widetilde{c}(\xi,t) - \widetilde{p}(\xi,t)\frac{\partial}{\partial t}, \quad (\xi,t) \in \overline{\widetilde{G}}^{\,r},\\
\widetilde{l}_{(9.2)}\widetilde{u}(\xi,t) &\equiv \varepsilon\Big[\widetilde{a}(\xi,t)\frac{\partial}{\partial\xi}\widetilde{u}(\xi,t)\Big], \quad (\xi,t) \in \widetilde{\gamma}; \quad B(\xi,t) = B^r(\xi,t), \quad (\xi,t) \in \overline{\widetilde{G}}^{\,r},\\
B^1(\xi,t) &= -\widetilde{b}^1(\xi,t) + \beta'(t)\widetilde{p}^1(\xi,t), \quad B^2(\xi,t) = \widetilde{b}^2(\xi,t) + \beta'(t)\widetilde{p}^2(\xi,t);\\
\widetilde{a}(\xi,t) &= \widetilde{a}^r(\xi,t),\ldots, \quad \widetilde{f}(\xi,t) = \widetilde{f}^r(\xi,t), \quad (\xi,t) \in \overline{\widetilde{G}}^{\,r}, \quad r = 1,2.
\end{aligned}$$

The domain

$$\overline{\widetilde{G}} = \overline{\widetilde{G}}^{\,1} \cup \overline{\widetilde{G}}^{\,2}, \quad \widetilde{G}^1 \cap \widetilde{G}^2 \neq \emptyset \tag{9.3}$$

is a domain, in general, with curvilinear lateral boundaries; the interface boundary $\widetilde{\gamma}$ is immovable, moreover, the distance between the lateral boundary $\widetilde{S}^L$ and the set γ is not less than $\min[d-\beta_0, d-\beta^0]$.

The solution of problem (9.2) can be differentiated with respect to t on $\overline{\widetilde{G}}$ and with respect to ξ on $\overline{\widetilde{G}}^{\,r}$ (see, e.g., [14]), and it is ε-uniformly bounded on $\overline{\widetilde{G}}$ together with its derivatives with respect to t (under the suitable smoothness condition for the data of problem (2.2), (2.1)). We write the function $\widetilde{u}(\xi,t)$ on the set $\overline{\widetilde{G}}^{\,r}$ as a sum of the functions

$$\widetilde{u}(\xi,t) = \widetilde{U}(\xi,t) + \widetilde{V}(\xi,t), \quad (\xi,t) \in \overline{\widetilde{G}}^{\,r}, \ \ r = 1,2, \tag{9.1b}$$

where $\widetilde{U}(\xi,t)$ and $\widetilde{V}(\xi,t)$ are the regular and singular (interior layer) components of the solution. The function $\widetilde{U}(\xi,t)$ is the restriction onto $\overline{\widetilde{G}}^{\,r}$ of the function $\widetilde{U}^{0r}(\xi,t)$, $(\xi,t) \in \overline{\widetilde{G}}^{\,0r}$, which is the (bounded) solution of the problem

$$\begin{aligned}
\widetilde{L}^{0r}\widetilde{U}^{0r}(\xi,t) &= \widetilde{f}^{0r}(\xi,t), \qquad (\xi,t) \in \widetilde{G}^{0r},\\
\widetilde{U}^{0r}(\xi,t) &= \widetilde{\varphi}^{0r}, \qquad (\xi,t) \in \widetilde{S}^{0r}, \ \ r = 1,2;
\end{aligned}$$

the set $\overline{\widetilde{G}}^{\,0r}$ is obtained by an extension of the set $\overline{\widetilde{G}}^{\,r}$ beyond the interface boundary $\widetilde{\gamma}$, $\widetilde{G}^{01} = \widetilde{G}^1 \cup \{[0,\infty) \times (0,T]\}$, $\widetilde{G}^{02} = \widetilde{G}^2 \cup \{(-\infty,0] \times (0,T]\}$; the operator $\widetilde{L}^{0r}$ and the functions $\widetilde{f}^{0r}(\xi,t)$, $\widetilde{\varphi}^{0r}(\xi,t)$ are continuations of the operator $\widetilde{L}_{(9.2)}$ and of the functions $\widetilde{f}^r(\xi,t)$, $\widetilde{\varphi}(\xi,t)$ from the sets $\overline{\widetilde{G}}^{\,r}$ and $\widetilde{S} \cap \widetilde{S}^{0r}$ onto the sets $\overline{\widetilde{G}}^{\,0r}$ and $\overline{\widetilde{S}}^{\,0r}$, which preserve the smoothness and boundedness properties, i.e.

$$a_0 \le \widetilde{a}^{0r}(\xi,t) \le a^0, \ \ B_0 \le |B^{0r}(\xi,t)| \le B^0, \ldots,$$

$$|\widetilde{f}^{0r}(\xi,t)| \le M, \ \ (\xi,t) \in \overline{\widetilde{G}}^{\,0r}, \ \ |\widetilde{\varphi}^{0r}(\xi,t)| \le M, \ \ (\xi,t) \in \widetilde{S}^{0r}.$$

The function $\widetilde{V}(\xi,t)$ is the solution of the problem

$$\begin{aligned}
\widetilde{L}\,\widetilde{V}(\xi,t) &= 0, \qquad (\xi,t) \in \widetilde{G}^r,\\
\widetilde{V}(\xi,t) &= \widetilde{u}(\xi,t) - \widetilde{U}(\xi,t), \qquad (\xi,t) \in \widetilde{S}^r, \ \ r = 1,2.
\end{aligned}$$

For the functions $\widetilde{U}(\xi,t)$, $\widetilde{V}(\xi,t)$ we obtain the estimates

$$\begin{aligned}
&\left|\frac{\partial^{k_1+k_0}}{\partial\xi^{k_1}\partial t^{k_0}}\widetilde{U}(\xi,t)\right| \le M, \\
&\left|\frac{\partial^{k_1+k_0}}{\partial\xi^{k_1}\partial t^{k_0}}\widetilde{V}(\xi,t)\right| \le M\varepsilon^{-k_1}\exp(-m_1\varepsilon^{-1}|\xi|),\\
&\quad (\xi,t) \in \overline{\widetilde{G}}^{\,r}, \quad k_1 + 2k_0 \le 4, \quad r = 1,2,
\end{aligned} \tag{9.4}$$

where $m_1 \in (0, m_0)$, $m_0 = \min_{\overline{G}}[a(x,t)^{-1} \, |b(x,t) - p(x,t)(d/dt)\beta(t)|]$. Returning to the variables x, t, we find

$$\left|\frac{\partial^{k_1+k_0}}{\partial x^{k_1}\partial t^{k_0}} U(x,t)\right| \le M, \tag{9.5}$$

$$\left|\frac{\partial^{k_1+k_0}}{\partial x^{k_1}\partial t^{k_0}} V(x,t)\right| \le M\varepsilon^{-k_1-k_0}\exp(-m_1\varepsilon^{-1}|x-\beta(t)|),$$

$$(x,t) \le \overline{G}^{\,r}, \quad k_1 + 2k_0 \le 4, \quad r = 1, 2, \quad m_1 = m_{1(9.4)}.$$

Theorem 9.1 *Let a, b, c, p, $f \in C^{4+\alpha}(\overline{G}^r)$, $\varphi \in C^{4+\alpha}(\overline{S}^L) \cap C^{4+\alpha}(S_0)$, $\beta \in C^{3+\alpha/2}([0,T])$, and also $u \in C^{4+\alpha,2+\alpha/2}(\overline{G}^{\,r})$, $\alpha > 0$, $r = 1, 2$, and let the condition* (2.4) *holds. Then the components of the solution to the problem* (2.2), (2.1) *from the representation* (9.1) *satisfy the estimates* (9.4), (9.5).

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