Solving Curved Linear Programs via the Shadow Simplex Method

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CWI Scientific Meeting 01/15

Outline

1. Introduction
   - Linear Programming and its Applications
   - The Simplex Method
   - Results
2. The Shadow Simplex Method
   - Pivot Rule
   - 3-Step Shadow Simplex Path
3. Conclusions

Outline

Linear Programming

- Linear Programming (LP): linear constraints & linear objective with continuous variables.

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$$\text{subject to } Ax \leq b, \quad x \in \mathbb{R}^n \quad (A \text{ has } m \text{ rows, } n \text{ columns})$$

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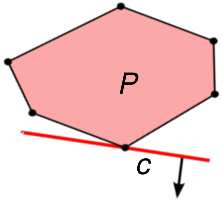

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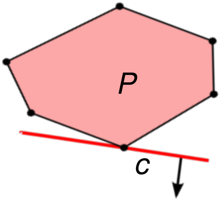

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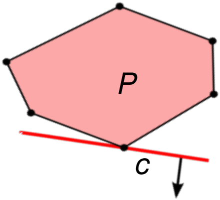

- Amazingly versatile modeling language.
- Generally provides a “relaxed” view of a desired optimization problem, but can be solved in polynomial time!

Mixed Integer Programming

- Mixed Integer Programming (MIP): models both continuous and discrete choices.

$$\max \quad c^T x + d^T y$$
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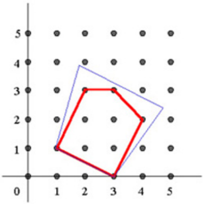

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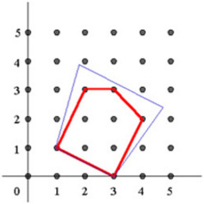

- One of the most successful modeling language for many real world applications. While instances can be extremely hard to solve (MIP is NP-hard), many practical instances are not.

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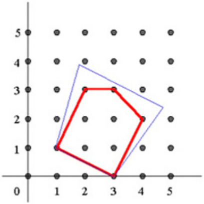
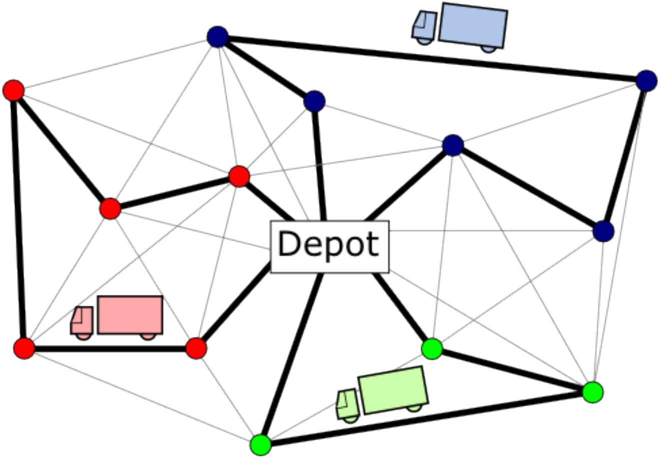


- Many sophisticated software packages exist for these models (CPLEX, Gurobi, etc.). MIP solving is now considered a *mature and practical* technology.

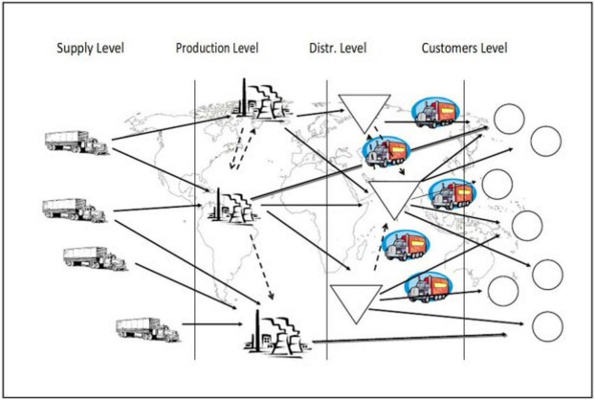
Sample Applications

- Routing delivery / pickup trucks for customers.



Sample Applications

- Optimizing supply chain logistics.



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- Add extra constraints to tighten the LP or “guess” the values of some of the integer variables. Repeat.
- Need to solve a lot of LPs quickly.

Linear Programming via the Simplex Method

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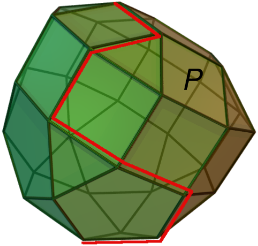
Simplex Method: move from vertex to vertex along the graph of P until the optimal solution is found.



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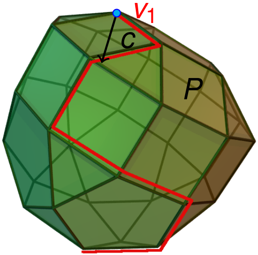

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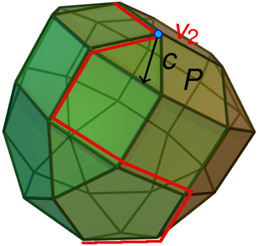

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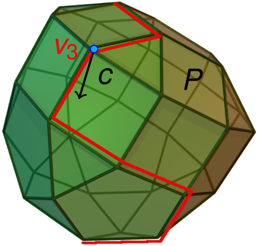

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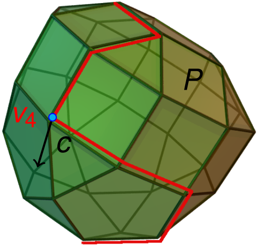

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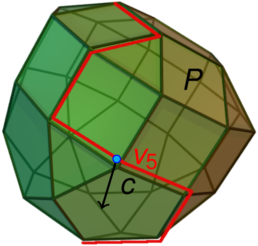

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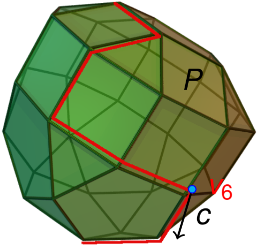

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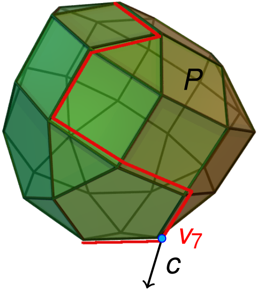

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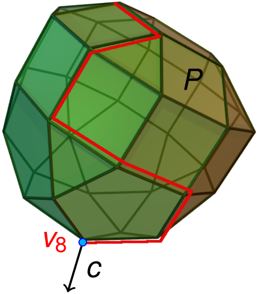

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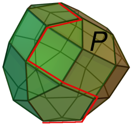

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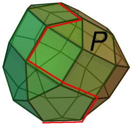

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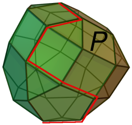

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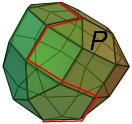

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- Simplex pivots implementable using "simple" linear algebra. Worst case $O(mn)$ time per iteration.

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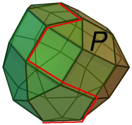

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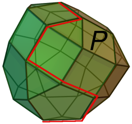

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Simplex lower bounds:

- Klee-Minty (1972): designed "deformed cubes", providing worst case examples for many pivot rules.
- Friedmann et al. (2011): systematically designed bad examples using Markov decision processes.
- In these examples, the pivot rule is tricked into taking an (sub)exponentially long path, even though short paths exists.

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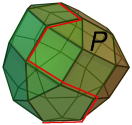

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Simplex upper bounds:

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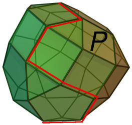

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Polynomial bounds for random or smoothed linear programs:

Borgwardt (82), Smale (83), Adler (83), Todd (83), Haimovich (83), Meggido (86), Adler-Shamir-Karp (86,87), Spielman-Teng (01,04), Spielman-Deshpande (05), Spielman-Kelner (06), Vershynin (06)

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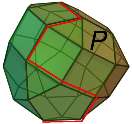

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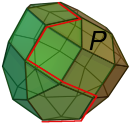
These works rely on the *shadow simplex method*.

Linear Programming and the Hirsch Conjecture

$$P = \{x \in \mathbb{R}^n : Ax \leq b\}, \quad A \in \mathbb{R}^{m \times n}$$



P lives in $\mathbb{R}^n$ (ambient dimension is n) and has m constraints.

Besides the computational efficiency of the simplex method, an even more basic question is not understood:

Question

How can we bound the length of paths on the graph of P? I.e. how to bound the ***diameter*** *of P?*

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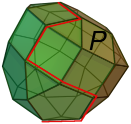

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Conjecture (Polynomial Hirsch Conjecture)

The diameter of P is bounded by a polynomial in the dimension n and number of constraints m.

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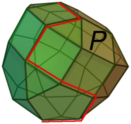

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Diameter upper bounds:

- Barnette, Larman (1974): $\frac{1}{3}2^{n-2}(m - n + \frac{5}{2})$.
- Kalai, Kleitman (1992), Todd (2014): $(m - n)^{\log n}$.

Well-Conditioned Polytopes

$$P = \{x \in \mathbb{R}^n \ : \ Ax \leq b\}, \ \ A \in \mathbb{Z}^{m \times n}$$

Definition (Bounded Subdeterminants)

An integer matrix A has subdeterminants bounded by Δ if every square submatrix B of A satisfies $|\det(B)| \leq \Delta$.

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  - Bonifas, Di Summa, Eisenbrand, Hähnle, Niemeier (2012): non-constructive $O(n^{3.5}\Delta^2 \log n\Delta)$ bound.
  - Brunsch,Röglin (2013): shadow simplex method finds paths of length $O(mn^3\Delta^4)$.

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- Optimization:
  - Dyer, Frieze (1994): $\Delta = 1$ (totally unimodular) random walk simplex uses $O(m^{16}n^6 \log(mn)^3)$ pivots.
  - Eisenbrand, Vempala (2014): random walk simplex uses $O(m \, \mathrm{poly}(n, \Delta))$ pivots.

Faster Shadow Simplex

$$P = \{x \in \mathbb{R}^n \ : \ Ax \leq b\}, \ \ A \in \mathbb{Z}^{m \times n}$$

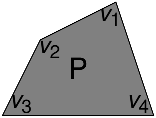
Subdeterminants of A bounded by Δ.

Theorem (D., Hähnle 2014+)

- *Diameter is bounded by* $O(n^3 \Delta^2 \ln(n\Delta))$.
- *Can solve LP using* $O(n^5 \Delta^2 \ln(n\Delta))$ *pivots on expectation.*

Based on a new analysis and variant of the **shadow simplex method.**

τ-wide Polyhedra



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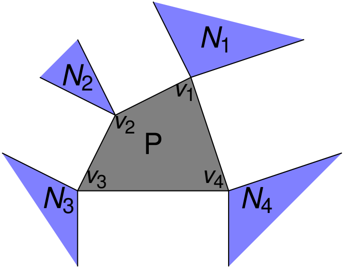

- Normal cone at v is all objectives maximized at v.

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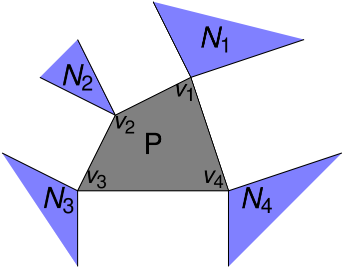

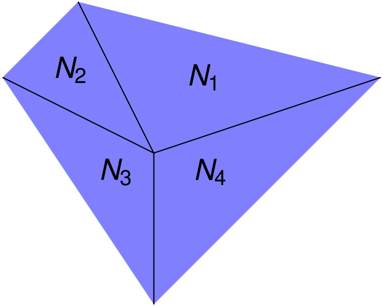

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- Normal fan is the set of normal cones.

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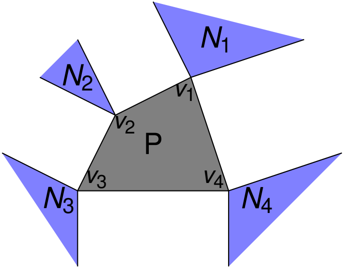
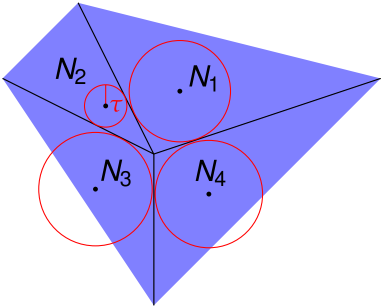
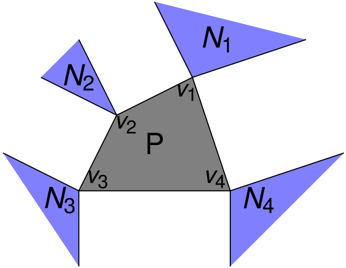
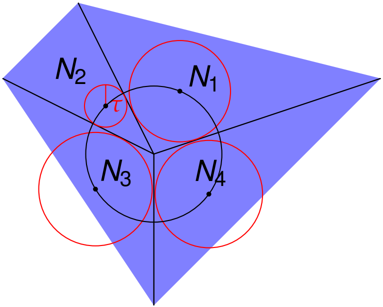




- Normal cone at *v* is all objectives maximized at *v*.
- Normal fan is the set of normal cones.
- *P* is τ-wide if each normal cone contains a unit ball of radius τ centered on the unit sphere.

τ-wide Polyhedra





- Normal cone at v is all objectives maximized at v.
- Normal fan is the set of normal cones.
- P is τ-wide if each normal cone contains a unit ball of radius τ centered on the unit sphere.

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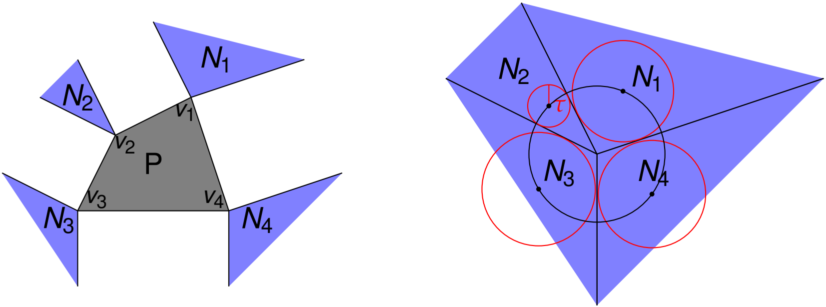

- P has subdeterminant bound $\Delta \Rightarrow P$ is τ-wide for $\tau = 1/(n\Delta)^2$.
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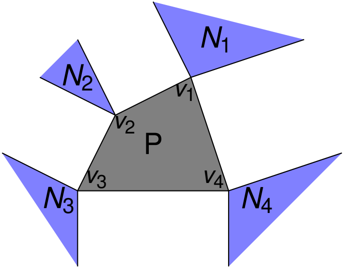

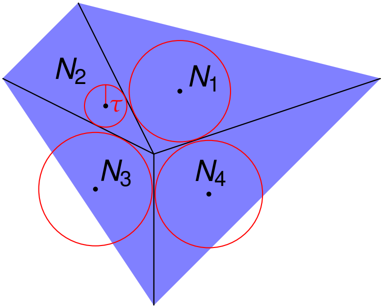

In 2 dimensions all interior angles are at most $\pi - 2\tau$, i.e. *sharp*.

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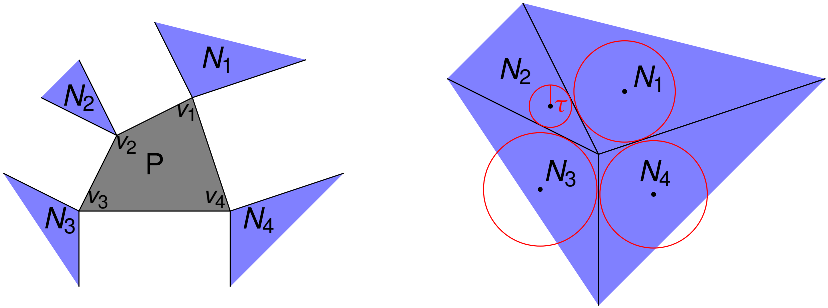


In 2 dimensions all interior angles are at most $\pi - 2\tau$, i.e. *sharp*.

Theorem (D.-Hähnle 2014)

If P is τ-wide then the diameter of P is bounded by $O(n/\tau \ln(1/\tau))$.

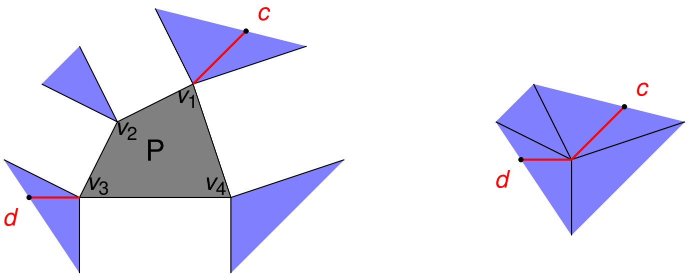
Outline

1. Introduction
   - Linear Programming and its Applications
   - The Simplex Method
   - Results

2. The Shadow Simplex Method
   - Pivot Rule
   - 3-Step Shadow Simplex Path

3. Conclusions

Pivot Rule



- Move from v_1 to v_3 by following $[c, d]$ through the normal fan.
- Pivot steps correspond to crossing facets of the normal fan.

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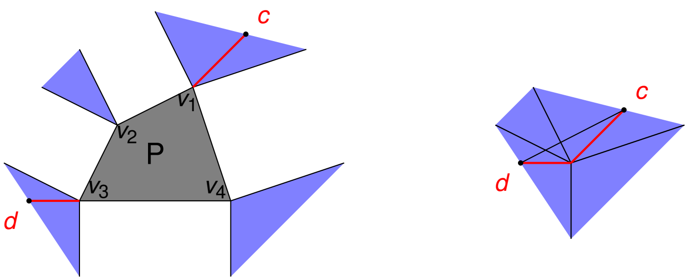

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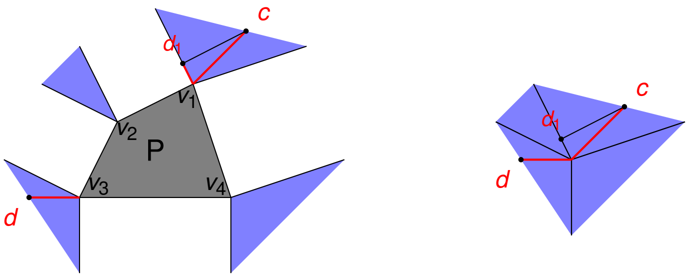

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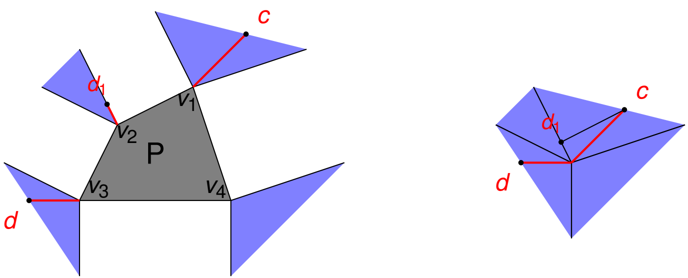

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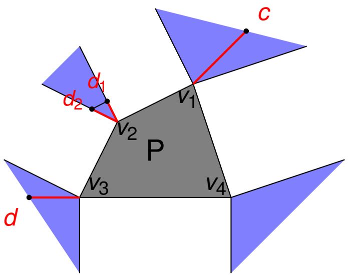

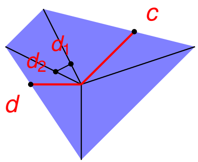

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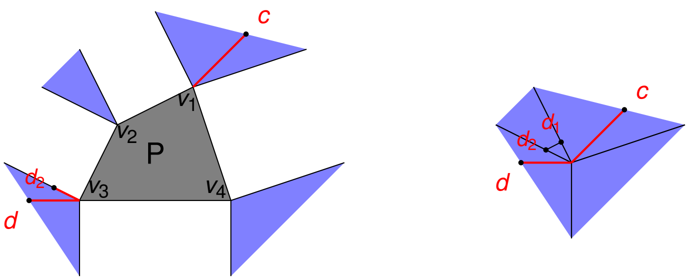

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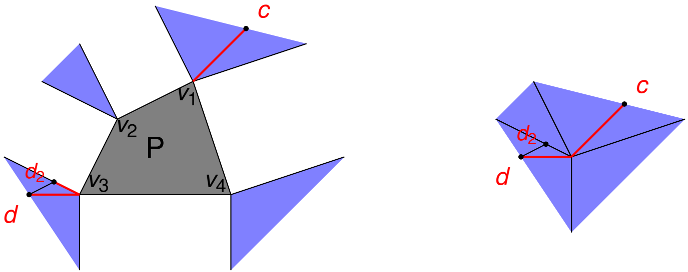

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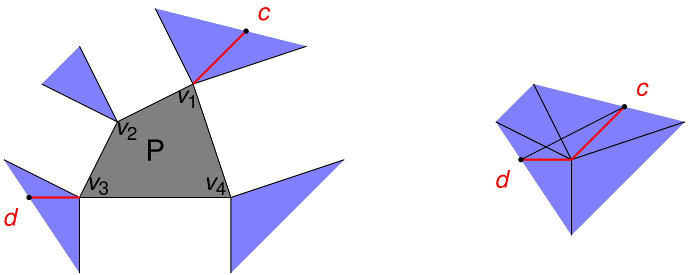

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Question

How can we bound the number of intersections with the normal fan?

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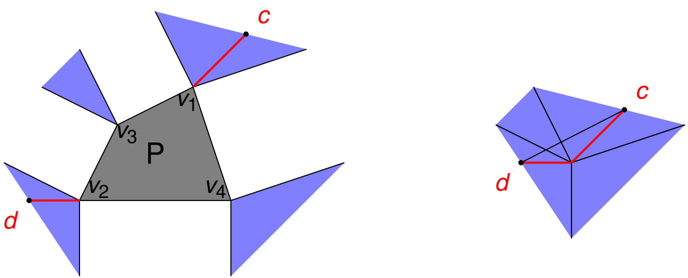

Question

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Polynomial bounds for random and smoothed linear programs:

Borgwardt (82), Smale (83), Adler (83), Todd (83), Haimovich (83), Meggido (86), Adler-Shamir-Karp (86,87), Spielman-Teng (01,04), Spielman-Deshpande (05), Spielman-Kelner (06), Vershynin (06)

3-Step Shadow Simplex Path

- We use a 3-step shadow simplex path:

$$c \xrightarrow{(a)} c + X \xrightarrow{(b)} d + X \xrightarrow{(c)} d$$

where X is distributed proportional to $e^{-\|x\|}$.

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Theorem (D.-Hähnle 2014)

Assume P is an n-dimensional τ-wide polytope.

- *Phase (b). Expected number of crossings of $[c + X, d + X]$ with normal fan is $O(\|c - d\| / \tau)$.*

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Main ingredient for all our results.

Outline

Navigation over the Voronoi Graph

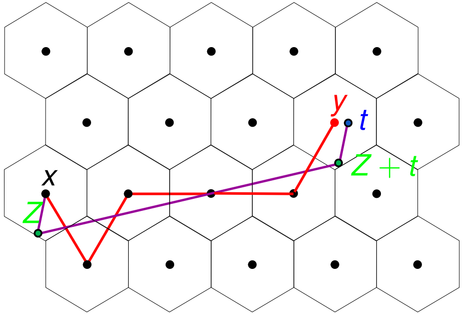

Figure: Randomized Straight Line algorithm

- Closest Vector Problem (CVP): Find closest lattice vector y to t.

Navigation over the Voronoi Graph

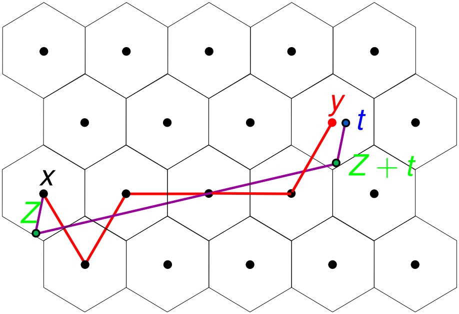

Figure: Randomized Straight Line algorithm

- Closest Vector Problem (CVP): Find closest lattice vector y to t.
- Can reduce CVP to efficient navigation over the Voronoi graph (Sommer,Feder,Shalvi 09; Micciancio,Voulgaris 10-13).

Navigation over the Voronoi Graph

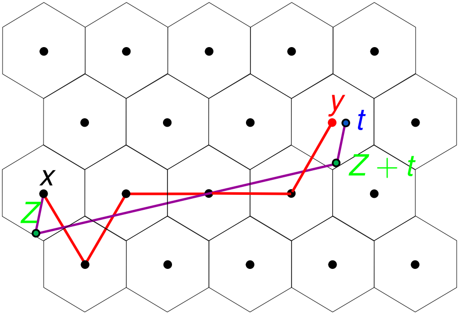

Figure: Randomized Straight Line algorithm

- Closest Vector Problem (CVP): Find closest lattice vector y to t.
- Can move between "nearby" lattice points using a polynomial number of steps (Bonifas, D. 14).

Summary

- New and simpler analysis and variant of the Shadow Simplex method.
- Improved diameter bounds and simplex algorithm for *curved polyhedra*.

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Thank you!