Can linear programs solve NP-hard problems?

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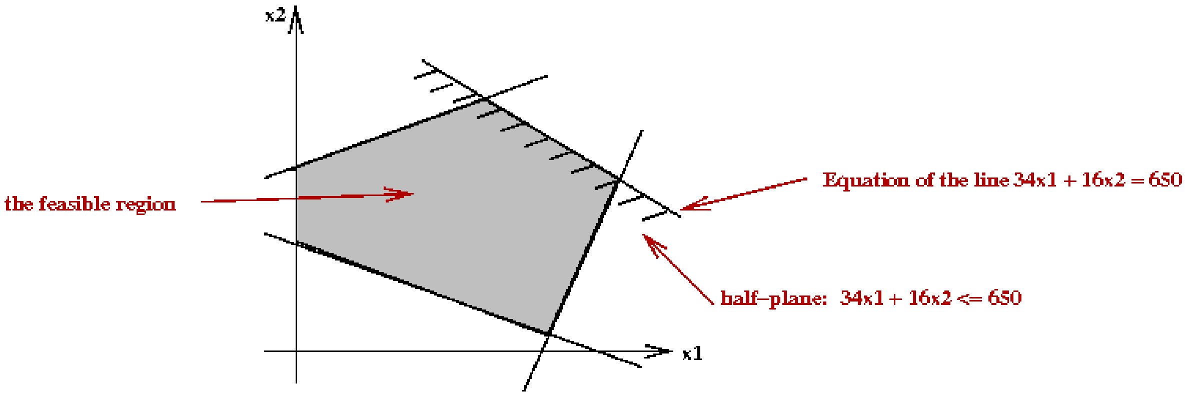
- Feasible region is a polytope



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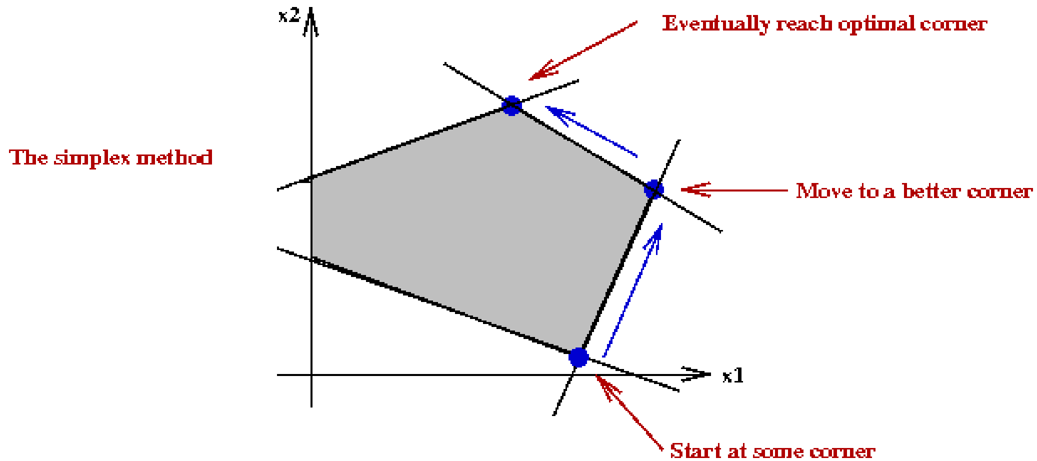

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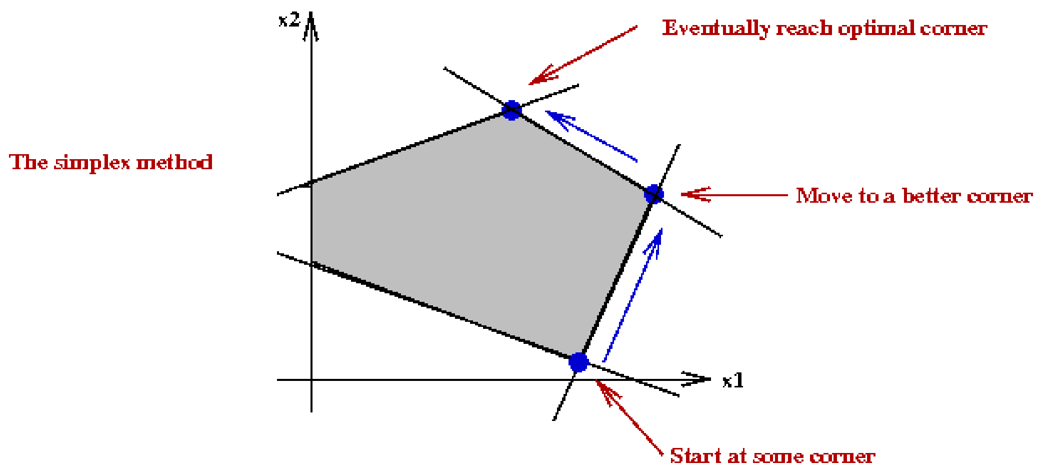

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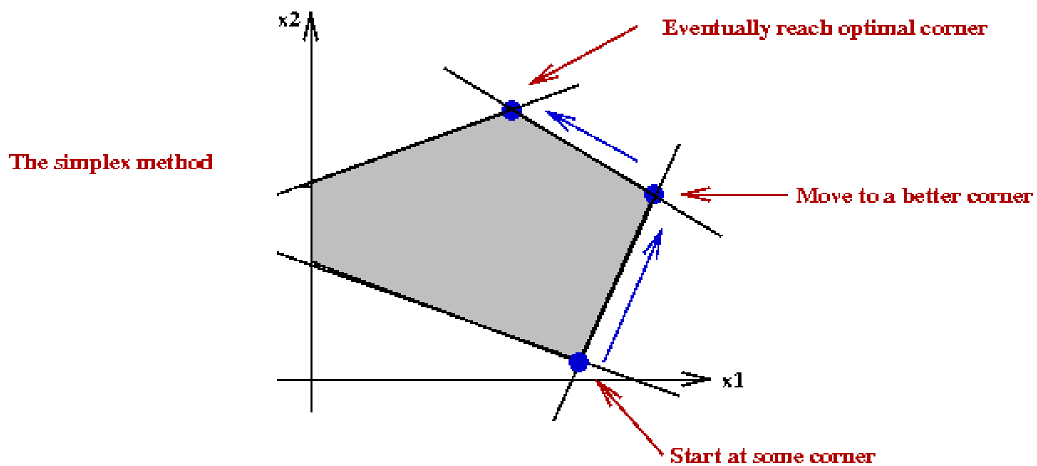

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  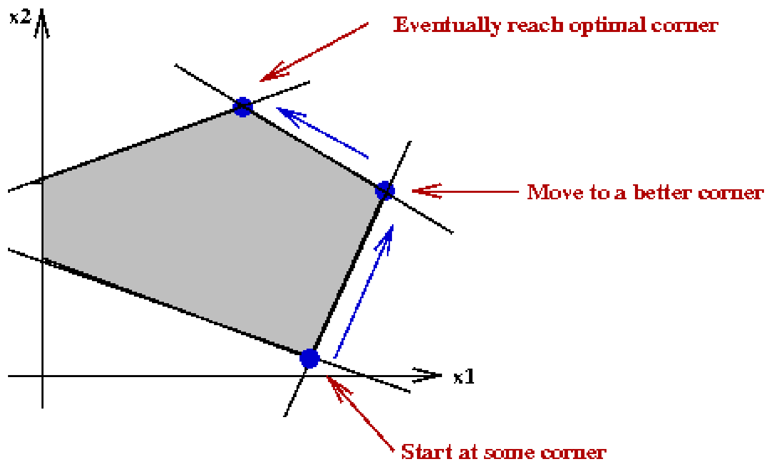

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- Ellipsoid method (Khachiyan'79): takes polynomial time in the worst case, but is not practical
- Interior point method (Karmarkar'84): reasonably efficient in practice

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- Fiorini, Massar, Pokutta, Tiwary, dW (STOC'12):
  any LP for TSP needs exponential size

Polytopes and optimization problems

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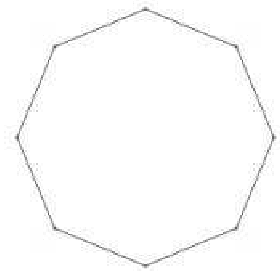
- Polytope P: convex hull of finite set of points in $\mathbb{R}^d$



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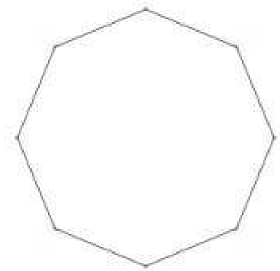

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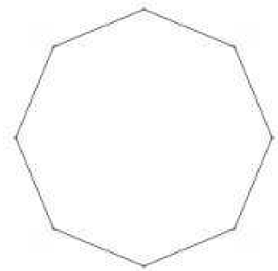

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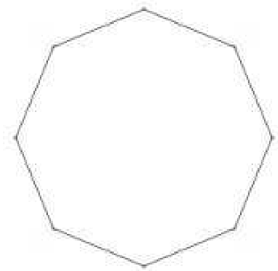

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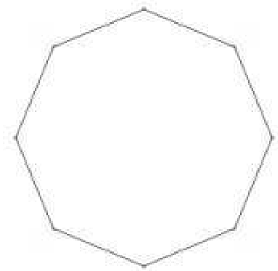

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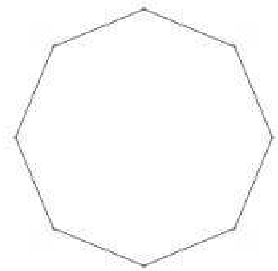

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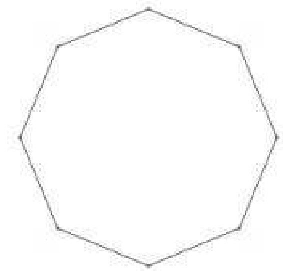

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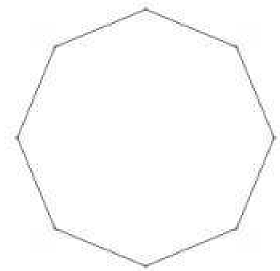

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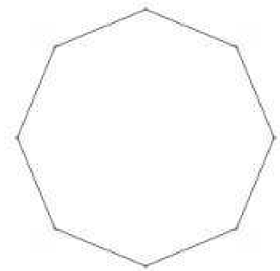

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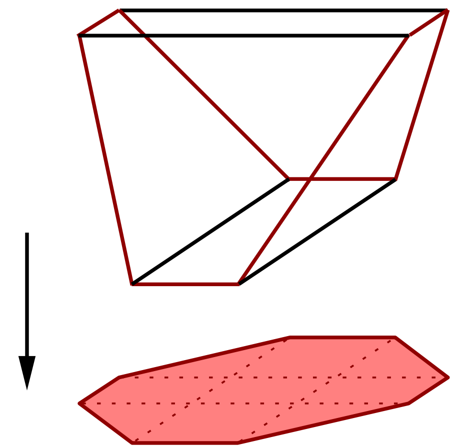

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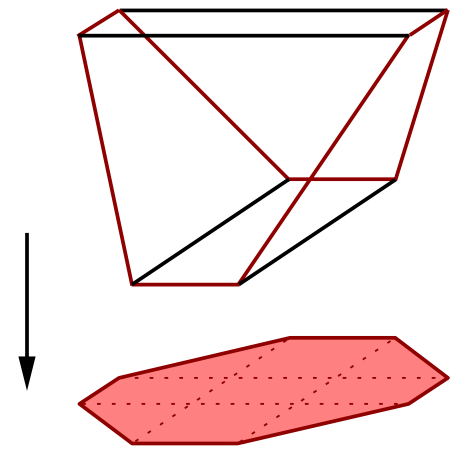

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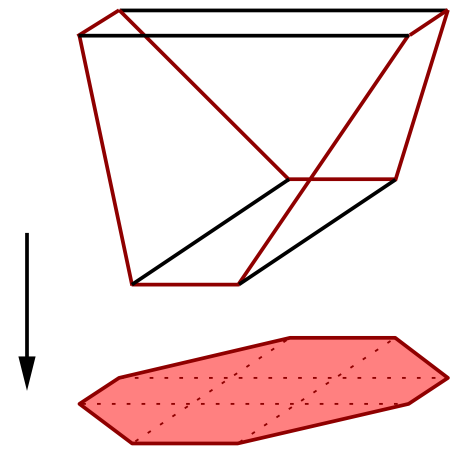

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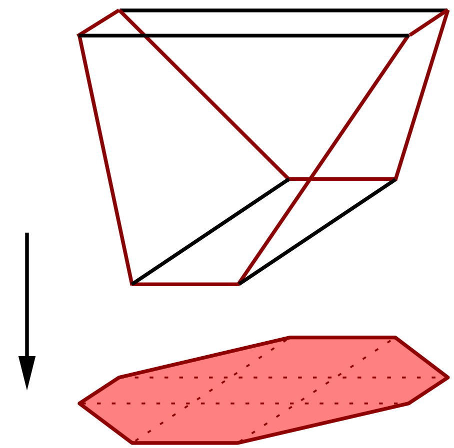

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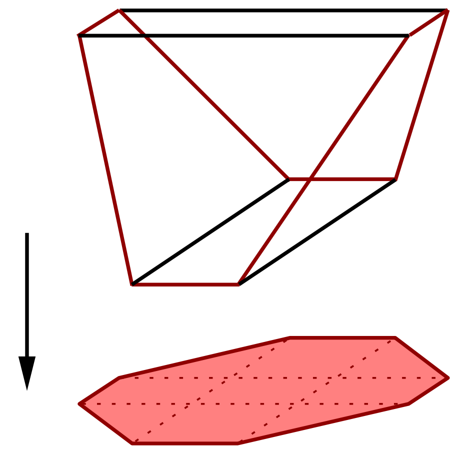

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- Our goal: strong lower bounds on $xc(P)$ for interesting P

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- Big problem until now: which polytope to analyze, and how to analyze its slack matrix?

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- So every linear program based on extended formulations needs exponentially many constraints
- This rules out many efficient algorithms for NP-hard problems, and refutes all P=NP “proofs” à la Swart