From linear to semidefinite optimization: Some selected applications

Networks and Optimization

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What is semidefinite programming?

Semidefinite programming (SDP) is linear optimization over the cone of positive semidefinite matrices.

LP		SDP
vector variable $x \in \mathbb{R}^n$	$\rightsquigarrow$	symmetric matrix variable X
$x \geq 0$		$X \succeq 0$ [positive semidefinite]

LP

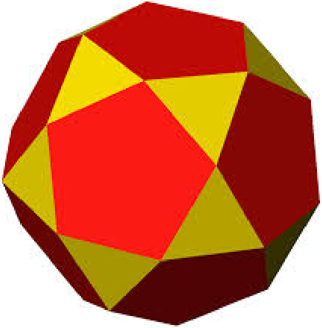
$$\boxed{\begin{array}{ll} \max_x & \langle c, x\rangle \\ \text{s.t.} & \langle a_j, x\rangle = b_j \quad (j = 1, \ldots, m) \\ & x \geq 0 \end{array}}$$

SDP

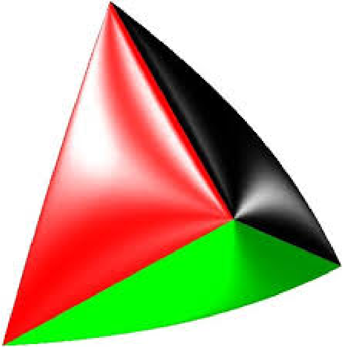
$$\boxed{\begin{array}{ll} \sup_X & \langle C, X\rangle \\ \text{s.t.} & \langle A_j, X\rangle = b_j \quad (j = 1, \ldots, m) \\ & X \succeq 0 \end{array}}$$

There are efficient algorithms to solve SDP (up to any precision).

Geometrically



LP



SDP

LP vs. SDP

1940's: Dantzig simplex algorithm for LP.

Works well in practice, but is it **efficient (= poly-time)**?

From the 1980's: first **efficient algorithms**:

Khachiyan: ellipsoid method (not practical)

Karmarkar, Nemirovski-Nesterov: interior-point algorithms (practical)

LP is widely used, also in industrial applications.

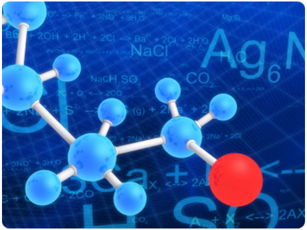
SDP has a greater modeling power:

- sensor network localization [SDP with rank constraint]
- statistics, finance [matrix completion]
- combinatorial optimization [best known approximation algorithms]
- sums of squares of polynomials [real algebraic geometry]
- quantum information

. . . but still needs to be upgraded for large scale problems.

Sensor network localization

Reconstruct the positions of n objects in (say) the **3-dimensional space** from **partial information** on their pairwise distances d_{ij} $(ij \in E)$.



Molecular conformation problem

Find vectors $u_1, \cdots, u_n \in \mathbb{R}^3$ such that $\|u_i - u_j\|_2 = d_{ij} \;\; \forall ij \in E$.

Equivalently: Find a **positive semidefinite matrix** X such that

$$X_{ii} + X_{jj} - 2X_{ij} = d_{ij}^2 \;\; \forall ij \in E \;\; \text{and } \mathrm{rank}\, X \leq 3.$$

$\rightsquigarrow$ SDP with a rank constraint

Matrix completion

Can one **complete a given partial matrix** to a fully specified **positive semidefinite matrix**?

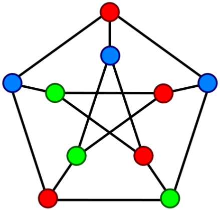
$$\begin{pmatrix} 1 & 0 & ? & -1 \\ 0 & 1 & 1 & ? \\ ? & 1 & 1 & 0 \\ -1 & ? & 0 & 1 \end{pmatrix} \text{Yes: } ? = 0 \qquad \begin{pmatrix} 1 & 0 & ? & -1 \\ 0 & 1 & 1 & ? \\ ? & 1 & 1 & 1 \\ -1 & ? & 1 & 1 \end{pmatrix} \text{No!}$$

of **specified maximum rank?**

- Applications in statistics, finance
- Gives bounds for ranks of optimal solutions of arbitrary SDP's
- Links to topological graph parameters

PhD thesis of Antonios Varvitsiotis, 25 November 2013

Some combinatorial problems over graphs



$\chi = 3$ $\omega = 2$
$\alpha = 4$

- **Chromatic number** $\chi(G)$: minimum number of colors needed to properly color the nodes of G.
- **Clique number** $\omega(G)$: maximum cardinality of a set of pairwise adjacent nodes (**clique**).
- **Independence number** $\alpha(G)$: maximum cardinality of a set of pairwise non-adjacent nodes (**independent set**).

$$\omega(G) \leq \chi(G) \quad \alpha(G) \leq \chi(\overline{G})$$

χ, α, ω are **NP-hard.**

LP vs. SDP approach

Polytope P_G: convex hull of characteristic vectors $\chi^S \in \{0,1\}^V$ of independent sets

$$\alpha(G) = \max\left\{\sum_{i\in V} x_i : x \in P_G\right\}$$

$$\alpha(G) \le \mathrm{lp} = \max\left\{\sum_{i\in V} x_i : x \ge 0,\ \sum_{i\in C} x_i \le 1 \ \forall \text{ cliques } C\right\}$$

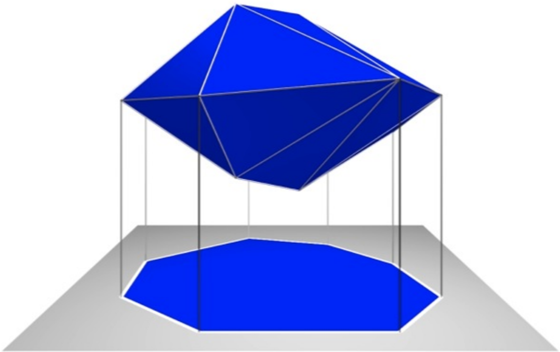
$$\alpha(G) \le \mathrm{sdp} = \max\left\{\sum_{i\in V} x_i : \begin{pmatrix} 1 & x^{\mathsf{T}} \\ x & X \end{pmatrix} \succeq 0,\ X_{ij} = 0 \ \forall ij \in E, X_{ii} = x_i \ \forall i\right\}$$

"Sandwich inequalities" of Lovász [1979]: [$\mathrm{sdp}(G) = \vartheta(G)$]

$$\alpha(G) \le \mathrm{sdp} \le \mathrm{lp} \le \chi(\overline{G})$$

Only known efficient algorithm for graphs with $\alpha(G) = \chi(\overline{G})$!

Fundamental idea: Lift to higher dimensional space



- New variables X_{ij} modeling pairwise products $x_i x_j$ of original variables
- Linearize higher products $x_i x_j x_k$, $x_i x_j x_k x_l$, ...

$\rightsquigarrow$ **hierarchy** of tighter SDP relaxations

$\rightsquigarrow$ best bounds for graph coloring, geometric sphere packing, codes.
[Gijswijt, Gvozdenović, Laurent, Regts, Schrijver, Vallentin]

Classical and quantum information

Zero-error source-channel communication over a noisy channel:

- **Shannon capacity:** $C(G) = \sup_m \frac{1}{m} \log \alpha(G^m)$
  G: confusability graph of the channel
- **Witsenhausen rate:** $R(G) = \inf_m \frac{1}{m} \log \chi(G^m)$
  G: characteristic graph of the source

[Lovász'79, Nayak et al.'06]: $C(G) \leq \log \vartheta(G) \leq R(\overline{G})$. Equality for C_5.

Does quantum entanglement help?

Entangled parameters: α^*, χ^*, C^*, R^*, defined by
replacing $0/1$ *valued variables by positive operator valued variables.*

Sandwich inequalities and separation results:

$$C(G) \leq C^*(G) \;\leq\; \log \vartheta(G) \leq R^*(\overline{G}) \leq R(\overline{G})$$

joint work with **Algorithms and Complexity** group
[Briët, Buhrman, de Wolf, Gijswijt, Laurent, Piovesan, Scarpa]

Positive polynomials and sums of squares

Polynomial optimization problem:

$$\min_{x \in K} p(x) = \max\{\lambda : p - \lambda \text{ is positive on } K\},$$

$$K = \{x : q_1(x) \geq 0, \cdots, q_m(x) \geq 0\} \text{ is a semi-algebraic set.}$$

1. *Testing whether a polynomial p is* **positive** *is* **NP-hard.**
2. *If p is a* **sum of squares of polynomials** *then p is positive.*
3. *One can* **test** *whether p is a sum of squares* **with SDP.**

[Schmüdgen 91, Putinar 93] show s.o.s. positivity certificates on K:

If p is **strictly positive on** K compact, then $\mathbf{p} = \mathbf{s_0} + \mathbf{s_1 q_1} + \cdots + \mathbf{s_m q_m}$ for some $s_0, \cdots, s_m$ sums of squares of polynomials.

$\rightsquigarrow$ hierarchies of SDP relaxations, computing the **global optimum**

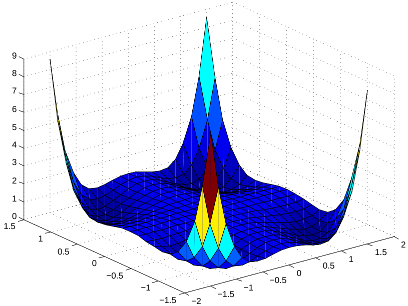
$\rightsquigarrow$ algorithms for computing real roots of polynomial equations [Lasserre, Laurent, Rostalski]

This goes back to Hilbert

Hilbert [1888]: *Every positive polynomial in n variables and even degree d is a* **sum of squares of polynomials** *if and only if $n = 1$, or $d = 2$, or ($n = 2$ and $d = 4$).*

Hilbert's 17th problem [1900]: *Is every positive polynomial is a* **sum of squares of rational functions***?*

Artin [1927]: **Yes**



Motzkin [1960]:

$$p = x^4y^2 + x^2y^4 - 3x^2y^2 + 1$$

is positive, but **not a sum of squares**.

Some new directions

- Use sums of squares of polynomials in **non-commutative variables** to design efficient approximations for **quantum** graph parameters.

  Joint project N&O with Algorithms and Complexity

- Deal with **integral variables** in polynomial optimization.

  Starting MINO (Mixed Integer Nonlinear Optimization) EU Initial Training Network.

  Planned collaboration with Life Sciences.

- Joint seminar on the use of SDP hierarchies in combinatorial optimization (with Nikhil Bansal, TUE/CWI).