From linear to semidefinite optimization:
Some selected applications

Networks and Optimization

Monique Laurent
What is semidefinite programming?

Semidefinite programming (SDP) is linear optimization over the cone of positive semidefinite matrices.

**LP**

vector variable $x \in \mathbb{R}^n$  $\leadsto$  symmetric matrix variable $X$

$x \geq 0$  $\implies$  $X \succeq 0$ [positive semidefinite]

\[
\begin{align*}
\text{LP} & \quad \text{max}_x \langle c, x \rangle \\
\text{s.t.} & \quad \langle a_j, x \rangle = b_j \quad (j = 1, \ldots, m) \\
& \quad x \geq 0
\end{align*}
\]

\[
\begin{align*}
\text{SDP} & \quad \text{sup}_X \langle C, X \rangle \\
\text{s.t.} & \quad \langle A_j, X \rangle = b_j \quad (j = 1, \ldots, m) \\
& \quad X \succeq 0
\end{align*}
\]

There are efficient algorithms to solve SDP (up to any precision).
Geometrically
1940’s: Dantzig simplex algorithm for LP.
Works well in practice, but is it efficient (= poly-time)?

From the 1980’s: first efficient algorithms:
Khachiyan: ellipsoid method (not practical)
Karmarkar, Nemirovski-Nesterov: interior-point algorithms (practical)

LP is widely used, also in industrial applications.

SDP has a greater modeling power:

- sensor network localization [SDP with rank constraint]
- statistics, finance [matrix completion]
- combinatorial optimization [best known approximation algorithms]
- sums of squares of polynomials [real algebraic geometry]
- quantum information

... but still needs to be upgraded for large scale problems.
Sensor network localization

Reconstruct the positions of \( n \) objects in (say) the 3-dimensional space from partial information on their pairwise distances \( d_{ij} \) \((ij \in E)\).

Molecular conformation problem

Find vectors \( u_1, \cdots, u_n \in \mathbb{R}^3 \) such that \( \|u_i - u_j\|_2 = d_{ij} \) \( \forall ij \in E \).

Equivalently: Find a positive semidefinite matrix \( X \) such that

\[
X_{ii} + X_{jj} - 2X_{ij} = d_{ij}^2 \quad \forall ij \in E \quad \text{and} \quad \text{rank } X \leq 3.
\]

\( \Rightarrow \) SDP with a rank constraint
Matrix completion

Can one complete a given partial matrix to a fully specified positive semidefinite matrix?

\[
\begin{pmatrix}
1 & 0 & ? & -1 \\
0 & 1 & 1 & ? \\
? & 1 & 1 & 0 \\
-1 & ? & 0 & 1 \\
\end{pmatrix}
\]

Yes: ? = 0

\[
\begin{pmatrix}
1 & 0 & ? & -1 \\
0 & 1 & 1 & ? \\
? & 1 & 1 & 1 \\
-1 & ? & 1 & 1 \\
\end{pmatrix}
\]

No!

of specified maximum rank?

- Applications in statistics, finance
- Gives bounds for ranks of optimal solutions of arbitrary SDP’s
- Links to topological graph parameters

PhD thesis of Antonios Varvitsiotis, 25 November 2013
Some combinatorial problems over graphs

- **Chromatic number** $\chi(G)$: minimum number of colors needed to properly color the nodes of $G$.
- **Clique number** $\omega(G)$: maximum cardinality of a set of pairwise adjacent nodes (clique).
- **Independence number** $\alpha(G)$: maximum cardinality of a set of pairwise non-adjacent nodes (independent set).

$$\omega(G) \leq \chi(G) \quad \alpha(G) \leq \chi(\overline{G})$$

$\chi$, $\alpha$, $\omega$ are NP-hard.
LP vs. SDP approach

**Polytope** $P_G$: convex hull of characteristic vectors $\chi^S \in \{0, 1\}^V$ of independent sets

\[
\alpha(G) = \max \left\{ \sum_{i \in V} x_i : x \in P_G \right\}
\]

\[
\alpha(G) \leq \text{lp} = \max \left\{ \sum_{i \in V} x_i : x \geq 0, \sum_{i \in C} x_i \leq 1 \ \forall \text{cliques } C \right\}
\]

\[
\alpha(G) \leq \text{sdp} = \max \left\{ \sum_{i \in V} x_i : \begin{pmatrix} 1 \\ x^T \\ X \end{pmatrix} \succeq 0, X_{ij} = 0 \ \forall ij \in E, X_{ii} = x_i \ \forall i \right\}
\]

“Sandwich inequalities” of Lovász [1979]: \[
[\text{sdp}(G) = \vartheta(G)]
\]

\[
\alpha(G) \leq \text{sdp} \leq \text{lp} \leq \chi(G)
\]

**Only known efficient** algorithm for graphs with $\alpha(G) = \chi(\overline{G})$!
Fundamental idea: Lift to higher dimensional space

- New variables $X_{ij}$ modeling pairwise products $x_i x_j$ of original variables
- Linearize higher products $x_i x_j x_k, x_i x_j x_k x_l, \ldots$

$\Rightarrow$ hierarchy of tighter SDP relaxations

$\Rightarrow$ best bounds for graph coloring, geometric sphere packing, codes.

[Gijswijt, Gvozdenović, Laurent, Regts, Schrijver, Vallentin]
Classical and quantum information

Zero-error source-channel communication over a noisy channel:

- **Shannon capacity:**
  \[ C(G) = \sup_m \frac{1}{m} \log \alpha(G^m) \]
  \( G \): confusability graph of the channel

- **Witsenhausen rate:**
  \[ R(G) = \inf_m \frac{1}{m} \log \chi(G^m) \]
  \( G \): characteristic graph of the source

[Lovász’79, Nayak et al.’06]: \( C(G) \leq \log \vartheta(G) \leq R(\overline{G}) \). Equality for \( C_5 \).

Does quantum entanglement help?

**Entangled parameters:** \( \alpha^*, \chi^*, C^*, R^* \), defined by replacing 0/1 valued variables by positive operator valued variables.

Sandwich inequalities and separation results:

\[ C(G) \leq C^*(G) \leq \log \vartheta(G) \leq R^*(\overline{G}) \leq R(\overline{G}) \]

joint work with Algorithms and Complexity group

[Briët, Buhrman, de Wolf, Gijswijt, Laurent, Piovesan, Scarpa]
Positive polynomials and sums of squares

Polynomial optimization problem:

\[
\min_{x \in K} p(x) = \max\{\lambda : p - \lambda \text{ is positive on } K\},
\]

\[K = \{x : q_1(x) \geq 0, \cdots, q_m(x) \geq 0\}\] is a semi-algebraic set.

1. Testing whether a polynomial \( p \) is positive is NP-hard.
2. If \( p \) is a sum of squares of polynomials then \( p \) is positive.
3. One can test whether \( p \) is a sum of squares with SDP.

[Schmüdgen 91, Putinar 93] show s.o.s. positivity certificates on \( K \):

If \( p \) is strictly positive on \( K \) compact, then \( p = s_0 + s_1 q_1 + \cdots + s_m q_m \) for some \( s_0, \cdots, s_m \) sums of squares of polynomials.

\( \leadsto \) hierarchies of SDP relaxations, computing the global optimum

\( \leadsto \) algorithms for computing real roots of polynomial equations

[Lasserre, Laurent, Rostalski]
This goes back to Hilbert

Hilbert [1888]: Every positive polynomial in \( n \) variables and even degree \( d \) is a sum of squares of polynomials if and only if \( n = 1 \), or \( d = 2 \), or \( (n = 2 \text{ and } d = 4) \).

Hilbert’s 17th problem [1900]: Is every positive polynomial is a sum of squares of rational functions?

Artin [1927]: Yes

Motzkin [1960]:

\[ p = x^4 y^2 + x^2 y^4 - 3x^2 y^2 + 1 \]

is positive, but not a sum of squares.
Some new directions

- Use sums of squares of polynomials in **non-commutative variables** to design efficient approximations for **quantum** graph parameters.
  
  Joint project N&O with Algorithms and Complexity

- Deal with **integral variables** in polynomial optimization.

  Starting MINO (Mixed Integer Nonlinear Optimization) EU Initial Training Network.

  Planned collaboration with Life Sciences.

- Joint seminar on the use of SDP hierarchies in combinatorial optimization (with Nikhil Bansal, TUE/CWI).