Queueing in a random environment with time-scale separation

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Queueing

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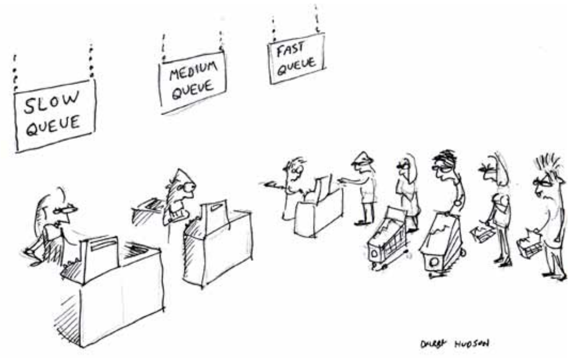
- arrivals
- service time
- # servers (1, many or ∞)

Queueing



- arrivals
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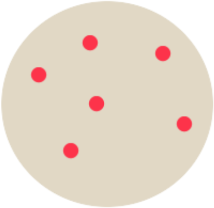
$\#\ \text{jobs} = M,$

$P(M = 7)?$

Motivation

Generation and decay of molecules



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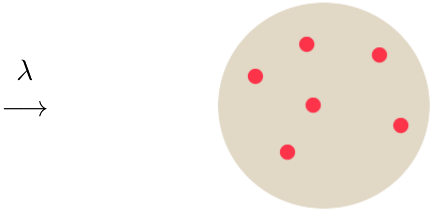

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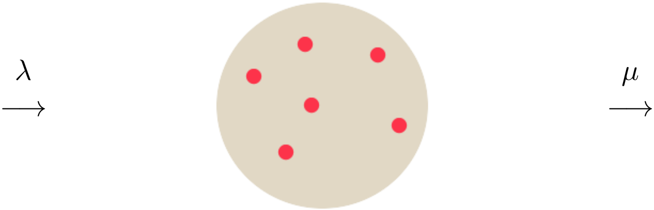
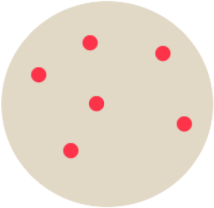


Motivation

Generation and decay of molecules

$\xrightarrow{\lambda}$



$\xrightarrow{\mu}$

mRNA

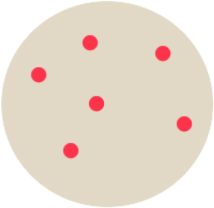
Motivation

Generation and decay of molecules

$\xrightarrow{\lambda}$

Burstiness

$\lambda_1, \lambda_2, \ldots$



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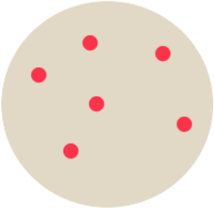

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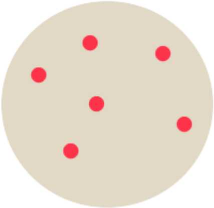

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Decay independent
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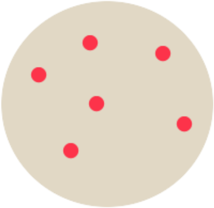

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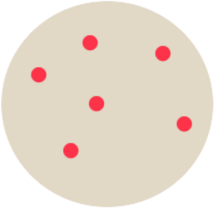

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Mod. M/M/∞

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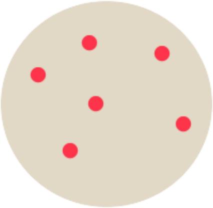

$\xrightarrow{\mu}$

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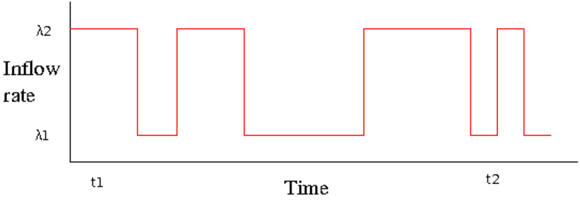
How many molecules?

Outline

Random environment: Markov-modulation

A Markovian background process $J(t)$ determines the Poisson arrival (inflow) rate λ_i to the queue

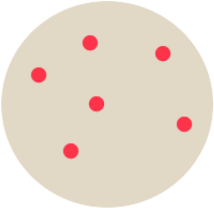


- $J(t_1) = 2$, $\Rightarrow$ arrival rate at time t_1 is λ_2
  $J(t_2) = 1$, $\Rightarrow$ arrival rate at time t_2 is λ_1

Modulated model

Modulating background process $J(t)$

$\lambda_{J(t)}$
$\longrightarrow$
Poisson with
varying rate
Modulation



μ
$\longrightarrow$
Exponential decay
indep. of others
Infinite servers

mRNA
Mod. M/M/∞

Time-scales

Different processes move at different speeds

$$J(t): \quad i \xrightarrow{q_{ij}} j$$

$$M \xrightarrow{\lambda_{J(t)}} M+1$$

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- exploit to get approximations for the main process
  - e.g. Ball, Kurtz, Popovic, Rempala (2006) on stochastic models for chemical reaction networks
- scaling & asymptotics
  - can simplify the equations

Scaling background process

$\pi_i := P(J = i)$ in equilibrium

If $J(t) = i$, then

- arrival rate λ_i
- changes state $i \xrightarrow{q_{ij}} j$

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When $N \to \infty$

- J^N is much faster than the λ's
- rapidly switch $\lambda_i \to \lambda_j \to \ldots$
- $\lambda_\infty := \sum_i \pi_i \lambda_i$

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Uniform Poisson arrivals in the limit!

What happens if we...

speed up $J(t)$ with N^α, $N \to \infty$?

- $Q \mapsto N^\alpha Q$

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also speed up λ_i with N?

- $\lambda_i \mapsto N\lambda_i, \quad Q \mapsto N^\alpha Q, \quad \alpha > 1,$ background $>$ arrivals

J. Blom, O. Kella, M. Mandjes, and H. Thorsdottir. Markov-modulated infinite-server queues with general service times. Queueing Systems, 2013.

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$$\frac{M^N(t) - N\varrho(t)}{\sqrt{N}} \to \mathcal{N}(0, \sigma)$$

Central limit theorem!

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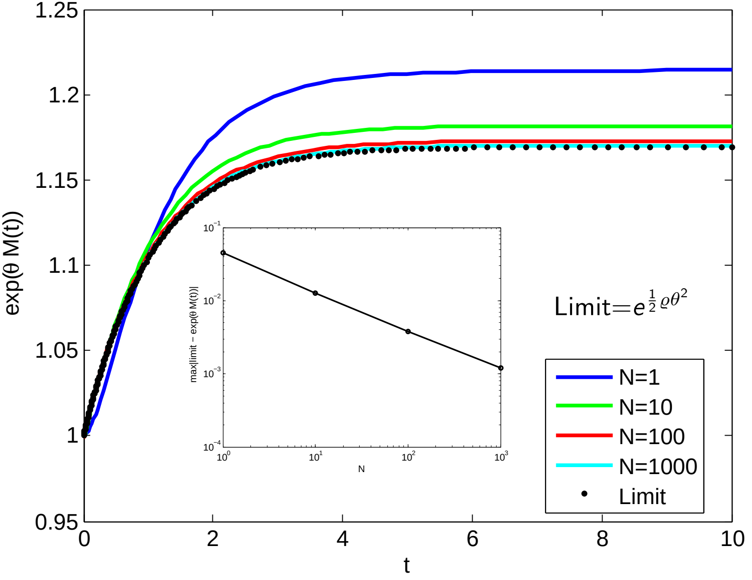
Numerical convergence to Normal MGF



Results

CLT for main process M:

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Extends to:

- $(t_1, \ldots, t_K)$ time points
- weak convergence to an OU-process
- background slower than arrivals
  - divide by $N^{1-\alpha/2}$
- modulated, general μ's

D. Anderson, J. Blom, M. Mandjes, H. Thorsdottir, K. de Turck. A functional CLT for a Markov-modulated infinite-server queue. MCAP, 2014.