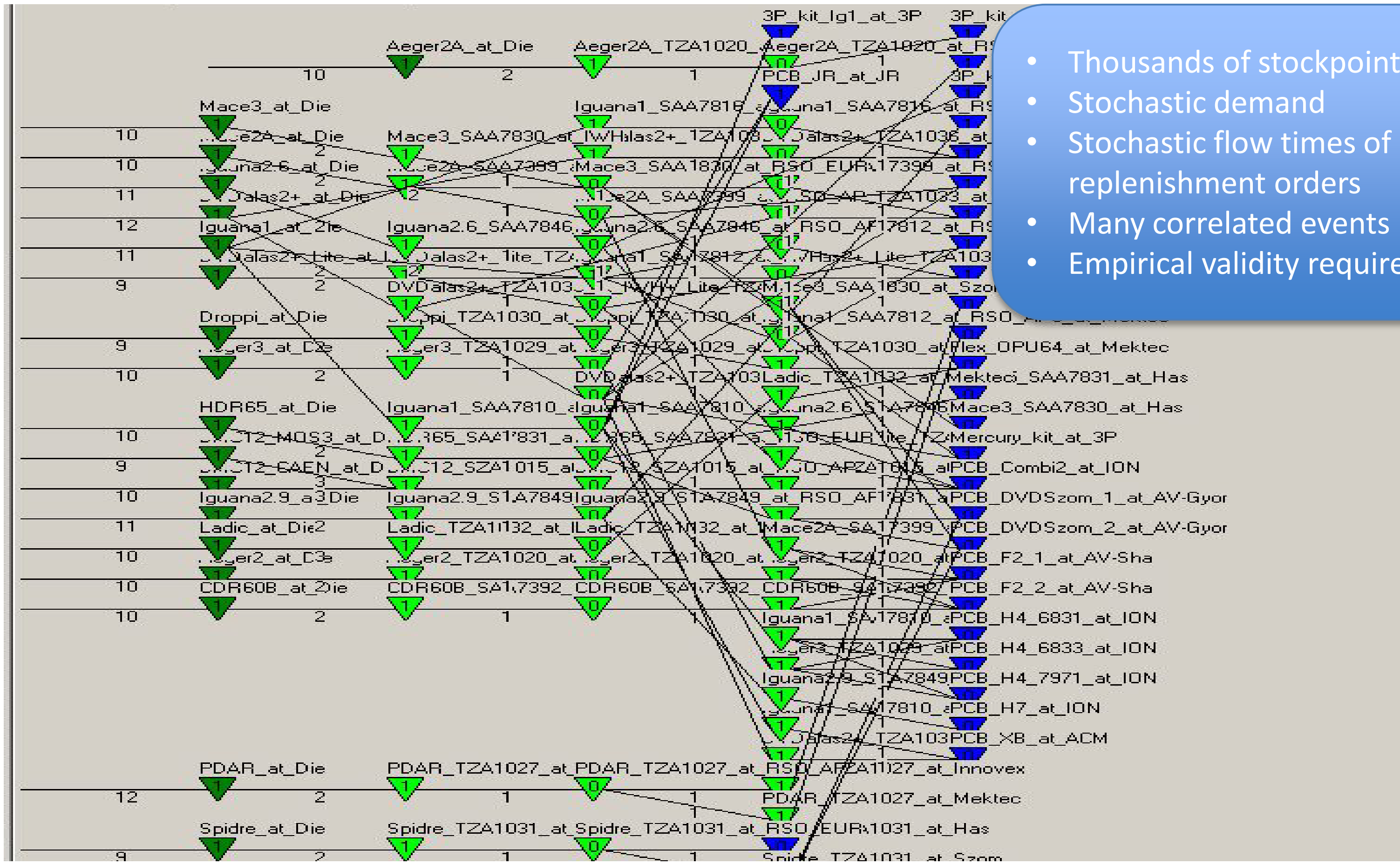
CWI Centrum Wiskunde & Informatica

Decision making under uncertainty in manufacturing and distribution systems

Ton de Kok

Real-life manufacturing and distribution systems: value networks



- Thousands of stockpoints
- Stochastic demand
- Stochastic flow times of replenishment orders
- Many correlated events
- Empirical validity required

Modelling value networks

N	Number of items
E	Collection of items with exogenous demand
a_{ij}	Number of items i needed to create one item j, $i=1,\dots,N$, , $j=1,\dots,N$
T	Decision horizon
$D_i(t)$	Exogenous demand for item i in period t, $t=1,\dots,T$, $i=1,\dots,N$
$F_{t,i}(t+s)$	Prediction made at beginning of period t of exogenous demand for item i in period $t+s$, $t=1,\dots,T$, $s=0,\dots,T-t$, $i=1,\dots,N$
L_i	Lead time of item i, $i=1,\dots,N$
$X_i(t)$	Net stock of item i at the end of period t, $t=0,\dots,\mathrm{T}$, $i=1,\dots,N$
$r_i(t)$	Quantity released from item i at the beginning of period t, $t=1,\dots,\mathrm{T}$, $i=1,\dots,N$
h_i	Inventory cost per item i in stock at the end of a period, $i=1,\dots,N$
p_k	Penalty costs per item k shortfall at the end of a period, $k \in E$
υ_i	Safety stock of item i, $i=1,\dots,\mathrm{N}$

Problem formulation

Looks like a simple LP

$$\min_{\{r_i(t) \mid 1\le i\le N, 1\le t\le T\}} \sum_{t=1}^{T}\sum_{i=1}^{N} h_i X_i^+(t) + \sum_{t=1}^{T}\sum_{k\in E} p_k X_k^-(t)$$

$s.t.$

$$\sum_{j=1}^{N} a_{ij} r_j(t) \le X_i(t-1), i=1,\ldots N, t=1,\ldots,T$$ material availability

$$X_i(t) = X_i(t-1) - \sum_{j=1}^{N} a_{ij} r_j(t) - D_i(t) + r_i(t-L_i), i=1,\ldots N, t=1,\ldots,T$$ inventory balance

$$r_i(t) \ge 0$$

However, $D_i(t)$ is a random variable

Unsatisfied demand is backlogged

Is this a good idea for determining a policy under stochastic demand

Replace demand by forecast and set a planning horizon

$$\min_{\{r_i(t) | 1 \le i \le N, 1 \le t \le T\}} \sum_{s=0}^{T-1} \sum_{i=1}^{N} h_i \left(X_i \left(t+s\right) - \upsilon_i \right)^+ + \sum_{s=1}^{T-1} \sum_{k \in E} p_k \left(X_k \left(t+s\right) - \upsilon_k \right)^-$$

$s.t.$

$$\sum_{j=1}^{N} a_{ij} r_j \left(t+s\right) \le X_i \left(t+s-1\right), i = 1, \ldots N, \quad = 0, \ldots, T-1$$

$$X_i \left(t+s\right) = X_i \left(t+s-1\right) - \sum_{j=1}^{N} a_{ij} r_j \left(t+s\right) - \quad \left(t+s\right) + r_i \left(t+s-L_i\right), i = 1, \ldots N, s = 0, \ldots, T-1$$

$$r_i \left(t+s\right) \ge 0, i = 1, \ldots, N, s = 0, \ldots, T-1$$

Introduction of exogenous safety stocks to cope with uncertainty

But how to set these safety stocks?

Indeed this is an LP, can be solved easily and is the basis for decision support systems

Finite horizon SDP formulation

$$\min_{\{r_i(t)|1\le i\le N,1\le t\le T\}} E\left[\sum_{t=1}^{T}\sum_{i=1}^{N} h_i X_i^+(t) + \sum_{t=1}^{T}\sum_{k\in E} p_k X_k^-(t)\right]$$

$s.t.$

$$\sum_{j=1}^{N} a_{ij} r_j(t) \le X_i(t-1), i=1,\ldots N, t=1,\ldots,T$$ material availability

$$X_i(t) = X_i(t-1) - \sum_{j=1}^{N} a_{ij} r_j(t) - D_i(t) + r_i(t-L_i), i=1,\ldots N, t=1,\ldots,T$$ inventory balance

$$r_i(t) \ge 0$$

Infinite horizon SDP formulation

$$\min_{\{r_i(t)|1\le i\le N, 1\le t\le T\}} E\left[\lim_{T\to\infty}\frac{1}{T}\sum_{t=1}^{T}\sum_{i=1}^{N} h_i X_i^+(t) + \sum_{t=1}^{T}\sum_{k\in E} p_k X_k^-(t)\right]$$

$s.t.$

$$\sum_{j=1}^{N} a_{ij} r_j(t) \le X_i(t-1), i=1,\ldots N, t\ge 0$$ material availability

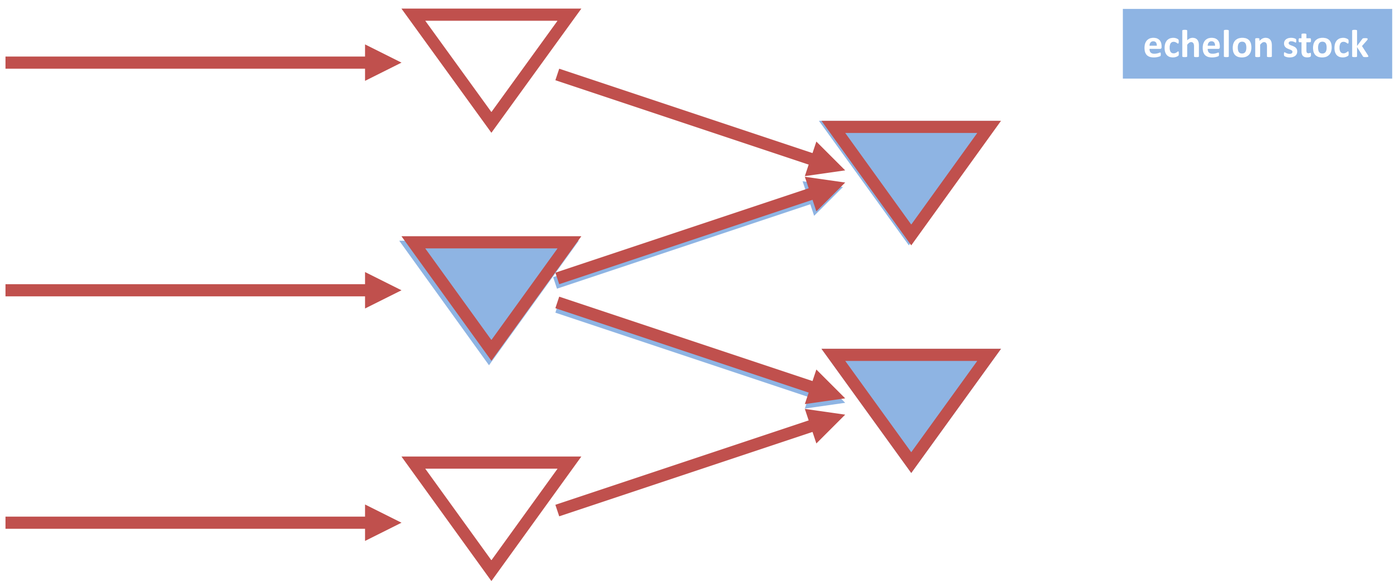
$$X_i(t) = X_i(t-1) - \sum_{j=1}^{N} a_{ij} r_j(t) - D_i(t) + r_i(t-L_i), i=1,\ldots N, t\ge 0$$ inventory balance

$$r_i(t)\ge 0$$

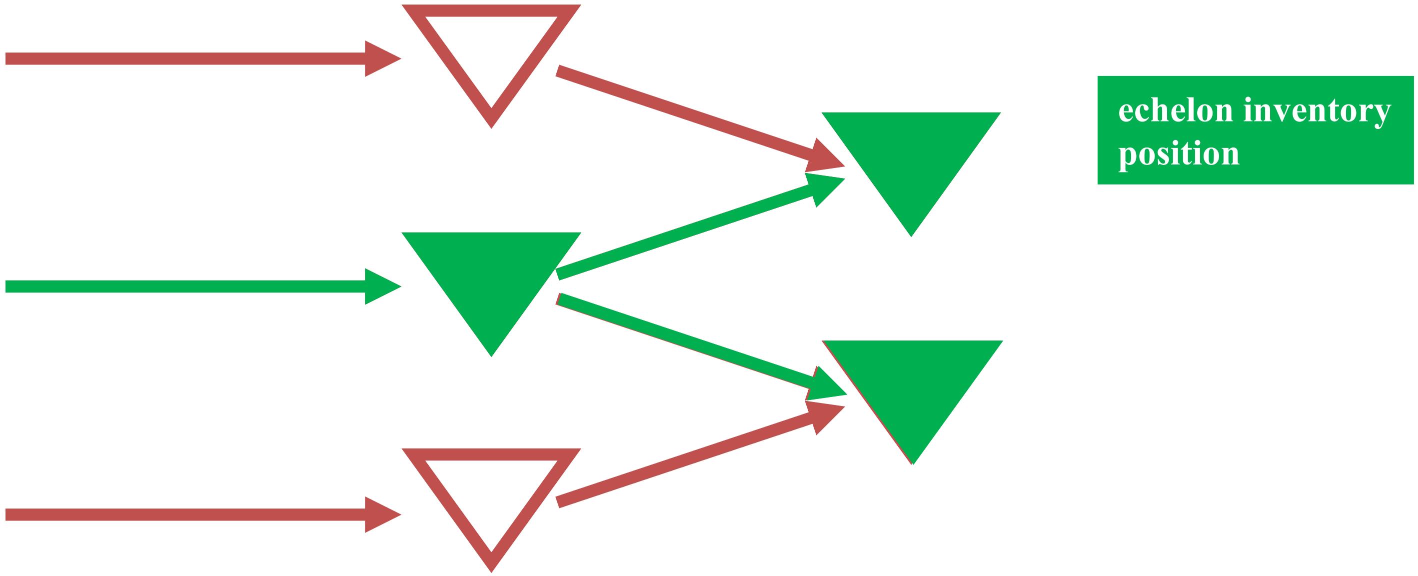
We need 2 things:

1. Optimal policy structure
2. Algorithm to compute the optimal policy

Echelon concept for general assembly systems

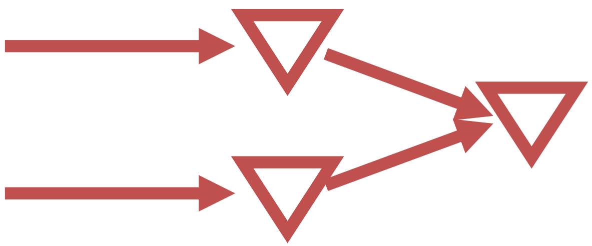


Echelon concept for general assembly systems



Convergent (assembly) systems

- Each item has at most one successor
- Constant flow times (flow time = lead time)
- I.i.d. demand per period
- Linear holding and penalty costs
- No lot sizing costs and constraints



- Echelon base stock policies are optimal
  - Order such that echelon stock equals fixed level
- Base stock levels can be determined recursively

$$\left(p_k + h_{p(i)}\right) = \left(p_k + h_k\right) P\left\{I_{ki} \geq 0\right\}$$

I_{ki} Net stock of end-item k in system consisting of the subsystem with the i shortest cumulative lead times

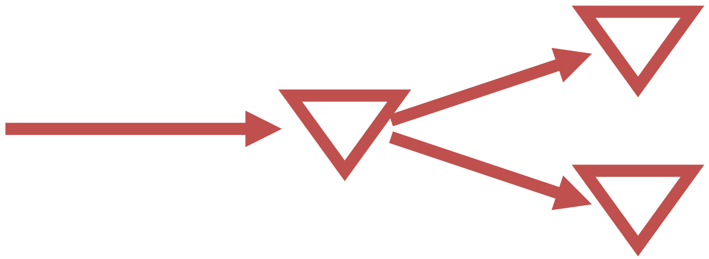
$p(i)$ Predecessor of i when items are ordered according to increasing cumulative lead times

Divergent (distribution) systems

- Each item has at most one predecessor
- Constant flow times (flow time = lead time)
- I.i.d. demand per period
- Linear holding and penalty costs
- No lot sizing costs and constraints
- Relaxation of

  $r_i(t) \geq 0$

- Echelon base stock policies are optimal, optimal allocation functions determined implicitly using Lagrange multipliers
- Base stock levels can be determined recursively, under optimal policy



$$\left(p_k + h_{p(i)}\right) = \left(p_k + h_k\right) P\left\{I_{ki} \geq 0\right\}$$

I_{ki} Net stock of end-item k in system consisting of the subsystem with root node i

Computational issues

- Implicit allocation functions
  - Multiple equations for a single variable, i.e. as many equations as end-items in the echelon of item, for which base stock level must be determined
  - Implicit functions computationally intractable
- Multi-dimensional integrals involving pdf's

Computational issues

- Implicit allocation functions
  - Explicit allocation functions: linear allocation rules
    - In case of a shortage, allocate this shortage according to fixed fractions among successors
  - Numerical study revealed that linear allocation rules are close-to-optimal
  - Linear allocation rules yield recursive optimality equations, too

$$\sum_{k \in E_i} \left(p_k + h_{p(i)} \right) = \sum_{k \in E_i} \left(p_k + h_k \right) P\left\{ I_{ki} \geq 0 \right\}$$

- Multi-dimensional integrals involving pdf’s
  - Recursive expressions for the non-stockout probabilities for end-items
  - “Aggregation of random variables”
  - Applying two-moment fits under assumption of gamma distributions for demand during lead times and for subsequently arising “aggregate” random variables

Finite horizon ruin probabilities

- Non-stockout probabilities $P\{I_{ki}\geq 0\}$ can be written as finite horizon ruin probabilities

$$P\{I_{ki} \geq 0\} = P\left\{\sum_{k=1}^{j} X_k \leq \xi_j, j = 1,\ldots,i\right\}$$

- Expressions can be written recursively

$$G_i(\xi_1,\ldots,\xi_i) = P\left\{\sum_{k=1}^{j} X_k \leq \xi_j, j = 1,\ldots,i\right\}$$

Recursive computations

- Define random variables Y_i

$$P\{Y_1 \le x\} = P\{X_1 \le x\}$$

$$P\{Y_i \le x\} = \frac{G_i(\xi_1,\ldots,\xi_{i-1},x)}{G_{i-1}(\xi_1,\ldots,\xi_{i-1})}, x \ge 0,\ i=2,\ldots,N.$$

- Theorem

$$P\{Y_i \le x\} = \frac{P\{X_i + Y_{i-1} \le x, Y_{i-1} \le \xi_{i-1}\}}{P\{Y_{i-1} \le \xi_{i-1}\}}, i = 2,\ldots,N$$

Two-moment recursion

- Fit mixture of Erlang distributions on $E[Y_i]$ and $\sigma^2(Y_i)$

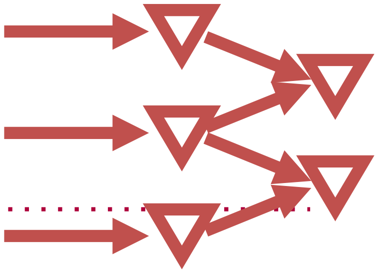
$$E[Y_i] = E[X_i] + E[Y_{i-1} | Y_{i-1} \le \xi_{i-1}],\ i = 2,\dots,N-1$$
$$\sigma^2(Y_i) = \sigma^2(X_i) + \sigma^2(Y_{i-1} | Y_{i-1} \le \xi_{i-1}),\ i = 2,\dots,N-1$$

- Compute $P\{I_{ki} \ge 0\} = G_i(\xi_1,\dots,\xi_i)$ recursively

$$G_1(\xi_1) = P\{Y_1 \le \xi_1\}$$
$$G_i(\xi_1,\dots,\xi_i) = P\{Y_i \le \xi_i\} G_{i-1}(\xi_1,\dots,\xi_{i-1}), i = 2,\dots,N$$

Application of conditional random variable recursion

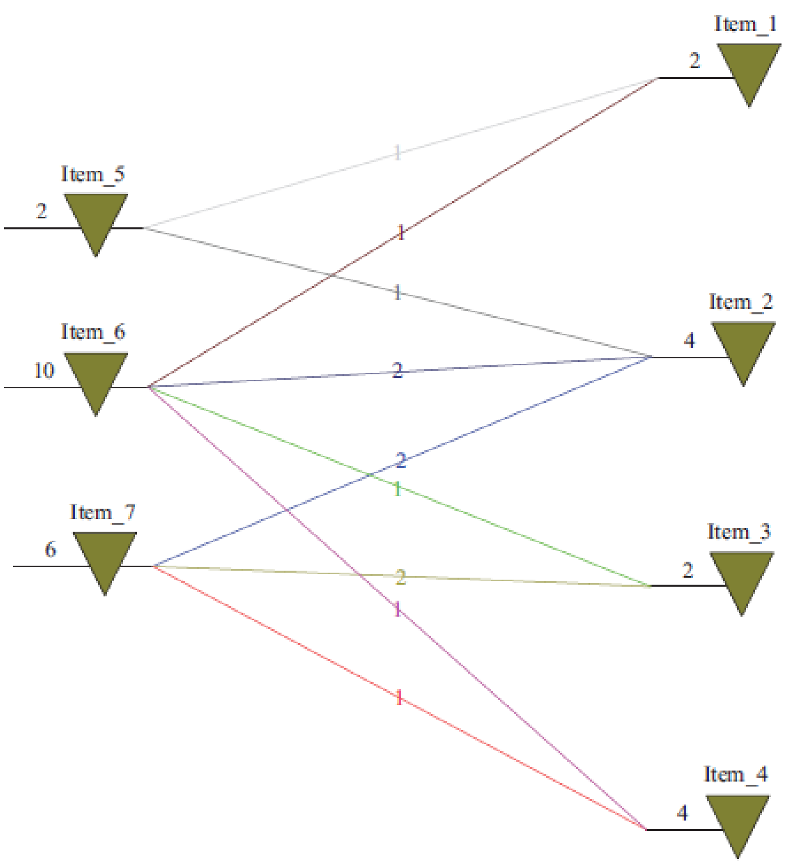
- Multi-echelon models
- Ruin probabilities
- Lost-sales model
  - Approximation very accurate for gamma-distributed demand
  - Fixed non-stockout probability policy outperforms all policies known to date and yields asymptotically optimal policy for high penalty costs and long lead times
- Queueing models
  - Lindley's integral equation over a finite horizon
- Vehicle routing with stochastic travel times
- Appointment scheduling
- Project networks with stochastic activity durations
  - Problem related to the problem discussed here
  - Similar optimality equations

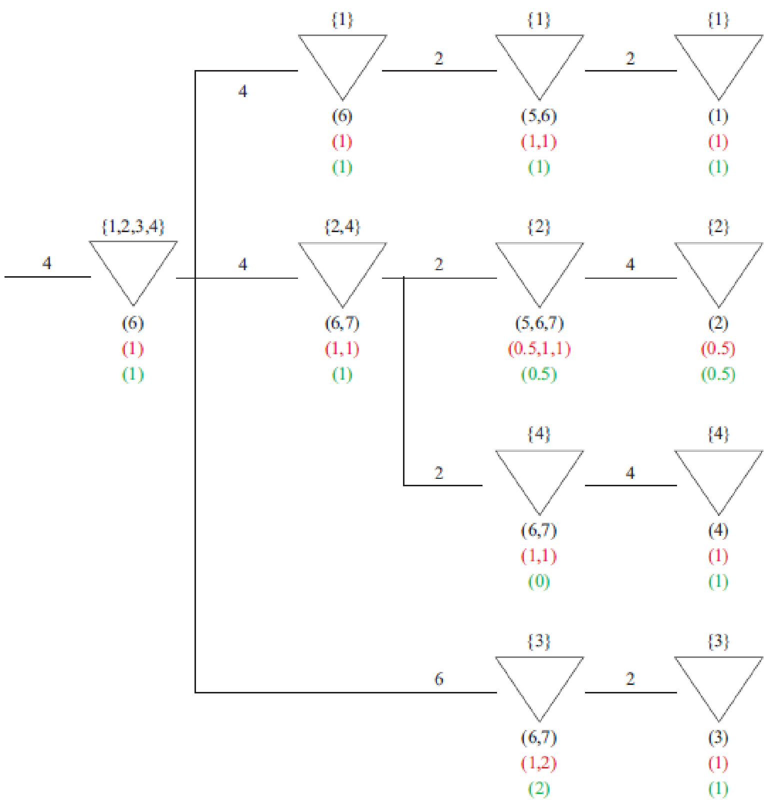


General value networks

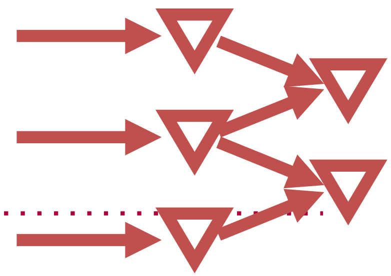
- Optimal policy only known for specific small structures and under specific item cost assumptions
  - N-model
- Synchronized Base Stock policy enables analysis and optimization of general structures
  - Allocate before ordering
  - Synchronize orders based on constraints from earlier ordering decisions
  - Derive a set of divergent structures from the network structure determined by $\{L_i\}$ and (a_{ij})

General network and associated decision node structure





General value networks



- Optimal policy only known for specific small structures and under specific item cost assumptions
  - N-model
- Synchronized Base Stock (SBS) policy enables analysis and optimization of general structures
  - Allocate before ordering
  - Synchronize orders based on constraints from earlier ordering decisions
  - Derive a set of divergent structures from the network structure determined by $\{L_i\}$ and (a_{ij})
  - Optimize divergent systems base stock levels
- Ordering decisions follow from adding orders for an item from the divergent *decision node structures*
- SBS policies outperform rolling scheduling LP- and QP-based policies
- SBS policies enable to compu … data on average inventory levels, showing em … variety of value networks
- Simplified version of SBS has … ply chain to weekly release orders across the sup … oduct
  - Network-wide plan computed i…

For most policies applicable we proof that optimal policy satisfies Newsvendor fractiles for end-items

$$P\{X_k \ge 0\} = \frac{p_k}{p_k + \sum_{m \in U_k} h_m}, k \in E$$

Open problems

- Lot sizing
  - Lot-sizing affects synchronization principle, as lot-sizes cover demand over future periods
  - Heuristic based on deriving nested review periods seems to preserve empirical validity regarding investment in inventory versus customer service
  - Some toy problems have been analyzed as potential building block
- Finite capacity
  - Only results for serial systems
    - Complicated policy structure
  - Empirical validity of model suggests that lead times effectively model capacity constraints
  - But operational order release decisions should satisfy resource constraints
- Simulation-based optimization
  - Proposed recursive two-moment approximation scheme deteriorates as number of echelons increase
  - Monte-Carlo simulation can be used to solve the optimality equations

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Thank you