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Omnibus Tests of Multinormality Based on Skewness and Kurtosis

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SUMMARY

Measures of univariate skewness and kurtosis have long been used as a test of univariate normality, several omnibus test procedures based on a combination of the measures having been proposed, see Pearson, D'Agostino and Bowman (1977) and Mardia (1979). Mardia (1970) proposed measures of multivariate skewness and kurtosis, and constructed a test of multinormality based on these measures. We obtain the correlation between these measures and propose four omnibus tests using the two measures. The performances of these tests are compared by means of a Monte Carlo study.

1. INTRODUCTION

Measures of univariate skewness and kurtosis have long been used as a test for univariate normality, several omnibus test procedures based on a combination of the measures having been proposed. With the testing of multinormality in mind, we propose four omnibus tests based on the following measures of multivar*ate

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skewness and kurtosis of Mardia (1970).

Let $X_i' = (X_{1i},..,X_{pi})$, i=1,..,n be n independent observations on a p-variate random variable X and let $\bar{X}' = (\bar{X}_1,..,\bar{X}_p)$ and $S=(S_{rs})$ denote the sample mean vector and covariance matrix respectively. The sample measures of multivariate skewness and kurtosis are then given by

$$b_{1,p} = \sum_{r,s,t=1}^{p} \sum_{r',s',t'=1}^{p} S^{rr'} S^{ss'} S^{tt'} M_{111}^{(rst)} M_{111}^{(r's't')} \qquad (1.1)$$

and

$$b_{2,p} = \sum_{r,s=1}^{p} \sum_{t,u=1}^{p} S^{rs} S^{tu} M_{1111}^{(rstu)} \qquad (1.2)$$

where

$$S^{-1} = (S^{rs}) , \quad M_{i_1..i_s}^{(r_1..r_s)} = \frac{1}{n} \sum_{i=1}^{n} \prod_{k=1}^{s} (X_{r_k i} - \bar{X}_{r_k})^{i_k}$$

In addition to their use in testing multinormality, Davis (1979) has shown that both these measures are involved in first order approximations to the sampling distributions of test statistics used in connection with the general linear model under moderate non-normality.

2. THE CORRELATION BETWEEN $b_{1,p}$ AND $b_{2,p}$

Let $X_1,..,X_n$ be a random sample from a $N_p(\mu,\Sigma)$ population. Since both $b_{1,p}$ and $b_{2,p}$ are invariant under linear transformations, it can be assumed that $\mu = 0$ and $\Sigma = I$. Now S converges to Σ in probability and so, from (1.1)

$$b_{1,p} \to \sum_{r,s,t=1}^{p} \left\{ M_{111}^{(rst)} \right\}^2 \qquad (2.1)$$

in probability. Following a method employed by Lawley (1959) in the context of canonical correlation analysis, it is possible to express S as $S = I + S^*$ so that, to order n^{-1}, $E(S^*) = 0$. On expanding S^{-1}, and after some simplification (1.2) becomes

$$b_{2,p} = \frac{1}{n} \sum_{i=1}^{n} \{(X_i - \bar{X})'(X_i - \bar{X})\}^2 - \frac{2}{n} \sum_{i=1}^{n} (X_i - \bar{X})'(X_i - \bar{X})(X_i - \bar{X})' S^* (X_i - \bar{X}) + \ldots \qquad (2.2)$$

which can also be written as

$$b_{2,p} = 3 \sum_{r=1}^{p} M_4^{(r)} + 3 \sum_{r=1}^{p} \sum_{s \neq r=1}^{p} M_{22}^{(rs)} - \frac{2n}{n-1} \{ \sum_{r=1}^{p} M_2^{(r)} M_4^{(r)}$$

$$+ \sum_{r=1}^{p} \sum_{r \neq s=1}^{p} M_2^{(r)} M_{31}^{(rs)} + \sum_{r=1}^{p} \sum_{s=1}^{p} \sum_{t \neq s=1}^{p} M_{211}^{(rst)} M_{11}^{(st)} \} + \ldots \qquad (2.3)$$

The following moments for $b_{1,p}$ and $b_{2,p}$ were obtained by Mardia (1970)

$$E(b_{1,p}) = p(p+1)(p+2)n^{-1}, \quad Var(b_{1,p}) = 12p(p+1)(p+2)n^{-2}$$

$$E(b_{2,p}) = (n-1)p(p+2)/(n+1), \; Var(b_{2,p}) = 8p(p+2)n^{-1} \qquad (2.4)$$

Applying the standard method of obtaining standard errors, see for example Kendall and Stuart (1977, pp 243-58), and using the asymptotic results

$$Cov\,(M_4^{(r)}, M_3^{(r)^2}) = 432n^{-2}, \quad Cov\,(M_{22}^{(rs)}, M_3^{(r)^2}) = 36n^{-2},$$

$$Cov\,(M_2^{(r)} M_4^{(r)}, M_3^{(r)^2}) = 540n^{-2}, \quad Cov\,(M_2^{(r)} M_{22}^{(rs)}, M_3^{(r)^2}) = 72n^{-2},$$

$$Cov\,(M_4^{(r)}, M_{21}^{(rs)^2}) = 72n^{-2}, \quad Cov\,(M_4^{(r)}, M_{21}^{(sr)^2}) = 24n^{-2},$$

$$\mathrm{Cov}\,(M_{22}^{(rs)}\ ,\ M_{21}^{(rs)^2}) = 28n^{-2}\ ,$$

$$\mathrm{Cov}\,(M_{22}^{(rs)}\ ,\ M_{21}^{(rt)^2}) = 8n^{-2}\ ,\quad \mathrm{Cov}\,(M_{22}^{(rs)}\ ,\ M_{21}^{(tr)^2}) = 4n^{-2}\ ,$$

$$\mathrm{Cov}\,(M_2^{(r)}\ M_4^{(r)}\ ,\ M_{21}^{(rs)^2}) = 96n^{-2}\ ,\quad \mathrm{Cov}\,(M_2^{(r)}\ M_4^{(r)},\ M_{21}^{(sr)^2}) = 36n^{-2}\ ,$$

$$\mathrm{Cov}\,(M_2^{(r)}\ M_{22}^{(rs)},\ M_{21}^{(rs)^2}) = 36n^{-2}\ ,\quad \mathrm{Cov}\,(M_2^{(r)}\ M_{22}^{(rs)},\ M_{21}^{(sr)^2}) = 32n^{-2}\ ,$$

$$\mathrm{Cov}\,(M_2^{(r)}\ M_{22}^{(rs)},\ M_{21}^{(rt)^2}) = 16n^{-2}\ ,\quad \mathrm{Cov}\,(M_2^{(r)}\ M_{22}^{(rs)},\ M_{21}^{(tr)^2}) = 8n^{-2}\ ,$$

$$\mathrm{Cov}\,(M_4^{(r)},\ M_{111}^{(rst)^2}) = 12n^{-2}\ ,\quad \mathrm{Cov}\,(M_{22}^{(rs)},\ M_{111}^{(rst)^2}) = 8n^{-2}\ ,$$

$$\mathrm{Cov}\,(M_{22}^{(rs)},\ M_{111}^{(rtu)^2}) = 2n^{-2}\ ,\quad \mathrm{Cov}\,(M_2^{(r)}\ M_4^{(r)},\ M_{111}^{(rst)^2}) = 18n^{-2}\ ,$$

$$\mathrm{Cov}\,(M_2^{(r)}\ M_{22}^{(rs)}\ ,\ M_{111}^{(rst)^2}) = 10n^{-2}\ ,$$

$$\mathrm{Cov}\,(M_2^{(r)}\ M_{22}^{(rs)}\ ,\ M_{111}^{(rtu)^2}) = 4n^{-2}\ ,$$

$$\mathrm{Cov}\,(M_2^{(r)}\ M_{22}^{(rs)}\ ,\ M_{111}^{(stu)^2}) = 2n^{-2}\ .$$

It can be shown, after some algebra, that to order n^{-2}

$$\mathrm{Cov}\,(b_{1,p},b_{2,p}) = 12p\ (8p^2 - 13p + 23)n^{-2} \qquad (2.5)$$

and so

$$\mathrm{Corr}\,(b_{1,p},b_{2,p}) = \frac{3(8p^2 - 13p + 23)}{(p+2)\{6(p+1)\}^{\frac{1}{2}}}\,n^{-\frac{1}{2}}\ . \qquad (2.6)$$

In the univariate case, Fisher (1930) showed that the correlation between b_1 and b_2 is given by

$$\text{Corr}\ (b_1, b_2) = \left\{ \frac{54n(n^2-9)}{(n-2)(n+5)(n+7)^2 \{\beta_2(\sqrt{b_1})-1\}} \right\}^{\frac{1}{2}} \qquad (2.7)$$

where $\beta_2(\sqrt{b_1}) = 3 + 36n^{-1} - 864n^{-2} + 12096n^{-3} - \ldots$. Asymptotically this becomes $\sqrt{27}n^{-\frac{1}{2}}$, the value to which (2.6) reduces when $p=1$.

More emphasis has been placed on the use of $\sqrt{b_1}$ rather than b_1 itself, both as a measure of skewness and as a test for normality in the univariate situation, and so it is natural to consider the multivariate $\sqrt{b_{1,p}}$. Since there is no obvious way of attributing a sign to $\sqrt{b_{1,p}}$ it would normally be considered as a positive definite quantity and therefore not a direct generalisation of $\sqrt{b_1}$, but only of the modulus of this statistic. One would therefore not expect to regain the usual univariate results by the substitution $p=1$ in expressions for the moments of $\sqrt{b_{1,p}}$.

Once again applying the method of obtaining standard errors and using (2.4) the following asymptotic moments associated with $\sqrt{b_{1,p}}$ may be obtained for large a,

$$E\ (\sqrt{b_{1,p}}) = \{a^{\frac{1}{2}} - \frac{3}{2}a^{-\frac{1}{2}} + \frac{9}{8}a^{-3/2} + O\ (a^{-5/2})\}\ n^{-\frac{1}{2}}, \qquad (2.8)$$

$$\text{Var}\ (\sqrt{b_{1,p}}) = \{3 - \frac{9}{2}a^{-1} - \frac{27}{2}a^{-2} + O\ (a^{-3})\}\ n^{-1}, \qquad (2.9)$$

$$\text{Cov}(\sqrt{b_{1,p}}\ ,\ b_{2,p}) = \{6pa^{-\frac{1}{2}}\ (8p^2 - 13p + 23)\}n^{-3/2}, \qquad (2.10)$$

where $a = p(p+1)(p+2)$.

An alternative method of calculating the moments of $\sqrt{b_{1,p}}$ is to utilise the distributional result obtained by Mardia (1970)

$$\frac{n}{6} b_{1,p} \sim \chi_{\nu}^2 \text{ where } \nu = \frac{1}{6}p(p+1)(p+2) \qquad (2.11)$$

and the moments of a χ_{ν} distribution. Thus, we have asymptotically

$$E(\sqrt{b_{1,p}}) = \sqrt{12}\,\Gamma\{\tfrac{1}{2}(\nu+1)\}\{\Gamma(\tfrac{1}{2}\nu)\}^{-1} n^{-\frac{1}{2}} \qquad (2.12)$$

$$\mathrm{Var}(\sqrt{b_{1,p}}) = [6\nu - 12\Gamma^2\{\tfrac{1}{2}(\nu+1)\}\{\Gamma(\tfrac{1}{2}\nu)\}^{-2}]\, n^{-1} \qquad (2.13)$$

which yield very similar values to those given by (2.8) and (2.9).

3. OMNIBUS TESTS OF MULTINORMALITY INVOLVING $b_{1,p}$ and $b_{2,p}$.

Even for moderately large n the correlation between $b_{1,p}$ and $b_{2,p}$ is not negligible and so the two measures should be combined in some way to form a single test statistic. In some situations it may be possible to specify the way in which data is likely to depart from normality but generally this will not be the case and so an omnibus test, sensitive as far as possible to any form of departure is required.

3.1 A Test with Rectangular Contours

Using the approximations to the probability integrals of $b_{1,p}$ and $b_{2,p}$ of Mardia (1974) in normal sampling one can construct the following simple test. Let $b_{1,p}(\alpha')$ be the upper $100\alpha'$ % point of $b_{1,p}$ and ${}_Ub_{2,p}(\beta')$, ${}_Lb_{2,p}(\beta')$ be the upper and lower $100\beta'$ % points, respectively, of $b_{2,p}$. Then the four points with co-ordinates $\{0, {}_Ub_{2,p}(\beta')\}$, $\{b_{1,p}(\alpha'), {}_Ub_{2,p}(\beta')\}$, $\{b_{1,p}(\alpha'), {}_Lb_{2,p}(\beta')\}$ and $\{0, {}_Lb_{2,p}(\beta')\}$ form a rectangle as shown in Figure 1. Were $b_{1,p}$ and $b_{2,p}$ independent in sampling from a normal population the overall probability of a $(b_{1,p}, b_{2,p})$ point falling outside the rectangle would be

$$\alpha = \alpha' - 2\alpha'\beta' + 2\beta' . \qquad (3.1)$$

It is arbitrary, but clearly convenient, to take $\alpha' = 2\beta'$ since $b_{1,p}$ is a one tailed test, in which case (3.1) becomes

$$\alpha = 4(\alpha' - \alpha'^2) \text{ or } \alpha' = \tfrac{1}{2}\{1 - (1-\alpha)^{\frac{1}{2}}\} . \qquad (3.2)$$

Because the measures are not independent a smaller proportion of points fall outside the rectangle than the relations (3.2) allow and so the test is conservative. Using the Monte Carlo method we can correct the α' level to obtain conventional values for α as done by Pearson et al (1977) for the univariate case.

3.2 Tests Based on Elliptic Regions.

In the univariate case D'Agostino and Pearson (1973) proposed an omnibus test of normality using standardised normal equivalent deviates, the test statistic having a χ^2_2 distribution. However, it was subsequently realised by the authors that the use of this statistic was not justified, at least in small samples, due to the dependence between $\sqrt{b_1}$ and b_2. As an alternative Bowman and Shenton (1975) proposed using the test statistic

$$K^2 = X_s^2(\sqrt{b_1}) + X_s^2(b_2), \tag{3.3}$$

where $X_s(\sqrt{b_1})$ and $X_s(b_2)$ are normal variates obtained by the use of Johnson's S_u transformation, Johnson (1949). The statistic is again assumed to have a χ^2_2 distribution.

Mardia (1970) derived the following asymptotic null distributions for $b_{1,p}$ and $b_{2,p}$.

$$\frac{n}{6}\, b_{1,p} \sim \chi^2_{p(p+1)(p+2)/6} \quad , \; b_{2,p} \sim N\{p(p+2),\quad 8p(p+2)n^{-1}\}. \tag{3.4}$$

Since $b_{1,p}$ is always positive the lognormal transformation is more appropriate than Johnson's S_u for bringing $b_{1,p}$ to approximate normality, where $X_L(b_{1,p})$ is defined by

$$X_L(b_{1,p}) = \gamma + \delta \log(b_{1,p} - \xi) \tag{3.5}$$

the parameters being defined in Johnson (1949). Alternatively, using the normal approximation to a χ^2 variable, $U(b_{1,p})$ defined by

$$U(b_{1,p}) = \{b_{1,p} - p(p+1)(p+2)n^{-1}\}/\{12p(p+1)(p+2)n^{-2}\}^{\frac{1}{2}} \tag{3.6}$$

has an approximation standard normal distribution. Since $b_{2,p}$ has a normal distribution asymptotically it is straightforward to transform it to standard normality by using

$$U(b_{2,p}) = \{b_{2,p} - p(p+2)(n-1)/(n+1)\} / \{8p(p+2)n^{-1}\}^{\frac{1}{2}} . \quad (3.7)$$

Using (3.5) - (3.7) two statistics, analogous to that of (3.3) can be defined, these are

$$K_L^2 = X_L^2(b_{1,p}) + U^2(b_{2,p}) , \quad (3.8)$$

$$K_N^2 = U^2(b_{1,p}) + U^2(b_{2,p}) . \quad (3.9)$$

Neither of these statistics take account of the covariation between $b_{1,p}$ and $b_{2,p}$ which is not negligible, even for moderately large n, and so, using (3.6) and (3.7), a third statistic which accounts for the covariation between the two measures may be defined by

$$K_c^2 = b'V^{-1}b \quad (3.10)$$

where

$$b' = \{b_{1,p} - p(p+1)(p+2)n^{-1},\ b_{2,p} - p(p+2)(n-1)/(n+1)\}$$

$$V = \begin{bmatrix} 12p(p+1)(p+2)n^{-2} & 12p(8p^2-13p+23)n^{-2} \\ 12p(8p^2-13p+23)n^{-2} & 8p(p+2)n^{-1} \end{bmatrix} .$$

Due to the approximate normality of $X_L(b_{1,p})$, $U(b_{1,p})$ and $U(b_{2,p})$ all three test statistics will have a χ_2^2 distribution asymptotically. As can be easily seen from the form of the covariance matrix V, K_c^2 is likely to be dominated by the $b_{2,p}$ term, a problem which could be overcome by considering a similar test using $\sqrt{b_{1,p}}$ rather than $b_{1,p}$ since in this case both variances are of the same order.

The results of Monte Carlo simulations on $(b_{1,p}, b_{2,p})$ provide evidence of the validity of the assumptions made about transformations of these measures, Table 1 gives details of the results for the case

Table 1 and Figure 1 here

§4 Comparison of the Omnibus Test Procedures

Initially the adequacy of the χ_2^2 distribution as an approximation to the null distributions of the statistics K_L^2, K_N^2 and K_C^2 was examined using Monte Carlo methods. From Table 2, which gives the probabilities that a χ_2^2 variable takes a value less than the simulation percentage point for each test statistic and different values of p, n and α, it can be seen that, in general, the χ_2^2 distribution provides a better approximation to the null distribution of K_L^2 than that of K_N^2. This is to be expected since, from Table 1, it is clear that $X_L(b_{1,p})$ is closer to normality than $U(b_{1,p})$. For larger n the χ_2^2 approximation to the null distribution of K_C^2 appears reasonable, the missing values in the table indicating its total unsuitability for small n due to the asymptotic nature of the expressions (2.4) and (2.5).

Table 2 here

Using a range of both skew and symmetric non-normal populations the performances of $b_{1,p}$ $b_{2,p}$ and the four omnibus tests were compared for $p \leq 4$ and $n \leq 100$, the tests being applied at the 5% level. Looking at the results for the case $p = 2$ it is apparent that all the tests perform well against clearly non-normal alternatives, even for small n, and consistently badly against those alternatives which are very similar to the standard multinormal distribution. Table 3 gives the results for two such populations when $p = 2$. For the remaining symmetric alternatives

Table 3 here

$p = 2$. For comparison the true values of the mean and variance of $b_{1,2}$ and $b_{2,2}$ are given; the agreement being quite satisfactory except for $Var(b_{2,2})$. Notice also that the approximate normality assumption is supported for $X_L(b_{1,p})$. The 95% confidence regions for $b_{1,p}$, $b_{2,p}$ and the four omnibus tests with $p = 2$, $n = 75$ are illustrated in Figure 1. Also included are the $(b_{1,2}, b_{2,2})$ values for 50 observations from a $N_2(0, I)$ population, and it can be seen that either two or three of these fall outside each region, the number to be expected if the χ_2^2 assumption is correct.

$b_{2,p}$ proved to be the most powerful test, in general, whilst the rectangle test was the best of the omnibus tests. The K_C^2 test statistic performed better than either K_L^2 or K_N^2, which appeared very similar, and $b_{1,p}$, as would be expected, was consistently poor against such alternatives. For the outstanding asymmetric alternatives K_C^2 appears to be the most powerful of all the tests when an appreciable difference in power, not accountable to Monte Carlo error, occurs. The power of the rectangle test lies between those of $b_{1,p}$ and $b_{2,p}$ in all cases, as one would expect, $b_{1,p}$ being more powerful against skew alternatives. The performances of both K_L^2 and K_N^2 are again similar and not as good as that for the rectangle test.

For larger values of p the results appear similar to those for $p = 2$, the ordering of the tests by power being, in general, consistent. The distributional assumption made about $b_{2,p}$ appears to be poor for smaller values of n since the estimated significance level in this case is considerably larger than the nominal level. As noted in connection with Table 2 the use of K_C^2 is hampered by the essentially asymptotic nature of the results used in its derivation.

§5 Conclusions

The results outlined in the previous section suggest that, of Mardia's two measures, $b_{2,p}$ is preferable for use as a test of normality whilst there appears to be justification for using both the test with rectangular contours and K_C^2 from the omnibus tests. K_C^2 would seem to be especially useful against suspected asymmetric alternatives. At present the use of this test statistic is constrained by the asymptotic nature of the results (2.4) and (2.5), especially the variance of $b_{2,p}$, but with the small samples, or the use of the Monte Carlo results based on a large scale study these would be lifted.

The results of a study of some tests for multinormality by Giorgi and Fattorini (1976) suggest that $b_{1,p}$ and $b_{2,p}$ should be used to test the null hypothesis in large samples. Consequently it is to be hoped that the performances of both R and K_C^2 will also compare favourably with those for other test statistics. This is currently under investigation.

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Sample size n	Variable	μ'	σ^2	$\sqrt{\beta_1}$	β_2
25	$b_{1,2}$	.778241 (.96)	.412093 (.4608)	2.20987	11.5412
	$X_L(b_{1,2})$	-.29467	.909092	.588772	3.19526
	$U(b_{1,2})$	-.267756	.894299	2.20987	11.5412
	$b_{2,2}$	7.35651 (7.385)	1.38469 (2.56)	1.22079	5.285
	$U(b_{2,2})$	.014483	.540895	1.22079	5.285
50	$b_{1,2}$	.433125 (.48)	.105312 (.1152)	1.64509	7.46811
	$X_L(b_{1,2})$	-.141635	.92093	.381379	2.7323
	$U(b_{1,2})$	-.138106	.914164	1.64509	7.46811
	$b_{2,2}$	7.65841 (7.686)	.835322 (1.28)	.927695	4.61619
	$U(b_{2,2})$	-.013309	.652595	.927695	4.61619
75	$b_{1,2}$	.301664 (.32)	.05015 (.0512)	1.44319	5.64766
	$X_L(b_{1,2})$	-.085889	.956789	.357133	2.6245
	$U(b_{1,2})$	-.081033	.9795	1.44319	5.64766
	$b_{2,2}$	7.78923 (7.789)	.648863 (.853)	.741365	3.60901
	$U(b_{2,2})$	.005891	.760387	.741365	3.60901
100	$b_{1,2}$	.227339 (.24)	.028463 (.0288)	1.62016	6.8309
	$X_L(b_{1,2})$	-.075568	.937786	.371659	2.78157
	$U(b_{1,2})$	-.074606	.98831	1.62016	6.8309
	$b_{2,2}$	7.85574 (7.842)	.549768 (.64)	.753997	4.07686
	$U(b_{2,2})$	.021694	.859012	.753997	4.07686

Table 1 Moments of $b_{1,2}$, $b_{2,2}$ and Related Variates Based on 2000 Simulations

Parenthetic entries refer to theoretical values of the moment parameters.

p	n	α / T	.01	.05	.1	.5	.9	.95	.99
χ_2^2 percentage points			.0201	.103	.2107	1.386	4.61	5.99	9.21
2	25	K_L^2	.0051	.0296	.0565	.3686	.8113	.8833	.9901
		K_N^2	.0052	.0296	.0638	.3403	.7133	.8695	.9979
		K_C^2	-	-	-	-	-	-	-
	50	K_L^2	.0077	.0391	.0728	.416	.8436	.929	.9971
		K_N^2	.0081	.0361	.0786	.3893	.7996	.9533	.9997
		K_C^2	.0091	.0567	.1118	.4527	.9077	.9798	.9999
	75	K_L^2	.0096	.0415	.0866	.4202	.8435	.9064	.9965
		K_N^2	.0084	.0385	.0792	.3957	.8056	.9226	.9996
		K_C^2	.0106	.0491	.0959	.4316	.8643	.9551	.9994
	100	K_L^2	.011	.0439	.0836	.4462	.8475	.9097	.993
		K_N^2	.0118	.0441	.0843	.4166	.8103	.9152	.9995
		K_C^2	.0146	.0528	.0995	.4291	.838	.9389	.9986
3	25	K_L^2	.007	.0263	.0505	.3733	.8644	.9282	.9896
		K_N^2	.0067	.0313	.0574	.3682	.7757	.8697	.9938
		K_C^2	-	-	-	-	-	-	-
	50	K_L^2	.0061	.0348	.069	.4018	.881	.9465	.9957
		K_N^2	.0054	.033	.068	.391	.8218	.9181	.9997
		K_C^2	-	-	-	-	-	-	-
	75	K_L^2	.0083	.0411	.0849	.4603	.8829	.9522	.9946
		K_N^2	.007	.0407	.0861	.4379	.845	.9341	.9991
		K_C^2	.015	.0627	.1269	.5772	.967	.9914	.9998
	100	K_L^2	.0056	.0379	.0821	.4593	.8956	.9489	.9945
		K_N^2	.0085	.0381	.0789	.4414	.8673	.9566	.9986
		K_C^2	.01	.0536	.1052	.5237	.9336	.9828	.9997

Table 2 Probabilities of χ_2^2 Variables Taking Values less than Empirical Percentage Points

Sample Size	Test Population	$b_{1,2}$	$b_{2,2}$	R	K_L	K_N	K_C
25	1	.812	.676	.766	.788	.816	.046
	2	.03	.142	.054	.018	.014	0
50	1	.986	.94	.972	.984	.988	.98
	2	.046	.1	.026	.042	.04	.056
75	1	.996	.992	.996	1.0	1.0	.996
	2	.032	.058	.026	.028	.03	.042
100	1	1.0	1.0	1.0	1.0	1.0	1.0
	2	.05	.07	.044	.048	.05	.053

Population 1 : $N_2((5,5)',I)$: Population 2 : $N_2(0,\begin{bmatrix} 1 & 0.5 \\ 0.5 & 1 \end{bmatrix})$

Table 3 : Empirical Power of Omnibus Tests Against Two Non-normal Alternatives

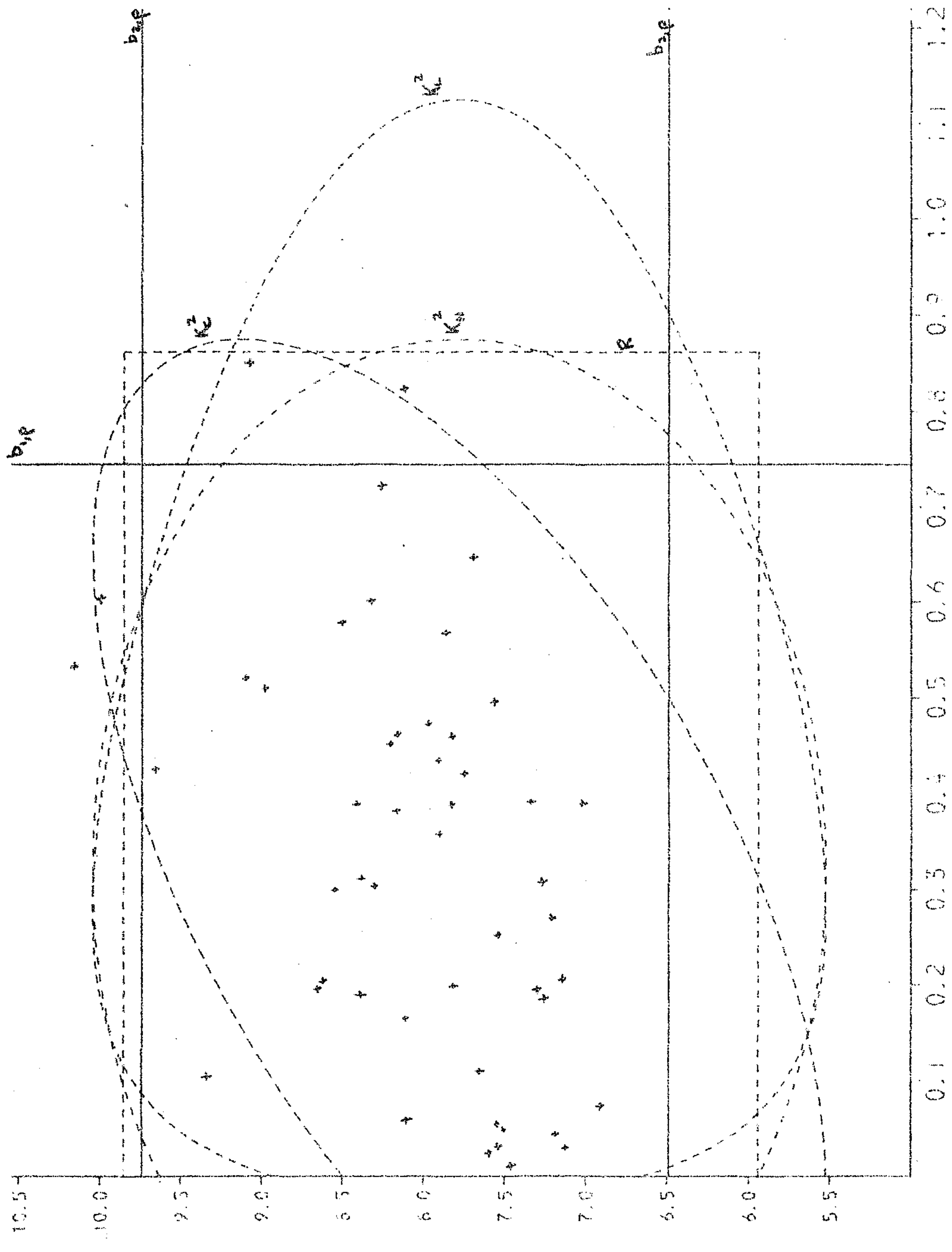

Figure 1. Confidence Regions for Omnibus Tests, $p = 2$, $n = 75$.