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by

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SUMMARY

The paper reviews new directional distributions motivated by applications since Mardia (1975 a,b). The major new developments have taken place in three broad topics. Firstly, in modelling data which is clustered around a small circle on a sphere, and secondly in defining appropriate multivariate directional distributions and thirdly, weighted distributions on a rotating sphere. Various other families of distributions have also appeared. In addition, various distributions on generalized spaces such as cylindrical and shape distributions have been constructed. These distributions will be discussed in the context of their practical relevance to various areas of scientific application.

1. INTRODUCTION

Let θ be a circular random variable. The most important distribution on a circle is von Mises whose p.d.f. is

$$f(\theta) = \{2\pi I_o(\kappa)\}^{-1} \exp\{\kappa \cos(\theta-\mu_o)\}, \; 0<\theta\leqslant 2\pi, \; 0<\mu_o\leqslant 2\pi, \kappa>0.$$

We will say θ is $M(\mu_o,\kappa)$. Let θ and ϕ be colatitude and longitude respectively. The Fisher distribution is important on the sphere and it has p.d.f. given by $f(\theta,\phi) = \{\kappa/4\pi\sinh\kappa\} \exp[\kappa\{\cos\mu_o\cos\theta + \sin\mu_o\sin\theta$

$$\times \cos(\phi-\nu_o)\}]\sin\theta,$$

$$0<\theta\leq\pi, \quad 0<\phi\leq 2\pi, \kappa>0.$$

These distributions and other important distributions such as Bingham's axial distribution, Down's distribution on a Stiefel manifold are defined and reviewed in Mardia (1972) and Mardia (1975 a,b). Since these papers, Bingham (1974), Khatri and Mardia (1977), Mardia and Khatri (1977), Mardia and Zemroch (1977), Jupp and Mardia (1979) give their further properties and results. However, we will concentrate on the new distributions which have since then appeared from practical considerations in directional statistics. Here, directional statistics stands for any distribution on non-Euclidean space.

2. SMALL CIRCLE DISTRIBUTIONS

Often there is the need to model a small circle distribution on the sphere, that is, observations are concentrated near a parallel of latitude relative to $\underset{\sim}{\mu}$, the axis of symmetry. Interest in the problem arose from some data in the field of plate tectonics, (see Mardia and Gadsden, 1977).

2.1. Fitting a small circle by minimization.

Let $\underset{\sim}{x}_i' = (x_i, y_i, z_i)$, i=1, ..., n be n observations on the unit sphere, and let the small circle be defined by

$$\lambda x + \mu y + \nu z = \cos\alpha, \quad \underset{\sim}{x}'\underset{\sim}{x} = 1, \quad \underset{\sim}{\mu}'\underset{\sim}{\mu} = 1, \quad \underset{\sim}{\mu}' = (\lambda,\mu,\nu). \tag{2.1}$$

By minimizing the sum of the squared deviations

$$V = 1 - \frac{\cos\alpha}{n}\sum_{i=1}^{n} \underset{\sim}{\mu}'\underset{\sim}{x}_i - \frac{\sin\alpha}{n}\sum_{i=1}^{n} \{1 - (\underset{\sim}{\mu}'\underset{\sim}{x}_i)^2\}^{\frac{1}{2}}, \tag{2.2}$$

Mardia and Gadsden (1977) show that $\hat{\alpha}$ and $\hat{\underset{\sim}{\mu}}$ are solutions of the equations

$$\sum_{i=1}^{n} \underset{\sim}{\mu}' \underset{\sim}{x}_i = \frac{\cos\alpha}{\sin\alpha} \sum_{i=1}^{n} \{1-(\underset{\sim}{\mu}'\underset{\sim}{x}_i)^2\}^{\frac{1}{2}}, \tag{2.3}$$

$$\cos\alpha \sum_{i=1}^{n} \underset{\sim}{x}_i = \sin\alpha \sum_{i=1}^{n} \underset{\sim}{x}_i' \underset{\sim}{\mu}' \underset{\sim}{x}_i \{1-(\underset{\sim}{\mu}' \underset{\sim}{x}_i)^2\}^{\frac{1}{2}} - \rho\underset{\sim}{\mu}, \tag{2.4}$$

where ρ is a Lagrange multiplier introduced for the constraint $\underset{\sim}{\mu}'\underset{\sim}{\mu}=1$. (2.3) and (2.4) have to be solved numerically. An iterative method is given in Mardia and Gadsden (1977).

Let $F(\underset{\sim}{\mu}, \kappa)$ denote the Fisher distribution with mean directional vector $\underset{\sim}{\mu}$ and concentration parameter κ. Consider the model

$$\underset{\sim}{x}_i \sim F\{\underset{\sim}{\mu}(\phi_i), \kappa\}, \quad i=1, \ldots, n,$$

where

$$\underset{\sim}{\mu}(\phi_i) = \underset{\sim}{\Gamma}(\cos\alpha, \ \sin\alpha \ \sin\phi_i, \ \sin\alpha \ \cos\phi_i)'$$

with $\underset{\sim}{\Gamma}$ as an orthogonal matrix with the first column of $\underset{\sim}{\Gamma}$ as $\underset{\sim}{\mu}$. Now the least square estimators of $\underset{\sim}{\mu}$ and α are the same as the maximum likelihood estimators for the model. Note that the m.l.e. of ϕ_i is ϕ_i^* which is the longitude for $\underset{\sim}{z}_i = \hat{\underset{\sim}{\Gamma}} \underset{\sim}{x}_i$, $i=1, \ldots, n$. The model is still under investigation.

2.2 <u>Maximum Entropy Small Circle Distribution.</u>

Let θ and ϕ denote colatitude and longitude. From maximum entropy considerations, Mardia and Gadsden (1977) proposed the following small circle distribution with probability element (p.e.) of θ, ϕ as

$$dF(\theta,\phi) = C(\kappa, \alpha) \exp\{\kappa \cos(\theta-\alpha)\} \sin\theta \, d\theta d\phi, \quad 0\leqslant\theta\leqslant\pi, \ 0\leqslant\phi\leqslant 2\pi \tag{2.5}$$

when the pole is given by the z-axis. In terms of direction cosines

$\underset{\sim}{x}$, the p.e.

$$= C(\kappa,\alpha) \exp\left(\kappa\left[\beta(\underset{\sim}{\mu}'\underset{\sim}{x}) + \gamma\{1-(\underset{\sim}{\mu}'\underset{\sim}{x})^2\}^{\frac{1}{2}}\right]\right)dS, \tag{2.6}$$

where $\beta=\cos\alpha$, $\gamma=\sin\alpha$ and dS is the uniform measure. The normalising constant is given by

$$C(\kappa,\alpha)^{-1} = 2\pi \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (\kappa\beta)^{2i} (\kappa\gamma)^{j} \, B(i+\tfrac{1}{2}, \tfrac{1}{2}j+1)\{(2i)!\ j!\}^{-1}. \tag{2.7}$$

For $\alpha=0$, (2.5) reduces to the Fisher distribution and for $\alpha = \pi/2$ we have

$$dF(\theta,\phi) = C \exp(\kappa\sin\theta)\sin\theta\, d\theta d\phi,$$

a girdle distribution which has been investigated by Selby (1964). For $\alpha\neq 0$, the distribution has a dimple at the pole.

Maximum likelihood estimation in general is complicated, but simplifies for large κ. For given $\underset{\sim}{\mu}$ and α we have

$$\hat{\kappa}^{-1} = 2V + 3V^2, \tag{2.8}$$

where V is defined in equation (2.2).

Mardia and Gadsden (1977) analyse two sets of small-circle data: (a) 66 hot spots and areas of vulcanism in the North Pacific, known as the Hawaiian Trend and (b) 15 data points as to the local directions of the earth's magnetic field in Australia about 100 million years ago.

2.3 Generalized Dimroth-Watson Small Circle Distribution.

Because of the intractibility of (2.6), Bingham and Mardia (1978) proposed the distribution with p.e.

$$f(\underset{\sim}{x};\ \tau,\ \nu,\ \underset{\sim}{\mu})\ \frac{dS}{4\pi} = C(\tau,\ \nu)^{-1} \exp\{-\tau(\underset{\sim}{\mu}'\underset{\sim}{x}-\nu)^2\}\ \frac{dS}{4\pi}, \tag{2.9}$$

$$0\leq\nu<\infty, \quad -\infty<\tau<\infty, \quad \underset{\sim}{\mu}'\underset{\sim}{\mu}=1, \quad \underset{\sim}{x}'\underset{\sim}{x}=1.$$

For $\nu=0$, (2.9) becomes the Dimroth-Watson distribution, while for $\tau\to 0$ such that $2\tau\nu\to\kappa$, it becomes the Fisher distribution. Let $P(\theta) = -\tau(\cos\theta-\nu)^2$. When $\nu\geq 1$ and $\tau>0$, $P(\theta)$ has a maximum at $\theta=0$ and a minimum at $\theta=\pi$ and no other extrema. When $\nu\geq 1$ and $\tau<0$, the extrema are reversed. Thus when $\nu\geq 1$ the distribution is of polar type concentrated about $\underset{\sim}{\mu}(\tau>0)$, $-\underset{\sim}{\mu}(\tau<0)$. When $0<\nu<1$, $\tau>0$, we have a small circle distribution, because $P(\theta)$ has a maximum at $\theta = \alpha \equiv \arccos \nu$ and minima at $\theta=0$ and $\theta=\pi$. When $0<\nu<1$, $\tau<0$, there is a minimum at $\theta=\alpha$ and maxima at $\theta=0$ and $\theta=\pi$.

The normalizing constant in (2.9) is given by

$$C(\tau,\nu) = \tfrac{1}{2}(1-\nu)\,{}_1F_1\{\tfrac{1}{2};\tfrac{3}{2};-\tau(1-\nu)^2\}+\tfrac{1}{2}(1+\nu)\,{}_1F_1\{\tfrac{1}{2};\tfrac{3}{2};-\tau(1+\nu)^2\}, \quad (2.10)$$

where ${}_1F_1$ is a confluent hypergeometric function.

Bingham and Mardia (1978) give an iterative method of finding the maximum likelihood estimators for $0<\nu<1$ and $\tau>0$, the case of interest. Define S_1, S_2, and $\underset{\sim}{S}$ by

$$\left.\begin{aligned} S_1 &= \underset{\sim}{\mu}'\underset{\sim}{x}, \quad S_2 = n^{-1}\sum_{i=1}^{n}(\underset{\sim}{\mu}'\underset{\sim}{x}_i-S_1)^2, \\ \underset{\sim}{S} &= n^{-1}\sum_{j=1}^{n}\underset{\sim}{x}_j\underset{\sim}{x}_j' - \bar{\underset{\sim}{x}}\,\bar{\underset{\sim}{x}}', \end{aligned}\right\} \quad (2.11)$$

and let $t_1\geq t_2\geq t_3$ be the eigenvalues of $\underset{\sim}{S}$ with corresponding eigenvectors $\underset{\sim}{u}_1$, $\underset{\sim}{u}_2$, $\underset{\sim}{u}_3$.

When $6t_3 < (1-\underset{\sim}{u}_3'\bar{\underset{\sim}{x}})^2$

$$\text{or } \frac{S_2}{(1-S_1)^2} < \frac{1}{6},$$

approximate maximum likelihood estimates are given by

$$\hat{\underset{\sim}{\mu}} = \underset{\sim}{u}_3, \quad \hat{\nu} = \underset{\sim}{u}_3'\bar{\underset{\sim}{x}}, \quad \hat{\tau} = (2t_3)^{-1}. \quad (2.12)$$

Bingham and Mardia (1978) give various tests of hypothesis and confidence regions when τ is large.

2.4 Rotated Fisher Small Circle.

In studying the arrival directions of cosmic rays it is necessary to model a small-circle distribution (Edwards, 1980). A suitable model for the z-axis as the pole is given by

$$f(\theta,\phi) = \frac{\kappa}{4\pi\sinh\kappa} e^{\kappa\cos\alpha\cos\theta} I_o(\kappa\sin\alpha\sin\theta)\sin\theta, \qquad (2.13)$$

The model is derived by rotating a Fisher distribution around a small circle of constant colatitude, $\theta=\alpha$. For general pole, $\underset{\sim}{\mu}$,

$$f(\underset{\sim}{\ell}) = \frac{\kappa}{4\pi\sinh\kappa} e^{\kappa\cos\alpha (\underset{\sim}{\ell}'\underset{\sim}{\mu})} I_o\left[\kappa\sin\alpha\{1-(\underset{\sim}{\ell}'\underset{\sim}{\mu})^2\}^{\frac{1}{2}}\right] \qquad (2.14)$$

(2.13), unlike (2.9) and (2.5), is not in the exponential family, however the normalizing constant is simple and does not depend on α. Note the close comparison between (2.6) and (2.14).

For known $\underset{\sim}{\mu}$, maximum likelihood estimates of α and κ can be easily found.

3. MULTIVARIATE DIRECTIONAL DISTRIBUTIONS

From maximum entropy considerations, Mardia (1975a) constructed a suitable distribution when two unit random vectors $\underset{\sim}{\ell}_1$ and $\underset{\sim}{\ell}_2$ are correlated, of the form,

$$\text{const} \times \exp\{\underset{\sim}{a}_1{}'\ \underset{\sim}{\ell}_1 + \underset{\sim}{a}_2{}'\ \underset{\sim}{\ell}_2 + \text{tr}\ \underset{\sim}{A}\ \underset{\sim}{\ell}_1\ \underset{\sim}{\ell}_2{}'\},\ \underset{\sim}{\ell}_1,\ \underset{\sim}{\ell}_2 \in S_p, \qquad (3.1)$$

i.e. a family of bivariate von Mises-Fisher distributions. Kent (1979) explores a particular case which has ellipse-shaped probability contours about the pole. Mardia (1979) discusses general problems of testing dependence. The marginal distributions of $\underset{\sim}{\ell}_1$, $\underset{\sim}{\ell}_2$ however, are not of the von Mises-Fisher form except for trivial cases.

Mardia (1975) in the author's reply also introduced a family of bivariate matrix Bingham-von Mises-Fisher distributions. For $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ on the Stiefel Manifolds $V_m(R^p)$ and $V_n(R^q)$, the density of $(\underset{\sim}{X}, \underset{\sim}{Y})$ has the form

$$C \exp \{tr(\underset{\sim}{F}\underset{\sim}{X}) + tr(\underset{\sim}{G}\underset{\sim}{Y}) + tr(\underset{\sim}{X}'\underset{\sim}{A}\underset{\sim}{X}\underset{\sim}{B}) + tr(\underset{\sim}{Y}'\underset{\sim}{S}\underset{\sim}{Y}\underset{\sim}{T}) + tr(\underset{\sim}{Y}'\underset{\sim}{C}\underset{\sim}{Y}\underset{\sim}{D})$$

$$+ tr(\underset{\sim}{X}'\underset{\sim}{S}\underset{\sim}{Y}\underset{\sim}{T}) + tr(\underset{\sim}{Y}'\underset{\sim}{U}\underset{\sim}{X}\underset{\sim}{V})\}, \qquad (3.2)$$

where the matrices are appropriately constrained. Jupp and Mardia (1980) consider the generalized exponential family of densities given by

$$\exp \{\alpha(\underset{\sim}{A}, \underset{\sim}{t}_1) + \underset{\sim}{t}_1'\underset{\sim}{A}\,\underset{\sim}{t}_2 + \beta(\underset{\sim}{A}, \underset{\sim}{t}_2) - \kappa(\underset{\sim}{A})\}, \qquad (3.3)$$

where $\underset{\sim}{t}_1 \equiv \underset{\sim}{t}_1(\underset{\sim}{X})$, $\underset{\sim}{t}_2 \equiv \underset{\sim}{t}_2(\underset{\sim}{X}\ \underset{\sim}{Y})$, and $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ take values on general Riemannian manifolds M and N. The interest lies in a correlation coefficient ρ^2, for bidirectional distributions and it is shown that if $\underset{\sim}{X}$ and $\underset{\sim}{Y}$ are independent on M and N, then under suitable regularity conditions,

$$\rho^2 = tr(\underset{\sim}{\Sigma}_{11}\ \underset{\sim}{A}\ \underset{\sim}{\Sigma}_{22}\ \underset{\sim}{A}') + O(||\underset{\sim}{A}||^3), \qquad (3.4)$$

where $\underset{\sim}{\Sigma}_{11}$ and $\underset{\sim}{\Sigma}_{22}$ are the covariance matrices of $\underset{\sim}{t}_1$ and $\underset{\sim}{t}_2$ for the distribution with $\underset{\sim}{A} = \underset{\sim}{0}$, and where $||\underset{\sim}{A}||^2 = tr(\underset{\sim}{A}\ \underset{\sim}{A}')$.

Both (3.1) and a sub-class of (3.2) are contained in the family defined by (3.3). By this approach, Jupp and Mardia (1980) obtain various important correlation coefficients including that of Mardia and Puri (1978). Further, it is shown that the likelihood ratio for the test of independence is asymptotically equivalent to the sample counterpart of ρ^2. Stephens (1979) gives spherical correlation coefficients following ideas of Mackenzie (1957) and Downs (1974).

4. MISCELLANEOUS CONTRIBUTIONS

4.1. Off-set normal distribution (Cairns-Fraser model).

Fraser (1979) and Cairns (1975) propose a model for directional data which allows for skewness and is called the projected normal distribution but it is really the off-set normal distribution with the parameters identified.

Consider the bivariate normal distribution with mean $(\lambda,0)'$ and covariance matrix $\underset{\sim}{I}$, the p.d.f. of which we denote $\phi_2\{\underset{\sim}{x};\ (\lambda,0)',\ \underset{\sim}{I}\}$. The p.d.f. of $\underset{\sim}{\ell}$ is given by

$$f(\underset{\sim}{\ell}) = \int_{r=o}^{\infty} \phi_2\{r\ell_1,\ r\ell_2;\ (\lambda,0)',\ \underset{\sim}{I}\}rdr, \tag{4.1}$$

where $\underset{\sim}{\ell}' = (\ell_1,\ \ell_2)$, $\quad \underset{\sim}{\ell}'\underset{\sim}{\ell} = 1$.

On performing the integration ,(4.1) becomes

$$f(\underset{\sim}{\ell}) = \frac{1}{\sqrt{2\pi}}\ \phi(\lambda) + \lambda\ell_1\Phi(\lambda\ell_1)\phi(\lambda\ell_2), \tag{4.2}$$

where Φ (x) is the distribution function and $\phi(x)$ the p.d.f. of a N(0,1) variable.

From the standard form given by (4.2), the vector $\underset{\sim}{\ell}$ is subjected to the matrix $\underset{\sim}{A}$ defined by

$$\underset{\sim}{A} \equiv \underset{\sim}{A}_1\ \underset{\sim}{A}_2\ \underset{\sim}{A}_3 = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} 1 & \tau \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix}, \tag{4.3}$$

to give the generalized variable on the unit circle,

$$\underset{\sim}{A}\underset{\sim}{\ell} / ||\underset{\sim}{A}\underset{\sim}{\ell}||. \tag{4.4}$$

Note that $\underset{\sim}{A}_1$ is a rotation matrix, $\underset{\sim}{A}_2$ skews the plane parallel to the first axis and $\underset{\sim}{A}_3$ scales the first axis by σ and the second axis by

σ^{-1}. Thus it is the distribution of $\theta|r=1$ from $N_2\left[\underset{\sim}{A}\binom{\lambda}{0}, \underset{\sim}{A}\ \underset{\sim}{A}'\right]$. Fraser (1979) fits the projected normal distribution to the famous Turtle data, obtaining maximum likelihood estimates numerically.

This work is related to the offset Normal distribution of Mardia (1972) and Saw (1978).

4.2 Weighted Distributions.

A problem in Cosmic Rays had led us to explore weighted distributions on the sphere.* Using the terminology of Rao (1965), a random variable x has a probability of being observed proportional to u(x), where x has true underlying p.d.f. g(x), the observed p.d.f. of x being defined by

$$f(x) = \frac{u(x)g(x)}{\int u(x)g(x)dx} . \tag{4.5}$$

We consider the case where the weights depend only on one variable, i.e. $f(\theta,\phi) = C\ u(\theta)\ g(\theta,\phi)$. Assuming $u(\theta)$ can be approximately written in the form of a step function, closed forms for the maximum likelihood estimators can be found for small concentration, as well as approximations to the likelihood ratio tests of uniformity against a Fisher or Dimroth-Watson alternative. The case of truncation is a special case of equal weights on part of the sphere and zero weights elsewhere. Unweighted variables simply have equal weights on the complete sphere.

Let the weighted Fisher distribution be defined by

$$f_\kappa(\theta,\phi) = \frac{1}{2\pi C(\kappa,\mu_o)} \quad u_j \ e^{\kappa\{\cos\mu_o\cos\theta + \sin\mu_o\sin\theta\cos(\phi-\nu_o\}}\sin\theta,$$

$$\theta_{j-1}<\theta<\theta_j,\ 0<\phi<2\pi, \quad j=1, \quad \ldots, m, \tag{4.6}$$

where u_j is constant over the range (θ_{j-1}, θ_j). Asymptotically the likelihood ratio test of uniformity,

*Work with Rob Edwards in sections 3.2, 3.3, 3.4, (see Edwards, 1980).

$$f_o(\theta,\phi) = \frac{u_j \sin\theta}{2\pi \sum_{j=1}^{m} u_j\{\cos\theta_{j-1} - \cos\theta_j\}}, \qquad (4.7)$$

against a Fisher alternative reduces to

$$n\bar{R}^2 = n\left[\frac{\bar{R}_x^{\,2}}{\mathrm{var}\,\bar{R}_x} + \frac{\bar{R}_y^{\,2}}{\mathrm{var}\,\bar{R}_y} + \frac{\{\bar{R}_z - E(\bar{R}_z)\}^2}{\mathrm{var}\,\bar{R}_z}\right] \sim \chi^2{}_3, \qquad (4.8)$$

where $\bar{R}_x = \frac{1}{n}\sum_{i=1}^{n} \sin\theta_i \cos\phi_i$,

$$\bar{R}_y = \frac{1}{n}\sum_{i=1}^{n} \sin\theta_i \sin\phi_i,$$

$$\bar{R}_z = \frac{1}{n}\sum_{i=1}^{n} \cos\theta_i,$$

$$E(\bar{R}_z) = \frac{D_2(u)}{4D_1(u)},$$

$$\mathrm{var}\,(\bar{R}_x) = \mathrm{var}\,(\bar{R}_y) = \frac{1}{24n}\{9 - \frac{D_3(u)}{D_1(u)}\},$$

$$\mathrm{var}\,(\bar{R}_z) = \frac{1}{48n}\{12 + \frac{4D_3(u)}{D_1(u)} - \frac{3D_2{}^2(u)}{D_1{}^2(u)}\},$$

$$D_k(u) = \sum_{j=1}^{m} u_j\{\cos k\theta_{j-1} - \cos k\theta_j\},\ k=1,2,\ldots$$

Similar results can be found for a Dimroth-Watson distribution.

4.3. Some Conditional Forms.

The inference for weighted distributions can be based on conditional distributions of $\phi|\theta$ if weights are known. Downs (1966) investigated the distribution on the sphere of the sub-vector (ℓ,m) constrained to lie on the unit circle. For the Fisher distribution this is equivalent to the conditional distribution of $\phi|\theta$, namely

$$f(\phi|\theta) = \frac{1}{2\pi I_o(\kappa\sin\mu_o\sin\theta)} \exp\{\kappa\sin\mu_o\sin\theta\cos(\phi-\nu_o)\}. \qquad (4.9)$$

Maximum likelihood estimation is possible only of ν_o and $\kappa\sin\mu_o$.

The Dimroth-Watson distribution gives

$$g(\phi|\theta) = \frac{1}{2\pi} \; C(\kappa,\mu,\theta) \; \exp\Big[-\frac{\kappa}{2}\{\sin^2 \mu_o\sin^2\theta\cos^2(\theta-\nu_o)$$

$$+ \sin 2\mu_o\sin 2\theta \cos(\phi-\nu_o)\}\Big], \qquad (4.10)$$

where $\{C(\kappa,\mu,\theta)\}^{-1} = I_o(-\frac{\kappa}{2}\sin^2 \mu_o\sin^2\theta)I_o(-\frac{\kappa}{2}\sin 2\mu_o\sin 2\theta)$

$$+ 2 \sum_{r=1}^{\infty} I_r(-\frac{\kappa}{2}\sin^2\mu_o\sin^2\theta)\; I_{2r}(-\frac{\kappa}{2}\sin 2\mu_o\sin 2\theta).$$

Unlike the Fisher distribution, provided $\mu_o \neq 0$, maximum likelihood estimators of all 3 parameters can be found.

4.4 Distribution on a Truncated Sphere from Uniformly Rotating Caps with Uniform Arrivals.

Suppose that (μ_o, ν_o) are the coordinates of a recording station at a particular time on the earth, i.e. μ_o is latitude and fixed whereas ν_o is an instant of time, i.e. we can take $0<\nu_o\leq 2\pi$ for a day. In various phenomena such as in high energy particles, we can record observations only C_o degrees from μ_o. Let

$$\alpha_1 = \min(C_o-\mu_o, \mu_o+C_o), \; \alpha_2 = \max(C_o-\mu_o, \mu_o+C_o).$$

We have assumed $0<C_o<\pi/2$, $0<\mu_o<\pi/2$ without any loss of generality.

The equation of the small circle of the boundary of the cap can be seen as

$$\cos(\phi-\nu_o) = (\sin\mu_o\sin\theta)^{-1}\cos C_o - \cot\mu_o\cot\theta. \qquad (4.11)$$

Above this, the observations will be restricted to this cap. Hence, if there is a uniform distribution on the cap, we have

$$f(\theta,\phi|\mu_o,\nu_o) = C \sin\theta, \qquad 0<\theta<\alpha_1, \quad 0<\phi<2\pi,$$

$$= C \sin\theta, \qquad \alpha_1<\theta<\alpha_2, \quad \nu_o-a(\theta)<\phi<\nu_o+a(\theta), \quad (4.12)$$

where

$$a(\theta) = \cos^{-1}\{(\sin\mu_o \sin\theta)^{-1} \cos C_o - \cot\mu_o \cot\theta\}.$$

Note that the zone $0<\theta<\alpha_1$, $0<\phi<2\pi$ is the intersection of the caps with respect to rotation along the north axis.

The distribution of uniformly rotating caps is

$$g_{\mu_o}(\theta,\phi|\mu_o) = \int_0^{2\pi} f(\theta,\phi|\mu_o,\nu_o)\, d\nu o.$$

It is found that

$$g(\theta,\phi|\mu_o) = 2\pi C \sin\theta, \qquad 0<\theta<\alpha_1,$$

$$= 2Ca(\theta)\sin\theta, \qquad \alpha_1<\theta<\alpha_2,$$

with $C = 1/\{2\pi(1-\cos C_o)\}$.

As expected ϕ is uniformly distributed, and the marginal p.d.f., $h(\theta,\mu_o)$, of θ can easily be written down. These provide u_j's for (3.6) above.

This idea can be extended to other distributions, such as the Fisher distribution.

5. FAMILIES OF DISTRIBUTIONS

5.1. Johnson and Wehrly's Bivariate Models.

As we have seen, the maximum entropy densities do not in general lead to marginal distributions of types commonly used in directional data. To derive bivariate distributions with given marginals, Johnson and Wehrly (1978), Wehrly and Johnson (1980) use the following construction.

Let f_1 (θ) and f_2 (ϕ) be specified densities on the circle and F_1 (θ) and F_2 (ϕ) be their distribution functions defined with respect to fixed, arbitrary origins. Also let g(.) be a density on the circle. Then

$$f(\theta,\phi) = 2\pi g\left[2\pi\{F_1(\theta) \pm F_2(\phi)\}\right] f_1(\theta) f_2(\phi), \tag{5.1}$$

where $0 \leq \theta, \phi < 2\pi$ are densities on the torus having the specified marginal densities $f_1(\theta)$ and $f_2(\phi)$.

These distributions also lead to a family of distributions for a Markov process.

5.2 Generalised Exponential Models.

Beran (1979) proposes as exponential models for directional data,

$$f_h(\underset{\sim}{x}) = \exp\{h(\underset{\sim}{x}) - d(h)\}, \qquad \underset{\sim}{x} \in S_p, \quad h \in M, \tag{5.2}$$

where M is a subspace of $C(S_p)$, the set of all real-valued continuous functions whose domain is S_p, which is invariant under every rotation g in R^p such that $h(.) \in M$ entails $h(g) \in M$. d(h) is chosen to make f_h integrate to one. Both the von Mises-Fisher and the Bingham distributions are included as special cases.

In canonical exponential form (5.2) can be written in the form

$$f_{\underset{\sim}{\beta}}(\underset{\sim}{x}) = \exp\{\underset{\sim}{\beta}'\underset{\sim}{v}(\underset{\sim}{x}) - C(\underset{\sim}{\beta})\}, \quad \underset{\sim}{\beta} \in R^q, \ \underset{\sim}{x} \in \underset{\sim}{S}_p, \tag{5.3}$$

where the $\{v_i : 1 \leq i \leq q\}$ are functions in $C(S_p)$ such that $\{1, v_1(\underset{\sim}{x}), \ldots, v_q(\underset{\sim}{x})\}$ are linearly independent and $\underset{\sim}{v}(\underset{\sim}{x})$ is the vector $\left[v_1(\underset{\sim}{x}), \ldots, v_q(\underset{\sim}{x})\right]'$ and $\underset{\sim}{\beta} = \left[\beta_1, \ldots, \beta_q\right]'$.

Beran discusses both maximum likelihood estimation and also a regression estimator. Briefly, it represents on the circle a model such as

$$\exp\{\sum_{i=1}^{k}(\alpha_i \cos i\theta + \beta_i \sin i\theta) - C(\underset{\sim}{\alpha},\underset{\sim}{\beta})\}.$$

However, the problem is complicated on the sphere.

5.3 Additive Family.

Suppose that θ_1, θ_2, θ_3 are independently distributed.

Let $\theta = (\theta_1+\theta_2) \bmod 2\pi$,

$\phi = (\theta_1+\theta_3) \bmod 2\pi$.

An important member is the bivariate von Mises with $\theta_i \sim M(\mu_i, \kappa_i)$,

$i = 1, 2, 3$. Using the bivariate normal case, we can define the correlation between θ and ϕ as

$$\rho = \sigma_1^2/\{(\sigma_1^2+\sigma_2^2)(\sigma_1^2+\sigma_3^2)\}^{\frac{1}{2}},$$

where

$$\sigma_i^2 = A(\kappa_i).$$

Another member is the wrapped normal where the marginals are again wrapped normal. For large κ_1, κ_2 and κ_3, we have the bivariate normal situation. These have been used by Holmes (1980) in various simulation studies.

5.4 Precision Family.

A family which works for Euclidean as well as spherical space is given by

$$\exp\{\kappa\ \phi\ \frac{(x-\mu)}{\sigma} - u(\kappa)\},$$

where ϕ is a strictly concave function and κ is a precision parameter. Thus it contains normal, hyperbolic and von Mises distributions among others.

Let $\phi(x)$ be a concave function. From exponential family theory, $u(\kappa)$ is strictly convex. For $\sigma=1$, the maximum likelihood estimators of μ and κ are the solutions of

$$\sum_{i=1}^{n} \phi'(x_i-\hat{\mu}) = 0, \quad u'(\hat{\kappa}) = \frac{1}{n} \sum_{i=1}^{n} \phi(x_i-\hat{\mu}). \tag{5.4}$$

These are unique under the conditions given. Further, for large n, $\hat{\mu}$ and $\hat{\kappa}$ are independently distributed as $N(\mu, \frac{a}{n})$ and $N(\kappa, \frac{b}{n})$ where

$$a^{-1} = \kappa^2 \int\{\phi'(x-\mu)\}^2 f(x;\kappa,\mu)\,dx, \quad b^{-1} = u''(\kappa).$$

For higher dimensions, we have

$$\exp\left[\kappa\ \phi\{(\underset{\sim}{x}-\underset{\sim}{\mu})'\ \underset{\sim}{\Sigma}^{-1}(\underset{\sim}{x}-\underset{\sim}{\mu})\} - b(\underset{\sim}{\Sigma},\kappa)\right].$$

The Fisher and Bingham distributions are members of this class. One important member is

$$C(\kappa)\,|\underset{\sim}{\Sigma}|^{-\frac{1}{2}} \exp\left[-\frac{\kappa}{2}\{(\underset{\sim}{x}-\underset{\sim}{\mu})'\underset{\sim}{\Sigma}^{-1}(\underset{\sim}{x}-\underset{\sim}{\mu}) - 1\}^2\right], \quad \underset{\sim}{x}\in R^2, \ \kappa>0,$$

where $C(\kappa) = (\kappa/2\pi)^{\frac{1}{2}}/\{\pi\Phi(\kappa^{\frac{1}{2}})\}$, $\Phi(.)$ = d.f. of $N(0,1)$. This has been used in a critical analysis of megalithic data where elliptic pattern or circular pattern is suspected, (see Mardia and Holmes, 1980).

6. GENERALIZED SPACES

6.1 Distributions on a cylinder.

Mardia and Sutton (1978) proposed a model for cylindrical variables (x,θ), $-\infty<x<\infty$, $0<\theta\leq 2\pi$, applications of which are found in rhythmometry, medicine, demography, biology and climatology. For example, one measurement related to wind direction and another to ozone concentration in pollution. The model has p.d.f. given by

$$f(x,\theta) = \{2\pi\, I_o(\kappa)\}^{-1} \quad \exp\{\kappa \cos(\theta-\mu_o)\}$$

$$\times\ (2\pi\, \sigma_c^{\,2})^{-\frac{1}{2}} \quad \exp\left[-\{(x-\mu_c)^2/2\sigma_c^{\,2}\right], \qquad (6.1)$$

$\kappa>0$, $0<\mu_o<2\pi$, $I_o(\kappa)$ is the modified Bessel function of the first kind and order zero and

$$\mu_c = \mu + \sigma\kappa^{\frac{1}{2}}\{\rho_1(\cos\theta - \cos\mu_o) + \rho_2(\sin\theta - \sin\mu_o)\},$$

$$\sigma_c^{\,2} = \sigma^2(1-\rho^2) \text{ and } \rho = (\rho_1^{\,2}+\rho_2^{\,2})^{\frac{1}{2}}, \quad 0\leqslant\rho\leqslant1.$$

The marginal distribution of θ is von Mises with mean direction μ_o and concentration parameter κ, but the marginal distribution of x is complicated.

For $\rho = 0$, x and θ are independently distributed as $N(\mu, \sigma^2)$ and $M(\mu_o, \kappa)$ respectively. $\rho = 1$ implies perfect correlation.

Johnson and Wehrly (1977) proposed forming a model from considering a bivariate normal random variable (Y_1, Y_2) with means μ_1, μ_2, variances $\sigma_1^{\,2}$, $\sigma_2^{\,2}$ and covariance σ_{12}. Defining $\theta = Y_1$ (mod 2π), $X = Y_2$, (X,θ) has a characteristic function

$$\phi(p,t) = \exp\{-\tfrac{1}{2}(p^2\sigma_1^{\,2}+2pt\sigma_{12}+t^2\sigma_2^{\,2}) + i(p\mu_1+t\mu_2)\}, \qquad (6.2)$$

where p is an integer and t a real number.

The marginal distribution of x is normal and that of θ, wrapped normal.

6.2. Shape Distributions (Triangle).

Investigation of central place theory, (Mardia, et al, 1977), and Ley lines, (Kendall, D.G. and Kendall, W.S., 1980), lead to appropriate distributions on triangles.

Consider three points $\underset{\sim}{x}_1$, $\underset{\sim}{x}_2$, $\underset{\sim}{x}_3$ in R^2. Miles (1970) gives the almost sure distribution of a random Delaunay triangle formed from a Poisson process in the plane. Let α_1, α_2, α_3 be the interior angles of the triangle formed from $\underset{\sim}{x}_1$, $\underset{\sim}{x}_2$, $\underset{\sim}{x}_3$. Miles finds the asymptotic distribution of a random triangle to be given by

$$f(\alpha_1, \alpha_2) = \frac{8}{3\pi} \sin \alpha_1 \sin \alpha_2 \sin (\alpha_1+\alpha_2), \ \alpha_1>0, \ \alpha_2>0,$$

$$\alpha_1+\alpha_2<\pi. \qquad (6.3)$$

Mardia et al (1977) find the joint distribution of the interior angles of a triangle when θ_1, θ_2, θ_3 have independent distributions $M(\mu_j, \kappa)$, $j=1, 2, 3$, where the means are spread $2\pi/3$ apart. In this case ,

$$f(\alpha_1, \alpha_2) = \frac{1}{\pi^2 I_o{}^3(\kappa)} \left[I_o\left[\kappa\{3+2 \sum_{j=1}^{3} \cos (2\alpha_j-2\tfrac{\pi}{3})\}^{\frac{1}{2}}\right]\right.$$

$$\left. + I_o\left[\kappa\{3+2 \sum_{j=1}^{3} \cos (2\alpha_j+2\tfrac{\pi}{3})\}^{\frac{1}{2}}\right]\right], \ \alpha_3=\pi-\alpha_1-\alpha_2. \qquad (6.4)$$

If $\underset{\sim}{x}_1$, $\underset{\sim}{x}_2$, $\underset{\sim}{x}_3$ are independently normally distributed with common mean and covariance matrix $\underset{\sim}{\Sigma} = \begin{vmatrix} \sigma^2 & 0 \\ 0 & \sigma^2 \end{vmatrix}$, then the joint p.d.f. of $\alpha_1, \alpha_2, \alpha_3$, the interior angles of the triangle thus formed, is given by

$$f(\alpha_1, \alpha_2) = 6S(3-C)^{-2}, \qquad (6.5)$$

where $C = \sum_{j=1}^{3} \cos 2\alpha_j$, $\quad S = \sum_{j=1}^{3} \sin 2\alpha_j$.

If we let $\underset{\sim}{x}_1$, $\underset{\sim}{x}_2$, $\underset{\sim}{x}_3$ be independently normally distributed with means at the vertices of an equilateral triangle and common covariance matrix $\underset{\sim}{\Sigma}$, then

$$f(\alpha_1, \alpha_2) = \frac{3S}{4\pi(3-C)^2}\{1 + \frac{3}{4\tau^2}(1 + \frac{\sqrt{3S}}{3-C})\} \exp\{-\frac{3}{4\tau^2}(1 - \frac{\sqrt{3S}}{3-C})\}$$

$$+ \frac{3S}{4\pi(3-C)^2}\{1 + \frac{3}{4\tau^2}(1 - \frac{\sqrt{3S}}{3-C})\}\exp\{-\frac{3}{4\tau^2}(1 + \frac{\sqrt{3S}}{3-C})\}, \quad (6.6)$$

where $\tau = \sigma/\rho$, ρ being the radius of the circumcircle on which the means lie, As $\tau\to\infty$, so (6.6) tends to (6.5). See Mardia (1980).

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