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Directional Distributions

by

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DIRECTIONAL DISTRIBUTIONS

(K.V.Mardia)

1. Basic Terminology.

Let X be a random angle which takes values on the circumference of the circle of unit radius, $0 < X \leqslant 2\pi$.

A given function f is the probability density function (p.d.f.) of an absolutely continuous circular distribution if and only if

$$f(x) \geqslant 0, \quad f(x+2\pi) = f(x), \; -\infty < x < \infty$$

and
$$\int_0^{2\pi} f(x)dx = 1.$$

We can define the pth trigonometric moments about the origin as $\alpha_p = E(\cos px)$ and $\beta_p = E(\sin px)$.

Let ϕ_p be the characteristic function of the random variable X. This is defined by

$$\phi_p = E(e^{ipx}) = \alpha_p + i\beta_p \quad , \; p = 0, \pm 1, \pm 2, \ldots$$

A circular distribution is always uniquely defined by its moments. For $p=1$, we write

$$\phi_1 = \rho e^{i\mu_0} ,$$

where μ_0 is the mean direction and ρ is the resultant length. We define the sample counterparts of μ_0 and ρ as $\bar{x}_0$ and $\bar{R}$ respectively.

Let $x_1, x_2, \ldots, x_n$ be n observations of the circular random angle X.

Let

$$\bar{C} = \frac{1}{n} \sum_{i=1}^{n} \cos x_i \quad , \quad \bar{S} = \frac{1}{n} \sum_{i=1}^{n} \sin x_i \quad .$$

Then $\bar{x}_o$ and $\bar{R}$ are defined by

$$\bar{C} = \bar{R} \cos \bar{x}_o \, , \; \bar{S} = \bar{R} \sin \bar{x}_o .$$

We have $\bar{R} = (\bar{C}^2 + \bar{S}^2)^{\frac{1}{2}}$, $0 \leqslant \bar{R} \leqslant 1$. These statistics play the same role as $\bar{x}$ and s^2 on the line except that $\bar{R}$ is a measure of precision rather than variance.

2. <u>Circular Models</u>.

<u>Uniform distribution</u>. A circular random variable X, is uniformly distributed on the circle if its probability density function is given by,

$$f(x) = 1/(2\pi), \quad 0 < x \leqslant 2\pi \quad .$$

Its characteristic function is given by

$$\phi_p = \begin{cases} 1 & \text{if } p = 0, \\ 0 & \text{if } p \neq 0. \end{cases}$$

<u>Von Mises Distribution</u>. A circular random variable X is said to have a <u>von Mises distribution</u> (von Mises, 1918) if its probability density function is given by

$$f(x) = \frac{1}{2\pi I_o(\kappa)} e^{\kappa \cos (x-\mu_o)}, \quad 0< x\leqslant 2\pi, \kappa>0, 0<\mu_o< 2\pi \quad ,$$

where $I_r(\kappa)$ is the modified Bessel function of the first kind and order r. The parameter μ_o is the mean direction and κ is called the concentration parameter. The von Mises distribution can be considered as the circular analogue to the Normal distribution on the line. The distribution is unimodal and symmetrical about $x=\mu_o$. For large κ, the random variable X is distributed as $N(\mu_o, 1/\kappa^{\frac{1}{2}})$, while for $\kappa=0$, the von Mises distribution reduces to the Uniform distribution. (See the figure below.)

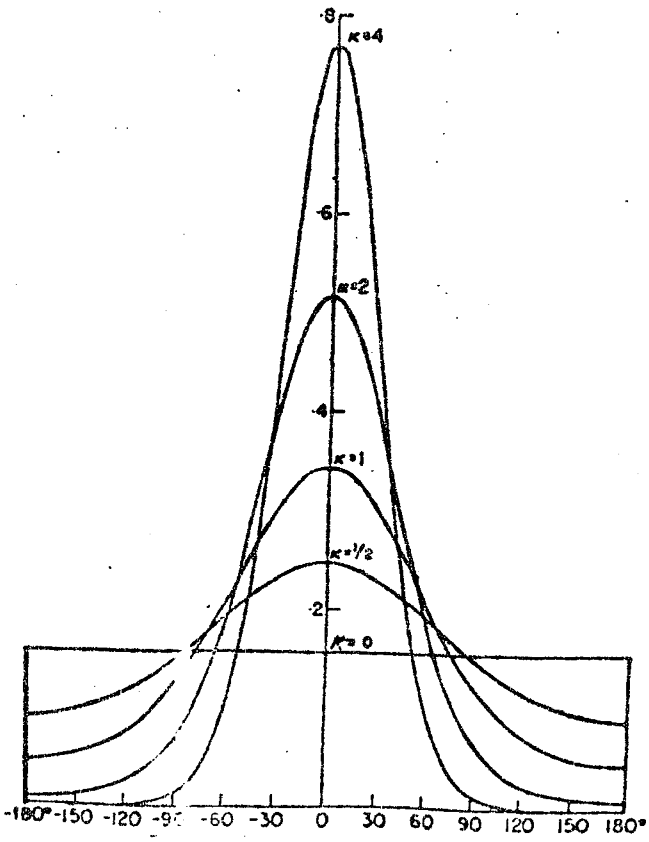

Fig. 1. Density of the von Mises distribution for $\mu_o = 0^o$ and $\kappa = 0, \frac{1}{2}, 1, 2, 4$.

The von Mises distribution has the Maximum Likelihood and Maximum Entropy Characterizations, both of which produce the Normal distribution on the line (see, Mardia 1975b). Kent (1975) gives a diffusion process leading to the von Mises distribution and also shows that the distribution is infinitely divisible.

The trigonometric moments are given by

$$\alpha_p = A_p(\kappa)\cos p\mu_o, p=1,2,\ldots,$$

$$\beta_p = A_p(\kappa)\sin p\mu_o, p=1,2,\ldots,$$

where $$A_p(\kappa) = I_p(\kappa)/I_o(\kappa).$$

Maximum likelihood estimates (m.l.e.) $\hat{\mu}_o, \hat{\kappa}$ of μ_o, κ are given by

$$\hat{\mu}_o = \bar{x}_o \quad ,$$

and $$A(\hat{\kappa}) = \bar{R} \ ,$$

where $$A(\hat{\kappa}) = A_1(\hat{\kappa}) \ .$$

A test of the null hypothesis of uniformity against the alternative of a von Mises distribution, with unknown μ_o and κ, is given by the Rayleigh Test, where the null hypothesis is rejected if

$$\bar{R} > K,$$

where K is a constant depending on n and the significance level.

For large n, we have the χ^2 approximation that under H_0,

$$2n\bar{R}^2 \sim \chi_2^2 \ .$$

For other tests on the circle and an extensive bibliography of the topic, see Mardia (1972,1975a).

Wrapped Distributions.

Given a distribution on the line, we can wrap it around the circumference of the circle of unit radius. If Y is the random variable on the line, the random variable X of the wrapped distribution is given by

$$X = Y(\text{mod } 2\pi) \ .$$

In particular we can wrap the normal distribution around the unit circle. Let Y be $N(0,\sigma^2)$; then the probability density function of the wrapped normal distribution is given by

$$f(x) = \frac{1}{\sigma\sqrt{(2\pi)}} \sum_{k=-\infty}^{\infty} \exp\left\{-\frac{1}{2}\frac{(x+2\pi k)^2}{\sigma^2}\right\}, \quad 0 < x \leqslant 2\pi \ .$$

The wrapped normal and the von Mises distributions can be made to approximate each other very closely by equating their first trigonometric moments to give (Stephens, 1963, Kent, 1978)

$$e^{-\frac{1}{2}\sigma^2} = A(\kappa) \ .$$

Although the two distributions are very similar, for inference purposes it is much more convenient to work with the von Mises distribution. We can also wrap other distributions on the line. (See, Mardia, 1972).

Let the random vector (Z_1, Z_2) have a bivariate normal distribution with mean vector $\underset{\sim}{\mu}$ and covariance matrix $\underset{\sim}{\Sigma}$. Then the distribution of the random angle X defined by

$$Z_1 = R\cos X, \; Z_2 = R\sin X$$

is called the Offset Normal Distribution. For details, see Mardia (1972).

Axial distributions.

In some situations, we have random axes (lines) rather than random angles, i.e. X and $(X+\pi) \bmod 2\pi$ represent the same line. Such data can be modelled by using antipodally symmetric distributions, i.e. the p.d.f. satisfies $f(x) = f(x+\pi)$. This procedure is identical to doubling the angle x. For details of axial distributions, see Mardia (1972).

3. Spherical Data.

Let θ and ϕ be colatitude and longitude respectively on the unit sphere. Define the direction cosines (l,m,n) of the point (θ, ϕ), by

$$l = \cos\phi \sin\theta , \quad m = \sin\phi \sin\theta , \quad n = \cos\theta .$$

Let (l_i, m_i, n_i), $i=1,\ldots,n$ be n observations from a continuous distribution on the sphere. The direction cosines of the mean direction $(\bar{l}_o, \bar{m}_o, \bar{n}_o)$ are therefore

$$\bar{l}_o = \sum_{i=1}^{n} l_i/R, \quad \bar{m}_o = \sum_{i=1}^{n} m_i/R, \quad \bar{n}_o = \sum_{i=1}^{n} n_i/R,$$

where R is the length of the resultant given by

$$R = \{(\Sigma l_i)^2 + (\Sigma m_i)^2 + (\Sigma n_i)^2\}^{\frac{1}{2}}.$$

The extension of the von Mises distribution is given by the Fisher distribution (Fisher, 1953) which has the p.d.f.

$$f(l,m,n) = c \exp\{\kappa(l\lambda+m\mu+n\nu)\}, \ \kappa > 0,$$

where (λ, μ, ν) are the direction cosines of the mean direction, and κ is the concentration parameter. Note that (l, m, n) is a unit random vector. In polar co-ordinates with (μ_o, ν_o) as the mean direction for (θ, ϕ), the p.d.f. becomes

$$f(\theta,\phi) = c \exp[\kappa\{\cos\mu_o\cos\theta + \sin\mu_o\sin\theta\cos(\phi-\nu_o)\}]\sin\theta,$$

where $\qquad 0<\theta<\pi, \quad 0<\phi\leqslant 2\pi, \quad \kappa\geqslant 0,$

and $\qquad c = \kappa/(4\pi\sinh\kappa).$

For $\kappa=0$, the Fisher distribution reduces to the uniform distribution on the sphere. Like the von Mises distribution on the circle, the Fisher distribution has the maximum likelihood and maximum entropy characterizations on the sphere, (see Mardia, 1975b). The Brownian Motion distribution on the sphere can be closely approximated by the Fisher distribution (see Roberts and Ursell, 1960). The distribution is unimodal with mode at (λ,μ,ν) and antimode at $(-\lambda,-\mu,-\nu)$. Further the distribution is rotationally symmetric about the mean direction.

The sample mean direction is the m.l.e. of (λ,μ,ν), while the m.l.e. of κ is the solution $\hat{\kappa}$ of

$$\coth \hat{\kappa} - 1/\hat{\kappa} = \bar{R} .$$

Now consider a situation where one observes not directions but axes, which are random variables on a projective hemisphere. It is convenient to represent such a random variable by an antipodally symmetric distribution on the sphere, i.e. the p.d.f. of $\underset{\sim}{l}' = (l,m,n)$ has antipodal-symmetry, i.e.

$$f(\underset{\sim}{l}) = f(-\underset{\sim}{l}) .$$

Note that the procedure of doubling the angles on the circle has no analogue here.

The unit random vector $\underset{\sim}{l}$ is said to have the Bingham distribution (Bingham, 1974) if its p.d.f. has the form

$$f(\underset{\sim}{l}) = d(\underset{\sim}{\kappa}) \exp\{\mathrm{tr}(\underset{\sim}{\kappa}\underset{\sim}{\mu}'\underset{\sim}{l}\underset{\sim}{l}'\underset{\sim}{\mu})\} ,$$

where $\underset{\sim}{\mu}$ now denotes an orthogonal matrix, $\underset{\sim}{\kappa} = \mathrm{diag}(\kappa_1, \kappa_2, \kappa_3)$ is a matrix of constants, and $d(\underset{\sim}{\kappa})$ is the normalizing constant depending only on $\underset{\sim}{\kappa}$. Since $\mathrm{tr}(\underset{\sim}{\mu}'\underset{\sim}{l}\underset{\sim}{l}'\underset{\sim}{\mu}) = 1$, the sum of the κ_i is arbitrary and it is usual to take $\kappa_3=0$.

The distribution contains a number of different forms, such as symmetric axial and girdle distributions of Watson (1965) and Dimroth (1962), as well as asymmetric axial and girdle distributions. For its various characterizations, see Mardia (1975b). For a discussion on the Fisher and Bingham distributions see Mardia (1972). A small circle distribution on the sphere is given by Bingham and Mardia (1978).

4. Extensions.

Generalizations to p dimensions can readily be made from the preceeding discussion. Let S_p denote the unit sphere in R^p. A unit random vector $\underset{\sim}{l}$ is said to have a p-variate von Mises-Fisher distribution if its p.d.f. is given by

$$f(\underset{\sim}{l}) = c_p(\kappa) e^{\kappa \underset{\sim}{\mu}'\underset{\sim}{l}}, \ \kappa > 0, \underset{\sim}{\mu}'\underset{\sim}{\mu} = 1, \ \underset{\sim}{l} \in S_p ,$$

where $$c_p(\kappa) = \kappa^{\frac{1}{2}p-1} / \{(2\pi)^{\frac{1}{2}p} I_{\frac{1}{2}p-1}(\kappa)\} .$$

This distribution was first introduced by Watson and Williams (1956). For a discussion of the distribution see Mardia (1975a).

So far, we have assumed that each orientation is characterized by a single distinguishable direction in p-dimensions. Downs (1972) has dealt with the problem where each orientation is characterized by a rigid configuration of k distinguishable directions in p-space with fixed angles between them. Let $\underset{\sim}{X}$ be a $p\times k$ random matrix lying on the Stiefel C-manifold $\{\underset{\sim}{X}:\underset{\sim}{X}'\underset{\sim}{X} = \underset{\sim}{C}\}$ where $\underset{\sim}{C}$ is a $k\times k$ constant matrix. Downs (1972) introduced the distribution with p.d.f. given by

$$f(\underset{\sim}{X}) = a(\underset{\sim}{C},\underset{\sim}{F})\exp\{tr(\underset{\sim}{F}'\underset{\sim}{X})\} \quad ,$$

where $a(\underset{\sim}{C},\underset{\sim}{F})$ is the normalizing constant and $\underset{\sim}{F}$ is a $p\times k$ parameter matrix. For $k=1$, the distribution reduces to the von Mises-Fisher distribution. For a further discussion see Mardia and Khatri (1977).

For the concept of circular dependence and correlation coefficients for bivariate circular variables, see Mardia (1975a). For distributions on a cylinder $(x,\theta), -\infty < x < \infty, 0 < \theta \leq 2\pi$, see Mardia and Sutton (1978) and Johnson and Wehrly (1978). For an important process involving orientation, see Kendall (1974). Various applications of directional data analysis can be found in Batschelet (1965) and Mardia (1972). Pearson and Hartley (1972, chapter 9), contain a useful set of tables and introductory material.

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