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PRODUCTS OF IDEMPOTENTS IN FINITE FULL TRANSFORMATION SEMIGROUPS: SOME IMPROVED BOUNDS

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SYNOPSIS

Let E be the set of idempotents in S_n, the semigroup of all singular selfmaps of $\{1,\dots,n\}$. For each α in S_n there is a unique $k(\alpha) \geqslant 1$ such that $\alpha \in E^{k(\alpha)}$, $\alpha \notin E^{k(\alpha)-1}$. It is known that $k(\alpha) \leqslant n + \text{cycl}\ \alpha - \text{fix}\ \alpha$, where cycl α is the number of cyclic orbits of α and fix α is the number of fixed points. Equality holds only in the case where α is of rank $n - 1$. An improved upper bound is obtained for $k(\alpha)$, applying to elements of arbitrary rank. A lower bound is obtained also.

1. INTRODUCTION

For standard terms in semigroup theory see [2]. Let X be the finite set $\{1,2,\dots,n\}$ and let $S_n = \mathcal{T}(X) \setminus \mathcal{G}(X)$ be the semigroup (under composition of mappings) of all singular mappings from X into itself. Let E be the set of idempotents of S_n. It is known [1] that E generates S_n, and from [3] one has the more exact result that $E^k = S_n$, $E^{k-1} \subset S_n$, where $k = [\frac{3}{2}(n-1)]$. The method of proof of this latter result involves showing that each α in S_n is expressible as a product of $n + \text{cycl}\ \alpha - \text{fix}\ \alpha$ *idempotents of rank* $n - 1$, where cycl α is the number of cyclic orbits of α and fix α is the number of fixed points of α, and that no smaller number of idempotents of rank $n - 1$

will suffice. For an element α of rank $n - 1$ we thus have a complete answer to the problem of finding the integer $k(\alpha)$ for which $\alpha \in E^{k(\alpha)}$, $\alpha \notin E^{k(\alpha)-1}$: in expressing α as a product of idempotents from S_n we must in fact make exclusive use of idempotents of rank $n - 1$, and so in this case $k(\alpha) = n + \text{cycl}\ \alpha - \text{fix}\ \alpha$.

For elements of rank less than $n - 1$ the position is less satisfactory: certainly

$$\alpha \in E^{n + \text{cycl}\ \alpha - \text{fix}\ \alpha},$$

but a much smaller number will in many cases suffice. The extreme case is an element α of rank 1, where $\alpha \in E$ and so $k(\alpha) = 1$, but where

$$n + \text{cycl}\ \alpha - \text{fix}\ \alpha = n + 0 - 1 = n - 1.$$

The first main result of this paper (Theorem 3.1) is an improved upper bound for $k(\alpha)$:

$$k(\alpha) \leqslant \text{rank}\ \alpha + \text{orb}\ \alpha - \text{fix}\ \alpha,$$

where rank α and fix α are as before and where orb α is the number of non-singleton orbits of α. This bound is improved further in Theorem 4.3, but at the cost of a much more complicated set of definitions.

In the final section it is shown (Theorem 5.1) that

$$k(\alpha) \geqslant (n - \text{fix}\ \alpha)/(n - \text{rank}\ \alpha).$$

REMARK. The central result in [3], namely that each α in S_n is a product of $n + \text{cycl}\ \alpha - \text{fix}\ \alpha$ idempotents of rank $n - 1$, had already appeared in a paper by Iwahori [4]. I am grateful to Professor Klaus Leeb of Erlangen for bringing this to my attention.

2. PRELIMINARIES

Let $\alpha \in S_n$. In the standard way we define im $\alpha = \{x\alpha : x \in X\}$, rank $\alpha = |\text{im } \alpha|$, ker $\alpha = \{(x,y) \in X \times X : x\alpha = y\alpha\}$, fix $\alpha = |\{x \in X : x\alpha = x\}|$. As in [3] the key idea is that of an *orbit* of α, i.e. an equivalence class in X under the equivalence

$$\omega = \{(x,y) \in X \times X : (\exists \ell, m \geq 0)\, x\alpha^{\ell} = y\alpha^{m}\}.$$

It is shown in [3] that each orbit Ω has a *kernel* $K(\Omega)$ characterised by the property that (for each x in Ω)

$$x \in K(\Omega) \text{ if and only if } x \in x\alpha^{-\underset{\sim}{N}},$$

where

$$x\alpha^{-\underset{\sim}{N}} = \{y \in X : y\alpha^{i} = x \text{ for some } i > 0\}.$$

Orbits are then classified into various types

standard orbits : $|\Omega| > |K(\Omega)| > 1$;

acyclic orbits : $|\Omega| > |K(\Omega)| = 1$;

cyclic orbits : $|\Omega| = |K(\Omega)| > 1$;

singleton orbits : $|\Omega| = |K(\Omega)| = 1$.

Notice that in all cases $\alpha|\Omega$ belongs to $\textcircled{T}(\Omega)$. In the case of standard and acyclic orbits $\alpha|\Omega \in \textcircled{T}(\Omega) \setminus \textcircled{G}(\Omega)$, while for cyclic (and indeed singleton) orbits $\alpha|\Omega \in \textcircled{G}(\Omega)$.

Let us denote the number of non-singleton orbits of α by orb α.

In [3] we used the notation

$$\begin{pmatrix} i \\ j \end{pmatrix},$$

(where i,j are distinct members of $\{1,2,\ldots,n\}$) for the idempotent ε of rank n - 1 defined by

$$i\varepsilon = j, \quad x\varepsilon = x \quad (x \neq i).$$

It will be convenient here to generalise this notation. First, if A is a subset of $X = \{1,2,\ldots,n\}$ and $b \in X\setminus A$ then the idempotent ε (of rank $n - |A|$) defined by

$$x\varepsilon = b \quad (x \in A), \qquad x\varepsilon = x \quad (x \notin A)$$

will be denoted by

$$\begin{pmatrix} A \\ b \end{pmatrix}.$$

More generally, if $A_1,\ldots,A_k$ are disjoint subsets of X and if $b_1,\ldots,b_k$ are distinct elements of X such that

$$(A_1 \cup \ldots \cup A_k) \cap \{b_1,\ldots,b_k\} = \emptyset,$$

then the idempotent ε (of rank $n - \sum |A_i|$) defined by

$$x\varepsilon = b_i \text{ if } x \in A_i \ (i = 1,\ldots,k); \quad x\varepsilon = x \text{ otherwise}$$

will be denoted by

$$\begin{pmatrix} A_1 & A_2 & \ldots & A_k \\ b_1 & b_2 & \ldots & b_k \end{pmatrix}.$$

If two idempotents

$$\varepsilon = \begin{pmatrix} A_1 & A_2 & \ldots & A_k \\ b_1 & b_2 & \ldots & b_k \end{pmatrix}, \quad \eta = \begin{pmatrix} C_1 & C_2 & \ldots & C_\ell \\ d_1 & d_2 & & d_\ell \end{pmatrix}$$

are such that the sets $A_1 \cup \ldots \cup A_k$, $\{b_1,\ldots,b_k\}$, $C_1 \cup \ldots \cup C_\ell$, $\{d_1,\ldots,d_\ell\}$ are pairwise disjoint, we shall write the idempotent

$$\begin{pmatrix} A_1 & A_2 & \ldots & A_k & C_1 & C_2 & \ldots & C_\ell \\ b_1 & b_2 & \ldots & b_k & d_1 & d_2 & \ldots & d_\ell \end{pmatrix}$$

as $\varepsilon \cup \eta$.

3. AN UPPER BOUND

We prove the following result.

THEOREM 3.1. *Let* $\alpha \in S_n = \textcircled{T}(X)\setminus\textcircled{G}(X)$, *where* $X = \{1,2,\ldots,n\}$ *and let* E *be the set of idempotents of* S_n. *Let* $k(\alpha)$ *be the unique positive integer for which* $\alpha \in E^{k(\alpha)}$, $\alpha \notin E^{k(\alpha)-1}$. *Then*

$$k(\alpha) \leqslant \text{rank } \alpha + \text{orb } \alpha - \text{fix } \alpha.$$

Proof. Let us suppose that α has

standard orbits $\Omega_1,\ldots, \Omega_t$,

acyclic orbits $\Omega_{t+1}, \ldots, \Omega_{t+a}$,

cyclic orbits $\Omega_{t+a+1},\ldots, \Omega_{t+a+c}$,

and singleton orbits $\Omega_{t+a+c+1},\ldots, \Omega_{t+a+c+s}$.

Then

$$\text{orb } \alpha = t + a + c, \quad \text{fix } \alpha = a + s,$$

$$\text{rank } \alpha = s + \sum_{i=1}^{t+a} \text{rank}(\alpha|\Omega_i) + \sum_{i=t+a+1}^{t+a+c} |\Omega_i|.$$

We consider α orbit by orbit. To avoid excessive use of subscripts let us simplify the notation and consider first a standard orbit Ω with kernel

$$k(\alpha) = \{x, x\alpha, \ldots, x\alpha^{m-1}\},$$

where $x\alpha^{m-1} = x$. Since the 'starting point' x of this enumeration of $K(\Omega)$ is in effect arbitrary, we may assume that there exists y in $\Omega\setminus K(\Omega)$ such that $y\alpha = x$. Let

$$\pi_0 = \begin{pmatrix} x\alpha^{m-1} \\ y \end{pmatrix}\begin{pmatrix} (x\alpha^{m-1})\alpha^{-1} \\ x\alpha^{m-1} \end{pmatrix}\begin{pmatrix} (x\alpha^{m-2})\alpha^{-1} \\ x\alpha^{m-2} \end{pmatrix}\cdots\begin{pmatrix} x\alpha^{-1}\setminus\{x\alpha^{m-1}\} \\ x \end{pmatrix},$$

a product of idempotents in S_n.

We consider the orbit Ω in 'layers' working outwards from the kernel. For each p in Ω there is a least n such that $p\alpha^n \in K(\Omega)$, i.e. such that

$$p \in K(\Omega)\alpha^{-n} \setminus K(\Omega)\alpha^{-(n-1)}.$$

If we denote this n by $n(p)$ we can then define $s(\Omega)$, the *spread* of Ω, by

$$s(\Omega) = \max\{n(p) : p \in \Omega\},$$

and we can write Ω as a disjoint union of $K(\Omega)$ and of the 'layers'

$$K(\Omega)\alpha^{-i} \setminus K(\Omega)\alpha^{-(i-1)} \qquad (i = 1,\ldots,s(\Omega)).$$

We may think of the product π_0 as taking care of the kernel $K(\Omega)$. Suppose now that

$$(K(\Omega)\alpha^{-1} \setminus K(\Omega)) \cap \operatorname{im}\alpha = \{y_1,\ldots,y_j\},$$

and consider the product

$$\pi_1 = \begin{pmatrix} y_1\alpha^{-1} \\ y_1 \end{pmatrix} \cdots \begin{pmatrix} y_j\alpha^{-1} \\ y_j \end{pmatrix}$$

of idempotents in S_n. More generally, for each i in $\{1,\ldots,s(\Omega)\}$ we take the set

$$(K(\Omega)\alpha^{-i} \setminus K(\Omega)\alpha^{-(i-1)}) \cap \operatorname{im}\alpha = \{z_1,\ldots,z_k\}$$

and consider

$$\pi_i = \begin{pmatrix} z_1\alpha^{-1} \\ z_1 \end{pmatrix} \cdots \begin{pmatrix} z_k\alpha^{-1} \\ z_k \end{pmatrix},$$

a product of idempotents in S_n.

Now let us examine the product

$$\pi_0\pi_1\cdots\pi_{s(\Omega)}\,. \tag{3.2}$$

With possibly one exception each element of (im α) $\cap$ Ω appears exactly once as a lower entry in one of $\pi_0,\pi_1,\ldots,\pi_{s(\Omega)}$. The exception is the element y, which appears as a lower entry in π_0, and may also (if $y \in \text{im}\,\alpha$) appear again in π_1. Hence the total number of idempotents in the product (3.2) is either $\text{rank}(\alpha|\Omega)$ or $\text{rank}(\alpha|\Omega) + 1$.

Having examined the lower entries let us look now at the upper entries. Each element of Ω appears exactly once in the union

$$\{x\alpha^{m-1}\} \mathbin{\dot\cup} (x\alpha^{m-1})\alpha^{-1} \cup \ldots \cup x\alpha^{-1}\backslash\{x\alpha^{m-1}\} \cup y_1\alpha^{-1} \cup \ldots$$

$$\cup\, y_j\alpha^{-1} \cup \ldots \cup z_1\alpha^{-1} \cup \ldots \cup z_k\alpha^{-1} \cup \ldots$$

and so is mapped non-trivially by one and only one of the idempotents in the product (3.2). Let $p \in \Omega$, $p \neq x\alpha^{m-1}$ and suppose that $p\alpha = q$. Then p appears in the upper entry of the idempotent

$$\begin{pmatrix} q\alpha^{-1} \\ q \end{pmatrix}.$$

Now q never appears *subsequently* in any upper entry in the product (3.2) and so the total effect of $\pi_0\pi_1\cdots\pi_{s(\Omega)}$ on p is to map it to q (= $p\alpha$). If $p = x\alpha^{m-1}$ then π_0 maps it first to y and then (by the last factor of π_0) to x. Since x does not then appear as an upper entry in any of $\pi_1,\ldots,\pi_{s(\Omega)}$ we conclude that the total effect of $\pi_0\pi_1\cdots\pi_{s(\Omega)}$ on $x\alpha^{m-1}$ is to map it to x $(=(x\alpha^{m-1})\alpha)$.

In summary, then, our conclusion is that the product (3.2) comprises either $\text{rank}(\alpha|\Omega)$ or $\text{rank}(\alpha|\Omega) + 1$ idempotents, that it coincides with α on Ω and that it maps $X\backslash\Omega$ identically.

Suppose now that Ω is an acyclic orbit of α, with $K(\Omega) = \{z\}$. We use a simplified version of our technique for dealing with a standard orbit, beginning by splitting Ω into layers as follows:

$$\Omega = \{z\} \cup (z\alpha^{-1}\setminus\{z\}) \cup (z\alpha^{-2}\setminus z\alpha^{-1}) \cup \ldots \cup (z\alpha^{-s(\Omega)}\setminus z\alpha^{-(s(\Omega)-1)}).$$

We then consider the product $\pi_1\pi_2\ldots\pi_{s(\Omega)}$, where

$$\pi_1 = \begin{pmatrix} z\alpha^{-1} \\ z \end{pmatrix}.$$

and where π_i $(i = 2,\ldots,s(\Omega))$, a product of idempotents arising from the set

$$(z\alpha^{-i}\setminus z\alpha^{-(i-1)}) \cap \operatorname{im}\alpha = \{z_1,\ldots,z_k\},$$

is given by

$$\pi_i = \begin{pmatrix} z_1\alpha^{-1} \\ z_1 \end{pmatrix} \cdots \begin{pmatrix} z_k\alpha^{-1} \\ z_k \end{pmatrix}.$$

Each element p of Ω appears exactly once as an upper entry in $\pi_1\ldots\pi_{s(\Omega)}$ and is mapped by the idempotent in question to $p\alpha$. Since $p\alpha$ does not then subsequently appear as an upper entry we conclude that $\pi_1\ldots\pi_{s(\Omega)}$ coincides with α on Ω and with the identity function on $X\setminus\Omega$. Since every element of $\operatorname{im}(\alpha|\Omega)$ appears exactly once as a lower entry, we conclude that the total number of idempotents in the product $\pi_1\ldots\pi_{s(\Omega)}$ is $\operatorname{rank}(\alpha|\Omega)$.

Suppose now that $\Omega = \{x, x\alpha, \ldots, x\alpha^{m-1}\}$, with $x\alpha^m = x$ and $m \geqslant 2$, is a cyclic orbit. Then as in [3] we choose y in $X\setminus\operatorname{im}\alpha$ (outside Ω) and obtain a product

$$\pi = \begin{pmatrix} x\alpha^{m-1} \\ y \end{pmatrix}\begin{pmatrix} x\alpha^{m-2} \\ x\alpha^{m-1} \end{pmatrix} \cdots \begin{pmatrix} x \\ x\alpha \end{pmatrix}\begin{pmatrix} y \\ x \end{pmatrix} \qquad (3.3)$$

of $|\Omega| + 1$ idempotents with the property that π coincides with α on Ω.

It is clear also that $y\pi = x$, but this turns out to be no disadvantage.

Let us look now at the global picture for α. Considering each non-singleton orbit in turn, and keeping the cyclic orbits to the last, we express α as a product of idempotents, the total number being at most

$$\sum_{i=1}^{t} (\mathrm{rank}(\alpha|\Omega_i) + 1) + \sum_{i=t+1}^{t+a} \mathrm{rank}(\alpha|\Omega_i) + \sum_{i=t+a+1}^{t+a+c} (|\Omega_i| + 1)$$

$$= \left(\sum_{i=1}^{t+a} \mathrm{rank}(\alpha|\Omega_i) + \sum_{i=t+a+1}^{t+a+c} |\Omega_i| + s \right) + t + c - s$$

$$= \mathrm{rank}\ \alpha + (t+a+c) - (a+s)$$

$$= \mathrm{rank}\ \alpha + \mathrm{orb}\ \alpha - \mathrm{fix}\ \alpha,$$

as required.

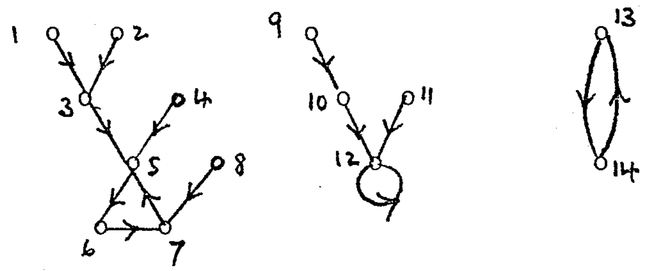
NOTE (3.4). By keeping the cyclic orbits to the last we ensure that no difficulty arises from the fact that the product (3.3) does not coincide with the identity mapping on $X \setminus \Omega$. For the element y, being in $X \setminus \mathrm{im}\ \alpha$, appears in some standard or acyclic orbit and so has already been mapped to $y\alpha$. The effective domain for π is in fact the image of the product of all the idempotents that have gone previously, and y does not appear in this image.

NOTE (3.5). In the next section we shall require to be a little more careful in dealing with cyclic orbits. We shall refer to the element y from $X \setminus \mathrm{im}\ \alpha$ appearing in the product (3.3) as the *attached element*.

EXAMPLE (3.6). Let $n = 14$ and

$$\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 \\ 3 & 3 & 5 & 5 & 6 & 7 & 5 & 7 & 10 & 12 & 12 & 12 & 14 & 13 \end{pmatrix}.$$

The orbits of α can be depicted thus:



Also rank α = 8, orb α = 3, fix α = 1. The routine described in the proof of the theorem gives

$$\alpha = \begin{pmatrix}7\\3\end{pmatrix}\begin{pmatrix}\{8,6\}\\7\end{pmatrix}\begin{pmatrix}5\\6\end{pmatrix}\begin{pmatrix}\{3,4\}\\5\end{pmatrix}\begin{pmatrix}\{1,2\}\\3\end{pmatrix}\begin{pmatrix}\{10,11\}\\12\end{pmatrix}\begin{pmatrix}9\\10\end{pmatrix}\begin{pmatrix}13\\11\end{pmatrix}\begin{pmatrix}14\\13\end{pmatrix}\begin{pmatrix}11\\14\end{pmatrix},$$

a product of 10 idempotents. It is clear from this example that the bound obtained in Theorem 3.1 is not best possible, for it is easy in this case to handle orbits simultaneously, obtaining that

$$\alpha = \begin{pmatrix}7 & \{10,11\}\\3 & 12\end{pmatrix}\begin{pmatrix}\{8,6\} & 9 & 13\\7 & 10 & 11\end{pmatrix}\begin{pmatrix}5 & 14\\6 & 13\end{pmatrix}\begin{pmatrix}\{3,4\} & 11\\5 & 14\end{pmatrix}\begin{pmatrix}\{1,2\}\\3\end{pmatrix},$$

a product of just 5 idempotents.

REMARK (3.7). If rank $\alpha = n - 1$ then all but one of the non-singleton orbits must be cyclic and so

$$\begin{aligned}\text{rank } \alpha + \text{orb } \alpha - \text{fix } \alpha &= (n-1) + (\text{cycl } \alpha + 1) - \text{fix } \alpha\\ &= n + \text{cycl } \alpha - \text{fix } \alpha,\end{aligned}$$

which is exactly the bound obtained in [4] and [3]. From [4] and [3] we thus have that

$$k(\alpha) = \text{rank } \alpha + \text{orb } \alpha - \text{fix } \alpha$$

for all elements α of rank n-1. We also have equality at the other extreme: if rank $\alpha = 1$ then $\alpha \in E$ and so $k(\alpha) = 1$. Also orb $\alpha = 1$, fix $\alpha = 1$ and so

$$\text{rank } \alpha + \text{orb } \alpha - \text{fix } \alpha = 1.$$

4. AN IMPROVED UPPER BOUND

The example (3.6) considered at the end of the last section suggests that a refined upper bound for $k(\alpha)$ might be obtained by handling orbits simultaneously as far as possible. To this end we define the *power* of a non-singleton orbit Ω of α by

$$p(\Omega) = \operatorname{rank}(\alpha|\Omega) - \operatorname{fix}(\alpha|\Omega) + 1.$$

Thus

$$p(\Omega) = \begin{cases} \operatorname{rank}(\alpha|\Omega) + 1 & \text{for a } \textit{standard} \text{ orbit,} \\ \operatorname{rank}(\alpha|\Omega) & \text{for an } \textit{acyclic} \text{ orbit,} \\ |\Omega| + 1 & \text{for a } \textit{cyclic} \text{ orbit,} \end{cases}$$

and if we use the routine described in the proof of Theorem 3.1 to express α as a product of idempotents the contribution from Ω is at most $p(\Omega)$ idempotents.

Let Ⓑ be the set of orbits that are either standard or acyclic, and let Ⓒ be the set of all cyclic orbits: thus orb α = |Ⓑ| + |Ⓒ|. Let

$$M_b(\alpha) = \max\{p(\Omega) : \Omega \in \text{Ⓑ}\}. \tag{4.1}$$

Suppose that the cyclic orbits

$$\Omega_0, \Omega_1, \ldots, \Omega_c$$

(with c = |Ⓒ| $- 1$) are ordered in such a way that

$$|\Omega_0| \geqslant |\Omega_1| \geqslant \ldots \geqslant |\Omega_c|.$$

(Beginning the count at Ω_0 rather than Ω_1 is purely a technical device to simplify the definitions and statements ahead.) Let

$$d = n - \operatorname{rank} \alpha \qquad (\geqslant 1)$$

and write

$$c = qd + r$$

by the division algorithm, with $q \geqslant 0$, $0 \leqslant r < d$. Let

$$M_c(\alpha) = \sum_{i=0}^{q} (|\Omega_{id}| + 1). \tag{4.2}$$

Then we have

THEOREM 4.3. *Let* $\alpha \in S_n = \textcircled{T}(X) \setminus \textcircled{G}(X)$, *where* $X = \{1,2,\ldots,n\}$, *and let* E *be the set of idempotents of* S_n. *Let* $k(\alpha)$ *be the unique positive integer for which* $\alpha \in E^{k(\alpha)}$, $\alpha \notin E^{k(\alpha)-1}$. *Then*

$$k(\alpha) \leqslant M_b(\alpha) + M_c(\alpha),$$

where $M_b(\alpha)$ *and* $M_c(\alpha)$ *are defined respectively by* (4.1) *and* (4.2).

Proof. Supplementing the notations introduced above, we write

$$X \setminus \operatorname{im} \alpha = \{z_0, \ldots, z_{d-1}\}$$

and denote the standard or acyclic orbits of α by $\Phi_1, \ldots, \Phi_b$.

By the method of Theorem 3.1 we express α as a product

$$\xi_1 \ldots \xi_b \tau_0 \ldots \tau_c,$$

where each ξ_i is a product of at most $p(\Phi_i)$ idempotents and each τ_j is a product of $|\Omega_j| + 1$ idempotents. Each τ_j involves the choice of an attached element (see Note (3.5)) from the list $z_0, \ldots, z_{d-1}$, and we may suppose that this is done so that τ_j has attached element z_ℓ, where ℓ is the unique element in $\{0, \ldots, d-1\}$ for which $j \equiv \ell \pmod d$.

Suppose now that

$$\xi_i = \varepsilon_{i1}\varepsilon_{i2}\ldots\varepsilon_{it_i} \qquad (i = 1, \ldots, b)$$

where $t_i = p(\Phi_i)$, a product of idempotents, and let us write

$$\varepsilon_{ij} = \begin{pmatrix} P_{ij} \\ q_{ij} \end{pmatrix},$$

where $P_{ij} \subseteq \Phi_i$, $q_{ij} \in \Phi_i$, $q_{ij} \notin P_{ij}$. Then

$$(P_{11} \cup \ldots \cup P_{b1}) \cup \{q_{11}, \ldots, q_{b1}\} = \emptyset$$

and so we may define the idempotent

$$\eta_1 = \varepsilon_{11} \cup \varepsilon_{21} \cup \ldots \cup \varepsilon_{b1} = \begin{pmatrix} P_{11} & P_{21} & \ldots & P_{b1} \\ q_{11} & q_{21} & \ldots & q_{b1} \end{pmatrix}.$$

Similarly, for $2 \leqslant j \leqslant M_b(\alpha)$ we define

$$\eta_j = \varepsilon_{1j} \cup \varepsilon_{2j} \cup \ldots \cup \varepsilon_{bj},$$

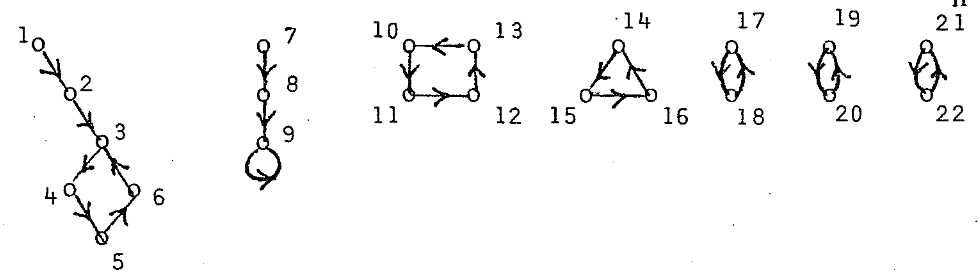
where if $j > t_i$ the term ε_{ij} is simply left out. We see then that

$$\xi_1 \xi_2 \ldots \xi_b = \eta_1 \eta_2 \ldots \eta_{M_b(\alpha)},$$

a product of $M_b(\alpha)$ idempotents.

A similar manoeuvre may be carried out on each of the products $\pi_0 \ldots \pi_{d-1}, \pi_d \ldots \pi_{2d-1}, \ldots, \pi_{qd} \ldots \pi_c$, giving product of idempotents of length $|\Omega_0| + 1, |\Omega_d| + 1, \ldots, |\Omega_{qd}| + 1$ respectively. (The arranging of the cyclic orbits in descending order of size ensures that the maximum orbital size is obtained at the beginning of each of the subdivisions of $\{0, \ldots, c\}$.) We obtain a product of idempotents of length $M_c(\alpha)$ and the result of the theorem is now immediate.

EXAMPLE (4.4). Let $n = 22$ and let α be the element of S_n with orbits



Here we have $b = 2$, $d = 2$, $c = 4$, $q = 2$. Also

$\Phi_1 = \{1,2,3,4,5,6\}, \quad p(\Phi_1) = 6,$

$\Phi_2 = \{7,8,9\}, \quad p(\Phi_2) = 2,$

rank α + orb α - fix α = 20 + 7 - 1 = 26,

$M_b(\alpha) + M_c(\alpha) = 6 + (5 + 3 + 3) = 17,$

and α is expressible as a product

$$\begin{pmatrix}6 & 8\\2 & 9\end{pmatrix}\begin{pmatrix}5 & 7\\6 & 8\end{pmatrix}\begin{pmatrix}4\\5\end{pmatrix}\begin{pmatrix}3\\4\end{pmatrix}\begin{pmatrix}2\\3\end{pmatrix}\begin{pmatrix}1\\2\end{pmatrix}\begin{pmatrix}10 & 14\\1 & 7\end{pmatrix}\begin{pmatrix}13 & 16\\10 & 14\end{pmatrix}\begin{pmatrix}12 & 15\\13 & 16\end{pmatrix}\begin{pmatrix}11 & 7\\12 & 15\end{pmatrix}$$

$$\begin{pmatrix}1\\11\end{pmatrix}\begin{pmatrix}17 & 19\\1 & 7\end{pmatrix}\begin{pmatrix}18 & 20\\17 & 19\end{pmatrix}\begin{pmatrix}1 & 7\\18 & 20\end{pmatrix}\begin{pmatrix}21\\1\end{pmatrix}\begin{pmatrix}22\\21\end{pmatrix}\begin{pmatrix}1\\22\end{pmatrix}$$

of 17 idempotents.

REMARK (4.5). The bound $M_b(\alpha) + M_c(\alpha)$ is still capable of improvement. If we look again at Example 3.6 we see that

$$M_b(\alpha) + M_c(\alpha) = 5 + 3 = 8,$$

but we have already seen that α is expressible as a product of 5 idempotents. The lower figure was obtained by dealing simultaneously with a cyclic and a non-cyclic orbit. The circumstances in which this can be done are rather hard to describe, and any improvement on Theorem 4.3 based on such considerations would probably be too complicated to be of interest unless some important application were in view.

REMARK (4.6). The new bound has at least the advantage of coinciding with $k(\alpha)$ for every idempotent. If α is idempotent then all the orbits are acyclic or singleton and $p(\Omega) = 1$ for every acyclic orbit Ω. Thus $M_b(\alpha) + M_c(\alpha) = 1$.

5. A LOWER BOUND

Let $\alpha \in S_n$. As in [1] we define

$$S(\alpha) = \{x \in X : x\alpha \neq x\};$$

by shift α we shall mean the cardinality of the set $S(\alpha)$. Again as in [1] we define

$$Z(\alpha) = X \setminus \operatorname{im} \alpha;$$

by def α (the *defect* of α) we shall mean the cardinality of the set $Z(\alpha)$. Notice that

$$\text{shift } \alpha = n - \text{fix } \alpha, \qquad \text{def } \alpha = n - \text{rank } \alpha.$$

THEOREM 5.1. *Let* $\alpha \in S_n = \textcircled{T}(X) \setminus \textcircled{G}(X)$, *where* $X = \{1,2,\ldots,n\}$ *and let* E *be the set of idempotents of* S_n. *Let* $k(\alpha)$ *be the unique positive integer for which* $\alpha \in E^{k(\alpha)}$, $\alpha \notin E^{k(\alpha)-1}$. *Then*

$$k(\alpha) \geqslant \text{shift } \alpha \; / \text{def } \alpha.$$

Proof. First notice that for every ε in E we have $x\varepsilon = x$ if and only if $x \in \operatorname{im} \varepsilon$. Hence $x\varepsilon \neq x$ if and only if $x \in X \setminus \operatorname{im} \varepsilon$, which amounts to saying that $S(\varepsilon) = Z(\varepsilon)$. Thus

$$\text{shift } \varepsilon = \text{def } \varepsilon \tag{5.2}$$

for every ε in E. Next, for α, β in S_n,

$$S(\alpha\beta) \subseteq S(\alpha) \cup S(\beta),$$

giving

$$\text{shift}(\alpha\beta) \leqslant \text{shift } \alpha + \text{shift } \beta. \tag{5.3}$$

Since $\operatorname{im}(\alpha\beta) \subseteq \operatorname{im} \beta$ we certainly have

$$\text{def}(\alpha\beta) \geqslant \text{def } \beta. \tag{5.4}$$

Also, since $\ker(\alpha\beta) \supseteq \ker \alpha$ we have

$$\text{rank}(\alpha\beta) = |X/\ker(\alpha\beta)| \leqslant |X/\ker \alpha| = \text{rank } \alpha,$$

and hence

$$\text{def}(\alpha\beta) \geqslant \text{def } \alpha. \tag{5.5}$$

Suppose now that $\alpha = \varepsilon_1\varepsilon_2\ldots\varepsilon_{k(\alpha)}$, a product of $k(\alpha)$ elements of E, and that def $\alpha = d$. Then by (5.4) and (5.5) def $\varepsilon_i \leqslant d$ for $i = 1,\ldots,k(\alpha)$. Hence shift $\varepsilon_i \leqslant d$ by (5.2). Finally, by (5.3), it follows that

$$\text{shift } \alpha \leqslant k(\alpha)d,$$

giving the required result.

REMARK (5.6). Applying this theorem to Example 4.4 gives shift $\alpha = 21$, def $\alpha = 2$ and so $k(\alpha) \geqslant 21/2$. Since $k(\alpha)$ must be an integer we may conclude that $k(\alpha) \geqslant 11$. This compares with the value of 17 for $M_b(\alpha) + M_c(\alpha)$. Applying the theorem to Example 3.6, where we already know that $k(\alpha) \leqslant 5$, gives $k(\alpha) \geqslant 13/6$ and hence $k(\alpha) \geqslant 3$.

REMARK (5.7). The upper and lower bounds given respectively by Theorems 4.3 and 5.1 are still potentially some distance apart. They are moreover somewhat unsatisfactory in that they are expressed in terms of such different parameters: it is not even obvious directly that

$$\text{shift } \alpha/\text{def } \alpha \leqslant M_b(\alpha) + M_c(\alpha).$$

It is possible that a better lower bound could be obtained by an analysis along the lines of [3] of the way in which the orbital structure of (with $\alpha \in S_n, \alpha \in E$) is determined by the orbital structure of α and the way that ε relates to it.

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