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Department of Mathematics

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# SOME AUXILIARY OPERATORS IN AUT-Π.

by

N.G. de Bruijn.

University of Technology
Department of Mathematics
P.O.Box 513 Eindhoven.
The Netherlands.

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For AUT-Π we refer to Zucker [2]. If we omit all those features that the languages of the AUTOMATH family have in common (cf. the description of AUT-QE in D. van Daalen [1]), the basic rules are the following (i) - (vii). Two simplifications are made here. First, we use a symbol $\tau$ which may be either type or prop. Secondly, we omit all Π's in expressions of degree 1, which does not make any essential difference. And we use the notation $(x : \alpha) \vdash$ in order to indicate that something is valid in the context extended by x (of type $\alpha$). As in [1], $[\![x/A]\!]\, Z$ means that in Z we have to replace x by A.

The rules are

(i) $\quad \vdash^1 \tau$

(ii) $$\frac{\vdash^2 \alpha : \tau \qquad (x : \alpha) \vdash^1 P}{\vdash^1 [x : \alpha]\, P}$$

(iii) $$\frac{\vdash^2 \alpha : \tau \qquad (x : \alpha) \vdash^2 Q : P}{\vdash^2 [x : \alpha] Q \; : \; [x : \alpha] P}$$

(iv) $$\frac{\vdash^3 A : \alpha : \tau \qquad \vdash^2 Q : [x : \alpha] P}{\vdash^2 \{A\} Q \; : \; [\![x/A]\!]\, P}$$

(v) $$\frac{\vdash^2 \alpha : \tau \qquad \vdash^2 Q : [x : \alpha]\tau}{\vdash^2 \Pi Q \; : \; \tau}$$

(vi) $$\frac{\vdash^2 \alpha : \tau \qquad (x : \alpha) \vdash^3 R : Q : \tau}{\vdash^3 [x : \alpha] R \; : \; [x : \alpha] Q}$$

(vii) $$\frac{\vdash^3 A : \alpha : \tau \qquad \vdash^3 R : \Pi Q \qquad \vdash^2 Q : [x : \alpha]\tau}{\vdash^3 \{A\} R \; : \; \{A\}\, Q}$$

We shall now define operators $\theta_1, \theta_2, \ldots, \theta_m$, acting on Q's with

$$\vdash^2 Q : [x_1 : \alpha_1] \ldots [x_m : \alpha_m]\tau$$

The symbols are metalinguistic: $\theta_j Q$ is used in the metalanguage to indicate a certain expression in the language, viz.

$$\theta_1 Q = [x_1 : \alpha_1] \ldots [x_{m-1} : \alpha_{m-1}] \Pi \{x_{m-1}\} \{x_{m-2}\} \ldots \{x_1\} Q$$

$$\theta_2 Q = [x_1 : \alpha_1] \ldots [x_{m-2} : \alpha_{m-2}] \Pi [x_{m-1} : \alpha_{m-1}] \Pi \{x_{m-1}\} \ldots \{x_1\} Q$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\theta_{m-1} Q = [x_1 : \alpha_1] \Pi [x_2 : \alpha_2] \Pi \ldots \Pi [x_{m-1} : \alpha_{m-1}] \Pi \{x_{m-1}\} \ldots \{x_1\} Q$$

$$\theta_m Q = \Pi [x_1 : \alpha_1] \Pi [x_2 : \alpha_2] \Pi \ldots \Pi [x_{m-1} : \alpha_{m-1}] \Pi \{x_{m-1}\} \ldots \{x_1\} Q$$

Note that $\theta_j$ is built by starting from the expression just given for $\theta_m Q$ and then omitting the first m-j $\Pi$'s. If m=1 we just have $\theta_1 Q = \Pi Q$. If m=2 then $\theta_1 Q =$ $= [x_1 : \alpha_1] \Pi \{x_1\} Q$ and $\theta_2 Q = \Pi [x_1 : \alpha_1] \Pi \{x_1\} Q$. If m=0 none of the $\theta_j$'s are defined.

We can now prove the validity of a new rule, viz:

(viii) $$\frac{\vdash^2 \alpha : \tau \qquad \vdash^2 Q : [x_1 : \alpha_1] \ldots [x_m : \alpha_m]\tau}{\vdash^2 \theta_j Q : [x_1 : \alpha_1] \ldots [x_{m-j} : \alpha_{m-j}]\tau}$$

for $1 \leq j \leq m$. If j=m-1 it is just the old rule (v).

For shortness we shall write $[x_i]$ and $(x_i)$ instead of $[x_i : \alpha_i]$ and $(x_i : \alpha_i)$.

Let us start from

$$\vdash^2 \alpha : \tau \qquad \vdash^2 Q : [x_1] \ldots [x_m]\tau \tag{1}$$

Applying (iv) we get

$$(x_i) \vdash^2 \{x_1\} Q : [x_1] \ldots [x_m]\tau$$

and m-2 more applications of the same rule leads to

$$(x_1) \ldots (x_{m-1}) \vdash^2 \{x_{m-1}\} \ldots \{x_1\} Q : [x_m]\tau$$

Next we apply (v):

$$(x_1) \ldots (x_{m-1}) \vdash^2 \Pi \{x_{m-1}\} \ldots \{x_1\} Q : \tau$$

and by (iii) this gives

$$(x_1)\dots(x_{m-2}) \overset{2}{\vdash} [x_{m-1}]\,\Pi\,\{x_{m-1}\}\dots\{x_1\}Q : [x_{m-1}]\tau \qquad (2)$$

Now m-2 further applications of (iii) gives

$$\overset{2}{\vdash} \theta_1 Q : [x_1]\dots[x_{m-1}]\,\tau$$

On the other hand, if we apply (v) to (2) followed by a single application of (iii) we get

$$(x_1)\dots(x_{m-3}) \overset{2}{\vdash} [x_{m-2}]\,\Pi\,[x_{m-1}]\,\Pi\,\{x_{m-1}\}\dots\{x_1\}Q : [x_{m-2}]\tau \qquad (3)$$

Now m-3 more applications of (iii) lead to

$$\overset{2}{\vdash} \theta_2 Q : [x_1]\dots[x_{m-2}]\tau .$$

On the other hand, if we apply (v) followed by (iii) to (3) we get

$$(x_1)\dots(x_{m-4}) \overset{2}{\vdash} [x_{m-3}]\,\Pi\,[x_{m-2}]\,\Pi\,[x_{m-1}]\,\Pi\,\{x_{m-1}\}\dots\{x_1\}Q : [x_{m-3}]\tau .$$

This way we get, indeed

$$\overset{2}{\vdash} \theta_j Q : [x_1]\dots[x_{m-j}]\,\tau \qquad (4)$$

for all j ($1 \le j \le m$).

We shall also show that $\theta_i\theta_j \overset{D}{=} \theta_{i+j}$. More precisely, if $\overset{2}{\vdash} Q : [x_1 : \alpha_1]\dots$ $\dots[x_m : \alpha_m]\tau$, and if $i \ge 1$, $j \ge 1$, $i+j \le m$, then $\theta_i\theta_j\,Q$ reduces to $\theta_{i+j}\,Q$ by means of repeated β-reduction. First we have (4), i.e.

$$\overset{2}{\vdash} [x_1]\dots[x_{m-j}]\,\Pi\,[x_{m-j+1}]\,\Pi\dots\Pi[x_{m-1}]\,\Pi\,\{x_{m-1}\}\dots\{x_1\}Q$$

$$: [x_1]\dots[x_{m-j}]\tau .$$

Applying $\theta_i$ to this we get

$$\overset{2}{\vdash} \theta_i\theta_j\,Q : [x_1]\dots[x_{m-j-i}]\,\tau$$

and

$$\theta_i\,\theta_j\,Q = [y_1]\ldots[y_{m-j-1}]\;\Pi\;[y_{m-j-i+1}]\;\Pi\ldots\Pi\,[\,y_{m-j-1}]\;\Pi\;\{\,y_{m-j-1}\}\ldots$$

$$\ldots\;\{y_1\}\;[x_1]\ldots[x_{m-j}]\;\Pi\;[x_{m-j+1}]\Pi\ldots\Pi[x_{m-1}]\;\Pi\;\{x_{m-1}\}\ldots\{x_1\}Q.$$

The sequence $\{y_{m-j-1}\}\ldots\{y_1\}\;[x_1]\ldots[x_{m-j-1}]$ is annihilated by m applications of β-reduction. After that, we change the names $y_1,\ldots y_{m-j-1}$ into $x_1,\ldots,x_{m-j-1}$, thus arriving at $\theta_{i+j}\,Q$.

Above we extended rule (v) to rule (viii). Similarly we shall extend rule (vii) to the following rule (ix) for $m \geq 1$:

$$\text{(ix)}\qquad \frac{\vdash^3 A : \alpha_1 \qquad \vdash^3 R : \theta_m\,Q \qquad \vdash^2 Q : [x_1 : \alpha_1]\ldots[x_m : \alpha_m]\tau}{\vdash^3 \{A\}\;R \quad : \quad \{A\}\,\theta_{m-1}\,Q}$$

If m=1 we have $\theta_m Q = \Pi\,Q$, and $\theta_{m-1}\,Q$ has to be explained as Q itself ($\theta_o$ was not defined before).

Rule (ix) is not hard to derive. Noting that $\theta_m\,Q = \Pi\theta_{m-1}\,Q$, and $\vdash^2 \theta_{m-1}\,Q$ : : $[x_1 : \alpha_1]\tau$ by (viii), we can apply (vii) with Q replaced by $\theta_{m-1}\,Q$, which leads to $\vdash^3 \{A\}\;R : \{A\}\theta_{m-1}\,Q$.

We note that in all rules formulas of the type $\vdash^3 R : Q$ lead to $\vdash^2 Q : \tau$ . Indeed, in (vi) we have $\vdash^2 \Pi\,[x : \alpha]Q : \tau$ by (v), and in (ix) we have $\vdash^2 \{A\}\theta_{m-1}Q : \tau$ by (iv), according to the typing of $\theta_{m-1}\,Q$ just derived.

Instead of the lower kind of (ix) we may as well get

$$\vdash^3 \{A\}\;R : \theta_{m-1}\;\{A\}\;Q$$

since $\{A\}\theta_{m-1}\,Q$ reduces to $\theta_{m-1}\,\{A\}\;Q$ by a single beta reduction.

More generally we observe that

$$\{A\}\theta_j\,Q \text{ reduces to } \theta_j\;\{A\}\;Q$$

by a single beta reduction if $j < m$.

The symbols $\theta_j$ also commute with abstraction : if

$$\vdash^2 Q : [x_1 : \alpha_1]\ldots[x_m : \alpha_m]\tau \text{ then}$$

$$[y : \beta]\;\theta_j\;Q \text{ reduces to } \theta_j\;[y : \beta]Q$$

if $j \leq m$.

These observations mean that in composite expressions like

{ } θ [ ] θ { } { } θ [ ] Q the θ's may all be shifted to the extreme left

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