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By : P.E. Jupp and B.D. Spurr

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Mathematical Institute, University of St Andrews, North Haugh, St Andrews KY16 9SS, Scotland.

SUMMARY

For testing a probability distribution on a compact Riemannian manifold for symmetry under the action of a given group of isometries, two classes of invariant tests are proposed and their properties studied. These tests are based on Sobolev norms and generalize Giné's Sobolev tests of uniformity. For general compact manifolds randomization tests analogous to Wellner's tests for the two-sample case are suggested. For the circle, distribution-free tests of symmetry based on uniform scores are provided. It is shown that every Sobolev test of uniformity can be decomposed into a test of symmetry and a test of uniformity on a quotient space.

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SOBOLEV TESTS FOR SYMMETRY OF DIRECTIONAL DATA

By: P.E. JUPP and B.D. SPURR
University of St. Andrews

Running head: SOBOLEV TESTS FOR SYMMETRY

1. Introduction. There are situations in the analysis of directional data where a natural hypothesis about the underlying distributions is that it has a specified type of symmetry. For example, one might ask about a probability distribution on the circle whether it is antipodally symmetric. If so, then it is in fact a distribution on the set of axes, or equivalently on the doubled angles. It would be prudent to test bimodal circular data for such symmetry before attempting to fit an axial distribution. More generally, one should test circular data for ℓ-fold symmetry before fitting a multi-modal distribution of the type introduced by Mardia and Spurr (1973). Similarly, one might ask about a distribution on the sphere whether it is symmetric about a given axis, whether it is antipodally symmetric, or indeed whether it is the uniform distribution. Although the circle and the sphere are the sample spaces of most practical interest for directional data,

those based on the empirical distribution function (a concept which seems unavailable for compact manifolds other than the circle and tori). Rank and permutation tests can be found in Hájek and Šidák (1967). A test of Kolmogorov-Smirnov type was proposed by Butler (1969) and tests of Cramér-von Mises type have been introduced by Orlov (1972) and Rothman and Woodroofe (1972), Srinavasan and Godio (1974), and Hill and Rao (1977). One of our uniform-scores tests (Example 2 of Section 4) is an analogue for the circle of that given by Rothman and Woodroofe. Feuerverger and Mureika (1977) suggested testing symmetry on the line by a Cramér-von Mises type statistic based on the empirical characteristic function. This is very similar to the test statistics T_n of our Section 3 which are based on Sobolev semi-norms. There appears to be little previous work on testing for symmetry (other than uniformity) on compact manifolds. This work includes the tests of symmetry about an axis of a circular distribution given by Schach (1969a) and Mardia (1972, page 195) and the likelihood ratio tests for axial symmetry of a Bingham distribution on a sphere by Bingham (1974). Our tests have little connection with these.

In Section 2 of this paper we review the material on Riemannian manifolds and Sobolev norms which we shall need. For details of this background material we refer the reader to Giné (1975) and for the background to directional statistics we recommend Mardia (1972). Proposition 2.1 provides the test is that ν is invariant under this action, i.e. that $g_*\nu = \nu$ for all $g \in G$. If ν is invariant under G, then it is also invariant under the closure of G. Therefore we shall assume that G is closed in the isometry group. It follows that G is a compact Lie group. (See Theorem 3.4 on page 239 of Kobayashi and Nomizu, 1963.) Thus we can use normalised Haar measure λ on G to average the actions on $C(\mathfrak{X})$ and $\mathcal{M}(\mathfrak{X})$. For $f \in C(\mathfrak{X})$

we define $$\bar{f} = \int_G g^*f \, d\lambda(g)$$

i.e. $$\bar{f}(x) = \int_G f(g.x) \, d\lambda(g)$$

and for $\nu \in \mathcal{M}(\mathfrak{X})$

we define $$\bar{\nu} = \int_G g_*\nu \, d\lambda(g)$$

i.e. $$\int_{\mathfrak{X}} f \, d\bar{\nu} = \int_{\mathfrak{X}} \bar{f} \, d\nu.$$

Now $C(\mathfrak{X})$ and $\mathcal{P}(\mathfrak{X})$ have the important direct-sum decompositions

$$C(\mathfrak{X}) = C(\mathfrak{X})_+ \oplus C(\mathfrak{X})_-$$

$$\mathcal{P}(\mathfrak{X}) = \mathcal{P}(\mathfrak{X})_+ \oplus \mathcal{M}(\mathfrak{X})_-$$

we can define $\|\nu\|_A^2 = \|\nu\|^2$ by

$$(2.1) \qquad \|\nu\|^2 = \sum_{k=1}^{\infty} \alpha_k^2 \sum_{f_i \in E_k} \left(\int f_i \, d\nu \right)^2 .$$

Thus $\|\nu\|^2$ is a weighted sum of squares of Fourier coefficients of ν. A neat expression for $\|\nu\|^2$ can be obtained from the function $\underset{\sim}{t} : \mathcal{X} \to L^2(\mathcal{X},\mu)$ defined by $\underset{\sim}{t}(x) = \sum_{k=1}^{\infty} \alpha_k \underset{\sim}{t}_k(x)$. If $\{\alpha_k\}_{k=1}^{\infty}$ satisfies condition C then

$$(2.2) \qquad \|\nu\|^2 = \left\| \int \underset{\sim}{t} \, d\nu \right\|_2^2$$

where $\|\cdot\|_2$ is the L^2 norm on $L^2(\mathcal{X},\mu)$. Giné's (1975) tests for uniformity can be related to those of Beran (1968, 1969a) using the equation

$$(2.3) \qquad \|\nu\|^2 = \int_{\mathcal{X}} \left[\int_{\mathcal{X}} g(x,y) \, d\nu(x) \right]^2 d\mu(y)$$

where $g(x,y) = \sum_{k=1}^{\infty} \alpha_k \sum_{f_i \in E_k} f_i(x) \, f_i(y)$.

We also have

$$(2.4) \qquad \|\nu\|^2 = \int_{\mathcal{X} \times \mathcal{X}} h(x,y) \, d\nu(x) \, d\nu(y)$$

where $h(x,y) = \sum_{k=1}^{\infty} \alpha_k^2 \sum_{f_i \in E_k} f_i(x) f_i(y) = \langle \underset{\sim}{t}(x), \underset{\sim}{t}(y) \rangle$, with $\langle \cdot,\cdot \rangle$ denoting the L^2 inner-product. The $\|\cdot\|$ given by (2.1), (2.2), (2.3) or (2.4) were introduced by Giné (1975) and form the basis of his tests for uniformity. They are semi-norms on $\mathcal{M}(\mathcal{X})$ which we shall call <u>Sobolev semi-norms</u>.

uniformity on the circle splits into a multiple of Ajne's (1968) A_n and a multiple of the U^2-statistic for testing uniformity of the doubled angles. Although A_n is usually regarded as a test of uniformity, it may usefully be considered as a test of symmetry under the antipodal action ($\theta \to \theta+\pi$) of $G = \mathbb{Z}_2$. The familiar "doubling of the angles" can be interpreted as the quotient map from the circle to the projective line. A similar partitioning of a particular Sobolev test of uniformity on the sphere, S^2, is given in Proposition 6.4 of Giné (1975).

If only finitely many α_k are non-zero then $\underset{\sim}{t}(\mathcal{X})$ is contained in a finite-dimensional subspace of $L^2(\mathcal{X},\mu)$. One method of testing for symmetry is to take a standard multivariate approach and to consider the Hotelling T^2-statistic.

$$T^2 = n\left(\int \underset{\sim}{t}_- \, d\varepsilon_n\right)' S^{-1} \left(\int \underset{\sim}{t}_- \, d\varepsilon_n\right)$$

where S is the sample covariance matrix of $\underset{\sim}{t}_-$. The asymptotic null distribution of T^2 is that of a chi-squared random variable with degrees of freedom equal to the dimension of the smallest affine subspace containing $\underset{\sim}{t}_-(\mathcal{X})$ (at least for distributions with support equal to $\mathcal{X}$). The sequence of tests which reject the hypothesis of symmetry for large values of T^2 is consistent against an alternative ν if and only if $\|\nu\|_-^2 > 0$. Thus no test of this type can be consistent against all alternatives. It is partly for this reason that we consider instead the randomization tests of the next section.

which reduces to

$$T_n = n\|\varepsilon_n\|_-^2 . \qquad (3.2)$$

Our tests use T_n as test-statistic and reject the null hypothesis of symmetry for large values of T_n. The observed value of T_n is compared with its null distribution conditional on $\bar{\varepsilon}_n$, the symmetrized distribution of the sample. More precisely, for a test at significance level α, the test functions ϕ are given by

$$\begin{aligned} \phi(\varepsilon_n) &= 1 \quad \text{if} \quad T_n(\varepsilon_n) > c_n \\ &= \gamma_n \quad \text{if} \quad T_n(\varepsilon_n) = c_n \\ &= 0 \quad \text{if} \quad T_n(\varepsilon_n) < c_n \end{aligned}$$

where $c_n = c(\varepsilon_n,\alpha)$ and $\gamma_n = \gamma(\varepsilon_n,\alpha)$ are the unique smallest non-negative numbers such that

$$E\{\phi(\varepsilon_n) \mid \bar{\varepsilon}_n\} = \alpha. \qquad (3.3)$$

A difficulty with these tests as with all randomization tests is that of determining the null distribution of T_n conditional on $\bar{\varepsilon}_n$. If G is finite, of order ℓ say, this can in principle be done by enumeration but involves $O(\ell^n)$ operations. Accordingly, we suggest following Dwass (1957) and Wellner (1979) in simulating the distribution by sampling from the distribution under H_0 of $(X_1,\ldots,X_n)$ conditional on $\bar{\varepsilon}_n$, where $X_1,\ldots,X_n$ are i.i.d. random variables on $\mathcal{X}$.

THEOREM 3.2 (Local alternatives). Suppose that $\nu \in \mathcal{P}(\mathcal{X})_+$ and that $\{\nu_n\}_{n=1}^{\infty}$ is a sequence in $\mathcal{P}(\mathcal{X})$ satisfying $\nu_n \to_{w^*} \nu$ and

$$\lim_{n\to\infty} n^{\frac{1}{2}} \int_{\mathcal{X}} f_i \, d(\nu_n - \nu) = d_i \text{ for } f_i \in E_{k-} \text{ with } \sum_{k=1}^{\infty} \alpha_k^2 \sum_{f_i \in E_{k-}} d_i^2 < \infty.$$

Then

$$T_n \to_d \sum_{k=1}^{\infty} \alpha_k^2 \sum_{f_i \in E_{k-}} \left\{ Z^{(\nu)}(f_i) + d_i \right\}^2 .$$

In particular, this gives the asymptotic distribution under the null hypothesis.

THEOREM 3.3 (Fixed alternatives). If ε_n is the empirical distribution of a random sample from $\nu \in \mathcal{P}(\mathcal{X})$ and if $\|\nu\|_-^2 > 0$, then

$$n^{-\frac{1}{2}} (T_n - n\|\nu\|_-^2) \to_d \mathcal{N}(0, \mathrm{Var}_\nu(u))$$

where $u(x) = 2 \sum_{k=1}^{\infty} \alpha_k^2 \sum_{f_i \in E_{k-}} \left(\int f_i d\nu \right) f_i(x)$ i.e. $u = 2E_\nu[\underset{\sim}{t}]$.

(4.2) $$T_n^* = n\|\eta_n\|_-^2.$$

The isometry group of the circle is $O(2)$, the orthogonal group of $\mathbb{R}^2$, consisting of rotations and reflections. The only closed subgroups G are:

(i) $G = O(2)$ or $G = SO(2)$, the rotation group. In either case, $N(G) = O(2)$ and G-invariance is the same as uniformity. The functions $\{\sqrt{2}\cos k\theta, \sqrt{2}\sin k\theta\}$ form an orthonormal basis of $E_k = E_{k-}$.

(ii) $G = \mathbb{Z}_\ell$, a cyclic group of rotations, generated by $\theta \to \theta + 2\pi/\ell$ for some positive integer ℓ. Then $N(G) = O(2)$ and $L^2(S^1,\mu)_-$ has $\{\sqrt{2}\cos k\theta, \sqrt{2}\sin k\theta\}_{k \not\equiv 0 (\mathrm{mod}\ \ell)}$ as an orthonormal basis.

(iii) G is generated by a cyclic group of rotations, $\mathbb{Z}_\ell$, and by a reflection (which we may assume to be about the axis through 0 and π). Then $N(G)$ is generated by this reflection and by $\mathbb{Z}_{2\ell}$. An orthonormal basis of $L^2(S^1,\mu)_-$ is given by

$$\{\sqrt{2}\cos k\theta, \sqrt{2}\sin k\theta\}_{k\not\equiv 0(\mathrm{mod}\ \ell)} \cup \{\sqrt{2}\sin k\theta\}_{k\equiv 0(\mathrm{mod}\ \ell)}.$$

In particular, if G is generated by a single reflection, $\{\sqrt{2}\sin k\theta\}_{k=1}^{\infty}$ is an orthonormal basis of $L^2(S^1,\mu)_-$. Tests for symmetry under reflection in a given axis have been considered by Schach (1969a) and by Mardia (1972, page 195). Their tests are not of our type.

used by Beran (1969b)

$$(4.3) \qquad T_n^* = \frac{1}{n} \sum_{i=1}^{n} \sum_{j=1}^{n} h_-(\beta_i - \beta_j)$$

where $h_-(\theta) = 2 \sum_{k \not\equiv 0 (\text{mod } \ell)} \alpha_k^2 \cos k\theta$. Equivalence of the two forms follows from Proposition 5.2 of Giné (1975). Note that, as $\|\cdot\|_-^2$ is invariant under $O(2)$, T_n^* is well-defined independently of the origin and orientation of S^1. Note also that T_n^* is invariant under $O(2)$.

The case where G is a finite group which includes a reflection is similar except that the origin may be taken as any point on an axis of symmetry and that no analogue of the computational formula (4.3) seems to be available. In the case when G is $O(2)$ or $SO(2)$, $\eta_n = \varepsilon_n$ and our tests coincide with those of Giné (1975).

An important property of the uniform scores tests is that they are distribution-free for sampling from continuous distributions. This is because under the null hypothesis of symmetry and in the absence of ties, the distribution of η_n conditional on $\bar{\eta}_n$ is the image of a uniform distribution on G^n. The null distribution of T_n^* may be determined by enumeration or by sampling. This null distribution is not in general the same as that of the corresponding test for uniformity or two-sample tests. However the following theorem combined with Theorem 4.1 of Giné (1975) and Theorem 1 of Beran (1969b), shows that the asymptotic distributions are the same.

If $\{\alpha_k k^s\}$ is bounded for some $s > 3/2$, then the sequence of tests based on T_n^* is consistent against ν if and only if $\|H_* \nu\|_-^2 > 0$.

The proofs of Theorems 4.1 and 4.2 are given in the Appendix.

We conclude this section with two examples.

EXAMPLE 1.

If $\alpha_1 = 1$ and $\alpha_k = 0$, for $k \geq 2$, then $T_n^* = 2n\bar{R}^2$ where $\bar{R}$ is the mean resultant length of the uniform scores. Also $T_n^* = 2n^{-1} \sum_{i=1}^{n} \sum_{j=1}^{n} \cos(\beta_i - \beta_j)$. The analogous test of uniformity is the Rayleigh test (see, for example, Mardia 1972, page 133) and the analogous two-sample test is the uniform scores test of Wheeler and Watson (1964) and Mardia (1967). The test based on T_n^* is appropriate for testing $\mathbb{Z}_\ell$-symmetry for any integer $\ell \geq 2$. Under the null hypothesis we have, asymptotically, $T_n^* \sim \chi_2^2$. Some critical values of T_n^* for $\ell = 2$ are given in Table 1.

EXAMPLE 2.

If $\alpha_k = k^{-1}$ for $k \geq 1$, then it can be show that, for testing $\mathbb{Z}_\ell$-symmetry contain misprints. Beran's equation (3.8) is correct in the case of equal sample sizes but the denominator in the term $(2j-1)/N$ should be $2m$. In Mardia's (7.4.13), the term $\{n(2i-1) - n_1\}/2n_1$ should be $n\{(2i-1) - n_1\}/2n_1$.]

The statistic T_n^* in (4.3) can also be expressed as

$$T_n^* = 4\pi^2 n \int_0^{2\pi} (F_\varepsilon(\theta) - F_{\bar{\varepsilon}}(\theta) - \mu)^2 \, dF_{\bar{\varepsilon}}(\theta)$$

where $$\mu = \int_0^{2\pi} (F_\varepsilon(\theta) - F_{\bar{\varepsilon}}(\theta))dF_{\bar{\varepsilon}}(\theta)$$

and F_ε and $F_{\bar{\varepsilon}}$ are the distribution functions of ε and $\bar{\varepsilon}$. It can be seen that T_n^* is a statistic of Cramér-von Mises type generalizing Watson's one-sample U^2. It is also an analogue on the circle of the statistic R_n introduced by Rothman and Woodroofe (1972) to test symmetry on the line.

When testing for $\mathbb{Z}_\ell$-symmetry, the same T_n^* as above is obtained on taking $\alpha_k = k^{-1}$ for $k \not\equiv 0 \pmod{\ell}$ and $\alpha_k = 0$ otherwise. For $\ell = 2$, we have

$$T_n^* = \pi^2 A_n$$

where A_n is Watson's (1967) A_n-statistic due to Ajne (1968) applied to the uniform scores. (In equation (6.5) of Giné, 1975, the coefficient of $T_n^{(1)}(\{\alpha_k\})$ should be π^{-2} instead of $4\pi^{-1}$.) A useful computing formula for A_n from Beran (1969b) is $A_n = n/4 - (n\pi)^{-1} \sum_{i} \sum_{< j} d(\beta_i, \beta_j)$ where

$$P(A_n > x) = P(T_n^* > \pi^2 x) \to 4\pi^{-1} \sum_{k=1}^{\infty} (-1)^{k-1}(2k-1)^{-1}\exp\{-(2k-1)^2\pi^2 x/2\}.$$

Upper tail percentage points are tabulated in Stephens (1969). It is interesting to note that this limiting distribution of A_n is also the distribution of the time to absorption of a particle starting at 1/2 and performing Brownian motion on [0,1] with an absorbing barrier at each end. (See Feller 1966, page 330.) It would be nice to have a direct probabilistic explanation for this connection. Some critical values of T_n^* for $\ell = 2$ are given in Table 2.

APPENDIX

We outline here the proofs of the theorems in Sections 3 and 4. The proofs parallel those of Giné (1975) and Wellner (1979) with the difference that one of our basic tools is the central limit theorem for triangular arrays rather than the usual central limit theorem or a permutational version. It will be useful to give also some auxiliary results.

We shall need to follow Giné and Wellner in considering the convergence properties of certain processes on $C(B_s)$, where B_s is the unit ball in the Sobolev space $H_s(\mathcal{X})$ endowed

For (i) it suffices to prove convergence of the one-dimensional distributions. For $f \in L^2(\mathfrak{X},\mu)$, we have $X_n(f) = \sum_{i=1}^n X_{i,n}$, where $X_{i,n} = n^{-\frac{1}{2}} f(g_i . x_{i,n})$. We can now apply the central limit theorem for triangular arrays, Theorem 2 on page 128 of Gnedenko and Kolmogorov (1954). By the Sobolev lemma, for $s > (\dim \mathfrak{X})/2$, $B_s \subset C(\mathfrak{X})$. As we consider only those f in B_s, we may assume that f is bounded. This ensures that the conditions of the theorem of Gnedenko and Kolmogorov hold. As $E\{X_n(f)\} = 0$ and $\mathrm{Var}\{X_n(f)\} = \int_{\mathfrak{X}} f^2 \, d\bar{\nu}_n \to \int_{\mathfrak{X}} f^2 \, d\nu$, convergence of $X_n(f)$ to $Z^{(\nu)}(f)$ follows.

Tightness of $\{\mathcal{L}(X_n | B_{s-})\}_{n=1}^{\infty}$ can be proved as in Giné (1975, page 1251), on replacing his F_n and $M_{n,\varepsilon}$ by their images in $C(B_{s-})$ and on using $\int_{\mathfrak{X}} f^2 \, d\bar{\nu}_n$ in the variance calculations.

PROOF OF THEOREM 3.1.

Giné (1975, Theorem 4.1) defines a seminorm h on $C(B_s)$ extending $\|\cdot\|$ on $\mathcal{P}(\mathfrak{X})$. If the sequence $\{\alpha_k\}_{k=1}^{\infty}$ satisfies condition C, then $h \in C(C(B_s))$. As $T_n = \{h(Z_n)\}^2$, we have $n^{-1}T_n = \{h(\varepsilon_n)\}^2 \to \|\nu\|_-^2$.

PROOF OF THEOREM 4.2.

Let H_n denote a probability integral transform of $\bar{\varepsilon}_n$. Then

$$n^{-1}T_n^* = \|\eta_n\|_-^2 = \|(H_n)_* \bar{\varepsilon}_n\|_-^2 \longrightarrow \|H_*\nu\|_-^2 ,$$

so that $\|H_*\nu\|_-^2 > 0$ implies consistency.

For the converse, we consider the space D of right-continuous functions on $[0,2\pi]$ which have left-hand limits. Let $B_{s,1}$ denote B_s with the topology of uniform convergence of functions and their first derivatives. Then we define $\Psi : D \to C(B_{s,1})$ by

$$\Psi(F)(f) = \int_0^{2\pi} f(2\pi\tilde{F}(t))dF(t) \qquad f \in B_s$$

where $\tilde{F}(t) = \ell^{-1} \sum_{j=0}^{\ell-1} \{F(t+j2\pi\ell^{-1}) - F(j2\pi\ell^{-1})\}$, $0 \leqslant t \leqslant \ell^{-1}$, and $\tilde{F}(t+2\pi\ell^{-1}) = \tilde{F}(t) + \ell^{-1}$ in the case $G = \mathbb{Z}_\ell \subset SO(2)$ with an analogous definition of $\tilde{F}(t)$ if G contains a reflection. Let F_{ε_n} and F_ν be the distribution functions of ε_n and ν. Then $\Psi(F_{\varepsilon_n})$ and $\Psi(F_\nu)$ are the images in $C(B_{s,1})$ of η_n and $H_*\nu$. As $s > {}^3{}_2$, the Sobolev lemma ensures that functions in B_s are differentiable. It can then be shown that Ψ is differentiable if D is given the uniform $(\|\cdot\|_\infty)$ topology. Now, $\|n^{\frac{1}{2}}(F_{\varepsilon_n} - F_\nu)\|_\infty$ has a limiting (Kolmogorov-Smirnov) distribution while $n^{\frac{1}{2}}(F_{\varepsilon_n} - F_\nu)$ has a limiting distribution in D when D is given the Skorohod topology (Billingsley, 1968, page 141).

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TABLE 1

Significance points of Rayleigh-type statistic T_n^* for $\mathbb{Z}_2$-symmetry (Example 1)

(Exact levels, $P(T_n^* \geq x)$, in parentheses)

sample size n	approximate significance level 0.10		0.05		0.025		0.01	
7	5.77	(0.11)						
8	4.72	(0.13)	6.57	(0.06)				
9	5.70	(0.07)	7.37	(0.04)				
10	4.42	(0.12)	6.65	(0.04)	8.17	(0.02)		
11	4.80	(0.10)	6.36	(0.04)	7.58	(0.02)	8.98	(0.01)
12	4.55	(0.10)	5.73	(0.05)	7.34	(0.025)	8.49	(0.01)
13	4.55	(0.10)	6.10	(0.05)	7.28	(0.02)	8.30	(0.01)
14	4.57	(0.10)	5.90	(0.05)	7.15	(0.025)	8.36	(0.01)
15	4.66	(0.10)	5.84	(0.05)	6.97	(0.025)	8.43	(0.01)
∞	4.61		5.99		7.38		9.21	