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A note on polynomiallike arithmetical functions.

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1. Introduction. Very often, not only in Number-Theory or Analysis, it is interesting to decide whether a given function is a polynomial. In the present note we show, that the polynomials are the only arithmetical functions $f : \mathbb{Z} \to \mathbb{Z}$ which satisfy a growth-condition and a certain divisibility-property.

Denote by $\mathcal{G}_t$ the set of functions $f : \mathbb{Z} \to \mathbb{Z}$, such that

$$(*) \qquad a-b \mid f(a) - f(b)$$

and

$$(**) \qquad f(n) = O(|n|^t)$$

hold.

Our aim is to prove the following

Theorem: Let $f \in \mathcal{G}_t$. Then f is an integer-valued polynomial.

For abbreviation denote by $\mathcal{P}_t$ the set of integer-valued polynomials $g : \mathbb{Z} \to \mathbb{Z}$ with degree $\leq t$ and by $\mathcal{P}^*_t$ the subset of such polynomials with integer coefficients. Clearly the theorem means, that $\mathcal{G}_t$ is included in $\mathcal{P}_t$.

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2. Auxiliary Lemmas

In order to prove the theorem, we list some useful lemmas.

Lemma 1: $\mathcal{G}_t$ is a Z-Module.

Proof: Clear.

Lemma 2: $\mathcal{P}_t^*$ is a submodule of $\mathcal{G}_t$.

Proof: Clear.

Lemma 3: For any $a \in \mathbf{Z}$ and any $k \in \mathbf{N}$, the integer

$$\gcd \quad (a\cdot(a+1)\cdot\ldots\cdot(a+k-1),\ a+k)$$

is a divisor of $k!$.

Proof: By induction. $k = 1$ is trivial.
Assume that for every $a \in \mathbf{Z}$ the relation

$$\gcd \quad (a\cdot(a+1)\cdot\ldots\cdot(a+k-1),\ a+k) \mid k!$$

holds and set

$$d := \gcd \quad (a\cdot(a+1)\cdot\ldots\cdot(a+k),\ a+k+1) .$$

Suppose $p^\alpha \parallel d$, $p^\beta \parallel a$, $p^\gamma \parallel (a+1)\cdot\ldots\cdot(a+k)$ and $p^\delta \parallel a+k+1$.

Then clearly $\alpha = \min(\beta+\gamma, \delta)$.

We have $p^{\min(\beta,\delta)} \mid k+1$ and $p^{\min(\gamma,\delta)} \mid k!$ by assumption,

and therefore $p^{\min(\beta,\delta) + \min(\gamma,\delta)} \mid (k+1)!$.

The inequality

$$\min(\beta+\gamma, \delta) \leq \min(\beta, \delta) + \min(\gamma, \delta)$$

implies

$$p^\alpha \mid (k+1)!$$

and this yields the assertion in the case $k+1$.

3. Proof of the theorem

a) We start by constructing functions f_j, $1 \leq j \leq t$, with the following properties:

(i) $f_j \in \mathcal{G}_t$

(ii) $f_j(m) = 0$ for $m = 0, \ldots, j-1$

(iii) $\prod_{m=0}^{j-1} (n-m) \mid f_j(n)$ for $n \notin \{0, \ldots, j-1\}$.

Clearly, the function

$$f_1(n) := f(n) - f(0)$$

satisfies the conditions (i), (ii) and (iii) .
Assume, that f_{j-1} has already been constructed.
Define

$$(3.1) \qquad \tilde{f}_j(n) := f_{j-1}(n) - n(n-1)\cdot\ldots\cdot(n-j+2)\,\frac{f_{j-1}(j-1)}{(j-1)!}$$

$$(3.2) \qquad = f_{j-1}(n) - f_{j-1}(j-1) - \frac{f_{j-1}(j-1)}{(j-1)!}\,\{n(n-1)\cdot\ldots\cdot(n-j+2)-(j-1)!\}$$

Abbreviating $n(n-1)\cdot\ldots\cdot(n-j+2)-(j-1)!$ by $q(n)$, we obtain

$$q(j-1) = 0$$

and therefore

$$(3.3) \qquad (n-j+1) \mid \tilde{f}_j(n)$$

because of (*) and (3.2).

It follows from (3.1), that

$$(3.4) \qquad n(n-1)\cdot\ldots\cdot(n-j+2) \mid \tilde{f}_j(n) .$$

In consideration of Lemma 3 and using (3.3) and (3.4), we arrive at

$$(3.5) \qquad n(n-1)\cdot\ldots\cdot(n-j+1) \mid (j-1)!\,\tilde{f}_j(n).$$

Hence, the function

$$f_j(n) := (j-1)!\, \tilde{f}_j(n)$$

has the desired properties.

b) Next, we consider the function

$$g(n) := \frac{f_t(n)}{n(n-1)\cdot\ldots\cdot(n-t+1)} \quad \text{for} \quad n \notin \{0,\ldots,t-1\} .$$

Then clearly g is integer-valued and bounded.

c) Let a be a fixed integer, $a \notin \{0,\ldots,t-1\}$.
Since

$$m \mid f_t(a+m) - f_t(a) \quad ,$$

it follows

$$m \mid a(a-1)\cdot\ldots\cdot(a-t+1)\ \{g(a+m)-g(a)\}$$

by definition of g and therefore

$$g(a+m) = g(a) =: C$$

for $a \notin \{0,\ldots,t-1\}$ and $|m|$ sufficiently large.

d) Repeating the argument used in c), we obtain

$$g(n) = C$$

for all $n \notin \{0,\ldots,t-1\}$.

e) From b) and d) it follows, that

$$f_t(n)=C\cdot n(n-1)\cdot\ldots\cdot(n-t+1) \quad \text{for } n \notin \{0,\ldots,t-1\} .$$

But $f_t(n) = 0$ for $n = 0,\ldots,t-1$, and so f_t is a polynomial with integer coefficients.
By construction it is clear, that f is an integer-valued polynomial, and the proof is complete.

4. Final remarks

From Lemma 2 we obtain, that every polynomial with integer coefficients fulfills the condition (*). Of course, this is not true for integer-valued polynomials in general. For example, the values of the polynomial

$$f(n) = \frac{1}{2} n^2 - \frac{1}{2} n = \frac{n(n-1)}{2}$$

are integers for $n \in \mathbb{Z}$, but

$$f(n) - f(m) = (n-m) \frac{n+m-1}{2} .$$

On the other hand, the polynomial

$$f(n) = \frac{1}{2} n^4 + \frac{1}{2} n^2 = \frac{n^2(n^2+1)}{2}$$

satisfies the condition (*). Therefore we have

$$\mathcal{P}_t^* \underset{\neq}{\subset} \mathcal{G}_t \underset{\neq}{\subset} \mathcal{P}_t .$$

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