A family of polynomials with concyclic zeros

by

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1. Introduction. For positive integers n denote by P_n the linear mapping from the exponential polynomials over $\mathbb{C}$ to the polynomials over $\mathbb{C}$ that replaces exp(az) by its approximation $(1 + \frac{az}{n})^n$. Further we call a set of points in the complex plane concyclic if each of its points lies on the same (possibly infinite) circle. K.B.Stolarsky [3] conjectured that all zeros of $P_n E(z)$, where

$$E(z) = \prod_{j=1}^{J} (e^{\lambda_j z} - 1)$$

with nonzero real numbers λ_j, for $n \geq J$ are concyclic. This conjecture is established by R.J.Evans and K.B.Stolarsky [1]. Following the same method we obtain an analogous result for a more general class of exponential polynomials. We show the

Theorem: Let $\lambda_1,\ldots,\lambda_J$ be nonzero real numbers and $\vartheta_1,\ldots,\vartheta_J$ complex numbers of modulus one. Denote by k the number of j, $1 \leq j \leq J$, with $\vartheta_j = 1$. Assume $n \geq k$. Then the zeros of $P_n E(z)$ with

$$E(z) = \prod_{j=1}^{J} (e^{\lambda_j z} - \vartheta_j)$$

are concyclic. For $\lambda := \sum_{j=1}^{J} \lambda_j = 0$, they are purely imaginary. In the other case, the zeros all lie on the circle with radius $|n/\lambda|$ centered at $-n/\lambda$.

In order to prove the theorem, we need some preliminaries.

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2. Auxiliary Lemmas. First of all, we quote a theorem of Obrechkoff about zeros of polynomials([2], p.95, cf. [1])

Lemma 1. Let $g(z)$ be a polynomial whose zeros all lie in a vertical strip S. Then for real h and complex α with $|\alpha| = 1$, the zeros of

$$f(z) = g(z + h) + \alpha\, g(z - h)$$

lie in the same strip S .

For fixed $\lambda_i \neq 0$, $|\vartheta_i| = 1$, define an operator Δ_i on the set of polynomials by

$$\Delta_i\, g(z) = g(z) - \frac{1}{\vartheta_i}\, g(z + \lambda_i) .$$

This operator has the following properties:

Lemma 2. (i) For any i,j we have $\Delta_i \Delta_j\, g(z) = \Delta_j \Delta_i\, g(z)$.
(ii) For a polynomial $g(z)$ of degree m, $\Delta_i\, g(z)$ is a polynomial with

$$\deg \Delta_i g(z) = \begin{cases} m, & \text{if } \vartheta_i \neq 1 \\ m-1, & \text{if } \vartheta_i = 1 \end{cases}$$

(iii) If all zeros of the nonconstant polynomial $g(z)$ lie on $\mathrm{Re}\, z = \sigma$, then all zeros of $\Delta_i\, g(z)$ lie on

$$\mathrm{Re}\, z = \sigma - \frac{\lambda_i}{2} .$$

Proof. (i) and (ii) are trivial.
(iii) By Lemma 1, the zeros of

$$f(s) := g\left(s - \frac{\lambda_i}{2}\right) - \frac{1}{\vartheta_i}\, g\left(s + \frac{\lambda_i}{2}\right)$$

all lie an $\mathrm{Re}\, s = \sigma$. With $z = s - \frac{\lambda_i}{2}$ we obtain

$$f(s) = \Delta_i\, g(z)$$

and therefore the assertion.

<u>3. Proof of the theorem.</u> Let $\vartheta := \prod_{j=1}^{J} \vartheta_j$

Then clearly

$$R_n(z) := \pm P_n E(z) = \vartheta - \sum_{j=1}^{J} \frac{\vartheta}{\vartheta_j} \left(1+ \frac{\lambda_j z}{n}\right)^n + \sum_{i<j} \frac{\vartheta}{\vartheta_i \vartheta_j} \left(1+ \frac{(\lambda_i+\lambda_j)z}{n}\right)^n - +.$$

Set $\omega := \frac{n}{z}$. Thus

$$\frac{1}{\vartheta} R_n\left(\frac{n}{\omega}\right) \omega^n = \omega^n - \sum_{j=1}^{J} \frac{1}{\vartheta_j} (\omega+\lambda_j)^n + \sum_{i<j} \frac{1}{\vartheta_i \vartheta_j} (\omega+\lambda_i+\lambda_j)^n - + \ldots$$

$$= \Delta_1 \Delta_2 \ldots \Delta_J \, \omega^n .$$

In view of Lemma 2(i), we can assume without loss of generality

$$\vartheta_1 = \ldots = \vartheta_k = 1 .$$

The condition $n \geq k$ insures, that for any j, $2 \leq j \leq J$,

$$\Delta_j \ldots \Delta_J \, \omega^n$$

is a polynomial of degree ≥ 1 . Applying Lemma 2(iii) J times with $\sigma = 0$,

$$R_n(z) = 0$$

implies

$$\mathrm{Re}\, \omega = -(\Sigma \lambda_j)/2 .$$

If $\sum_{j=1}^{J} \lambda_j$ equals 0 , then z lies on the imaginary axis. Otherwise, the zeros all lie on a circle of radius $|r|$ centered at $-r$, where

$$r = n / \sum_{j=1}^{J} \lambda_j .$$

References

[1] R.J.Evans and K.B.Stolarsky, A family of polynomials with concyclic zeros II, Proc.Amer.Math.Soc.92 (1984), 393-396.

[2] N.Obrechkoff, Sur les racines des équations algébriques, Tôhoku Math.J.38 (1933), 93-100.

[3] K.B.Stolarsky, Zeros of exponential polynomials and "reductionism", Topics in Classical Number Theory. Colloq.Math.Soc. János Bolyai, vol.34, Budapest 1981, 1539 - 1552.

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