

PROVABLY COMPUTABLE FUNCTIONS AND THE FAST  
GROWING HIERARCHY

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ABSTRACT. Our aim here is to give as direct a proof as seems possible of

- I. *The functions provably computable in formal Peano Arithmetic are just those which appear in the Fast Growing Hierarchy below level  $\epsilon_0$  and also to illustrate its use by deducing the following well-known independence result of Kirby-Paris [1982].*
- II. *The statement "every Goodstein sequence terminates" is true but not provable in Peano Arithmetic.*

§0. The history of I goes back to Kreisel [1952] who showed that the functions provably computable in Peano Arithmetic can all be defined by recursions over certain natural well orderings of order-types  $< \epsilon_0$ . Later, Schwichtenberg [1971] and the second author [1970,72 building on work with Löb] independently generalised earlier hierarchy results of Grzegorzcyk [1953, giving the primitive recursive functions below level  $\omega$ ] and Robbin [1965, giving the "multiply recursive" functions below level  $\omega^\omega$ ] to show that Kreisel's "ordinal recursive" functions could be characterised by means of the Fast Growing Hierarchy below  $\epsilon_0$ , thus completing I. These results have since been reworked and further extended by many others and in various ways, but for reference we mention especially Schwichtenberg [1977] and Rose [1984]. The proof of I set out here is due to the first author and is a simple base-case of his much more general [1984]. His crucial idea is that by careful use of direct ordinal assignments one can avoid completely any mention of "codes" for infinite proof-trees.



An immediate corollary of I is that the fast growing function occurring at level  $\varepsilon_0$  is not itself provably computable. It is the strong combinatorial connections between this function and the Finite Ramsey Theorem (see Ketonen-Solovay [1981]) and the Goodstein Theorem (see Cichon [1983] and also Abrusci, et al [1984]) which then give direct access to the independence results of Paris-Harrington [1977] and Kirby-Paris [1982]. It must be noted however, that their original proofs were by quite different model-theoretic methods. The proof of II given here is that of Cichon [1983].

### §1. THE FAST GROWING HIERARCHY

By "a fast growing" hierarchy we simply mean a transfinitely extended version of the Grzegorzczuk hierarchy i.e. a transfinite sequence of number-theoretic functions  $F_\alpha$  defined recursively by iteration at successor levels and diagonalisation over suitable fixed fundamental sequences at limit levels. The  $F_\alpha$ 's thus form a backbone which we flesh out by collecting, at each level  $\alpha$ , the class  $C(F_\alpha)$  of all functions computable within time or space bounded by some fixed iterate  $F_\alpha^k = F_\alpha \circ F_\alpha \circ \dots \circ F_\alpha$ . Of course the hierarchy obtained will depend upon the initial function  $F_0$  and more importantly, upon the choice of fundamental sequences at limit ordinals. We are concerned

with the ordinals below  $\varepsilon_0 = \omega^{\omega^{\omega^{\dots}}}$  and for these there is an obvious choice of fundamental sequences. First note that every ordinal  $\alpha < \varepsilon_0$  can be represented in a unique Cantor normal form

$$\alpha = \omega^{\beta_k} + \omega^{\beta_{k-1}} + \dots + \omega^{\beta_1} + \omega^{\beta_0}$$

where  $\alpha > \beta_k \geq \beta_{k-1} \geq \dots \geq \beta_1 \geq \beta_0$ .

If  $\beta_0 = 0$  then  $\alpha$  is a successor. Otherwise  $\alpha$  is a limit and we can assign to it a fundamental sequence  $\alpha_0 < \alpha_1 < \alpha_2 < \dots$  with supremum  $\alpha$  as follows

$$\alpha_n = \omega^{\beta_k} + \omega^{\beta_{k-1}} + \dots + \omega^{\beta_1} + \begin{cases} \omega^{\beta_0(n+1)} & \text{if } \beta_0 = \gamma + 1 \\ \omega^{(\beta_0)_n} & \text{if } \beta_0 \text{ is a limit.} \end{cases}$$



Thus for example,  $\omega$  is assigned the fundamental sequence  $1 < 2 < 3 < \dots$

and  $\omega^\omega$  is assigned the fundamental sequence  $\omega < \omega^2 < \omega^3 < \dots$  etc.

The version of the Fast Growing Hierarchy we shall choose is the following:

$$\begin{aligned} F_0(n) &= n + 1 \\ F_{\alpha+1}(n) &= F_\alpha^{n+1}(n) \\ F_\alpha(n) &= F_\alpha(n) \text{ if } \alpha \text{ is a limit.} \end{aligned}$$

It is closely related to the so-called Hardy Hierarchy (see Wainer [1972]):

$$\begin{aligned} H_0(n) &= n \\ H_{\alpha+1}(n) &= H_\alpha(n+1) \\ H_\alpha(n) &= H_\alpha(n) \text{ if } \alpha \text{ is a limit} \end{aligned}$$

for if  $\alpha = \omega^\beta + \dots + \omega^\gamma$  as above and  $\gamma < \omega^{\beta+1}$  then

$$H_{\alpha+\gamma} = H_\alpha \circ H_\gamma$$

and hence for every  $\alpha < \varepsilon_0$

$$H_\alpha = F_\alpha.$$

Ketonen-Solovay [1981] noted an immediate combinatorial property of the  $H_\alpha$ 's :

call an interval  $[n,m] = \{n,n+1,\dots,m-1,m\}$   $0$ -large if it is non-empty, i.e.

$n \leq m$ ;  $\alpha + 1$ -large if there are at least two  $k \in [n,m]$  such that  $[k,m]$  is

$\alpha$ -large; and  $\lambda$ -large (where  $\lambda$  is a limit) if  $[n,m]$  is  $\lambda_n$ -large. Then

$$\begin{aligned} H_\alpha(n) &= \text{least } m \text{ such that } [n,m] \text{ is } \alpha\text{-large} \\ F_\alpha(n) &= \text{least } m \text{ such that } [n,m] \text{ is } \omega^\alpha\text{-large.} \end{aligned}$$

The crucial properties of the H- and F-hierarchies are that each  $H_\alpha$  (hence

each  $F_\alpha$ ) is strictly increasing and majorises every  $H_\beta(F_\beta)$  for  $\beta < \alpha$ .

These properties are easily proved using

DEFINITION 1 For each fixed  $k$  write  $\beta <_k \alpha$  if there is a descending sequence of ordinals

$$\alpha = \gamma_0 > \gamma_1 > \dots > \gamma_i > \gamma_{i+1} > \dots > \gamma_r = \beta$$

where for each  $i$ ,  $\gamma_{i+1} = \gamma_i - 1$  if  $\gamma_i$  is a successor, and





$$\gamma_{i+1} = (\gamma_i)_k \text{ if } \gamma_i \text{ is a limit.}$$

(This is just the relation  $\alpha \xrightarrow[k]{\phantom{\alpha}} \beta$  of Ketonen-Solovay).

LEMMA 1 For each limit ordinal  $\lambda < \epsilon_0$ , if  $m < n$  then  $\lambda_m + 1 <_n \lambda$ .

Hence if  $\beta < \lambda$  there is a  $k$  such that  $\beta <_n \lambda$  for all  $n \geq k$ .

THEOREM 1 For each  $\alpha < \epsilon_0$

- (i)  $H_\alpha$  is strictly increasing,
- (ii) if  $\beta <_k \alpha$  then  $H_\beta(n) < H_\alpha(n)$  for every  $n > k$ ,
- (iii) if  $\beta < \alpha$  then  $H_\beta(n) < H_\alpha(n)$  for every  $n > \text{some } k$ ,
- (iv) the same properties hold for the  $F_\alpha$ 's.

PROOF (i) and (ii) are proved simultaneously by a straightforward induction on  $\alpha$ , using the first part of Lemma 1. (iii) then follows by the second part of Lemma 1 and (iv) follows from the fact that  $F_\alpha = H_{\omega^\alpha}$  and if  $\beta <_k \alpha$ ,  $\omega^\beta <_k \omega^\alpha$ .

DEFINITION 2 The *elementary* functions are those which can be explicitly defined from the zero, successor, subtraction, projection and addition functions using bounded sums and products. If an arbitrary but fixed function  $f$  is thrown in as an additional initial function then the resulting class of functions "elementary-in- $f$ " is denoted  $E(f)$ .

It is well-known that if  $f$  has at least exponential growth and is computable within time or space bounded by some fixed iterate of itself then  $E(f)$  coincides with the complexity class  $C(f)$  defined earlier. Since  $2^n \leq F_2(n) \leq 3^n$  we can therefore conclude this section by stating

THEOREM 2 If  $0 < \beta < \alpha < \epsilon_0$  then

- (i)  $C(F_\alpha) = E(F_\alpha)$ ,
- (ii)  $F_\alpha \in C(F_\alpha)$ ,
- (iii)  $F_\alpha$  majorises every function in  $C(F_\beta)$ .

## §2. PEANO ARITHMETIC

We formalise arithmetic in a manner suitable for proof-theoretic analysis. Thus we shall derive finite sets  $\Gamma = \{A_1, \dots, A_k\}$  of formulas  $A$  built up using the logical symbols  $\wedge, \vee, \forall, \exists$  from *elementary prime formulas*



" $f(t_1, \dots, t_n) = t_{n+1}$ " or " $f(t_1, \dots, t_n) \neq t_{n+1}$ " representing the graphs of elementary functions  $f$ . The terms  $t_i$  are built up from variables  $x, y, \dots$  and the constant  $0$  using only the successor function symbol  $S$ , so each term is either  $SS\dots Sx$  or a numeral  $SS\dots SO$ . The *negation*  $\neg A$  of a formula  $A$  is defined by:  $\neg(f(t_1, \dots, t_n) = t_{n+1})$  is  $f(t_1, \dots, t_n) \neq t_{n+1}$ ,  $\neg(f(t_1, \dots, t_n) \neq t_{n+1})$  is  $f(t_1, \dots, t_n) = t_{n+1}$ ,  $\neg(A \wedge B)$  is  $\neg A \vee \neg B$ ,  $\neg \forall x B(x)$  is  $\exists x \neg B(x)$  etcetera. Henceforth  $\Gamma$  will denote an arbitrary set  $\{A_1, \dots, A_k\}$ . The intended meaning of  $\Gamma$  is the disjunction  $A_1 \vee \dots \vee A_k$  and we write  $\Gamma, A$  for  $\Gamma \cup \{A\}$  etcetera.

The axioms of Peano Arithmetic (PA) are of three kinds:

- 1) Logical Axioms:  $\Gamma, \neg A, A$  for every formula  $A$ .
- 2) Elementary Axioms: all substitution instances of
  - (=)  $\Gamma, x = x$      $\Gamma, x \neq y, y = x$      $\Gamma, x \neq y, y \neq z, x = z$
  - (S)  $\Gamma, Sx \neq 0$      $\Gamma, Sx \neq Sy, x = y$      $\Gamma, x \neq y, Sx = Sy$
  - (f)  $\Gamma$ , "defining equations for each elementary function  $f$ ".

For example in the case of addition we have axioms

- (+)  $\Gamma, x + 0 = x$   
 $\Gamma, x + y \neq z, x + Sy = Sz$   
 $\Gamma, x \neq x', y \neq y', z \neq z', x + y \neq z, x' + y' = z'$   
 $\Gamma, x + y \neq z, x + y \neq z', z = z'$ .

- 3) Induction Axioms:  $\Gamma, \neg A(0), \exists x(A(x) \wedge \neg A(Sx)), \forall xA(x)$  for every  $A$ .

The logical rules of inference are of five kinds:

$$( \wedge ) \quad \frac{\Gamma, A_0 \quad \Gamma, A_1}{\Gamma, (A_0 \wedge A_1)} \quad ( \vee ) \quad \frac{\Gamma, A_i}{\Gamma, (A_0 \vee A_1)} \quad i = 0 \text{ or } 1 .$$

$$( \forall ) \quad \frac{\Gamma, A(x)}{\Gamma, \forall xA(x)} \quad x \text{ not free in } \Gamma . \quad ( \exists ) \quad \frac{\Gamma, A(t)}{\Gamma, \exists xA(x)}$$

$$( \text{Cut} ) \quad \frac{\Gamma, \neg A \quad \Gamma, A}{\Gamma} .$$

The theorems of PA are those  $\Gamma$  derivable from axioms by the rules.

DEFINITION 3 A number-theoretic function  $F$  is said to be *provably comput-*



able in PA if there are two elementary functions  $V$  and  $T$  such that

(i) for all arguments  $n_1, \dots, n_k,$

$$F(n_1, \dots, n_k) = V(\text{least } m \text{ such that } T(n_1, \dots, n_k, m) = 0)$$

(ii)  $\forall x_1 \dots \forall x_k \exists y (T(x_1 \dots x_k, y) = 0)$  is a theorem of PA.

REMARK Condition (i) is no real restriction since every computable function can be expressed in this way. However, condition (ii) demands essentially that there is a program for computing  $F$  whose "total correctness" can be expressed and verified in PA.

**THEOREM 3** Every function appearing in the Fast Growing Hierarchy below level  $\epsilon_0$  is provably computable in PA.

PROOF We argue very informally, merely indicating the points at which the essential principles built into PA are used.

First notice that the elementary axioms serve to prove the totality of all elementary definitions and so if we can prove the totality of  $F$  we can also prove the totality of every function elementary in  $F$ . It therefore suffices to show that each of the Hardy functions  $H_\alpha$  ( $\alpha < \epsilon_0$ ) and therefore each  $F_\alpha$ , is provably computable in PA.

Using say prime factor decomposition, one can code each ordinal  $\alpha < \epsilon_0$  as a number  $\ulcorner \alpha \urcorner$  so that the decisions whether  $\alpha$  is a successor,  $\alpha$  is a limit, or  $\alpha < \beta$  are all given by elementary functions of the codes for  $\alpha$  and  $\beta$ . Furthermore if  $\alpha$  is a limit the function  $\ulcorner \alpha \urcorner, n \mapsto \ulcorner \alpha_n \urcorner$  becomes elementary. Thus one can express in the language of PA the principle of transfinite induction below a given ordinal  $\beta < \epsilon_0$ :  $TI(\beta, A)$  is just the disjunction of

$$\neg A(0), \neg \forall \alpha < \beta (A(\alpha) \rightarrow A(\alpha+1)), \neg \forall \alpha < \beta (\text{Lim}(\alpha) \wedge \forall x A(\alpha_x) \rightarrow A(\alpha)), \forall \alpha < \beta A(\alpha).$$

Now  $TI(\omega, A)$  follows immediately from the induction axiom 3) of PA. Suppose we assume  $\forall \gamma (TI(\gamma, A) \rightarrow TI(\gamma + \omega^\alpha, A))$  and  $TI(\gamma, A)$ . Then again by the induction axiom we get  $\forall x TI(\gamma + \omega^\alpha x, A)$ . But from this follows  $TI(\gamma + \omega^{\alpha+1}, A)$  since  $\delta < \gamma + \omega^{\alpha+1} \rightarrow \exists x (\delta < \gamma + \omega^\alpha x)$ . Thus we have proved

$$\forall \gamma (TI(\gamma, A) \rightarrow TI(\gamma + \omega^\alpha, A)) \rightarrow \forall \gamma (TI(\gamma, A) \rightarrow TI(\gamma + \omega^{\alpha+1}, A)).$$



We can also prove  $\text{Lim}(\alpha) \wedge \forall x \text{TI}(\gamma + \omega^x, A) \rightarrow \text{TI}(\gamma + \omega^\alpha, A)$  since  
 $\delta < \gamma + \omega^\alpha \wedge \text{Lim}(\alpha) \rightarrow \exists x (\delta < \gamma + \omega^x)$ . Therefore using *transfinite induction on*  
 $\alpha < \beta$  we can deduce  $\forall \alpha < \beta \forall \gamma (\text{TI}(\gamma, A) \rightarrow \text{TI}(\gamma + \omega^\alpha, A))$  and hence  
 $\forall \gamma (\text{TI}(\gamma, A) \rightarrow \text{TI}(\gamma + \omega^\beta, A))$ . In other words, for a suitable formula  $B$   
 depending on  $A$

$$\text{TI}(\beta, B) \rightarrow \forall \gamma (\text{TI}(\gamma, A) \rightarrow \text{TI}(\gamma + \omega^\beta, A))$$

is a theorem of PA. By repeated use of this, starting with  $\text{TI}(\omega, A)$ , we see  
 that for every  $\alpha < \varepsilon_0$  and every formula  $A$ ,  $\text{TI}(\alpha, A)$  is a theorem. (This  
 result goes back to Gentzen and was since refined and developed by Schütte,  
 Feferman and others).

Finally we can return to the functions  $H_\alpha$ . There is an elementary  
 function  $L$  such that

$$\begin{aligned} L(\alpha, n, k, 0) &= 1 \\ L(\alpha, n, k, \ell + 1) &= 0 \text{ if } \alpha = 0 \text{ and } n \leq k \leq \ell. \\ &= L(\beta, n+1, k, \ell) \text{ if } \alpha = \beta + 1. \\ &= L(\alpha_n, n, k, \ell) \text{ if } \alpha \text{ is a limit.} \\ &= 1 \text{ otherwise.} \end{aligned}$$

" $L(\alpha, n, k, \ell) = 0$ " expresses the fact that it takes  $\ell$  steps to verify that  
 $[n, k]$  is  $\alpha$ -large. Now let  $W, V, U$  be elementary pairing functions so that  
 $V(W(k, \ell)) = k$  and  $U(W(k, \ell)) = \ell$ , and define  $T(\alpha, n, m) = L(\alpha, n, V(m), U(m))$ .  
 Then for each  $\alpha < \varepsilon_0$

$$(i) \quad H_\alpha(n) = V(\text{least } m \text{ such that } T(\alpha, n, m) = 0)$$

and (ii)  $\forall x \exists y (T(\alpha, x, y) = 0)$  is provable in PA by transfinite induction up  
 to  $\alpha$ . Hence Theorem 3.

### §3. BOUNDING THE PROVABLY COMPUTABLE FUNCTIONS

In this section we prove the converse of Theorem 3. The strategy is to  
 first embed PA in an infinitary system of arithmetic which replaces the  
 induction axioms and the  $\forall$ -rule by the so-called  $\omega$ -rule:





$$(\omega) \quad \frac{\Gamma, A(0) \quad \Gamma, A(1) \quad \Gamma, A(2) \quad \dots \quad \Gamma, A(n) \quad \dots}{\Gamma, \forall x A(x)}$$

Then by well-known proof-theoretic methods, all but the most trivial applications of the Cut-rule can be eliminated and from the resulting "simplified" proofs we can read off bounds on existential theorems. The cut-elimination method stems from Gentzen, then Schütte, Tait, Takeuti, Feferman, Prawitz and many other since. The proof here is due to Buchholz and is based on the treatment by Tait [1968]. See also Schwichtenberg [1977]. The essential new ingredient is a careful assignment of ordinals which not only measure the "lengths" of  $\omega$ -proofs but also give direct estimates of number-theoretic bounds for existential theorems.

The basic idea underlying cut-elimination is very simple. Consider an  $\omega$ -proof of  $\Gamma$  of the following form where the cut-formula  $A$  is either  $\exists x B(x)$  or  $B_0 \vee B_1$ , so  $\neg A$  is either  $\forall x \neg B(x)$  or  $\neg B_0 \wedge \neg B_1$ .

$$\begin{array}{c}
 (\omega \text{ or } \wedge) \quad \frac{\Gamma, \neg B_i \text{ all } i}{\Gamma, \neg A} \qquad (\exists \text{ or } \vee) \quad \frac{\Gamma, B_n \text{ some } n}{\Gamma, A} \\
 \text{(Cut)} \quad \frac{\Gamma, \neg A \qquad \Gamma, A}{\Gamma}
 \end{array}$$

Then this proof can be replaced by (reduced to)

$$\text{(Cut)} \quad \frac{\Gamma, \neg B_n \qquad \Gamma, B_n}{\Gamma}$$

where the cut-formula  $B_n$  is now a "subformula" of the original  $A$ .

Repetition of this idea should hopefully yield a proof of  $\Gamma$  in which all the cut-formulas are prime! For the technicalities see Lemma 4 below.

Now the language of the infinitary system of arithmetic we are about to define is that of PA but with one new relation " $x \in N$ " added. Its intended meaning is that  $x$  is a non-negative integer. However we can now restrict attention to closed formulas only (i.e. ones without any free variables), since the free variable  $x$  in the premise of the  $\forall$ -rule is now replaced by infinitely many premises  $\Gamma, A(n)$ ,  $n = 0, 1, 2, \dots$  in the  $\omega$ -rule.



Thus the only terms are *numerals*  $n = SS\dots SO$  and the *closed elementary prime formulas* (cepf) are just atomic relations stating that the value of a given elementary function on given arguments  $n_1, \dots, n_r$  is - or is not - equal to a given number  $m$ . Every cepf is therefore either true or false.

DEFINITION 4 The relation  $\vdash^\alpha \Gamma$ , meaning  $\Gamma$  is derivable in the infinitary system with ordinal bound  $\alpha < \epsilon_0$ , is defined inductively according to the rules:-

(Axioms)  $\vdash^\alpha \Gamma, A$  if  $A$  is any true cepf or " $0 \in N$ ".

$\vdash^\alpha \Gamma, n \notin N, n \in N$   $n$  any numeral.

(N) if  $\vdash^\alpha \Gamma, n \in N$  then  $\vdash^{\alpha+1} \Gamma, Sn \in N$ .

( $\vee$ ) if  $\vdash^\alpha \Gamma, A_i$  ( $i=0$  or  $1$ ) then  $\vdash^{\alpha+1} \Gamma, (A_0 \vee A_1)$ .

( $\wedge$ ) if  $\vdash^\alpha \Gamma, A_i$  ( $i=0$  and  $1$ ) then  $\vdash^{\alpha+1} \Gamma, (A_0 \wedge A_1)$

( $\exists$ ) if  $\vdash^\alpha \Gamma, A(n)$  (some  $n$ ) then  $\vdash^{\alpha+1} \Gamma, \exists xA(x)$ .

( $\forall$ ) if  $\vdash^\alpha \Gamma, A(n)$  (every  $n$ ) then  $\vdash^{\alpha+1} \Gamma, \forall xA(x)$ .

(Cut) if  $\vdash^\alpha \Gamma, \neg A$  and  $\vdash^\alpha \Gamma, A$  then  $\vdash^{\alpha+1} \Gamma$ .

(Accumulation) if  $\vdash^\alpha \Gamma$  and if  $\alpha <_k \beta$  where

$k = \max \{2\} \cup \{3n : "n \notin N" \text{ is in } \Gamma\}$  then  $\vdash^\beta \Gamma$ .

Notice that if  $\vdash^\alpha \Gamma$  and  $\Gamma \subset \Gamma'$  then  $\vdash^\alpha \Gamma'$ , for if  $\alpha <_k \beta$  and  $k \leq k'$  then  $\alpha <_{k'} \beta$  by a straightforward induction on  $\beta$  using Lemma 1.

DEFINITION 5 For each formula  $A$  of PA let  $A^N$  be the result of relativising all quantifiers in  $A$  to  $N$ , i.e. replace  $\exists x(\dots)$  by  $\exists x(x \in N \wedge \dots)$  and replace  $\forall x(\dots)$  by  $\forall x(x \notin N \vee \dots)$ . If  $\Gamma = \{A_1, \dots, A_k\}$  let  $\Gamma^N = \{A_1^N, \dots, A_k^N\}$ .

EMBEDDING LEMMA 2 If  $\Gamma$  is a theorem of PA, containing free variables  $x_1, \dots, x_r$  then there is an ordinal  $\alpha = \omega \ell$  for some integer  $\ell$ , such that for all  $n_1, \dots, n_r$ ,

$$\vdash^\alpha n_1 \notin N, \dots, n_r \notin N, \Gamma^N(n_1, \dots, n_r)$$

where  $\Gamma^N(n_1, \dots, n_r)$  is the result of substituting  $n_i$  at all free occurrences of  $x_i$  in  $\Gamma^N$ .



PROOF by induction over the generation of theorems in PA.

(i) For the logical axioms it will clearly be sufficient to show that for every closed formula  $A$  in the language augmented with " $N$ ",  $\vdash^\alpha \neg A, A$  for some  $\alpha$ . We do this by induction on the "length" of  $A$ . If  $A$  is a closed elementary prime formula or " $n \in N$ " then  $\vdash^0 \neg A, A$  because it's an axiom. Now suppose, for example that  $B$  is the formula  $\exists x A(x)$  where, for each numeral  $n$ ,  $\vdash^\alpha \neg A(n), A(n)$ . Then by the  $\exists$ - $\omega$ - and Accumulation rules we get  $\vdash^{\alpha+1} \neg A(n), \exists x A(x)$ ,  $\vdash^{\alpha+2} \forall x \neg A(x), \exists x A(x)$  and hence  $\vdash^{\alpha+\omega} \neg B, B$ . The other forms of  $B$  can be treated similarly to obtain  $\vdash^{\omega \ell} \neg A, A$  for each closed  $A$  where  $\ell$  is its "length".

(ii) The result follows immediately when  $\Gamma$  is an elementary axiom because, on substitution of numerals for the free variables, we obtain a true cepf.

(iii) Now consider an induction axiom

$$\Gamma, \neg A(0), \exists x(A(x) \wedge \neg A(Sx)), \forall x A(x).$$

Deleting mention of  $\Gamma$  and any variables other than  $x$  which may occur free, it is sufficient to show  $\vdash^{\alpha+\omega} \neg A^N(0), \exists x(x \in N \wedge A^N(x) \wedge \neg A^N(Sx)), \forall x(x \notin N \vee A^N(x))$ , where  $\alpha = \omega(\ell+1)$ ,  $\ell$  again denoting the length of  $A^N(x)$ .

This follows by Accumulation from

$\vdash^{\alpha+3} \neg A^N(0), \exists x(x \in N \wedge A^N(x) \wedge \neg A^N(Sx)), \forall x(x \notin N \vee A^N(x))$  which follows, by the  $\vee$  and  $\omega$ -rules, from  $\vdash^\alpha \neg A^N(0), \exists x(x \in N \wedge A^N(x) \wedge \neg A^N(Sx))$ ,  $n \notin N, A^N(n)$  which in turn follows by Accumulation from  $\vdash^{\omega \ell + 3n} \neg A^N(0), \exists x(x \in N \wedge A^N(x) \wedge \neg A^N(Sx)), n \notin N, A^N(n)$  because  $\omega \ell + 3n <_k \omega \ell + \omega = \alpha$  if  $k \geq 3n$ . For this last line we actually prove

$$\vdash^{\omega \ell + 3n} \neg A^N(0), \exists x(x \in N \wedge A^N(x) \wedge \neg A^N(Sx)), A^N(n)$$

by induction on  $n$  (remember that we can always add  $n \notin N$  and any other "side" formulas throughout a derivation without increasing the ordinal bound).

Now for  $n = 0$  we have  $\vdash^{\omega \ell} \neg A^N(0), A^N(0)$  by part (i) above. Assume  $\vdash^\beta \neg A^N(0), C, A^N(n)$  where  $C$  is the formula  $\exists x(x \in N \wedge A^N(x) \wedge \neg A^N(Sx))$  and  $\beta = \omega \ell + 3n$ . Then  $\vdash^\beta \neg A^N(0), C, A^N(n), A^N(Sn)$  and by part (i)



$\vdash^{\beta} \neg A^N(0), C, \neg A^N(Sn), A^N(Sn)$ , so by the  $\wedge$ -rule,  $\vdash^{\beta+1} \neg A^N(0), C,$   
 $A^N(n) \wedge \neg A^N(Sn), A^N(Sn)$ . By  $n$  applications of the  $N$ -rule, starting with  
 $\vdash^{\beta+1-n} \neg A^N(0), C, 0 \in N, A^N(Sn)$  we also obtain  $\vdash^{\beta+1} \neg A^N(0), C,$   
 $n \in N, A^N(Sn)$  and therefore  $\vdash^{\beta+2} \neg A^N(0), C, n \in N \wedge A^N(n) \wedge \neg A^N(Sn),$   
 $A^N(Sn)$ . One further application of the  $\exists$ -rule then gives  $\vdash^{\beta+3} \neg A^N(0),$   
 $\exists x(x \in N \wedge A^N(x) \wedge \neg A^N(Sx)), A^N(Sn)$ . Hence the result.

(iv) Now suppose  $\Gamma, \forall x A(x)$  is derived from  $\Gamma, A(x)$  in PA using the  
 $\forall$ -rule. Then again deleting mention of any free variables occurring in  $\Gamma$ ,  
 we can assume inductively that there is an ordinal  $\alpha = \omega \cdot k$  for some  $k$ ,  
 such that  $\vdash^{\alpha} n \notin N, \Gamma^N, A^N(n)$  and hence  $\vdash^{\alpha+2} \Gamma^N, n \notin N \vee A^N(n)$  for  
 every  $n$ . Therefore  $\vdash^{\alpha+\omega} \Gamma^N, \forall x(x \notin N \vee A^N(x))$  by  $\omega$ -rule followed by  
 Accumulation, since  $\alpha + 3 \underset{k}{<} \alpha + \omega$  if  $k \geq 2$ .

(v) Suppose  $\Gamma, \exists x A(x)$  is derived from  $\Gamma, A(t)$  in PA using the  
 $\exists$ -rule, where  $t$  is some term. The only possible forms of term  $t$  are  
 either a numeral  $m = S^m 0$  or  $S^m x$  for some variable  $x$ , where  $S^m$  denotes  
 $m$  iterations of the successor symbol. It may be that  $t$  is of the form  
 $S^m x$  and  $x$  also occurs free in  $\Gamma$ . In this case (again neglecting any  
 other free variables in  $\Gamma, A$ ) we have, by the induction hypothesis, an  
 ordinal  $\alpha = \omega \cdot k$  such that for every  $n \vdash^{\alpha} n \notin N, \Gamma^N(n), A^N(S^m n)$ .  
 Choosing  $\beta = \alpha + \omega \cdot m$  we then have  $\vdash^{\beta} n \notin N, \Gamma^N(n), A^N(n+m)$  and also  
 $\vdash^{\beta} n \notin N, \Gamma^N(n), n+m \in N$  by alternate applications of the  $N$ -rule and  
 Accumulation, starting from  $\vdash^{\alpha} n \notin N, \Gamma^N(n), n \in N$ . Then by the  $\wedge$ -rule,  
 $\vdash^{\beta+1} n \notin N, \Gamma^N(n), n+m \in N \wedge A^N(n+m)$  and by the  $\exists$ -rule,  $\vdash^{\beta+2} n \notin N,$   
 $\Gamma^N(n), \exists x(x \in N \wedge A^N(x))$ , so by Accumulation,  $\vdash^{\beta+\omega} n \notin N, \Gamma^N(n),$   
 $\exists x(x \in N \wedge A^N(x))$  for every  $n$ , as required. If  $t$  is  $S^m x$  where  $x$  does  
 not occur free in  $\Gamma$  we can safely replace  $x$  by  $0$  throughout the proof of  
 $\Gamma, A$  thereby reducing to the case where  $t$  is  $S^m 0$ . This is then dealt  
 with in a similar way to the above but with  $n = 0$ .

(vi) The remaining cases, corresponding to applications of the  $\wedge, \vee$  or  
 Cut-rules, follow immediately from the induction hypothesis, using Accumulation  
 to "match up" the ordinal bounds on premises of  $\wedge, \text{Cut}$ .





This completes the proof.

REMARK The reason for this somewhat tiresome Embedding of PA into the infinitary system, is that the Cut-rule - which is the only rule whose conclusion fails to determine its premises - cannot be eliminated from PA. It can however be eliminated almost entirely from the infinitary system, as is shown in the following. The point is that from "cut-free" proofs one can read off numerical bounds. The price one pays for cut-elimination is an iterated exponential increase in the length of the proof. Since  $\epsilon_0$  is the first ordinal closed under exponentiation  $\alpha \mapsto \omega^\alpha$  we therefore catch a glimpse of the significance of  $\epsilon_0$  in proof-theoretic studies of PA.

The remaining results of this section concern the infinitary system of Definition 4 and finally provide the converse to Theorem 3.

INVERSION LEMMA 3

- (i) If  $\vdash^\alpha \Gamma, A_0 \wedge A_1$  then  $\vdash^\alpha \Gamma, A_0$  and  $\vdash^\alpha \Gamma, A_1$ .  
 (ii) If  $\vdash^\alpha \Gamma, \forall x A(x)$  then  $\vdash^\alpha \Gamma, A(n)$  for every  $n$ .

PROOF by straightforward inductions over the definition of  $\vdash^\alpha \Gamma, A$ .

DEFINITION 5 Write " $\vdash^\alpha \Gamma$  with cut-rank  $\leq r$ " if for every application of the Cut-rule  $\frac{\Delta, \neg A \quad \Delta, A}{\Delta}$  which occurs in the derivation  $\vdash^\alpha \Gamma$ , the "cut-formula"  $A$  has "length"  $\leq r$ . Here "length" is defined by

length(A) = 0 if A is a prime formula, = length(B) + 1 if A is  $\exists x B$  or  $\forall x B$ , = max(length( $B_0$ ), length( $B_1$ )) + 1 if A is  $B_0 \vee B_1$  or  $B_0 \wedge B_1$ .

REMARK We shall only be concerned with those derivations which have a finite bound on their cut-rank (though there are derivations which do not). Note especially that when PA is embedded into the infinitary system by Lemma 2, the resulting derivations all have finitely bounded cut-ranks. Note also that Inversion does not change cut-rank.

REDUCTION LEMMA 4 Suppose  $\vdash^\alpha \Gamma_0, \neg A$  with cut-rank  $\leq r$  where  $\alpha = \omega^{\alpha_k} + \dots + \omega^{\alpha_0}$  in Cantor normal form and A is of either form  $\exists x B(x)$  or  $B_0 \vee B_1$  and of length  $r + 1$ . Then if  $\vdash^\beta \Gamma, A$  with cut-rank  $\leq r$ , where  $\beta < \omega^{\alpha_0 + 1}$  we have  $\vdash^{\alpha + \beta} \Gamma_0, \Gamma$  with cut-rank  $\leq r$ .



PROOF by transfinite induction on  $\beta$ .

If  $\Gamma, A$  is an axiom then  $\Gamma$  must be an axiom and hence so is  $\Gamma_0, \Gamma$ .

If  $\vdash^\beta \Gamma, A$  follows by Accumulation from  $\vdash^\delta \Gamma, A$  where  $\delta <_k \beta$  and  $k = \max\{2\} \cup \{3n : "n \notin N" \text{ occurs in } \Gamma, A\}$  then by the induction hypothesis,  $\vdash^{\alpha+\delta} \Gamma_0, \Gamma$  with cut-rank  $\leq r$ . But  $k \leq \max\{2\} \cup \{3n : "n \notin N" \text{ is in } \Gamma_0, \Gamma\} = k'$  because  $A$  is not of the form  $n \notin N$ . Therefore  $\delta <_{k'} \beta$  and  $\alpha + \delta <_{k'} \alpha + \beta$  so  $\vdash^{\alpha+\beta} \Gamma_0, \Gamma$  by Accumulation, and still with cut-rank  $\leq r$ .

Suppose  $\Gamma = \Gamma', C$  where  $C$  is some formula other than  $A$  and suppose  $\vdash^\beta \Gamma', C, A$  follows from premises  $\vdash^\delta \Gamma', C_i, A$  by a rule other than Accumulation. Then  $\beta = \delta + 1$  and by the induction hypothesis,

$\vdash^{\alpha+\beta} \Gamma_0, \Gamma', C_i$  with cut-rank  $\leq r$ . Therefore by applying the rule concerned - and noting that if it's a cut then the length of each  $C_i$  must be  $\leq r$  by assumption - we obtain  $\vdash^{\alpha+\delta+1} \Gamma_0, \Gamma', C$ , i.e.  $\vdash^{\alpha+\beta} \Gamma_0, \Gamma$  with cut-rank  $\leq r$ .

Suppose  $A$  is of the form  $\exists x B(x)$  and  $\vdash^\beta \Gamma, A$  follows from  $\vdash^\delta \Gamma', B(n)$  by the  $\exists$ -rule with  $\beta = \delta + 1$  and  $\Gamma', A = \Gamma, A$ . Then  $\vdash^\delta \Gamma, A, B(n)$  and hence  $\vdash^{\alpha+\delta} \Gamma_0, \Gamma, B(n)$  by the induction hypothesis, with cut-rank  $\leq r$ . Since  $\vdash^\alpha \Gamma_0, \neg A$  and  $\neg A$  is  $\forall x \neg B(x)$  we have by Inversion  $\vdash^\alpha \Gamma_0, \neg B(n)$  with cut-rank  $\leq r$  and hence  $\vdash^{\alpha+\delta} \Gamma_0, \Gamma, \neg B(n)$  with cut-rank  $\leq r$ , by Accumulation. Therefore as  $B(n)$  has length  $r$  we can apply Cut to obtain  $\vdash^{\alpha+\beta} \Gamma_0, \Gamma$  with cut-rank  $\leq r$ .

Finally if  $A$  is  $B_0 \vee B_1$  and  $\vdash^\beta \Gamma, A$  follows from  $\vdash^\delta \Gamma', B_i$  with  $i = 0$  or  $1$ ,  $\beta = \delta + 1$  and  $\Gamma', A = \Gamma, A$ , then  $\vdash^\delta \Gamma, A, B_i$  and so by the induction hypothesis,  $\vdash^{\alpha+\delta} \Gamma_0, \Gamma, B_i$  with cut-rank  $\leq r$ . Since  $\vdash^\alpha \Gamma_0, \neg A$  and  $\neg A$  is  $\neg B_0 \wedge \neg B_1$  we then obtain  $\vdash^\alpha \Gamma_0, \neg B_i$  by Inversion and  $\vdash^{\alpha+\delta} \Gamma_0, \Gamma, \neg B_i$  by Accumulation, both with cut-rank  $\leq r$ . So as  $B_i$  has length  $\leq r$  we can apply Cut to obtain  $\vdash^{\alpha+\delta+1} \Gamma_0, \Gamma$ , i.e.  $\vdash^{\alpha+\beta} \Gamma_0, \Gamma$  with cut-rank  $\leq r$ .

CUT-ELIMINATION THEOREM 4 If  $\vdash^\alpha \Gamma$  with cut-rank  $\leq r + 1$  then

$\vdash^\omega \Gamma$  with cut-rank  $\leq r$  and therefore  $\vdash^{\alpha^*} \Gamma$  with cut-rank  $0$  where

$\alpha^*$  is obtained from  $\alpha$  by  $r+1$ -times iterated exponentiation thus :



$$\alpha^* = \omega \left. \begin{matrix} \omega \\ \dots \\ \omega \end{matrix} \right\} r + 1.$$

PROOF We only need prove that if  $\vdash^\alpha \Gamma$  with cut-rank  $\leq r + 1$  then  $\vdash^{\omega^\alpha} \Gamma$  with cut-rank  $\leq r$ , since the rest follows by repeated applications of this. Proceed by induction on the definition of  $\vdash^\alpha \Gamma$  and note that all cases except application of the cut-rule itself are absolutely trivial, because in the case of Accumulation we have  $\omega^\beta <_k \omega^\alpha$  if  $\beta <_k \alpha$ , and in the other cases we have  $\vdash^{\omega^{\beta+1}} \Gamma$  if  $\vdash^{\omega^{\beta+1}} \Gamma$  by Accumulation since  $\omega^{\beta+1} <_k \omega^{\beta+1}$  for any  $k \geq 1$ . So suppose  $\vdash^\alpha \Gamma$  follows from  $\vdash^\beta \Gamma, \neg A$  and  $\vdash^\beta \Gamma, A$  by Cut where  $\alpha = \beta + 1$  and  $A$  has length  $\leq r + 1$ . Inductively we have  $\vdash^{\omega^\beta} \Gamma, \neg A$  and  $\vdash^{\omega^\beta} \Gamma, A$  both with cut-rank  $\leq r$ , so if  $A$  has length  $\leq r$  we obtain by Cut,  $\vdash^{\omega^{\beta+1}} \Gamma$  with cut-rank  $\leq r$  and then  $\vdash^{\omega^\alpha} \Gamma$  by Accumulation since  $\omega^{\beta+1} <_k \omega^{\beta+1}$  for any  $k \geq 1$ . If  $A$  has length  $r + 1$  then the Reduction Lemma applies to give  $\vdash^{\omega^\beta + \omega^\beta} \Gamma$  with cut-rank  $\leq r$  and hence  $\vdash^{\omega^\alpha} \Gamma$  by Accumulation since  $\omega^{\beta+1} <_k \omega^{\beta+1}$  for any  $k \geq 1$ .

DEFINITION 6 By a *positive*  $\Sigma_1(N)$  *formula* is meant any formula built up from elementary prime formulas and " $x \in N$ " using only  $\wedge, \vee$  and  $\exists$ . Note that " $x \notin N$ " is not allowed. A set  $\Gamma = \{A_1, \dots, A_m\}$  of closed positive  $\Sigma_1(N)$  formulas is said to be *true in*  $k$  if one of the  $A_i$ 's in  $\Gamma$  is true when  $N$  is interpreted as the finite set  $\{n : 3n < k\}$ . Note that if  $\Gamma$  is true in  $k$  and  $k < k'$  then  $\Gamma$  is also true in  $k'$ .

BOUNDING LEMMA 5 If  $\Gamma$  contains only closed positive  $\Sigma_1(N)$  formulas and  $\vdash^\alpha n_1 \notin N, \dots, n_r \notin N, \Gamma$  with cut-rank 0 then  $\Gamma$  is true in  $F_\alpha(k)$  where  $k = \max\{2\} \cup \{3n_1, \dots, 3n_r\}$ .

PROOF by induction over  $\vdash^\alpha n_1 \notin N, \dots, n_r \notin N, \Gamma$ .

If it's an axiom then  $\Gamma$  contains either a true ceph or " $0 \in N$ " or " $n_i \in N$ " for some  $i$ , so  $\Gamma$  is true in  $F_\alpha(k)$  since  $F_\alpha(k) > 3n_i$  for each  $i$ .

The  $\omega$ -rule is not applicable since  $\Gamma$  does not contain  $\forall$  and the cases where the  $\wedge$  or  $\vee$ -rules are applied follow trivially from the



induction hypothesis since  $F_\beta(k) < F_{\beta+1}(k)$ .

For an application of the  $\exists$ -rule suppose  $\Gamma$  contains  $\exists xB(x)$  and  $\vdash^\beta n_1 \notin N, \dots, n_r \notin N, \Gamma, B(m)$  for some numeral  $m$  where  $\alpha = \beta + 1$ . Then  $B(m)$  cannot be of the form " $m \notin N$ " so by the induction hypothesis  $\Gamma, B(m)$  is true in  $F_\beta(k)$  and therefore in  $F_\alpha(k)$ . Hence  $\Gamma = \Gamma, \exists xB(x)$  is true in  $F_\alpha(k)$ .

For the N-rule suppose  $\Gamma$  contains " $Sm \in N$ " and  $\vdash^\beta n_1 \notin N, \dots, n_r \notin N, \Gamma, m \in N$  where  $\alpha = \beta + 1$ . Then  $\Gamma, m \in N$  is true in  $F_\beta(k)$  and therefore  $\Gamma = \Gamma, Sm \in N$  is true in  $F_\alpha(k)$  as  $3m < F_\beta(k)$  implies  $3m + 3 < F_\alpha(k)$ .

Now suppose  $\vdash^\alpha n_1 \notin N, \dots, n_r \notin N, \Gamma$  follows from  $\vdash^\beta n_1 \notin N, \dots, n_r \notin N, \Gamma, \neg A$  and  $\vdash^\beta n_1 \notin N, \dots, n_r \notin N, \Gamma, A$  by Cut where  $\beta + 1 = \alpha$ . Since the cut has rank 0,  $A$  is either a cefp or of the form " $m \in N$ " for some numeral  $m$ . By the induction hypothesis, if  $A$  is a cefp then both  $\Gamma, \neg A$  and  $\Gamma, A$  are true in  $F_\beta(k)$  and so  $\Gamma$  is true in  $F_\beta(k)$  and hence in  $F_\alpha(k)$  because one of  $\neg A$  or  $A$  is false. If  $A$  is of the form " $m \in N$ " then  $\Gamma$  must be true in  $F_\beta(\max(k, 3m))$  and  $\Gamma, m \in N$  is true in  $F_\beta(k)$ , so either  $\Gamma$  is true in  $F_\beta(k)$  - hence also in  $F_\alpha(k)$  - or else  $3m < F_\beta(k)$  and  $\Gamma$  is true in  $F_\beta(\max(k, 3m))$  which is  $< F_\beta(F_\beta(k)) < F_\beta^{k+1}(k) = F_\alpha(k)$  and therefore again we have  $\Gamma$  true in  $F_\alpha(k)$ .

Finally suppose  $\vdash^\alpha n_1 \notin N, \dots, n_r \notin N, \Gamma$  comes from  $\vdash^\beta n_1 \notin N, \dots, n_r \notin N, \Gamma$  by Accumulation where  $\beta <_k \alpha$ . Then  $\Gamma$  is true in  $F_\beta(k)$  by the induction hypothesis and  $F_\beta(k) \leq F_\alpha(k)$  follows from  $\beta <_k \alpha$  by an easy induction on  $\alpha$ . Hence  $\Gamma$  is true in  $F_\alpha(k)$ .

**THEOREM 5** Every function provably computable in PA is elementary in some  $F_\alpha$  for  $\alpha < \epsilon_0$ , and is therefore majorised by  $F_{\alpha+1}$ .

**PROOF** Suppose  $f$  is defined from elementary  $V$  and  $T$  by

$$f(n_1, \dots, n_r) = V(\text{least } m \text{ such that } T(n_1, \dots, n_r, m) = 0)$$

where  $\forall x_1 \dots \forall x_r \exists y (T(x_1, \dots, x_r, y) = 0)$  is a theorem of PA - note that " $T(x_1, \dots, x_r, y) = 0$ " is an elementary prime formula. By the Embedding Lemma and the Cut-Elimination Theorem there is an ordinal





$\alpha = \omega^{\omega^{\dots^{\omega \cdot \ell}}}$   
 $\alpha < \epsilon_0$  such that

$$\vdash^\alpha \forall x_1 (x_1 \notin N \vee \dots \forall x_r (x_r \notin N \vee \exists y (y \in N \wedge T(x_1, \dots, x_r, y) = 0))..)$$

with cut-rank 0. By repeated use of the Inversion Lemma together with the simple fact that if  $\vdash^\alpha \Gamma, A \vee B$  then  $\vdash^\alpha \Gamma, A, B$  we then obtain for all  $n_1, \dots, n_r$ ,

$$\vdash^\alpha n_1 \notin N, \dots, n_r \notin N, \exists y (y \in N \wedge T(n_1, \dots, n_r, y) = 0)$$

with cut-rank 0 and by the Bounding Lemma  $\exists y (y \in N \wedge T(n_1, \dots, n_r, y) = 0)$  is therefore true in  $F_\alpha(\max(2, 3n_1, \dots, 3n_r))$ . Thus for every sequence of arguments  $n_1, \dots, n_r$  there is an  $m < F_\alpha(\max(2, 3n_1, \dots, 3n_r))$  such that  $T(n_1, \dots, n_r, m) = 0$ . Now it is well-known that the elementary functions are closed under *bounded* minimisation, so

$$V(\text{least } m < b \text{ such that } T(n_1, \dots, n_r, m) = 0)$$

is an elementary function of  $n_1, \dots, n_r$  and  $b$ . Therefore substituting  $F_\alpha(\max(2, 3n_1, \dots, 3n_r))$  for  $b$  we obtain an elementary-in- $F_\alpha$  definition of  $f$ . Since  $F_{\alpha+1}$  majorises all functions elementary in  $F_\alpha$ , it majorises  $f$ .

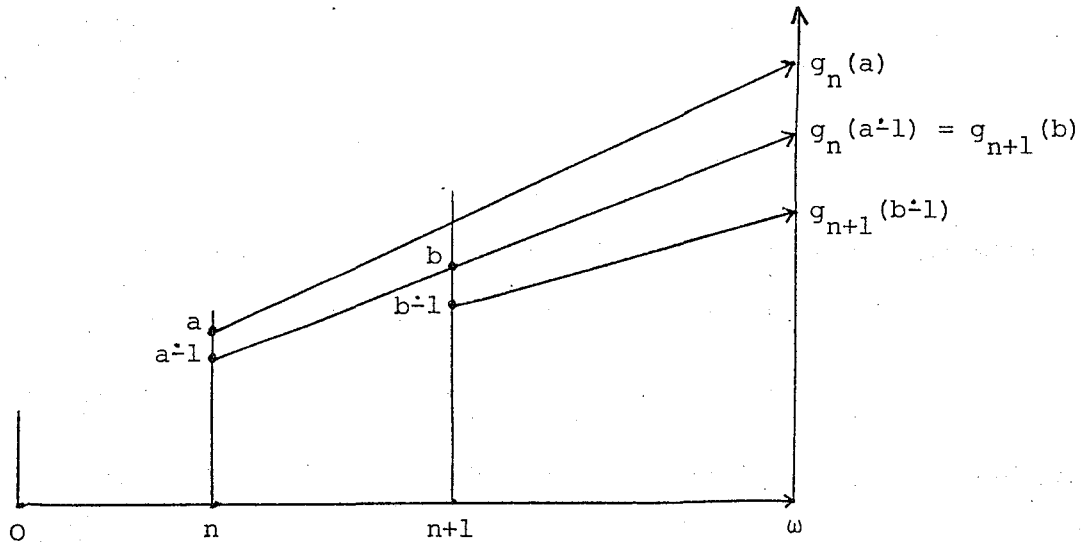
Theorems 3 and 5 together give the main result I.

§4. GOODSTEIN SEQUENCES

Fix  $n$  and consider any number  $a < (n+1)^{(n+1)^{\dots^{(n+1)}}}$ . Express  $a$  in complete base  $n+1$  form by decomposing  $a = \sum_{i \leq k} (n+1)^{a_i} m_i$  where each  $a_i > a_{i+1}$  and  $m_i \leq n$ ,  $a_i = \sum_{j \leq \ell} (n+1)^{a_{ij}} m_{ij}$  where each  $a_{ij} > a_{ij+1}$  and  $m_{ij} \leq n$ ,  $a_{ij} = \dots$  etcetera until, after  $n$  steps, all the  $a_{ij \dots k}$  are  $\leq n$ . Now replace  $(n+1)$  by  $\omega$  throughout the complete base  $n+1$  form of  $a$ , to obtain the Cantor normal form representation of some ordinal  $\alpha < \epsilon_0$ . Denote this ordinal  $\alpha$  by  $g_n(a)$ . Then  $g_n$  is a finite order-preserving embedding of the numbers less



than  $(n+1)^{\dots(n+1)}$  }  $n+1$  into the ordinals less than  $\epsilon_0$  and clearly every ordinal less than  $\epsilon_0$  is in the range of all but finitely-many  $g_n$ 's. Thus the  $g_n$ 's give a direct-limit representation of  $\epsilon_0$ :



For each  $n$  define  $g_{n,n+1} = g_{n+1}^{-1} \circ g_n$  so that  $g_{n,n+1}(a)$  is the result of replacing  $(n+1)$  by  $(n+2)$  throughout the complete base  $n+1$  form of  $a$ .

DEFINITION 7 The *Goodstein sequence starting with a* is the sequence of numbers  $\{a_k\}$  defined as follows. Choose the least number  $n$  such that

$a < (n+1)^{\dots(n+1)}$  }  $n+1$ . Set  $a_0 = a-1$  and  $a_{k+1} = g_{n+k,n+k+1}(a_k)-1$ .

The sequence *terminates* if  $\exists k (a_k = 0)$ .

NOTE that the statement "every Goodstein sequence terminates" is expressible by a formula

$$\forall x \exists y (T(x,y) = 0)$$

in the language of PA, where  $T$  is elementary. For  $a_k$  is a computable function of  $a$  and  $k$ , so there is an elementary function  $T'(a,k,\ell)$  such that  $T'(a,k,\ell) = 0$  if and only if within  $\ell$  steps it is possible to compute  $a_0, a_1, \dots, a_{k-1}, a_k$  and check that  $a_0 \neq 0, a_1 \neq 0, \dots, a_{k-1} \neq 0, a_k = 0$ . Now set  $T(a,y) = T'(a, V(y), U(y))$  where  $U, V$  are elementary pairing functions.



Then  $T(a,y) = 0$  if and only if  $V(y)$  is the least  $k$  such that  $a_k = 0$ , and  $U(y)$  bounds the number of steps needed to verify this.

Now with each Goodstein sequence  $\{a_k\}$  is associated a sequence  $\{g_{n+k}(a_k)\}$  of ordinals  $< \epsilon_0$ . It is not too difficult to convince oneself that for any

$$a < (n+1) \cdot \left. \begin{matrix} \dots \\ \dots \\ \dots \end{matrix} \right\} n+1,$$

$$g_n(a-1) = P_n(g_n(a))$$

where  $P_n(0) = 0$ ,  $P_n(\alpha+1) = \alpha$  and  $P_n(\alpha) = P_n(\alpha)$  if  $\alpha$  is a limit with fundamental sequence  $\{\alpha_n\}$ . Therefore by induction on  $k$ ,

$$g_{n+k}(a_k) = P_{n+k} P_{n+k-1} \dots P_{n+1} P_n(g_n(a))$$

and so the statement that "the Goodstein sequence starting with  $a$  terminates" is equivalent to

$$\exists m \geq n (P_m P_{m-1} \dots P_{n+1} P_n(g_n(a)) = 0).$$

LEMMA 6 For every  $\alpha < \epsilon_0$  and all  $n$ , if  $\alpha \neq 0$  then

$$H_\alpha(n) = \text{least } m \geq n (P_m P_{m-1} \dots P_{n+1} P_n(\alpha) = 0) + 1.$$

PROOF by straightforward induction over the definition of the Hardy functions  $H_\alpha$  given in §1.

THEOREM 6 (Kirby-Paris [1982]). The statement

"every Goodstein sequence terminates"

is true but not provable in Peano Arithmetic.

PROOF (Cichon [1983]). Since  $P_n(\alpha) < \alpha$  when  $\alpha \neq 0$  it follows that the sequence of ordinals

$$g_{n+k}(a_k) = P_{n+k} \dots P_{n+1} P_n(g_n(a))$$

associated with a given Goodstein sequence  $\{a_k\}$  is decreasing and therefore  $g_{n+k}(a_k)$  - and hence  $a_k$  - is eventually 0 (this was Goodstein's result).

Suppose, for a contradiction, that the formula  $\forall x \exists y (T(x,y) = 0)$  expressing that "every Goodstein sequence terminates" were a theorem of PA. Then the



function

$$\begin{aligned} \text{Goodstein}(a) &= V(\text{least } y \text{ such that } T(a,y) = 0) \\ &= \text{least } k \text{ such that } a_k = 0 \end{aligned}$$

would be provably computable in PA. But if we define  $H_{\epsilon_0}(n) = H_{(\epsilon_0)_n}(n)$

where  $(\epsilon_0)_n = \omega \cdot \omega \cdot \dots \cdot \omega$   $\left. \vphantom{(\epsilon_0)_n} \right\}^n$  then by the above Lemma and the comments preceding it

$$\begin{aligned} H_{\epsilon_0}(n) &= \text{least } m \geq n (P_m \dots P_{n+1} P_n ((\epsilon_0)_n) = 0) + 1 \\ &= \text{least } m \geq n (P_m \dots P_{n+1} P_n (g_n((n+1) \cdot \dots \cdot (n+1)) \left. \vphantom{g_n} \right\}^n)) = 0) + 1 \\ &= n + \text{Goodstein}((n+1) \cdot \dots \cdot (n+1) \left. \vphantom{(n+1)} \right\}^n) + 1. \end{aligned}$$

Thus if the Goodstein function were provably computable in PA so would be the function  $H_{\epsilon_0}$ . This is impossible however, because  $H_{\epsilon_0}(n) = F_{(\epsilon_0)_{n-1}}(n)$

majorises all provably computable functions of PA.

NOTES

- (1) Although we have not done so here, the techniques can be refined to produce independence results for various *fragments* of Peano Arithmetic, obtained by restricting the complexity of the formula  $A$  in the Induction Axiom. Parsons was the first to study the relationships between such fragments and his (and others') work in this area is developed further and unified in Sieg [1985].
- (2) An alternative approach to our Bounding Theorem 5, based on Gödel's Dialectica Interpretation of PA and Normalisation of infinitary terms, is set out in Rose [1984]. A streamlined version of this approach is in handwritten notes of Buchholz: "Three contributions to the conference on Recent Advances in Proof Theory", Oxford, April 1980.





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