The algebra $k(n)_*(k(n))$ for the prime 2

By

ROLF KULTZE and URS WÜRGLER

1986

1. Introduction. For any positive integer n let $k(n)^*(-)$ denote the (-1)-connected cover of the n-th Morava K-theory [3] associated to the prime p. k(n) is a BP-module spectrum with coefficients $k(n)_* = \pi_*(k(n)) \cong \mathbb{Z}_p[v_n]$, $|v_n| = 2(p^n-1)$. If p is odd, k(n) admits a unique admissible (with respect to BP) product and in [6] Yagita determined the algebra structure of the self homology $k(n)_*(k(n))$ for this case. For p = 2 k(n) admits exactly two admissible products m_n, $\bar{m}_n : k(n) \wedge k(n) \to k(n)$, both non-commutative, which are related by the formula

$$\bar{m}_n = m_n \cdot T = m_n + v_n m_n(q_{n-1} \wedge q_{n-1})$$

where T denotes the switch map and $q_{n-1} \in k(n)^*(k(n))$ the (n-1)-th Bockstein operation [4]. We choose one of these products and consider k(n) as a ring spectrum. The aim of the present paper is now to determine the structure of the algebra $k(n)_*(k(n))$ for the case p = 2 (Theorem 3.1). In particular it turns out that this algebra is commutative. Whereas Yagita uses an analysis of the Atiyah-Hirzebruch spectral sequence for $k(n)_*(BP\langle n\rangle)$ as his main tool ($BP\langle n\rangle$ denotes the Johnson-Wilson spectrum with coefficients $\mathbb{Z}_{(p)}[v_1,\dots, v_n]$, see [2]) we work with the spectra P(n) with coefficients $\mathbb{Z}_2[v_n,v_{n+1},\dots]$ [3] to be able to make use of our knowledge of the algebra $P(n)_*(P(n))$ for p = 2 [4].

2. Generators for the module $k(n)_*(k(n))$. For the rest of the paper we always suppose p = 2. Recall that $\mathfrak{A}(2)_* = H\mathbb{Z}_{2*}(H\mathbb{Z}_2) \cong \mathbb{Z}_2[\xi_1, \xi_2, \dots]$, $|\xi_i| = 2^i-1$. Let $Sq^{\Delta i} \in \mathfrak{A}(2)^*$ denote the dual of ξ_i and $c : \mathfrak{A}(2)_* \to \mathfrak{A}(2)_*$ the conjugation map. From [1] we know that

$$H\mathbb{Z}_{2*}(k(n)) \cong \mathbb{Z}_2[\bar{\xi}_1, \dots, \bar{\xi}_n, \bar{\xi}^2_{n+1}, \bar{\xi}_{n+2}, \dots]$$

where $\bar{\xi}_i = c(\xi_i)$. Let $\eta_n : P(n) \to k(n)$ and $\pi_n : k(n) \to H\mathbb{Z}_2$ be the canonical maps of ring spectra. Using the Baas-Sullivan exact sequence one deduces from [4], theorem 1.4, that there is an isomorphism of (commutative!) algebras

$$k(n)_*(P(n)) \cong k(n)_*[\alpha_0, \dots, \alpha_{n-1}, t_{n+1}, \dots], \qquad \alpha_i^2 = t_{i+1}$$

where $|\alpha_i| = 2^{i+1} - 1$ and $|t_j| = 2^{j+1} - 2$. Applying η_n we get elements $(1 \wedge \eta_n)_*(\alpha_i)$, $(1 \wedge \eta_n)_*(t_j) \in k(n)_*(k(n))$ which we denote by the same symbols. Let $\pi_{n*} : k(n)_*(k(n)) \longrightarrow H\mathbb{Z}_{2*}(k(n))$ be induced by π_n. Then $\pi_{n*}(\alpha_i) = \bar{\xi}_{i+1}$ $(0 \le i \le n-1)$ and $\pi_{n*}(t_j) = \bar{\xi}_j^2$. This is a consequence of a similar property of the corresponding elements in $P(n)_*(P(n))$ [4]. For exponent sequences $C = (c_0,\dots,c_{n-1})$, $E = (e_{n+1},e_{n+2},\dots)$ with almost all $e_i = 0$ we write as usual

$$\alpha^C t^E = \alpha_0^{c_0} \dots \alpha_{n-1}^{c_{n-1}} \cdot t_{n+1}^{e_{n+1}} \dots \ .$$

Again using the Baas-Sullivan exact sequence one sees that the kernel J_n of $(1 \wedge \eta_n)_*$: $k(n)_*(P(n)) \longrightarrow k(n)_*(k(n))$ is generated by the elements $\eta_R(v_{n+i})$ $(i \ge 1)$, so from Ravenel's formula [5] it follows that

$$J_n = (v_n \alpha_i^{2^{n+1}} - v_n^{2^{i+1}} \alpha_i^2,\ v_n t_j^{2^n} - v_n^{2^j} t_j)$$

and we see that $k(n)_*(k(n))$ contains the subalgebra

$$A_n = k(n)_*[\alpha_0, \dots, \alpha_{n-1}, t_{n+1}, \dots] \,/\, J_n.$$

The elements $\alpha^C t^E$ do not yet generate the $k(n)_*$-module $k(n)_*(k(n))$. To find the remaining generators we consider the cofibre sequence

$$\Sigma^{2^{n+1}-2} k(n) \xrightarrow{v_n} k(n) \xrightarrow{\pi_n} H\mathbb{Z}_2 \xrightarrow{\partial_n} \Sigma^{2^{n+1}-1} k(n)$$

which induces the exact sequence

$$\dots \longrightarrow k(n)_*(k(n)) \xrightarrow{\cdot v_n} k(n)_*(k(n)) \xrightarrow{\pi_{n*}} H\mathbb{Z}_{2*}(k(n)) \xrightarrow{\partial_{n*}} k(n)_*(k(n)) \longrightarrow \dots \ .$$

Let $\mathcal{E}_n$ denote the set of all exponent sequences $R = (r_{n+2}, r_{n+3}, \dots)$, $r_i = 0$ for almost all i and $r_i = 1$ otherwise. If $R \in \mathcal{E}_n$ we define $I_R = \{ i \in \mathbb{N} \mid r_{n+i} = 1 \}$ and we put $|R| = \text{card } I_R$. For $R \in \mathcal{E}_n$, $|R| > 0$, we define elements b^R by

$$b^R = \partial_{n*}(\bar{\xi}^R) \in k(n)_*(k(n)).$$

Observe that b^R is v_n-torsion and that

$$\pi_{n*}(b^R) = Sq^{\Delta_{n+1}}(\bar{\xi}^R) = \sum_{i\in I_R} \bar{\xi}_{i-1}^{2^{n+1}}\, \bar{\xi}^{R-\Delta_{n+1}} \qquad (1)$$

because $Sq^{\Delta_{n+1}}(\bar{\xi}_k) = \bar{\xi}_{k-n-1}^{2^{n+1}}$ and $Sq^{\Delta_{n+1}}$ is a derivation .

We would like now to prove that the elements $\alpha^C t^E b^R$ generate the $k(n)_*$-module $k(n)_*(k(n))$. For this we first look at the Atiyah-Hirzebruch spectral sequence $\{E^r, d^r\}$ for $k(n)_*(k(n))$. Let Γ_n denote the ring $\mathbb{Z}_2[\bar{\xi}_1, ..., \bar{\xi}_n, \bar{\xi}_{n+1}^2, \bar{\xi}_{n+2}^2, ...]$.

Lemma 2.1. The term $E_{*,0}^{\infty}$ of the AHSS for $k(n)_*(k(n))$ is a Γ_n- module generated by 1 and the elements

$$\sum_{i\in I_R} \bar{\xi}_{i-1}^{2^{n+1}}\, \bar{\xi}^{R-\Delta_{n+1}} \qquad (R \in \mathcal{E}_n).$$

P r o o f . A proof of this lemma may be given by adapting the method of the proof of [6], lemma 3.2 to the present situation and we will not give the details here. The only non-trivial differential in the spectral sequence is $d^{2^{n+1}-1} = Sq^{\Delta_{n+1}} \otimes v_n$. The triviality of d^r for $r \geq 2^{n+1}$ is seen by comparing $\{E^r, d^r\}$ with the AHSS for $k(n)_*(P(n))$ (which collapses) via $\eta_n : P(n) \longrightarrow k(n)$.

Corollary 2.2. The algebra $k(n)_*(k(n))$ is commutative. Moreover, as a $k(n)_*$-module, it is generated by the elements $\alpha^C t^E$ and $\alpha^C t^E b^R$, $|R| \geq 1$.

P r o o f . Lemma 2.1 implies that $k(n)_*(k(n))$ is generated by the elements $\alpha^C t^E$, $\alpha^C t^E b^R$ and $b^R \alpha^C t^E$. Now we observe that

$$\ker(\cdot\, v_n) \cap \operatorname{im}(.\, v_n) = 0 \quad . \qquad (2)$$

If $x_1 = \alpha^C t^E b^R - b^R \alpha^C t^E$, $x_2 = b^R b^S - b^S b^R$ $(R,S \in \mathcal{E}_n,\ |R| \geq 1,\ |S| \geq 1)$ then $x_i \in \ker \pi_{n*} = \operatorname{im}(\cdot v_n)$ for $i = 1,2$. But $v_n x_i = 0$, so by (2) we obtain $x_i = 0$. The corollary follows.

Remark 2.3. We have

$$b^{\Delta_{n+1}} = t_{i-1}^{2^n} - v_n^{2^{i-1}-1} t_{i-1}. \qquad (i \geq 2)$$

To see this put

$$y = b^{\Delta_{n+i}} - (t_{i-1}^{2^n} - v_n^{2^{i-1}-1}\, t_{i-1}).$$

It follows from (1) that $y \in \ker \pi_{n*} = \mathrm{im}(\cdot\, v_n)$. Moreover, looking at the structure of J_n, $y \in \ker(.\, v_n)$, hence $y = 0$ by (2).

3. The main result. Before determining the algebra structure of $k(n)_*(k(n))$ we introduce the following notations. Let Σ_n be the polynomial ring $\mathbb{Z}_2[\alpha_0, \ldots, \alpha_{n-1}, t_{n+1}, \ldots]$, M_n the free Σ_n-module generated by the elements $b^R (|R| \geq 2)$ and $N_n \subset M_n$ the Σ_n-submodule generated by the elements $\sum_{i \in I_R} t_{i-1}^{2^n} b^{R-\Delta_{n+i}}$ $(|R| \geq 3)$. For $R, S \in \mathcal{E}_n$ we define $I_{R,S} = I_R \cap I_S$.
If $I_{R,S} \neq \emptyset$ let I'_R denote the set $I'_R = I_R - I_{R,S}$ and define $R', R'' \in \mathcal{E}_n$ by

$$r'_{n+i} = \begin{cases} 1 & i \in I'_R \\ 0 & i \notin I'_R \end{cases}$$

$$R'' = R - R' .$$

Remark that $R'' = S''$ and $I_{R,S} = I_{R''} = I_{S''}$. Now we are ready to state our main result.

Theorem 3.1. Suppose $p = 2$. (a) There is an isomorphism of $k(n)_*$-modules

$$k(n)_*(k(n)) \cong k(n)_*[\alpha_0, \ldots, \alpha_{n-1}, t_{n+1}, \ldots] / J_n \oplus M_n/N_n.$$

(b) If $I_{R,S} = \emptyset$ then

$$b^R b^S = \sum_{i \in I_R} t_{i-1}^{2^n} b^{R+S-\Delta_{n+i}} = \sum_{j \in I_S} t_{j-1}^{2^n} b^{R+S-\Delta_{n+j}}.$$

(c) If $I_{R,S} \neq \emptyset$ then

$$b^R b^S = \sum_{i \in I_{R,S}} t_{i-1}^{2^n} t^{R''-\Delta_{n+i}} b^{R'+S'+\Delta_{n+i}} + t^{R''} \sum_{i \in I_{R'}} t_{i-1}^{2^n} b^{R'+S'-\Delta_{n+i}}.$$

P r o o f. (a) follows from the results of section 2 and an inspection of the map $M_n \to k(n)_*(k(n))$ (see also [6], p. 430).

(b) Let $\beta^R = \pi_{n*}(b^R) \in H\mathbb{Z}_{2*}(k(n))$. By (1) we have $\beta^R = \sum_{i \in I_R} \bar{\xi}_{i-1}^{2^{n+1}} \bar{\xi}^{R-\Delta_{n+i}}$, hence

$$\beta^{R+S} = \bar{\xi}^R \beta^S + \bar{\xi}^S \beta^R. \tag{3}$$

It follows from (a) that $\sum_{i \in I_R} t_{i-1}^{2^n} b^{R-\Delta_{n+i}} = 0$, thus

$$\sum_{i \in I_R} \bar{\xi}_{i-1}^{2^{n+1}} \beta^{R-\Delta_{n+i}} = 0. \tag{4}$$

Using (3) we see that

$$\bar{\xi}^{R-\Delta_{n+i}} \beta^S = \beta^{R+S-\Delta_{n+i}} + \bar{\xi}^S \beta^{R-\Delta_{n+i}}. \qquad (i \in I_R) \tag{5}$$

Hence we get from (4) and (5)

$$\beta^R \beta^S = \sum_{i \in I_R} \bar{\xi}_{i-1}^{2^{n+1}} \bar{\xi}^{R-\Delta_{n+i}} \beta^S = \sum_{i \in I_R} \bar{\xi}_{i-1}^{2^{n+1}} \beta^{R+S-\Delta_{n+i}} \tag{6}$$

and the assertion (b) follows from (2).

(c) From (3) we get

$$\bar{\xi}^{2(R''-\Delta_{n+i})} \beta^{R'+S'+\Delta_{n+i}} = \bar{\xi}_{i-1}^{2^{n+1}} \bar{\xi}^{R+S-2\Delta_{n+i}} + \bar{\xi}^{R+S''-\Delta_{n+i}} \beta^{S'} + \bar{\xi}^{S+R''-\Delta_{n+i}} \beta^{R'}. \qquad (i \in I_{R,S}) \tag{7}$$

Let T be R or S. It follows from (3) and (4) that

$$\sum_{i \in I_{R,S}} \bar{\xi}_{i-1}^{2^{n+1}} \beta^{T-\Delta_{n+i}} = \beta^{T'} \sum_{i \in I_{R,S}} \bar{\xi}_{i-1}^{2^{n+1}} \bar{\xi}^{T''-\Delta_{n+i}}. \tag{8}$$

Using (3), (6) and (8) we get

$$\beta^R\beta^S = (\bar{\xi}^{R'}\beta^{R''} + \bar{\xi}^{R''}\beta^{R'})(\bar{\xi}^{S'}\beta^{R''} + \bar{\xi}^{R''}\beta^{S'}) =$$

$$= \bar{\xi}^{R'+S'} \sum_{i \in I_{R,S}} \bar{\xi}_{i-1}^{2^{n+2}} \bar{\xi}^{2(R''-\Delta_{n+1})} + \bar{\xi}^{R} \sum_{i \in I_{R,S}} \bar{\xi}_{i-1}^{2^{n+1}} \beta^{S-\Delta_{n+1}} +$$

$$+ \bar{\xi}^{S} \sum_{i \in I_{R,S}} \bar{\xi}_{i-1}^{2^{n+1}} \beta^{R-\Delta_{n+1}} + \bar{\xi}^{2R''}\beta^{R'}\beta^{S'} =$$

$$= \sum_{i \in I_{R,S}} \bar{\xi}_{i-1}^{2^{n+1}} \left\{ \bar{\xi}_{i-1}^{2^{n+1}} \bar{\xi}^{R+S-2\Delta_{n+1}} + \bar{\xi}^{R+S''-\Delta_{n+1}}\beta^{S'} + \bar{\xi}^{S+R''-\Delta_{n+1}}\beta^{R'} \right\} +$$

$$+ \bar{\xi}^{2R''}\beta^{R'}\beta^{S'} .$$

Finally we obtain from (6) and (7)

$$\beta^R\beta^S = \sum_{i \in I_{R,S}} \bar{\xi}_{i-1}^{2^{n+1}} \bar{\xi}^{2(R''-\Delta_{n+1})} \beta^{R'+S'+\Delta_{n+1}} +$$

$$+ \bar{\xi}^{2R''} \sum_{i \in I_{R'}} \bar{\xi}_{i-1}^{2^{n+1}} \beta^{R'+S'-\Delta_{n+1}}$$

and our assertion (c) follows.

Remark 3.2. If $R = S$ then we obtain from Theorem 3.1 (c)

$$(b^R)^2 = \sum_{i \in I_R} t_{i-1}^{2^n} t^{R-\Delta_{n+1}} b^{\Delta_{n+1}} .$$

We can apply Remark 2.3 to see that $(b^R)^2 \in A_n$. This means that the direct summand M_n/N_n of $k(n)_*(k(n))$ is not invariant under multiplication.

References

[1] N. A. BAAS and I. MADSEN, On the realization of certain modules over the Steenrod algebra. Math. Scand. 31, 220 - 224 (1972).

[2] D. C. JOHNSON and S. WILSON, Projective dimension and Brown-Peterson homology. Top. 12, 327-353 (1973).

[3] D. C. JOHNSON and S. WILSON, BP-operations and Morava's extraordinary K-theories. Math. Z. 144, 55-75 (1975).

[4] R. KULTZE and U. WÜRGLER, A note on the algebra $P(n)_*(P(n))$ for the prime 2. Manuscr. math. 57, 195-203 (1987).

[5] D . C. RAVENEL, The structure of Morava stabilizer algebras. Invent. Math. 37, 109-120 (1976).

[6] N. YAGITA, On the Steenrod algebra of Morava K-theory. J. London Math. Soc. 22, 423-438 (1980).

Anschrift der Autoren:

Rolf Kultze
Fachbereich Mathematik
Universität Frankfurt
Robert-Mayer-Str. 6-10

D 6000 Frankfurt/Main

Urs Würgler
Mathematisches Institut
der Universität Bern
Sidlerstr. 5

CH 3012 Bern