Sum Formula for SL_2 over Imaginary Quadratic Number Fields

Somformule voor SL_2 over
imaginair quadratisch getallenlichamen
(met een samenvatting in het Nederlands)

Формула за сумирање за SL_2 над
имагинарно квадратично бројно поле
(со резиме на македонски јазик)

Proefschrift
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To Alek

for his patience

Contents

Introduction

The Bruggeman-Kuznetsov sum formula, see preprint [24], published in [25], and [2], obtained by the authors independently, gives a relation between the Fourier coefficients of cuspidal real-analytic modular forms on the upper half-plane and Kloosterman sums. We shall give a very short overview of it.

Let $\mathbb{H}^2$ be the upper half-plane with $\mathrm{PSL}_2(\mathbb{R})$ acting by fractional linear transformations, and let $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$. On $\mathbb{H}^2$ we put the standard metric of constant curvature -1. Let $-\Delta = -y^2(\partial_x^2 + \partial_y^2)$ be the corresponding Laplacian on $\mathbb{H}^2$. Let $\{\psi_j(z)\} \subset L^2(\Gamma\backslash\mathbb{H}^2)$ be a complete orthonormal system of cuspidal Maass forms with spectral parameter $\nu_j \in i(0,\infty)$, that is, ψ_j are eigenfunctions of $-\Delta$ which are also eigenfunctions for the Hecke operators, indexed according to increasing eigenvalue $\lambda_j = \frac{1}{4} - \nu_j^2$. Let the Fourier expansion of $\psi_j(z)$ be

$$\psi_j(x+iy) = \sum_{0\neq n\in\mathbb{Z}} \rho_j(n) y^{1/2} K_{\nu_j}(2\pi|n|y) e^{2\pi i n x}, \tag{1}$$

where K_ν is the K-Bessel function (1.30); this defines the $\rho_j(n) \in \mathbb{C}$. Let $E(\nu, z)$ be the non-analytic Eisenstein series with parameter ν. The Fourier coefficients of the Eisenstein series involve divisor sums $\sigma_\nu(n) = \sum_{d|n} d^\nu$ and the Riemann-zeta function $\zeta(\nu)$. Let h be an even holomorphic function on $\left\{\nu \in \mathbb{C} : |\operatorname{Re}\nu| < \frac{1}{2} + \varepsilon\right\}$ for some $\varepsilon > 0$ such that $|h(\nu)| \ll (1+|\nu|)^{-2-\delta} e^{-\pi|\operatorname{Im}\nu|}$ for some $\delta > 0$ and all ν with $|\operatorname{Re}\nu| \leqslant \frac{1}{2} + \varepsilon$. The sum formula states that for such a test function h the following equality holds for integers $n, m \geqslant 1$, with absolute convergence of all sums and integrals in the various terms:

$$\sum_{j=1}^{\infty} \rho_j(n)\overline{\rho_j(m)} \frac{h(\nu_j)}{\cos(\pi\nu_j)} + \frac{1}{\pi i}\int_{\operatorname{Re}s=0} h(s)\left(\frac{n}{m}\right)^{-s} \frac{\sigma_{2s}(n)\sigma_{-2s}(m)}{|\zeta(1+2s)|^2} ds =$$

$$= \delta_{n,m}\frac{i}{\pi^2}\int_{\operatorname{Re}s=0} s\tan(\pi s) h(s) ds + \sum_{c=1}^{\infty} \frac{S(n,m;c)}{c}\varphi\left(\frac{4\pi\sqrt{nm}}{c}\right), \tag{2}$$

where

$$\varphi(x) = \frac{2}{\pi i}\int_{\operatorname{Re}s=0} \frac{s\, J_{2s}(x)}{\cos(\pi s)} h(s)\, ds, \tag{3}$$

for $x > 0$, J_ν denotes the classical Bessel function (1.25), $S(n,m;c)$ is the classical Kloosterman sum (6.2), and $\delta_{n,m}$ is the Kronecker delta symbol.

The left hand side of a formula (2) comes from the spectral decomposition of the Hilbert space $L^2(\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}^2)$ and it is therefore called the spectral side. The two terms correspond to the discrete and continuous spectrum of the Laplace operator $-\Delta$. The right hand side is related to the geometry of the space $\{\left(\begin{smallmatrix}1 & *\\ 0 & 1\end{smallmatrix}\right)\}\backslash\mathrm{PSL}_2(\mathbb{Z})$ induced by the Bruhat decomposition of the group $\mathrm{PSL}_2(\mathbb{R})$ and it is therefore called the geometric side. Its first term, called delta term, comes from the representatives $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$ of the cosets in $\{\left(\begin{smallmatrix}1 & *\\ 0 & 1\end{smallmatrix}\right)\}\backslash\mathrm{PSL}_2(\mathbb{Z})$ with $c = 0$. The matrices with $c \neq 0$ coming from the big cell in the Bruhat decomposition of $\mathrm{PSL}_2(\mathbb{Z})\cap\mathrm{PSL}_2(\mathbb{R})$ give rise to the second term called for obvious reasons, the Kloosterman term.

Summation formulas of this type, like (2), may be used in two different ways. On one hand, in the given form (2) with the independent test function on the spectral side, it is a tool to obtain results concerning spectral data. For example, in Proposition 4.1 of [2] one finds the following distribution result

$$\sum_{j=1}^{\infty} e^{-v\lambda_j}\frac{\rho_j(n)\overline{\rho_j(m)}}{\cos(\pi\nu_j)} = \frac{\delta_{n,m}}{\pi^2}\left|\frac{n}{m}\right|^{1/2} v^{-1} + O(v^{-1/2-\varepsilon}), \tag{4}$$

as $v \downarrow 0$, with $\varepsilon > 0$.

On the other hand, if we know how to invert the Bessel transformation (3), we have the independent test function on the geometric side. We can then use the sum formula to obtain estimates for sums of Kloosterman sums. In [25] Kuznetsov gave a one-sided inverse of the Bessel transformation and proved that for $m, n \geqslant 1$ and $X \to \infty$,

$$\sum_{c=1}^{X}\frac{S(n,m;c)}{c} \ll_{n,m} X^{1/6}(\log X)^{1/3}. \tag{5}$$

The Weil bound for the Kloosterman sums implies the estimate $O(X^{1/2+\varepsilon})$ for the sum above (see [44]), and the Linnik hypothesis predicts that it is $O(X^{\varepsilon})$, see [29].

Generalizations of the sum formula (2) are obtained in various ways. Both Bruggeman and Kuznetsov consider the full modular group $\mathrm{SL}_2(\mathbb{Z})$, weight zero and trivial multiplier system. In [38] Proskurin used Kuznetsov approach to generalize the sum formula to cofinite discrete subgroups of $\mathrm{SL}_2(\mathbb{R})$, non-trivial multiplier system and general weights. He works with Fourier terms of positive order. Fourier terms of arbitrary order are treated by Bruggeman in [3], which in contrast to [25] and [38], takes more a representational point of view.

In [35] Miatello and Wallach give a formula of the same type for real connected semi-simple Lie groups of $\mathbb{R}$-rank one. There the upper half-plane is replaced by any complete Riemannian non-compact symmetric space of rank one, the discrete subgroup of isometries has finite co-volume, and only trivial K-types are considered. In [42], the authors extend the formula in [35] to products of rank one groups. The formula is very explicit when the group is a product of groups of the form $\mathrm{SL}_2(\mathbb{R})$ or $\mathrm{SL}_2(\mathbb{C})$. Bruggeman and Miatello [4] use the sum formula in [35] to study sums of generalized Kloosterman sums for this class of groups. By taking a suitable test function, they obtain an estimate of type (5) for those sums (see [4], Theorem 1 in 4.3.). In [5], the same authors give a sum formula for SL_2 over an arbitrary number field restricted to trivial K-types. The case of a totally real number field is described by Bruggeman, Miatello, and Pacharoni in [6] taking into account all K-types.

In his book [36], Motohashi gives an explicit formula for the fourth power moment of the Riemann zeta-function using the sum formula for $\mathrm{SL}_2(\mathbb{R})$. Analogous reasoning leads to an extension of these results to a higher-dimensional situation. Bruggeman and Motohashi [8] show how one can perform the same preparatory work with the group $\mathrm{SL}_2(\mathbb{C})$ in place of $\mathrm{SL}_2(\mathbb{R})$. Their ultimate goal is to give a spectral decomposition for the fourth power moment of the Dedekind zeta-function. The sum formula for SL_2 over the Gaussian number field including non-trivial K-types, as well as an explicit formula for the fourth power moment of the Dedekind zeta-function, are derived by the same authors in [9]. There the case of even functions is treated.

In this thesis we generalize the sum formula from [9] by considering a general imaginary quadratic field F and an arbitrary congruence subgroup $\Gamma = \Gamma_0(I)$, with $I \subset \mathcal{O}$ a non-zero ideal in the ring $\mathcal{O}$ of integers of F. We consider χ-automorphic functions with respect to Γ, where χ is a character of Γ trivial on $\Gamma_1(I) \subset \Gamma$. We also consider the case of odd functions.

In Chapters 1 and 2, we describe some fundamental facts about the geometry of the three-dimensional hyperbolic space, transformation groups on this space, and Bessel functions, as well as a small part of the representation theory of the Lie groups $\mathrm{SL}_2(\mathbb{C})$ and $\mathrm{SO}(2)$. In Chapter 3 we introduce automorphic functions and automorphic representations. Central in their Fourier expansion are Jacquet and Goodman-Wallach operators treated in Chapter 4. A more detailed description of the Fourier coefficients of Eisenstein series and cuspidal automorphic representations is given in Chapter 5. Chapters 6, 7, and 8 contain known results concerning Kloosterman sums, Poincaré series, and spectral decomposition of the space $L^2(\Gamma\backslash\mathrm{SL}_2(\mathbb{C}))$. They are stated in a form appropriate for our purposes.

In Chapter 9 we introduce the Lebedev transformation, its inverse on a certain class of test functions, and give some of its properties. This transform will be the building block for the Poincaré series used to derive the sum formula. My Lebedev transformation is an extension of the classical Lebedev transformation $f \mapsto \int_0^\infty f(r) K_\nu(r) \frac{dr}{r}$ which plays a significant rôle in the theory of sum formulas for rational Kloosterman sums. The square-integrability and boundedness of the chosen Poincaré series depend heavily on the results of Miatello and Wallach, [34], as well as on Lemma 5.2.1.

The derivation of the preliminary sum formula is explained in Chapter 10. It involves computation of the inner product of two special Poincaré series in two different ways: the spectral description where the Fourier coefficients $C_V(\omega; \nu_V, p_V)$ of cusp forms appear and the geometric description where the sums of Kloosterman sums appear. This is also the method used by Bruggeman [2], Kuznetsov [25] and Proskurin [38].

Our main result is the spectral sum formula, Theorem 11.3.3 in Chapter 11. It is obtained by extending the class of test functions in its preliminary version Proposition 10.3.1. The extension method is described in Section 11.2. It is analogue to the method of Miatello and Wallach, [].

In Section 11.5, we apply this formula to obtain weighted density results concerning the cuspidal automorphic representations in $L^2(\mathrm{SL}_2(\mathcal{O})\backslash\mathrm{SL}_2(\mathbb{C}))$ with eigenvalue λ_V not exceeding X and prescribed spectral parameter p_V. Namely, for arbitrary $\omega \in \mathcal{O}'\backslash\{0\}$, $p \in \frac{1}{2}\mathbb{Z}$ and $X \to \infty$ we have

$$\sum_{\substack{V: p_V = \pm p \\ \lambda_V \leqslant X}} |C_V(\omega; \nu_V, p_V)|^2 \sim \frac{2\epsilon_p}{3\pi^3\sqrt{|d_F|}} X^{3/2}, \tag{6}$$

where $\epsilon_0 = 1$ and $\epsilon_p = 2$ if $p \neq 0$.

In Chapter 12, Theorem 12.2.1, we give a one-sided inversion of the Bessel transformation (11.1), which allows us to prove the sum formula in a reversed form, see Theorem 12.3.2. Applications of this formula in obtaining estimates for the sum of Kloosterman sums (11.15) might be possible, but the limited time available for the present work forces us to postpone their derivation.

Chapter 1

Preliminaries

Convention. We use the notation $\mathbb{N} = \mathbb{Z}_{\geqslant 0}$ for the non-negative integers, and we call the elements of the set $\frac{1}{2} + \mathbb{Z}$ half-integers.

1.1 Three-Dimensional Hyperbolic Space

Three-dimensional hyperbolic space is the unique three-dimensional connected and simply connected Riemannian manifold with constant sectional curvature equal to -1. This space has certain concrete models which all have certain advantages. For our purposes, the upper half-space

$$\mathbb{H}^3 = \left\{(z,r) \mid z \in \mathbb{C},\ r > 0\right\} = \mathbb{C} \times (0,\infty) \tag{1.1}$$

in Euclidean three-space gives a convenient model of the three-dimensional hyperbolic space which in its properties closely resembles the well-known upper half-plane as a model of plane hyperbolic geometry. It is useful for computations to think of $\mathbb{H}^3$ as a subset of Hamilton's quaternions $\mathcal{H}(-1,-1)$. (See [11], §10.1.) As usual we write $1, i, j, k$ for the standard $\mathbb{R}$-basis of $\mathcal{H}(-1,-1)$. The notation for points in $\mathbb{H}^3$ is

$$(z,r) = z + rj = x + yi + rj \tag{1.2}$$

where $j = (0,0,1)$ and $x, y \in \mathbb{R}$. For a discussion of this space, other models, its Riemannian metric

$$r^{-2}\left(dx^2 + dy^2 + dr^2\right), \tag{1.3}$$

its $\mathrm{SL}_2(\mathbb{C})$-invariant measure

$$r^{-3} dx\, dy\, dr, \tag{1.4}$$

and its Laplace-Beltrami operator

$$L := r^2\left(\partial_x^2 + \partial_y^2 + \partial_r^2\right) - r\partial_r, \tag{1.5}$$

see [11], Chapter 1.

The group $G = \mathrm{SL}_2(\mathbb{C})$ acts on $\mathbb{H}^3$, and the action of the matrix $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$ is given by

$$g(z,r) = \left(\frac{(az+b)(\bar{c}\bar{z}+\bar{d}) + a\bar{c}r^2}{|cz+d|^2 + |c|^2r^2}, \frac{r}{|cz+d|^2+|c|^2r^2}\right). \tag{1.6}$$

This action looks complicated, but actually it is very natural if it is looked at in the spirit of the theory of topological transformation groups and of Lie theory. Considered as a Lie group over $\mathbb{R}$, G has dimension 6. The special unitary group $K = \mathrm{SU}(2)$ is one of its maximal compact subgroups. The symmetric space associated to G is the quotient space G/K with G acting by left multiplication. The general theory of this situation is contained in [16]. The map

$$\pi \;:\; G \longrightarrow \mathbb{H}^3, \qquad \pi(g) = g \cdot j, \tag{1.7}$$

gives rise to an isomorphism, as topological spaces, between the symmetric space of $\mathrm{SL}_2(\mathbb{C})$ and $\mathbb{H}^3$ given by the explicit formula

$$G/K \ni \begin{pmatrix} \sqrt{r} & z/\sqrt{r} \\ 0 & 1/\sqrt{r} \end{pmatrix} \longleftrightarrow (z,r) \in \mathbb{H}^3. \tag{1.8}$$

The equation (1.6) describes in fact the natural action of G on G/K by multiplication of cosets from the left. See [11], §1.6, [31], p. 6–7 or [23] for more details on the computations of this correspondence.

1.2 Some Discrete Subgroups

Let $F = \mathbb{Q}(\sqrt{D})$, with $D < 0$ a square-free integer, be an imaginary quadratic number field. We write $\mathcal{O} = \mathcal{O}_F$ for the ring of integers in F. The discriminant of F is denoted by d_F. The negative integer d_F satisfies

$$d_F = \begin{cases} 4D & , \quad D \equiv 2,3 \pmod 4 \\ D & , \quad D \equiv 1 \pmod 4, \end{cases} \equiv \begin{cases} 0 & \pmod 4 \\ 1 & \pmod 4, \end{cases} \tag{1.9}$$

The ring of integers $\mathcal{O}$ has the $\mathbb{Z}$-basis consisting of 1 and β, where

$$\beta = \frac{d_F + \sqrt{d_F}}{2} = \frac{d_F}{2} + i\frac{\sqrt{|d_F|}}{2}. \tag{1.10}$$

The group $\mathrm{SL}_2(\mathcal{O})$ is a discrete cofinite, but not cocompact subgroup of $G = \mathrm{SL}_2(\mathbb{C})$. (See [11], Theorem 7.1.1. The result stated is actually for $\mathrm{PSL}_2(\mathcal{O})$, but it holds for $\mathrm{SL}_2(\mathcal{O})$ as well.)

Given a non-zero ideal $I \subset \mathcal{O}$, the principal congruence subgroup of level I is defined by:

$$\Gamma(I) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}) \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod I \right\}. \tag{1.11}$$

Any discrete subgroup $\Gamma \subset \mathrm{SL}_2(\mathcal{O})$ which is G-conjugate to a group containing $\Gamma(I)$ for some non-zero ideal $I \subset \mathcal{O}$ is called a congruence subgroup with respect to $\mathrm{SL}_2(\mathcal{O})$. Examples of congruence subgroups are:

$$\Gamma_1(I) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}) \;\middle|\; c \in I,\ a, d \equiv 1 \bmod I \right\}, \tag{1.12}$$

$$\Gamma_0(I) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}) \;\middle|\; c \in I \right\}. \tag{1.13}$$

The smaller the subgroup $\Gamma \subset \mathrm{SL}_2(\mathcal{O})$ is, the bigger is the class of automorphic functions on $\Gamma\backslash\mathbb{H}^3$. Therefore we want to consider a principal congruence subgroup $\Gamma(I)$, for $I \subset \mathcal{O}$ a non-zero ideal. But, conjugation with an element $g = \left(\begin{smallmatrix} A & 0 \\ 0 & A^{-1} \end{smallmatrix}\right) \in G$ gives for $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_1(I)$:

$$g \begin{pmatrix} a & b \\ c & d \end{pmatrix} g^{-1} = \begin{pmatrix} a & A^2 b \\ A^{-2} c & d \end{pmatrix} \in \Gamma(I),$$

for a suitably chosen A in some principal ideal contained in I such that $A^2 b \in I$. Therefore, up to conjugation, it is enough for our purpose to consider a congruence subgroup $\Gamma_1(I)$.

On the other hand, one can easily see that $\Gamma_1(I) = \bigcap_{\chi_0} \ker\chi_0$, where the intersection runs over all characters χ_0 of $\Gamma_0(I)$ which are trivial on $\Gamma_1(I)$. We note that since P as in (1.16) is contained in $\mathrm{SL}_2(\mathcal{O}/I)$ and $\mathrm{SL}_2(\mathcal{O}) \to \mathrm{SL}_2(\mathcal{O}/I)$ is surjective (see [17], p. 249), any $\gamma \equiv \left(\begin{smallmatrix} a & b \\ o & d \end{smallmatrix}\right)$ (mod I) is congruent to a product $\left(\begin{smallmatrix} 1 & b/d \\ 0 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 1/d & 0 \\ 0 & d \end{smallmatrix}\right)$ (mod I), with $\left(\begin{smallmatrix} 1 & b/d \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma_1(I)$. Then $\chi_0(\gamma) = \chi_0\left(\left(\begin{smallmatrix} 1/d & 0 \\ 0 & d \end{smallmatrix}\right)\right)$, which means that any character χ_0 of $\Gamma_0(I)$ which is trivial on $\Gamma_1(I)$ actually depends only on d. Hence, considering $\Gamma_0(I)$ and characters on $\Gamma_0(I)$ of the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \chi(d), \tag{1.14}$$

with χ being a character of $(\mathcal{O}/I)^*$, we are actually dealing with $\Gamma_1(I)$.

From now on, we fix a congruence subgroup $\Gamma := \Gamma_0(I)$, with $I \subset \mathcal{O}$ a non-zero ideal, and consider characters on Γ of the form (1.14). We shall use the same notation χ both for a character of $(\mathcal{O}/I)^*$ and the corresponding character of Γ indicated in (1.14).

1.3 Cusps

We consider the Riemannian sphere $\mathbb{P}^1(\mathbb{C}) = \mathbb{C} \cup \{\infty\}$ as the boundary of the upper half-space $\mathbb{H}^3$. Elements in $\mathbb{P}^1(\mathbb{C})$ are represented by $[z_1, z_2]$, where $z_1, z_2 \in \mathbb{C}$ and $(z_1, z_2) \neq (0, 0)$. Here $\infty = [1, 0]$. The action of G on $\mathbb{P}^1(\mathbb{C})$ is given by:

$$g[z_1, z_2] = [az_1 + bz_2, cz_1 + dz_2], \tag{1.15}$$

for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$. See [11], §1.1 for more details.

For each element $\zeta \in \mathbb{H}^3 \cup \mathbb{P}^1(\mathbb{C})$ we call $\Gamma_\zeta = \{\gamma \in \Gamma \mid \gamma\zeta = \zeta\}$, the subgroup of Γ fixing ζ, the stabilizer of ζ in Γ.

Let

$$P = \left\{ \left(\begin{array}{cc} u & z \\ 0 & u^{-1} \end{array} \right) \middle| \ u \in \mathbb{C}^*, \ z \in \mathbb{C} \right\} \tag{1.16}$$

be the set of upper-triangular matrices in G, and

$$N = \left\{ \left(\begin{array}{cc} 1 & z \\ 0 & 1 \end{array} \right) \middle| \ z \in \mathbb{C} \right\}. \tag{1.17}$$

its unipotent radical. The group N is isomorphic to the additive group $\mathbb{C}^+$, and the group P is isomorphic to the semi-direct product of $\mathbb{C}^+$ and $\mathbb{C}^*$. The group P also appears as the stabilizer of ∞ in the action of G on $\mathbb{P}^1(\mathbb{C})$.

If $\zeta \in \mathbb{P}^1(\mathbb{C})$ and $g_\zeta \in G$ with $\zeta = g_\zeta \cdot \infty$ we further define

$$\Gamma'_\zeta = \Gamma \cap g_\zeta N {g_\zeta}^{-1} = \Gamma_\zeta \cap g_\zeta N {g_\zeta}^{-1}. \tag{1.18}$$

Note that Γ'_ζ consists of the parabolic elements in Γ_ζ together with the identity.

By Definition 2.1.10 in [11], the cusps are those elements in $\mathbb{P}^1(\mathbb{C})$ whose stabilizer in Γ under the action (1.15) contains a free abelian group of rank 2. Since $\Gamma \subset G$ is a discrete subgroup of finite covolume, it follows from Theorem 2.5.1 in [11], that there are only finitely many Γ-inequivalent cusps. We denote by $\mathcal{C}(\Gamma)$ the set of representatives of the equivalence classes of cusps for Γ.

For each representative $\kappa \in \mathcal{C}(\Gamma)$, we fix $g_\kappa \in G$ such that $\kappa = g_\kappa \cdot \infty$. For the class of ∞ we choose ∞ as the representative and $g_\infty = 1$.

For each $\kappa \in \mathcal{C}(\Gamma)$, we have the stabilizer of κ

$$\Gamma_\kappa = \Gamma \cap g_\kappa P {g_\kappa}^{-1}, \tag{1.19}$$

and its unipotent subgroup

$$\Gamma'_\kappa = \Gamma \cap g_\kappa N {g_\kappa}^{-1} = \Gamma_\kappa \cap g_\kappa N {g_\kappa}^{-1}. \tag{1.20}$$

For $\kappa = \infty$ we have

$$\Gamma_\infty = \Gamma \cap P = \left\{ \begin{pmatrix} \varepsilon^{-1} & \xi \\ 0 & \varepsilon \end{pmatrix} \;\middle|\; \xi \in \mathcal{O},\, \varepsilon \in \mathcal{O}^* \right\} =: \Gamma_P,$$
$$\Gamma'_\infty = \Gamma \cap N = \left\{ \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} \;\middle|\; \xi \in \mathcal{O} \right\} =: \Gamma_N.$$

The possibilities for Γ_P and Γ_N are summarized in [11], Theorem 1.8. The unipotent subgroup Γ'_κ has finite index in Γ_κ. (See [11], p.100). For $\kappa = \infty$ it is easily seen that

$$[\Gamma_P : \Gamma_N] = |\mathcal{O}^*| = \begin{cases} 2 & \text{if } F = \mathbb{Q}(\sqrt{D}),\ D \neq -1, -3 \\ 4 & \text{if } F = \mathbb{Q}(i) \\ 6 & \text{if } F = \mathbb{Q}(\sqrt{-3}). \end{cases}$$

If $\kappa \in \mathcal{C}(\Gamma)$ is a cusp for Γ, then the discrete subgroup $g_\kappa^{-1}\Gamma'_\kappa g_\kappa$ corresponds to a lattice Λ_κ in $\mathbb{C}$ in the following way:

$$g_\kappa^{-1}\Gamma'_\kappa g_\kappa = \left\{ \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} \;\middle|\; \lambda \in \Lambda_\kappa \right\}, \tag{1.21}$$

see Theorem 2.1.8, (3) in [11]. Let $\mathcal{R}_\kappa \subset \mathbb{C}$ be a fundamental domain for the action of $g_\kappa^{-1}\Gamma'_\kappa g_\kappa$ on $\mathbb{P}^1(\mathbb{C})\backslash\{\infty\} = \mathbb{C}$. We denote its Euclidean area by $|\Lambda_\kappa|$. For $Y > 0$, we define

$$\mathcal{F}_\kappa(Y) = g_\kappa \left\{ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{r} & 0 \\ 0 & 1/\sqrt{r} \end{pmatrix} \;\middle|\; z \in \mathcal{R}_\kappa, r \geqslant Y \right\} \tag{1.22}$$

to be the cusp sector corresponding to κ. Then there exists a compact polyhedron $\mathcal{F}_0 \subset \mathbb{H}^3$ such that

$$\mathcal{F} = \mathcal{F}_0 \cup \bigcup_{\kappa \in \mathcal{C}_\chi} \mathcal{F}_\kappa(Y) \tag{1.23}$$

is a fundamental domain for $\Gamma\backslash\mathbb{H}^3$.

We denote by $\mathcal{C}_\chi$ the set of cusps $\kappa \in \mathcal{C}(\Gamma)$ for which the character $\chi : \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \chi(d)$ is trivial on Γ'_κ. It is clear that $\infty \in \mathcal{C}_\chi$.

1.4 Bessel functions

The Bessel functions of order ν are solutions of Bessel's differential equation

$$z^2 f''(z) + z f'(z) + (z^2 - \nu^2) f(z) = 0, \tag{1.24}$$

where ν and z are arbitrary complex numbers. Since this equation is singular at $z = 0$, cutting the complex z-plane along the segment $(-\infty, 0]$ gives for $\nu \notin \mathbb{Z}$ two linearly independent solutions $J_\nu(z)$ and $J_{-\nu}(z)$ of (1.24) which are holomorphic in $z \in \mathbb{C}\backslash(-\infty, 0]$. Here $J_\nu(z)$ is given by the power series

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m (z/2)^{\nu+2m}}{m!\Gamma(\nu+1+m)}, \tag{1.25}$$

which converges absolutely in the whole complex plane. If $\nu = n \in \mathbb{Z}$, the solutions are no longer linearly independent, and we have the relation

$$J_{-n}(z) = (-1)^n J_n(z). \tag{1.26}$$

For fixed z, considered as function of ν, $J_\nu(z)$ represents an entire function, and $J_\nu(z)(z/2)^{-\nu}$ is an even entire function of z for given ν.

Rotating the variable by an angle $\pi/2$, that is $z \mapsto iz$, the equation (1.24) transforms into the so called modified Bessel's differential equation

$$z^2 f''(z) + zf'(z) - (z^2 + \nu^2) f(z) = 0. \tag{1.27}$$

The solutions of this equations are called modified Bessel functions. As before, for $\nu \notin \mathbb{Z}$, we have two linearly independent solutions holomorphic in $\mathbb{C}\backslash(-\infty, 0]$. One of these is given by the power series

$$I_\nu(z) = \sum_{m=0}^{\infty} \frac{(z/2)^{\nu+2m}}{m!\Gamma(\nu+1+m)}, \tag{1.28}$$

and the other one is $I_{-\nu}(z)$. If $\nu = n \in \mathbb{Z}$, we have the relation

$$I_{-n}(z) = I_n(z). \tag{1.29}$$

Setting

$$K_\nu(z) = \frac{\pi}{2\sin\pi\nu}\big\{I_{-\nu}(z) - I_\nu(z)\big\} \tag{1.30}$$

yields a pair $I_\nu(z)$, $K_\nu(z)$ of linearly independent solutions to (1.27) for all $\nu \in \mathbb{C}$. The K-Bessel function $K_\nu(z)$ has a holomorphic extension to integral values of the parameter ν, and it is even with respect to ν:

$$K_{-\nu}(z) = K_\nu(z). \tag{1.31}$$

The Bessel functions of different order are related by many recursion formulas. For a short overview of these formulas, other properties, integral representations or estimates of the Bessel functions we refer to [32]. A far more extended treatment of the Bessel functions can be found in [43].

Here we give selected results that are going to be used in our further work. The power series expansions (1.28)–(1.30) show that

$$I_\nu(z) \ll_{z_0,\sigma} z^{\operatorname{Re}\nu} e^{\frac{\pi}{2}|\operatorname{Im}\nu|}(1+|\operatorname{Im}\nu|)^{-\operatorname{Re}\nu-\frac{1}{2}}, \tag{1.32}$$

and

$$K_\nu(z) \ll_{z_0,\sigma} z^{-|\operatorname{Re}\nu|-\varepsilon} e^{-\frac{\pi}{2}|\operatorname{Im}\nu|}(1+|\operatorname{Im}\nu|)^{|\operatorname{Re}\nu|-\frac{1}{2}}, \tag{1.33}$$

uniformly for $|\operatorname{Re}\nu| \leqslant \sigma$, $z \in (0, z_0)$ with $\sigma > 0$, $z_0 > 0$, $\varepsilon > 0$. The ε in the exponent of z in (1.33) is added to take care of the logarithmic contributions at integral values of ν.

The asymptotic expressions for the functions $I_\nu(z)$ and $K_\nu(z)$ on page 139 in [32], for large argument z and fixed order, tell us that the function $z \mapsto I_\nu(z)$ is exponentially increasing as $z \to \infty$, while $z \mapsto K_\nu(z)$ is exponentially decreasing as $z \to \infty$. For real positive $y > 1 + |\nu|^2$ we have

$$I_\nu(y) = (2\pi y)^{-1/2} e^y \left(1 + O\left(\frac{1+|\nu|^2}{y}\right)\right), \tag{1.34}$$

and

$$K_\nu(y) = \left(\frac{\pi}{2y}\right)^{1/2} e^{-y} \left(1 + O\left(\frac{1+|\nu|^2}{y}\right)\right). \tag{1.35}$$

To get an estimate uniform in the parameter ν, we use Basset's integral for K_ν, see [43], §6.16:

$$K_\nu(z) = \frac{\Gamma\left(\nu+\frac{1}{2}\right)}{2\sqrt{\pi}(z/2)^\nu} \int_{-\infty}^{\infty} e^{-izt}(1+t^2)^{-\nu-\frac{1}{2}}\,dt. \tag{1.36}$$

The integral in (1.36) converges absolutely if $\operatorname{Re}\nu > -\frac{1}{2}$, and yields an estimate

$$K_\nu(z) \ll_{\sigma_1,\sigma_2} z^{-\operatorname{Re}\nu} e^{-\frac{\pi}{2}|\operatorname{Im}\nu|}(1+|\operatorname{Im}\nu|)^{\operatorname{Re}\nu}$$

uniformly for $\operatorname{Re}\nu \in [\sigma_1, \sigma_2] \subset (0, \infty)$. After k-fold partial integration, Basset's formula becomes

$$K_\nu(z) = \Gamma\left(\nu+\tfrac{1}{2}\right)(z/2)^{-\nu-k} \int_{-\infty}^{\infty} p_k(t,\nu) e^{-izt}(1+t^2)^{-\nu-\frac{1}{2}-k}\,dt$$

where $p_k(u,\nu)$ is a polynomial in u and ν with degree at most k in both variables. This new integral representation is valid for $\operatorname{Re}\nu > -\frac{k}{2}$, and yields an estimate

$$K_\nu(z) \ll_{\sigma_1,\sigma_2,k} z^{-\operatorname{Re}\nu-k} e^{-\frac{\pi}{2}|\operatorname{Im}\nu|}(1+|\operatorname{Im}\nu|)^{\operatorname{Re}\nu+k} \tag{1.37}$$

uniformly for $\operatorname{Re}\nu \in [\sigma_1, \sigma_2] \subset \left(-\frac{k}{2}, \infty\right)$ for each integer $k \geqslant 1$.

Chapter 2

Representation theory

2.1 Structure of $\mathrm{SL}_2(\mathbb{C})$

In this section we state certain facts concerning the structure of the Lie group $\mathrm{SL}_2(\mathbb{C})$, consisting of complex 2×2 matrices of determinant 1, as well as its Lie algebra $\mathfrak{sl}_2(\mathbb{C})$.

2.1.1 Lie group $\mathrm{SL}_2(\mathbb{C})$

We shall use the following notation for explicit matrix elements:

$$\mathrm{n}[z] = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}, \quad \mathrm{h}[u] = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix}, \quad \mathrm{k}[\alpha, \beta] = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix},$$
$$\mathrm{a}[r] = \mathrm{h}[\sqrt{r}], \qquad \mathrm{w} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$
$$\mathrm{v}[t] = \begin{pmatrix} \cosh \frac{t}{2} & \sinh \frac{t}{2} \\ \sinh \frac{t}{2} & \cosh \frac{t}{2} \end{pmatrix}, \qquad \mathrm{w}[t] = \begin{pmatrix} \cos \frac{t}{2} & \sin \frac{t}{2} \\ -\sin \frac{t}{2} & \cos \frac{t}{2} \end{pmatrix},$$

for $z, \alpha, \beta, t \in \mathbb{C}$, $u \in \mathbb{C}^*$, $r > 0$. The elements w and $\mathrm{n}[z]$, $z \in \mathbb{C}$ are generators for the group $\mathrm{SL}_2(\mathbb{C})$ (See [11], Proposition 1.2).

For $t \in \mathbb{R}$, we have $\sinh it = i \sin t$, $\cosh it = \cos t$, and therefore

$$\mathrm{v}[it] = \begin{pmatrix} \cos \frac{t}{2} & i \sin \frac{t}{2} \\ i \sin \frac{t}{2} & \cos \frac{t}{2} \end{pmatrix}, \qquad \mathrm{w}[it] = \begin{pmatrix} \cosh \frac{t}{2} & i \sinh \frac{t}{2} \\ -i \sinh \frac{t}{2} & \cosh \frac{t}{2} \end{pmatrix}.$$

We shall also use the following subgroups of G:

$$H = \{\mathrm{h}[u] \mid u \in \mathbb{C}^*\}, \qquad A = \{\mathrm{a}[r] \mid r > 0\},$$
$$M = H \cap K = \{\mathrm{h}[u] \mid |u| = 1\}.$$

The group P has the decomposition $P = NH = NAM$.

With the Euler angles $\varphi, \theta, \psi \in \mathbb{R}$, each element of $K = \mathrm{SU}(2)$ can be written as

$$\mathrm{k}[\alpha, \beta] = \mathrm{h}[e^{i\varphi/2}]\mathrm{v}[i\theta]\mathrm{h}[e^{i\psi/2}]. \tag{2.1}$$

The Iwasawa decomposition of the Lie group G is $G = NAK$ in the sense that each element $g \in G$ can be written in a unique way as

$$g = \mathrm{n}[z]\mathrm{a}[r]\mathrm{k}[\alpha, \beta], \tag{2.2}$$

for some $z, \alpha, \beta \in \mathbb{C}$, $r > 0$. By Iwasawa coordinates we mean

$$(z, r, \varphi, \theta, \psi) \in \mathbb{C} \times (0, \infty) \times \mathbb{R} \times [0, \pi] \times \mathbb{R}, \tag{2.3}$$

corresponding to $g = \mathrm{n}[z]\mathrm{a}[r]\mathrm{h}[e^{i\varphi/2}]\mathrm{v}[i\theta]\mathrm{h}[e^{i\psi/2}]$. On an open dense subset of G these coordinates are unique, provided that $\theta \in [0, \pi)$ and $\varphi \pm \psi \in [0, 4\pi)$.

The Haar measures on subgroups of G are given by:

$$\begin{aligned} N &: \; dn = d_+z = d\,\mathrm{Re}\, z \wedge d\,\mathrm{Im}\, z \quad &&, \text{ with } n = \mathrm{n}[z] \\ A &: \; da = r^{-1}dr &&, \text{ with } a = \mathrm{a}[r] \\ K &: \; dk = (16\pi^2)^{-1} \sin\theta \, d\varphi \, d\theta \, d\psi &&, \text{ with } k = \mathrm{h}[e^{i\varphi/2}]\mathrm{v}[i\theta]\mathrm{h}[e^{i\psi/2}]. \end{aligned} \tag{2.4}$$

Introducing the notation $|a| := |r|^2 = r^2$, for $a = \mathrm{a}[r] \in A$, Haar measure on G is given by:

$$G \; : \; dg = |a|^{-1} dn \, da \, dk, \qquad \text{with } g = nak. \tag{2.5}$$

We can easily calculate that

$$\int_K dk = \frac{1}{16\pi^2} \int\int_{\varphi \pm \psi \in [0, 4\pi)} d\varphi \, d\psi \int_0^\pi \sin\theta \, d\theta = 1. \tag{2.6}$$

Since $G/K \cong NA \cong \mathbb{H}^3$, we have

$$\int_{\Gamma\backslash G} dg = \int_{\mathcal{F}_G} dg = \int_{\mathcal{F}} |a|^{-1} dn \, da \int_{K_{1/2}} dk = \frac{\mathrm{vol}(\Gamma\backslash\mathbb{H}^3)}{2}, \tag{2.7}$$

where $\mathcal{F}$ is a fundamental domain for $\Gamma\backslash\mathbb{H}^3$, $K_{1/2}$ is a fundamental domain for $\{1, \mathrm{h}[-1]\}\backslash K$, and $\mathcal{F}_G = \{\mathrm{n}[z]\mathrm{a}[r]k \mid (z, r) \in \mathcal{F}, k \in K_{1/2}\}$ is a fundamental domain for $\Gamma\backslash G$.

In particular, for $\Gamma = \mathrm{PSL}_2(\mathcal{O})$, Theorem 7.1.1 in [11] yields

$$\int_{\Gamma\backslash G} dg = \frac{|d_F|^{3/2}}{8\pi^2} \zeta_F(2), \tag{2.8}$$

where ζ_F is the zeta function associated to the field F. Fundamental domains of $\mathrm{PSL}_2(\mathcal{O})\backslash\mathbb{H}^3$ for all imaginary quadratic number fields are explicitly described in [11], §7.3.

2.1.2 Lie algebra $\mathfrak{sl}_2(\mathbb{C})$

The real Lie algebra of G is the real vector space $\mathfrak{sl}_2(\mathbb{C})$ consisting of complex 2×2 matrices with trace equal to zero,

$$\mathfrak{sl}_2(\mathbb{C}) = \{ X \in M_{2\times 2}(\mathbb{C}) \mid \mathrm{Tr}(X) = 0\} \tag{2.9}$$

with the Lie bracket $[X,Y] := XY - YX$. It has a $\mathbb{R}$-basis consisting of six elements

$$\begin{aligned} \mathbf{H}_1 &= \tfrac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \ \mathbf{V}_1 = \tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \mathbf{W}_1 = \tfrac{1}{2}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \\ \mathbf{H}_2 &= \tfrac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \ \mathbf{V}_2 = \tfrac{1}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \ \mathbf{W}_2 = \tfrac{1}{2}\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}. \end{aligned} \tag{2.10}$$

We identify the elements in $\mathfrak{sl}_2(\mathbb{C})$ with real right differentiations on G, by

$$(Xf)(g) = \frac{d}{dt} f(g \exp(tX))\bigg|_{t=0}, \qquad X \in \mathfrak{sl}_2(\mathbb{C}), \quad f \in C^\infty(G), \quad g \in G.$$

The variable t is real. A short calculation gives, for example,

$$(\mathbf{H}_2 f)(g) = \partial_\psi f(g). \tag{2.11}$$

Tensoring with $\mathbb{C}$ over $\mathbb{R}$ gives an identification of the complex Lie algebra $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C}$ of G with all left-invariant first-order differential operators with complex coefficients. Hence the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$, as a $\mathbb{C}$-algebra, can be identified with the set of all left-invariant differential operators on G.

The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ can also be viewed as complexification of the real Lie algebra $\mathfrak{sl}_2(\mathbb{R})$ of $\mathrm{SL}_2(\mathbb{R})$. As such, it is embedded in $\mathfrak{g}$ in two ways: $X \mapsto X$ and $X \mapsto \bar{X}$. Here the bar corresponds to complex conjugation in $\mathfrak{sl}_2(\mathbb{C})$ with respect to $\mathfrak{sl}_2(\mathbb{R})$. In this way we see that

$$\mathfrak{g} \cong \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C}) \tag{2.12}$$

as complex Lie algebras. Hence, the center $\mathcal{Z}(\mathfrak{g})$ of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is generated by the Casimir elements of both factors. In the action of $\mathfrak{g}$ by right differentiation in $C^\infty(G)$, one of the summands corresponds to holomorphic differentiation

$$(Xf)(g) = \partial_t f(g \exp(tX))|_{t=0}, \qquad t \in \mathbb{C},$$

and the other summand then corresponds to anti-holomorphic differentiation

$$(\bar{X}f)(g) = \partial_{\bar{t}} f(g \exp(tX))|_{t=0}, \qquad t \in \mathbb{C}.$$

We fix a basis of $\mathfrak{sl}_2(\mathbb{C})$

$$\mathbf{H} = \mathbf{H}_1 - i\mathbf{H}_2, \qquad \mathbf{V} = \mathbf{V}_1 - i\mathbf{W}_2, \qquad \mathbf{W} = \mathbf{W}_1 - i\mathbf{V}_2, \tag{2.13}$$

where the factor i means the complexification of respective elements. The Killing form $B(X,Y)$ on $\mathfrak{sl}_2(\mathbb{C})$ is the non-degenerate symmetric bilinear form given by

$$B(X,Y)=\mathrm{Tr}(\mathrm{ad}(X)\mathrm{ad}(Y)),$$

where $\mathrm{ad}(X)Y=[X,Y]$. The chosen basis (2.13) is orthogonal for the Killing form, and

$$B(\mathbf{H},\mathbf{H})=B(\mathbf{V},\mathbf{V})=-B(\mathbf{W},\mathbf{W})=8$$

This leads to the Casimir element $\Omega_{\mathfrak{sl}_2(\mathbb{C})}=\frac{1}{8}\left(\mathbf{H}^2+\mathbf{V}^2-\mathbf{W}^2\right)$ of $\mathfrak{sl}_2(\mathbb{C})$. (See [22], Chapter VIII, §3). Hence the element

$$\Omega_+=\frac{1}{8}\left(\mathbf{H}^2+\mathbf{V}^2-\mathbf{W}^2\right) \tag{2.14}$$

is the Casimir element of the first summand in (2.12), and the Casimir element of the second summand is its complex conjugate

$$\Omega_-=\frac{1}{8}\left(\bar{\mathbf{H}}^2+\bar{\mathbf{V}}^2-\bar{\mathbf{W}}^2\right) \tag{2.15}$$

The center $\mathcal{Z}(\mathfrak{g})$ of $\mathcal{U}(\mathfrak{g})$ is the polynomial ring $\mathbb{C}[\Omega_+,\Omega_-]$.

As differential operators the Casimir elements $\Omega_\pm$ are given by the following formulas in Iwasawa coordinates

$$\begin{aligned}\Omega_+=&\frac{1}{2}r^2\partial_z\partial_{\bar z}+\frac{1}{2}re^{i\varphi}\cot\theta\partial_z\partial_\varphi-\frac{1}{2}ire^{i\varphi}\partial_z\partial_\theta-\\&-\frac{re^{i\varphi}}{2\sin\theta}\partial_z\partial_\psi+\frac{1}{8}r^2{\partial_r}^2-\frac{1}{4}ir\partial_r\partial_\varphi-\frac{1}{8}{\partial_\varphi}^2-\frac{1}{8}r\partial_r+\frac{1}{4}i\partial_\varphi.\end{aligned} \tag{2.16}$$

and

$$\begin{aligned}\Omega_-=&\frac{1}{2}r^2\partial_z\partial_{\bar z}+\frac{1}{2}re^{-i\varphi}\cot\theta\partial_{\bar z}\partial_\varphi+\frac{1}{2}ire^{-i\varphi}\partial_{\bar z}\partial_\theta-\\&-\frac{re^{-i\varphi}}{2\sin\theta}\partial_{\bar z}\partial_\psi+\frac{1}{8}r^2{\partial_r}^2+\frac{1}{4}ir\partial_r\partial_\varphi-\frac{1}{8}{\partial_\varphi}^2-\frac{1}{8}r\partial_r-\frac{1}{4}i\partial_\varphi.\end{aligned} \tag{2.17}$$

Real generators of the ring $\mathcal{Z}(\mathfrak{g})$ are $\Omega_++\Omega_-$ and $-i(\Omega_+-\Omega_-)$. It turns out that the Casimir element of the real Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ of G is $\Omega_++\Omega_-$. Applied to K-invariant functions on G, $\Omega_++\Omega_-$ gives a constant multiple of the Laplacian on $\mathbb{H}^3$:

$$(\Omega_++\Omega_-)|_{\Gamma\backslash\mathbb{H}^3}=\tfrac{1}{4}L.$$

We now consider the maximal compact subgroup $K=\mathrm{SU}(2)$. Its real Lie algebra $\mathfrak{su}(2)$ is generated by $\mathbf{H}_2$,$\mathbf{W}_1$ and $\mathbf{W}_2$. This basis is orthogonal for the Killing form of $\mathfrak{su}(2)$, and

$$B(\mathbf{H}_2,\mathbf{H}_2)=B(\mathbf{W}_1,\mathbf{W}_1)=B(\mathbf{W}_2,\mathbf{W}_2)=-2.$$

This leads to the Casimir element

$$\Omega_{\mathfrak{k}} = -\frac{1}{2}\left({\mathbf{H}_2}^2 + {\mathbf{W}_1}^2 + {\mathbf{W}_2}^2\right) = \frac{1}{2}\left({\mathbf{H}_2}^2 \pm i\mathbf{H}_2 + \mathbf{E}^{\pm}\mathbf{E}^{\mp}\right) \tag{2.18}$$

in $\mathcal{Z}(\mathfrak{k})$, where $\mathfrak{k} = \mathfrak{su}(2) \otimes_{\mathbb{R}} \mathbb{C}$ is the complex Lie algebra of K.

Here $\mathbf{E}^{\pm} = \mathbf{W}_1 \pm i\mathbf{W}_2$ in $\mathfrak{k}$. These two elements generate $\mathfrak{k}$, and satisfy $[\mathbf{H}_2, \mathbf{E}^{\pm}] = \mp i\mathbf{E}^{\pm}$. Note that $\overline{\mathbf{E}^{+}} = \mathbf{E}^{-}$.

In terms of Iwasawa coordinates we have

$$\mathbf{E}^{\pm} = e^{\mp i\psi}\left(-\frac{1}{\sin\theta}\partial_\varphi \pm i\partial_\theta + \cot\theta\partial_\psi\right) \tag{2.19}$$

and

$$\Omega_{\mathfrak{k}} = \frac{1}{2\sin^2\theta}\left(\partial_\varphi^2 + \sin^2\theta\partial_\theta^2 + \partial_\psi^2 - 2\cos\theta\partial_\varphi\partial_\psi + \sin\theta\cos\theta\partial_\theta\right). \tag{2.20}$$

The center $\mathcal{Z}(\mathfrak{k})$ of $\mathcal{U}(\mathfrak{k})$ is the polynomial ring $\mathbb{C}[\Omega_{\mathfrak{k}}]$.

2.2 Irreducible representations of K

The irreducible representations of the group $K = \mathrm{SU}(2)$ are described in detail in [41], §6.2 and §6.3. They are obtained as restrictions of finite-dimensional representations T_n on G to K, and are uniquely determined, up to equivalence, by the non-negative integer $n \in \mathbb{N}$. We shall use the same notation for the restrictions to K.

A representation T_n of G is given by the formula

$$T_n(g)f(z) = (bz+d)^n f\left(\frac{az+c}{bz+d}\right), \tag{2.21}$$

for all $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G$ and f running through the space V_n of polynomials in one variable of degree at most n. Each of these representations T_n is irreducible, and unitary for a suitably chosen scalar product.

In particular, $g = \mathrm{h}[-1]$ in (2.21) gives $T_n(\mathrm{h}[-1]) = (-1)^n$. As we shall see later in Section 3.1, a χ-automorphic function f of K-type T_n satisfies

$$f(-g) = \chi(-1)f(g) \overset{(3.1)}{=} (-1)^n f(g),$$

and therefore it is even (odd) if and only if n is even (odd). We would like to distinguish between the even and the odd case. We write $n = 2l$ and consider further $l \in \frac{1}{2}\mathbb{N}$ as a parameter to characterize the irreducible representations of K. We put $\sigma_l = T_{2l}$.

Let $l \in \frac{1}{2}\mathbb{Z}$. The set

$$\{z^{l-q} \ : \ q \equiv l \,(\mathrm{mod}\,1), \ |q| \leqslant l\} \tag{2.22}$$

is a basis for V_{2l}, and the representation σ_l on K is given by

$$\sigma_l\left(\mathrm{k}[\alpha,\beta]\right) z^{l-q} = (\alpha z - \bar{\beta})^{l-q}(\beta z + \bar{\alpha})^{l+q}, \qquad \mathrm{k}[\alpha,\beta] \in K \tag{2.23}$$

where z^{l-q} is element of the basis (2.22).

For the elements of the one-parameter subgroup $M = \{\mathrm{h}[e^{it}] \ : \ t \in \mathbb{R}\} \subset K$ we have

$$\sigma_l\left(\mathrm{h}[e^{it}]\right) z^{l-q} = e^{-2qit} z^{l-q}\ , \qquad q \equiv l \ (\mathrm{mod}\ 1), \quad |q| \leqslant l.$$

So, the space V_{2l} is the direct sum of $(2l+1)$ one-dimensional weight spaces for the subgroup M acting with integer or half-integer weights q:

$$\mathrm{h}[e^{it}] \ \overset{\sigma_l}{\longmapsto} \ e^{-2qit}, \qquad |q| \leqslant l. \tag{2.24}$$

The number l determines a K-type, that is, an eigenvalue class of irreducible representations of K. The representation σ_l represents this equivalence class. Within the underlying vector space V_{2l} of σ_l, there are one-dimensional weight spaces for M, parameterized by q ($q \equiv l$ (mod 1), $|q| \leqslant l$). So the number q describes the M-types of a vector in a particular K-type.

Since K is compact, there exists an invariant scalar product $(\cdot,\cdot)$ on the space V_{2l}. It can be normalized such that

$$(z^{l-q}, z^{l-k}) = \begin{cases} 0 & , \quad k \neq q \\ (l-q)!(l+q)! & , \quad k = q. \end{cases} \tag{2.25}$$

(See [41], §6.2.3).

The representation σ_l is completely determined by the derived action of the Lie algebra $\mathfrak{k}$ of K:

$$\sigma_l(X)f(z) = \partial_t \sigma_l(\exp(tX))f(z)\big|_{t=0} \qquad \text{for all } X \in \mathfrak{k},\ f \in V_{2l}. \tag{2.26}$$

A simple computation gives

$$\begin{aligned} \sigma_l(\mathbf{H}_2)z^{l-q} &= -iqz^{l-q}, \\ \sigma_l(\mathbf{E}^{\pm})z^{l-q} &= (q \mp l)z^{l-(q\pm 1)}, \\ \sigma_l(\Omega_{\mathfrak{k}})z^{l-q} &= -\frac{1}{2}(l^2+l)z^{l-q}. \end{aligned} \tag{2.27}$$

Let $L^2(K)$ be the Hilbert space of all functions on K which are square integrable over K with respect to the Haar measure dk. We denote by $(\cdot,\cdot)_K$ and

$\|\cdot\|_K$ the scalar product and the norm on K obtained by integration with respect to the Haar measure dk in (2.4). We want to describe the structure of $L^2(K)$, and hence the unitary representations of the compact group K.

Let $l, q, p \in \frac{1}{2}\mathbb{Z}$, $l \geqslant 0$, $l \equiv q \equiv p \pmod 1$. We define a function $\Phi^l_{p,q}$ on K as the coefficient of z^{l-p} in the polynomial expansion of $\sigma_l(\mathrm{k}[\alpha,\beta])z^{l-q}$, where σ_l is an irreducible representation of K given by (2.23), i.e.

$$\sum_{|p|\leqslant l} \Phi^l_{p,q}(\mathrm{k}[\alpha,\beta])z^{l-p} = (\alpha z - \bar{\beta})^{l-q}(\beta z + \bar{\alpha})^{l+q}, \qquad \text{for} \quad |q| \leqslant l, \tag{2.28}$$

and $\Phi^l_{p,q} \equiv 0$ whenever one of the numbers p or q violate the condition $|p|, |q| \leqslant l$.

The functions $\Phi^l_{p,q}$ are (by construction) matrix coefficients of the representation σ_l, since

$$\Phi^l_{p,q}(k) = \frac{1}{(l-p)!(l+p)!}\left(\sigma_l(k)z^{l-q}, z^{l-p}\right). \tag{2.29}$$

(See [22], p.16). In particular, we have

$$\Phi^l_{p,q}(k_1k_2) = \sum_{|m|\leqslant l} \Phi^l_{p,m}(k_1)\Phi^l_{m,q}(k_2), \qquad k_1, k_2 \in K. \tag{2.30}$$

One easily checks that the following properties are satisfied:

(P1) $\quad \Phi^l_{p,q}(\mathrm{v}[-i\theta]) = (-1)^{p+q}\Phi^l_{-p,-q}(\mathrm{v}[i\theta])$,

(P2) $\quad \Phi^l_{q,p}(\mathrm{v}[i\theta]) = \dfrac{(l-p)!(l+p)!}{(l-q)!(l+q)!}\Phi^l_{p,q}(\mathrm{v}[i\theta])$,

(P3) $\quad \Phi^l_{p,q}(\mathrm{k}[\alpha,\beta]) = e^{-ip\varphi - iq\psi}\Phi^l_{p,q}(\mathrm{v}[i\theta])$ with Euler angles (2.1),

(P4) $\quad \overline{\Phi^l_{p,q}} = (-1)^{p+q}\Phi^l_{-p,-q}$.

For any $X \in \mathfrak{k}$, and $k \in K$, we have

$$\begin{aligned}
X\Phi^l_{p,q}(k) &= \partial_t \Phi^l_{p,q}(k\exp(tX))|_{t=0} \\
&\overset{(2.29)}{=} \{(l-p)!(l+p)!\}^{-1}\,\partial_t\left(\sigma_l(k\exp(tX))z^{l-q}, z^{l-p}\right)|_{t=0} \\
&= \{(l-p)!(l+p)!\}^{-1}\left(\sigma_l(k)\partial_t\sigma(\exp(tX))z^{l-q}|_{t=0}, z^{l-p}\right) \\
&\overset{(2.26)}{=} \{(l-p)!(l+p)!\}^{-1}\left(\sigma_l(k)\sigma_l(X)z^{l-q}, z^{l-p}\right).
\end{aligned}$$

Using this and (2.27), we immediately get for any $k \in K$:

$$\begin{aligned}
&\mathbf{H}_2\Phi^l_{p,q}(k) = -iq\Phi^l_{p,q}(k) \\
&\mathbf{E}^{\pm}\Phi^l_{p,q}(k) = (q \mp l)\Phi^l_{p,q\pm1}(k), \\
&\Omega_{\mathfrak{k}}\Phi^l_{p,q}(k) = -\frac{1}{2}(l^2+l)\Phi^l_{p,q}(k).
\end{aligned} \tag{2.31}$$

This proves the following

Lemma 2.2.1. *The functions* $\Phi^l_{p,q}$ *with* $l,p,q \in \frac{1}{2}\mathbb{Z}$ *such that* $l \geqslant 0$, $l \equiv q \equiv p$ *(mod 1), are simultaneous eigenfunctions of the differential operators* $\mathbf{H}_2$ *and* $\Omega_{\mathfrak{k}}$ *with eigenvalues* $-iq$ *and* $-\frac{1}{2}(l^2+l)$*, respectively.*

Let us now calculate the scalar products $\left(\Phi^l_{p,q}, \Phi^{l_1}_{p_1,q_1}\right)_K$. We have

$$\begin{aligned}\left(\Phi^l_{p,q}, \Phi^{l_1}_{p_1,q_1}\right)_K &= \int_K \Phi^l_{p,q}(k)\overline{\Phi^{l_1}_{p_1,q_1}(k)}dk \\ &\overset{(2.29)}{=} \{(l-p)!(l+p)!(l_1-p_1)!(l_1+p_1)!\}^{-1}\cdot \\ &\quad \cdot \int_K \left(\sigma_l(k)z^{l-q}, z^{l-p}\right)\overline{\left(\sigma_{l_1}(k)z^{l_1-q_1}, z^{l_1-p_1}\right)}. \end{aligned} \tag{2.32}$$

Since the Haar measure dk on K is normalized so that $\int_K dk = 1$, see (2.6), we can apply Schur's orthogonality relations (Corollary 1.10 in [22], p.15), and conclude that $\left(\Phi^l_{p,q}, \Phi^{l_1}_{p_1,q_1}\right)_K = 0$ if $l \neq l_1$ or $q \neq q_1$ or $p \neq p_1$. Otherwise

$$\begin{aligned}\left(\Phi^l_{p,q}, \Phi^l_{p,q}\right)_K &= \{(l-p)!(l+p)!\}^{-2}\frac{(z^{l-q}, z^{l-q})\overline{(z^{l-p}, z^{l-p})}}{\dim V_{2l}} \\ &\overset{(2.25)}{=} \frac{1}{2l+1}\frac{(l-q)!(l+q)!}{(l-p)!(l+p)!} = \frac{1}{2l+1}\binom{2l}{l-p}\binom{2l}{l-q}^{-1}. \end{aligned} \tag{2.33}$$

So, we see that the set

$$\left\{\Phi^l_{p,q} \;\middle|\; l,p,q \in \tfrac{1}{2}\mathbb{Z},\ p \equiv q \equiv l \ (\mathrm{mod}\ 1),\ |p|,|q| \leqslant l\right\} \tag{2.34}$$

is an orthogonal system in $L^2(K)$, with norms

$$\|\Phi^l_{p,q}\|_K = \frac{1}{\sqrt{2l+1}}\binom{2l}{l-p}^{1/2}\binom{2l}{l-q}^{-1/2}. \tag{2.35}$$

It is actually an orthogonal basis of $L^2(K)$, see [41], §6.2.3.

If we denote by $L^2_{\mathrm{even}}(K)$, and $L^2_{\mathrm{odd}}(K)$ the subspaces of $L^2(K)$ consisting of even, and odd functions respectively, then

$$\left\{\Phi^l_{p,q} \;\middle|\; l,p,q \in \mathbb{Z},\ |p|,|q| \leqslant l\right\},$$

is an orthogonal basis of $L^2_{\mathrm{even}}(K)$, and

$$\left\{\Phi^l_{p,q} \;\middle|\; l,p,q \in \tfrac{1}{2}+\mathbb{Z},\ |p|,|q| \leqslant l\right\}$$

is an orthogonal basis of $L^2_{\mathrm{odd}}(K)$.

We arrange the basis (2.34) so that we have the following decomposition of $L^2(K)$ into irreducible subspaces

$$L^2(K) = \overline{\bigoplus_{\substack{l,q\in\mathbb{Z}\\ |q|\leqslant l}} L^2(K;l,q)} \oplus \overline{\bigoplus_{\substack{l,q\in\frac{1}{2}+\mathbb{Z}\\ |q|\leqslant l}} L^2(K;l,q)}, \tag{2.36}$$

$$L^2(K;l,q) = \bigoplus_{\substack{p\equiv l \ (\mathrm{mod}\ 1)\\ |p|\leqslant l}} \mathbb{C}\Phi^l_{p,q}. \tag{2.37}$$

The space

$$L^2(K;l,q) = \left\{ f \in L^2(K) \mid \Omega_{\mathfrak{k}} f = -\tfrac{1}{2}(l^2+l)f,\ \mathbf{H}_2 f = -iqf \right\},$$

is called the subspace of type (l,q).

In general, in a space in which K acts, we shall say an element is of type (l,q) if it is a simultaneous eigenvector of $\mathbf{H}_2$ and $\Omega_{\mathfrak{k}}$ with eigenvalues $-iq$ and $-\frac{1}{2}(l^2+l)$, respectively.

2.3 Irreducible unitary representations of G

The principal series of $\mathrm{SL}_2(\mathbb{C})$ is the family of representations $\mathcal{P}^{2p,2\nu}$ of G discussed in [22], Chap.II, §4, indexed by the spectral parameter $(\nu,p) \in \mathbb{C}\times\frac{1}{2}\mathbb{Z}$. The unitary principal series representations are then indexed by $(\nu,p)\in i\mathbb{R}\times\frac{1}{2}\mathbb{Z}$, while the complementary series are indexed by $(\nu,0)$ for $\nu\in(-1,1)\backslash\{0\}$.

Let $H^\infty(\nu,p)$ be the space of functions $f\in C^\infty(G)$ that satisfy

$$f(n\mathrm{a}[r]\mathrm{h}[e^{it}]g) = r^{1+\nu}e^{-2pit}f(g), \tag{2.38}$$

for all $g\in G$, $n\in N$, $\mathrm{a}[r]\in A$, $\mathrm{h}[e^{it}]\in M$. The Iwasawa decomposition shows that such functions are determined by their behavior on K. The vector space $H^\infty(\nu,p)$ is isomorphic to the space

$$C_p^\infty(K) := \left\{ f\in C^\infty(K) \ | f(\mathrm{h}[e^{it}]k) = e^{-2pit}f(k) \right\} \tag{2.39}$$

via $f\mapsto\tilde{f}$, where $\tilde{f}(n\mathrm{a}[r]k) := r^{1+\nu}f(k)$ for $f\in C_p^\infty(K)$.

The group G acts in $H^\infty(\nu,p)$ by right translation:

$$gf(x) = f(xg), \qquad \text{for } x,g\in G.$$

This is a representation of G in the space $H^\infty(\nu,p)$ and it is actually the induced representation $U(MAN,\psi_p,\nu,\cdot)$ given in Knapp [22], Chap.VII, §1, where $\psi_p(\mathrm{h}[e^{it}]) = e^{-2pit}$ and $\nu(\mathrm{h}[t]) = 2\nu t$. More precisely, our space $H^\infty(\nu,p)$ is a dense subspace of the space of functions F on G satisfying

$$F(gman) = e^{-(\nu_{\mathrm{knapp}}+\rho)\log a}\psi_p(m)^{-1}F(g)$$

for all $g \in G$, $m \in M$, $a \in A$, $k \in K$, given on p. 168 in [22], via $f \mapsto F(g) := f(g^{-1})$ with $f \in H^\infty(\nu,p)$. Here $\nu_{\text{knapp}} = \nu$, $v = -2i\nu$, and ρ is the half-sum of the positive roots of $(\mathfrak{g},\mathfrak{a})$ for N. Knapp works with left translations of G.

Hence, the "induced picture" in [22] corresponds to our space $H^\infty(\nu,p)$, the "compact picture" corresponds to the restrictions to K, i.e. the space $C_p^\infty(K)$, and in the "non-compact picture" a function $f \in H^\infty(\nu,p)$ corresponds to the function $z \mapsto f\left(\begin{smallmatrix} 1 & 0 \\ z & 1 \end{smallmatrix}\right)$ on $\mathbb{C}$.

There is a duality between the spaces $H^\infty(\nu,p)$ and $H^\infty(-\nu,-p)$ given by the bilinear form

$$\langle f_1, f_2 \rangle := \int_K f_1(k) f_2(k) dk$$

for $f_1 \in H^\infty(\nu,p)$, $f_2 \in H^\infty(-\nu,-p)$. This duality satisfies

$$\langle g f_1, g f_2 \rangle = \langle f_1, f_2 \rangle, \qquad \text{for all} \quad g \in G. \tag{2.40}$$

(See [22], Chap.VII §2).

The restriction to K of each $\varphi \in H^\infty(\nu,p)$ is square integrable on K. Conversely, each $f \in L^2(K)$ that satisfies $f(\mathrm{h}[e^{it}]k) = e^{-2pit} f(k)$ almost everywhere for $k \in K$, for some $p \in \frac{1}{2}\mathbb{Z}$, can be extended to a function on G satisfying (2.38). The space consisting of such functions we denote by $H^2(\nu,p)$. The action of G by right translation and the bilinear form $\langle \cdot,\cdot \rangle$ extend to $H^2(\nu,p)$. All the statements here about $H^2(\nu,p)$ are equivalent to the statements concerning $\mathcal{P}^{2p,2\nu}$. Theorem 16.2 in [22] gives a complete list of all the irreducible unitary representations of G up to unitary equivalence. In our terms it reads:

- the trivial representation,
- unitary principal series: $H^2(\nu,p)$ with $\nu \in i\mathbb{R}$, $p \in \frac{1}{2}\mathbb{Z}$,
- complementary series: completion of $H^\infty(\nu,0)$, with $\nu \in (0,1)$, with respect to another inner product.

The spaces $H^2(\nu,p)$ and $H^2(-\nu,-p)$, with $\nu \in i\mathbb{R}$, are isomorphic as G-modules.

Taking $g = \exp tX$, $X \in \mathfrak{g}$ in (2.40), and differentiating with respect to t, we get for $f_1 \in H^\infty(\nu,p)$, $f_2 \in H^\infty(-\nu,-p)$:

$$\langle X f_1, f_2 \rangle + \langle f_1, X f_2 \rangle = 0, \qquad \text{for all} \quad X \in \mathfrak{g}. \tag{2.41}$$

A vector in a representation space for G is called K-finite if its K translates span a finite-dimensional space. (See [22] for details.) We denote by $H(\nu,p)$ the space of K-finite vectors in $H^2(\nu,p)$. Right translations by elements of K leave $H(\nu,p)$ invariant, but the G-action does not. However, it is known that $H(\nu,p)$ is preserved by the action of the Lie algebra $\mathfrak{g}$. As K is connected, the action of

K is determined by the action of $\mathfrak{k} \subset \mathfrak{g}$. Furthermore, unitarity of the $\mathfrak{g}$-action in $H^\infty(\nu, p)$ means that there exists a scalar product $(\cdot,\cdot)$ on $H^\infty(\nu, p)$ such that

$$(Xf_1, f_2) + (f_1, \bar{X} f_2) = 0, \qquad \text{for all} \quad X \in \mathfrak{g} \tag{2.42}$$

holds. The space $H^2(\nu, p)$ is then the completion of $H^\infty(\nu, p)$ with respect to that scalar product, and the action of G in $H^2(\nu, p)$ is unitary. The irreducibility of the G-module $H^2(\nu, p)$ is equivalent to the irreducibility of the $\mathfrak{g}$-module $H^\infty(\nu, p)$, and hence the space $H(\nu, p)$ as a $\mathfrak{g}$-module.

For $\nu \in \mathbb{C}$, $l, p, q \in \frac{1}{2}\mathbb{Z}$, $l \geqslant 0$, $p \equiv q \equiv l \pmod 1$, we put

$$\varphi_{l,q}(\nu, p)(n\mathrm{a}[r]k) := r^{1+\nu} \Phi^l_{p,q}(k), \tag{2.43}$$

for $n \in N$, $r > 0$, $k \in K$. Clearly $\varphi_{l,q}(\nu, p)$ is a left N-invariant function on G of type (l, q).

Since $\mathbf{H}_2, \mathbf{E}^\pm, \Omega_\mathfrak{k} \in \mathfrak{k} \subset \mathfrak{g}$, it is clear from (2.31) and the definition (2.43), that

$$\begin{aligned} \mathbf{H}_2 \varphi_{l,q}(\nu, p) &= -iq\varphi_{l,q}(\nu, p), \\ \mathbf{E}^\pm \varphi_{l,q}(\nu, p) &= (q \mp l)\varphi_{l,q}(\nu, p), \\ \Omega_\mathfrak{k} \varphi_{l,q}(\nu, p) &= -\frac{l^2 + l}{2} \varphi_{l,q}(\nu, p). \end{aligned} \tag{2.44}$$

A direct but lengthy calculation, using (2.16)–(2.17), definition (2.43), and the properties of the functions $\Phi^l_{p,q}$, gives

$$\Omega_\pm \varphi_{l,q}(\nu, p) = \frac{1}{8}\left((\nu \mp p)^2 - 1\right) \varphi_{l,q}(\nu, p), \tag{2.45}$$

which means that $H(\nu, p)$ are simultaneous eigenspaces of $\Omega_\pm$. See [9], (3.29).

It follows from the discussion in Section 2.2 that for given $p \in \frac{1}{2}\mathbb{Z}$, the functions $\Phi^l_{p,q}$ with $l \equiv q \equiv p \pmod 1$, $l \geqslant |p|$, and $|q| \leqslant l$, form an orthogonal basis for the space

$$L^2_p(K) := \{ f \in L^2(K) \mid f(\mathrm{h}[e^{it}]k) = e^{-2pit} f(k) \}.$$

This immediately gives an orthogonal basis of the space of K-finite vectors in the unitary principal series representations

$$H(\nu, p) = \{ \text{ linear combination of } \varphi_{l,q}(\nu, p) \ \}, \tag{2.46}$$

consisting of the functions $\varphi_{l,q}(\nu, p)$ for $l \equiv q \equiv p \pmod 1$, $l \geqslant |p|$, and $|q| \leqslant l$. Their $H(\nu, p)$-norms are determined as follows: Assuming that we are not in the case $\nu \in \mathbb{Z}$, $|\nu| > |p|$, the irreducible spaces $H(\nu, p)$ and $H(-\nu, -p)$ are isomorphic via

$$\iota(\nu, p)\varphi_{l,q}(\nu, p) = \frac{\Gamma(l + 1 - \nu)}{\Gamma(l + 1 + \nu)} \varphi_{l,q}(-\nu, -p). \tag{2.47}$$

One $\mathfrak{g}$-morphism from $H(\nu,p)$ to $H(-\nu,-p)$ is given by the Jacquet integral $\mathbf{J}_0 = \mathbf{J}_0(\nu,p)$, to be described in (4.23), Section 4.1. It is an isomorphism of $\mathfrak{g}$-modules for $\nu \notin \mathbb{Z}$. Indeed, the integral gives a continuous function for $\mathrm{Re}\,\nu > 0$, it has a meromorphic continuation in $\nu \in \mathbb{C}$ with poles at $\nu \in \mathbb{Z}_{\leqslant 0}$, commutes with the action of $\mathfrak{g}$, and $\mathbf{J}_0(-\nu,-p)\mathbf{J}_0(\nu,p) : \varphi_{l,q}(\nu,p) \mapsto \frac{\pi^2}{p^2-\nu^2}\,\varphi_{l,q}(\nu,p)$. We choose $\iota(\nu,p)$ to be a multiple of $\mathbf{J}_0(\nu,p)$ such that $\iota(-\nu,-p)\iota(\nu,p)$ is the identity on $H(\nu,p)$.

Complex conjugation gives a linear map $^{-} : H(\nu,p) \to H(\bar\nu,-p)$ which satisfies $\overline{Xf} = \bar X \bar f$ for all $X \in \mathfrak{g}$.

We shall write $(\cdot,\cdot)_{\mathrm{ps}}$ for the scalar product in $H(\nu,p)$ with respect to which the completion is done, if (ν,p) parameterizes the principal series $H(\nu,p)$, and $(\cdot,\cdot)_{\mathrm{cs}}$ if (ν,p) parameterizes the complementary series $H(\nu,0)$.

In the case $\nu \in i\mathbb{R}$, we have $H(\bar\nu,-p) = H(-\nu,-p)$, and a scalar product on $H(\nu,p)$ is given by

$$(f_1,f_2)_{\mathrm{ps}} = \langle f_1, \overline{f_2}\rangle, \qquad \text{for all} \quad X \in \mathfrak{g}. \tag{2.48}$$

In particular, for $\varphi_{l,q}(\nu,p), \varphi_{l',q'}(\nu,p) \in H(\nu,p)$ we have

$$\begin{aligned}(\varphi_{l,q}(\nu,p), &\varphi_{l',q'}(\nu,p))_{\mathrm{ps}} = \\ &= \left\langle \varphi_{l,q}(\nu,p), \overline{\varphi_{l',q'}(\nu,p)}\right\rangle = \int_K \Phi^l_{p,q}(k)\overline{\Phi^{l'}_{p,q'}(k)}dk = \delta_{l,l'}\delta_{q,q'}\|\Phi^l_{p,q}\|^2_K.\end{aligned}$$

Hence

$$\|\varphi_{l,q}(\nu,p)\|_{\mathrm{ps}} = \|\Phi^l_{p,q}\|_K. \tag{2.49}$$

In the case $\nu \in (0,1)$, the space $H(\bar\nu,-p) = H(\nu,-p)$ is isomorphic to $H(-\nu,p)$ via $\iota(\nu,-p)$. For $p=0$, a scalar product that gives a unitary structure to $H(\nu,0)$ is given by

$$(f_1,f_2)_{\mathrm{cs}} = \langle f_1, \iota(\nu,0)\overline{f_2}\rangle, \qquad \text{for all} \quad X \in \mathfrak{g}. \tag{2.50}$$

In particular, for $\varphi_{l,q}(\nu,0), \varphi_{l',q'}(\nu,0) \in H(\nu,0)$ we have

$$\begin{aligned}(\varphi_{l,q}(\nu,0), \varphi_{l',q'}(\nu,0))_{\mathrm{cs}} &= \left\langle \varphi_{l,q}(\nu,0), \iota(\nu,0)\overline{\varphi_{l',q'}(\nu,0)}\right\rangle \\ &= \frac{\Gamma(l'+1-\nu)}{\Gamma(l'+1+\nu)}\int_K \Phi^l_{0,q}(k)\overline{\Phi^{l'}_{0,q'}(k)}dk = \delta_{l,l'}\delta_{q,q'}\frac{\Gamma(l+1-\nu)}{\Gamma(l+1+\nu)}\|\Phi^l_{0,q}\|^2_K.\end{aligned}$$

Hence

$$\|\varphi_{l,q}(\nu,0)\|_{\mathrm{cs}} = \sqrt{\frac{\Gamma(l+1-\nu)}{\Gamma(l+1+\nu)}}\|\Phi^l_{0,q}\|_K. \tag{2.51}$$

Chapter 3

Automorphic forms and automorphic representations

3.1 Automorphic forms

Let σ be a finite-dimensional representation of K on the vector space V_σ. Let χ denote a character of $(\mathcal{O}/I)^*$. Also let the corresponding character on Γ be given by (1.14).

Definition 3.1.1. *A χ-automorphic form of K-type σ is a smooth function $f : G \to V_\sigma$ satisfying*

(i) *$f(\gamma g) = \chi(d) f(g)$, for all $\gamma = \left(\begin{smallmatrix} * & * \\ * & d \end{smallmatrix}\right) \in \Gamma$,*

(ii) *$f(gk) = \sigma(k)^{-1} f(g)$, for all $k \in K$,*

(iii) *f is an eigenfunction of all elements in $\mathcal{Z}(\mathfrak{g})$.*

The transformation rule given in condition (i) is called χ-automorphic behavior of the function f with respect to Γ. We assume that σ is a left representation, and therefore the inverse in condition (ii) is necessary to obtain equivariance. In condition (iii), we identify elements of $\mathcal{Z}(\mathfrak{g})$ with the corresponding differential operator. As $\mathcal{Z}(\mathfrak{g})$ is a commutative ring, (iii) implies that there exists a character Υ of $\mathcal{Z}(\mathfrak{g})$ such that $Xf = \Upsilon(X)f$ for all $X \in \mathcal{Z}(\mathfrak{g})$. This character is determined by its values on $\Omega_\pm$, since Ω_+ and Ω_- generate the ring $\mathcal{Z}(\mathfrak{g})$.

We denote the space of all χ-automorphic forms for a given representation σ and a given character Υ of $\mathcal{Z}(\mathfrak{g})$ by $\mathcal{A}_\chi(\Upsilon; \sigma)$.

Definition 3.1.2. *A χ-automorphic form f has polynomial growth if it satisfies*

$$f(g_\kappa n \mathrm{a}[r] k) = O(r^{b_\kappa}) \qquad \text{as} \quad r \to \infty$$

for some $b_\kappa \in \mathbb{R}$, at each cusp $\kappa \in \mathcal{C}_\chi$, uniformly for $n \in N$ and $k \in K$.

By $\mathcal{A}_\chi^{\mathrm{pol}}(\Upsilon;\sigma)$ we denote the linear subspace of functions $f \in \mathcal{A}_\chi(\Upsilon;\sigma)$ that have polynomial growth.

Definition 3.1.3. *A χ-automorphic form f is square integrable if it determines an element of $L^2(\Gamma\backslash G;\chi) \otimes_{\mathbb{C}} V_\sigma$.*

The measure on $\Gamma\backslash G$ in the definition of $L^2(\Gamma\backslash G;\chi)$ is induced by the Haar measure dg on G, fixed in (2.5). The choice of the norm on V_σ is not important since the space V_σ is finite-dimensional and all the norms are equivalent.

The central element $\mathrm{h}[-1]$ is contained in $G\cap\Gamma$, hence the following consistency condition must be satisfied:

$$\chi(-1) = \sigma(\mathrm{h}[-1]). \tag{3.1}$$

Each finite-dimensional representation of K is the direct sum of irreducible ones. This reduces the study of χ-automorphic forms on G to those of type σ, where σ is an irreducible representation of K.

3.2 Automorphic functions

Definition 3.2.1. *Let $l, q \in \frac{1}{2}\mathbb{Z}$ such that $|q| \leqslant l$ and $q \equiv l \pmod 1$. Let χ be a character of $(\mathcal{O}/I)^*$ and Υ a character of $\mathcal{Z}(\mathfrak{g})$. A χ-automorphic function of type (l,q) with character Υ is a smooth function $f : G \to \mathbb{C}$ such that*

(i) $f(\gamma g) = \chi(d) f(g)$, *for all* $\gamma = \left(\begin{smallmatrix} * & * \\ * & d \end{smallmatrix}\right) \in \Gamma$,

(ii) $\Omega_{\mathfrak{k}} f = -\frac{l^2+l}{2} f, \qquad \mathbf{H}_2 f = -iqf,$

(iii) $\Omega_\pm f = \Upsilon(\Omega_\pm) f.$

The automorphic functions are just components of the vector valued automorphic forms for a suitably chosen basis of the space V_σ.

We denote by $\mathcal{A}_\chi(\Upsilon; l, q)$ the space of all χ-automorphic functions of type (l,q) with character Υ, and

$$\mathcal{A}_\chi^{\mathrm{pol}}(\Upsilon; l, q) = \{ f \in \mathcal{A}_\chi(\Upsilon; l, q) \mid f \text{ of polynomial growth} \}. \tag{3.2}$$

We define $\mathcal{A}_\chi^2(\Upsilon; l, q)$ as the space of $f \in \mathcal{A}_\chi(\Upsilon; l, q)$ that are square-integrable on $\Gamma\backslash G$, i.e. satisfy $\int_{\Gamma\backslash G} |f(g)|^2 dg < \infty$.

3.2.1 Fourier expansion of automorphic functions

The ring of integers $\mathcal{O} \subset \mathbb{C}$ is a lattice in $\mathbb{C}$. Let

$$\mathcal{O}' = \{z \in F \mid \mathrm{Tr}(zz') \in \mathbb{Z},\ \forall z' \in \mathcal{O}\}$$

be the dual lattice of $\mathcal{O}$. So, $\mathcal{O}'$ is a fractional ideal containing the ring of integers $\mathcal{O}$. The unitary characters on N are of the form

$$\chi_\omega \ :\ \mathrm{n}[z] \mapsto e^{2\pi i \mathrm{Tr}(\omega z)}, \qquad \omega \in \mathbb{C}. \tag{3.3}$$

The characters χ_ω with $\omega \in \mathcal{O}'$ are precisely the characters of N that are trivial on $\Gamma_N = N \cap \Gamma$. These will appear in the Fourier expansion of automorphic functions.

For any continuous function f on G having a χ-automorphic transformation behavior, the function $z \mapsto f(\mathrm{n}[z]g)$ is periodic on $\mathbb{C}$ for the lattice $\mathcal{O}$. Thus, there exists a Fourier expansion of f at the cusp ∞:

$$f(g) = \sum_{\omega \in \mathcal{O}'} F_\omega f(g), \tag{3.4}$$

where the Fourier term of order ω is given by

$$F_\omega f(g) = \frac{1}{\mathrm{vol}(\Gamma_N \backslash N)} \int_{\Gamma_N \backslash N} \chi_\omega(n)^{-1} f(ng) dn. \tag{3.5}$$

Here $\mathrm{vol}(\Gamma_N \backslash N) = \frac{\sqrt{|d_F|}}{2}$ is the Euclidean area of the fundamental parallelogram of $\mathcal{O}$.

The function $F_\omega f$ transforms via the character χ_ω with respect to the left action of N:

$$F_\omega f(ng) = \chi_\omega(n) F_\omega f(g), \qquad \forall n \in N. \tag{3.6}$$

For any $X \in \mathcal{U}(\mathfrak{g})$, we have

$$\begin{aligned} XF_\omega f(g) &= \partial_t F_\omega f(g \exp(tX))|_{t=0} \\ &= \partial_t \left\{ \frac{2}{\sqrt{|d_F|}} \int_{\Gamma_N \backslash N} \chi_\omega(n)^{-1} f(ng \exp(tX)) dn \right\} \Bigg|_{t=0} \\ &= \frac{2}{\sqrt{|d_F|}} \int_{\Gamma_N \backslash N} \chi_\omega(n)^{-1} \partial_t f(ng \exp(tX))|_{t=0}\, dn \\ &= \frac{2}{\sqrt{|d_F|}} \int_{\Gamma_N \backslash N} \chi_\omega(n)^{-1} X f(ng) dn = F_\omega X f(g), \end{aligned} \tag{3.7}$$

that is, the operator F_ω commutes with every element of $\mathcal{U}(\mathfrak{g})$.

For $\omega \in \mathcal{O}$, let $C^\infty(N\backslash G,\omega)$, be the space of all smooth functions f on G such that $f(ng) = \chi_\omega(n)f(g)$ for all $n \in N$. Identities (3.6) and (3.7) imply that if $f \in \mathcal{A}_\chi(\Upsilon; l, q)$, then $F_\omega f$ belongs to the space

$$W_{l,q}(\Upsilon,\omega) = \{h \in C^\infty(N\backslash G,\omega) \mid h \text{ is of type } (l,q) \text{ with character } \Upsilon\}\,. \tag{3.8}$$

In general, the functions occurring in the spectral decomposition of $L^2(N\backslash G,\omega)$, for some $\omega \in \mathbb{C}^*$, are called Whittaker functions, generalizing the Whittaker functions $W_{\varkappa,\mu}$ that turn up for $G = \mathrm{SL}_2(\mathbb{R})$. This explains the letter W in (3.8).

We already mentioned in Section 3.1, the character Υ is determined by its values on $\Omega_\pm$ since they generate the ring $\mathcal{Z}(\mathfrak{g})$. Not every point in $\mathbb{C}^2$ can occur as $\big(\Upsilon(\Omega_+), \Upsilon(\Omega_-)\big)$ for some character Υ of $\mathcal{Z}(\mathfrak{g})$. We shall now investigate which complex values can appear as such.

Lemma 3.2.2. *If $W_{l,q}(\Upsilon,\omega) \neq \{0\}$ then there exist $\nu \in \mathbb{C}$ and $p \in \frac{1}{2}\mathbb{Z}$, $|p| \leqslant l$, uniquely determined modulo $(\nu,p) \mapsto (-\nu,-p)$, such that $\Upsilon = \Upsilon_{\nu,p}$. Here $\Upsilon_{\nu,p}$ is the character of $\mathcal{Z}(\mathfrak{g})$ defined by:*

$$\Upsilon_{\nu,p}(\Omega_\pm) = \frac{1}{8}\left((\nu \mp p)^2 - 1\right).$$

Proof. Let $h \in W_{l,q}(\Upsilon,\omega)$ and $h \neq 0$. We note that for any fixed $g \in G$, the function $k \mapsto h(gk)$ belongs to $L^2(K; l, q)$. In particular, for $g = \mathrm{n}[z]\mathrm{a}[r]$,

$$h(\mathrm{n}[z]\mathrm{a}[r]k) = \sum_{|p|\leqslant l} \tilde{h}_p(z,r)\Phi^l_{p,q}(k). \tag{3.9}$$

The formulas (2.16), (2.17) and the property (P3) on page 15 imply that the condition $\Omega_\pm h = \Upsilon(\Omega_\pm)h$ is equivalent to the system of equations

$$\begin{cases} \Upsilon(\Omega_+)\tilde{h}_p = & \frac{1}{2}(l-p)r\partial_z\tilde{h}_{p+1} + \\ & +\frac{1}{8}\left(4r^2\partial_z\partial_{\bar z} + r^2\partial_r^2 - (2p+1)r\partial_r + p(p+2)\right)\tilde{h}_p \\ \Upsilon(\Omega_-)\tilde{h}_p = & -\frac{1}{2}(l+p)r\partial_{\bar z}\tilde{h}_{p-1} + \\ & +\frac{1}{8}\left(4r^2\partial_z\partial_{\bar z} + r^2\partial_r^2 + (2p-1)r\partial_r + p(p-2)\right)\tilde{h}_p \end{cases} \tag{3.10}$$

where $\tilde{h}_p \equiv 0$ if $|p| > l$.

For $\omega = 0$, the function h is N-invariant, which means that the functions $\tilde{h}_p = \tilde{h}_p(r)$ depend only on r. We write $\Upsilon(\Omega_\pm) = \frac{1}{8}(a_\pm - 1)$, with some $a_\pm \in \mathbb{C}$. Then (3.10) is equivalent to

$$\begin{cases} r^2\tilde{h}_p'' - r\tilde{h}_p' + \left(p^2 + 1 - \frac{a_+ + a_-}{2}\right)\tilde{h}_p = 0 \\ p\left(r\tilde{h}_p' - \tilde{h}_p\right) = \frac{a_- - a_+}{4}\tilde{h}_p \end{cases} \tag{3.11}$$

- If $p = 0$, then $a_+ = a_-$ or $\tilde{h}_0 = 0$. For non-zero $\tilde{h}_0$ the first equation of (3.11) is
$$r^2\tilde{h}_0'' - r\tilde{h}_0' + (1 - a_+)\tilde{h}_0 = 0.$$
  - If $a_+ = 0$, this equation has a fundamental system of solutions: r and $r\log r$. Then $\Upsilon(\Omega_+) = \Upsilon(\Omega_-) = -\frac{1}{8}$.
  - If $a_+ \neq 0$, this equation has a fundamental system of solutions: $r^{1+\nu}$ and $r^{1-\nu}$, where $\nu^2 = a_+$. Then $\Upsilon(\Omega_+) = \Upsilon(\Omega_-) = \frac{1}{8}\left(\nu^2 - 1\right)$.
- If $p \neq 0$, the second equation in (3.11) has a solution $r^{1+\nu}$, with $\nu = \frac{a_- - a_+}{4p}$, and the first equation therefore implies $a_+ + a_- = 2(\nu^2 + p^2)$. Hence $a_\pm = (\nu \mp p)^2$, and $\Upsilon(\Omega_\pm) = \frac{1}{8}\left((\nu \mp p)^2 - 1\right)$.

For $\omega \neq 0$, $\tilde{h}_p(z, r) = e^{2\pi i(\omega z + \bar{\omega}\bar{z})}h_p(r)$, and (3.10) becomes

$$\begin{cases} r^2h_p'' - (2p+1)rh_p' + \left((p+1)^2 + (4\pi i|\omega|r)^2 - \mu_+^2\right)h_p = \\ \qquad\qquad = 8(p-l)\pi i\omega r h_{p+1} \\ r^2h_p'' + (2p-1)rh_p' + \left((p-1)^2 + (4\pi i|\omega|r)^2 - \mu_-^2\right)h_p = \\ \qquad\qquad = 8(l+p)\pi i\bar{\omega} r h_{p-1}, \end{cases} \tag{3.12}$$

where we have written $\Upsilon(\Omega_\pm) = \frac{1}{8}\left(\mu_\pm^2 - 1\right)$, for some $\mu_\pm \in \mathbb{C}$. Without loss of generality, we may assume that

$$0 \leqslant \operatorname{Re}\mu_+ \leqslant \operatorname{Re}\mu_-. \tag{3.13}$$

Since $h_p \equiv 0$ if $|p| > l$, we see from the first equation for $p = l$, and from the second equation for $p = -l$ that $h_{\pm l}$ are solutions of the differential equations

$$r^2\frac{d^2w}{dr^2} + (1 - 2(l+1))r\frac{dw}{dr} + \left((4\pi i|\omega|r)^2 + (l+1)^2 - \mu_\pm^2\right)w = 0. \tag{3.14}$$

The equations (3.14) have solutions $h_{\pm l}(r) = r^{l+1}Z_{\mu_\pm}(4\pi i|\omega|r)$, with Z_μ being an arbitrary solution of the modified Bessel's differential equation of order μ. This fact can be found in [32], p. 77. The function $Z_\mu(4\pi i|\omega|r)$ can be written as a linear combination of the fundamental system $K_\mu(4\pi|\omega|r)$, $I_\mu(4\pi|\omega|r)$. So, the general solutions of (3.14) are

$$h_{\pm l}(r) = A_\pm r^{l+1}K_{\mu_\pm}(4\pi|\omega|r) + B_\pm r^{l+1}I_{\mu_\pm}(4\pi|\omega|r), \tag{3.15}$$

where $A_\pm$, $B_\pm$ are constants.

Let us assume that $\mu_\pm \notin \mathbb{Z}$. Inductively applying the second equation of (3.12) to $h_l(r)$, we see that in the expansion of $h_{-l}(r)$ all terms are multiples of either $r^{-l+1+\mu_+ + 2m}$ or $r^{-l+1-\mu_+ + 2n}$ with integers $m, n \geqslant 0$.

- If $A_- \neq 0$ in (3.15), h_{-l} contains a term equal to a multiple of $r^{l+1-\mu_-}$. Thus we have either $-l+1+\mu_+ + 2m = l+1-\mu_-$ or $-l+1-\mu_+ + 2n = l+1-\mu_-$. The first identity gives $\mu_+ + \mu_- = 2(l-m)$, which implies that $\mu_\pm = \pm\nu + (l-m)$ with some $\nu \in \mathbb{C}$, and $0 \leqslant m \leqslant l$ because of (3.13). The second identity gives $\mu_+ = \mu_- - 2(l-n)$, which implies that $\mu_\pm = \nu \mp (l-n)$ with some $\nu \in \mathbb{C}$, and $0 \leqslant n \leqslant l$ because of (3.13). In both situations we have $\Upsilon(\Omega_\pm) = \frac{1}{8}\left((\nu \mp p)^2 - 1\right)$, with some $p \in \frac{1}{2}\mathbb{Z}$, $|p| \leqslant l$.
- If $A_- = 0$ in (3.15), we have $h_{-l}(r) = B_- r^{l+1} I_{\mu_-}(4\pi|\omega|r)$. Inductively applying the first equation of (3.12) to $h_{-l}(r)$, in the same way as before, we obtain that there exist some $\nu \in \mathbb{C}$ and $p \in \frac{1}{2}\mathbb{Z}$, $|p| \leqslant l$, such that $\Upsilon(\Omega_\pm) = \frac{1}{8}\left((\nu \mp p)^2 - 1\right)$.

We note that $\mu_+ \in \mathbb{Z}$ implies $\mu_- \in \mathbb{Z}$, and vice versa. If $\mu_- \in \mathbb{Z}$, then by applying inductively the first equation of (3.12) to $h_{-l}(r)$, we see that all terms in the expansion of $h_l(r)$ are multiples of either $r^{-l+1+\mu_- +2m}\log r$ or $r^{-l+1+\mu_- +2n}$ with integers $m, n \geqslant 0$.

- If $A_+ \neq 0$, then h_l contains a term which is multiple of $r^{l+1+\mu_+}\log r$, so it must be $\mu_+ = \mu_- - 2(l-m)$, with $0 \leqslant m \leqslant l$.
- If $A_+ = 0$, then h_l contains a term which is multiple of $r^{l+1+\mu_+}$, so $\mu_+ = \mu_- - 2(l-n)$, with $0 \leqslant n \leqslant l$.

In both cases, we again arrive at the same conclusion. ■

Remark 1. The argument of the functions (3.15) is twice the corresponding argument in [9], p. 21. This is due to the use of $\mathrm{Tr}(\omega z) = \mathrm{Tr}_{F/\mathbb{Q}}(\omega z)$ as a scalar product on $\mathbb{C}$, instead of $\mathrm{Re}(\omega z) = \frac{1}{2}\mathrm{Tr}(\omega z)$ used in [9].

The above discussion for $\omega = 0$ actually shows that

$$\begin{aligned} W_{l,q}(\Upsilon_{0,0}, 0) &= \mathbb{C}\,\varphi_{l,q}(0,0) \oplus \mathbb{C}\,\partial_\nu \varphi_{l,q}(\nu,p)|_{\nu=0}, \\ W_{l,q}(\Upsilon_{\nu,p}, 0) &= \mathbb{C}\,\varphi_{l,q}(\nu,p) \oplus \mathbb{C}\,\varphi_{l,q}(-\nu,-p), \quad \text{if} \quad (\nu,p) \neq (0,0). \end{aligned} \tag{3.16}$$

Thus

$$\dim W_{l,q}(\Upsilon_{\nu,p}, 0) = 2. \tag{3.17}$$

From this discussion we only know that

$$\dim W_{l,q}(\Upsilon_{\nu,p}, \omega) \leqslant 2 \tag{3.18}$$

when $\omega \neq 0$.

If a function f is of polynomial growth, then its Fourier terms given by (3.5) inherit this growth property. We put

$$W_{l,q}^{\mathrm{pol}}(\Upsilon_{\nu,p}, \omega) = \left\{ h \in W_{l,q}(\Upsilon_{\nu,p}, \omega) \;\middle|\; \begin{array}{c} h \text{ of polynomial growth,} \\ \text{uniformly on } K \end{array} \right\}. \tag{3.19}$$

The function $K_\nu(x)$ decreases exponentially and $I_\nu(x)$ increases exponentially as $x \to \infty$. So, for any function $h \in W_{l,q}^{\text{pol}}(\Upsilon_{\nu,p}, \omega)$, the I-Bessel term in (3.15) does not occur. Recalling the asymptotic expansion of the K-Bessel function

$$K_\nu(x) = \sqrt{\frac{\pi}{2}}\, x^{-1/2} e^{-x}\big(1 + O(|x|^{-1})\big)$$

(see [32], p.139), we obtain the following

Lemma 3.2.3. *Let $\omega \neq 0$. If $W_{l,q}^{\text{pol}}(\Upsilon_{\nu,p}, \omega)$ is not empty, then it is spanned by a unique element h of exponential decay, and*

$$h(n\mathrm{a}[r]k) = O\left(|\omega r|^b e^{-4\pi|\omega|r}\right), \qquad \text{as } r \to \infty$$

for a certain $b \in \mathbb{R}$.

In the next chapter, we shall actually prove that $\dim W_{l,q}^{\text{pol}}(\Upsilon_{\nu,p}, \omega) = 1$ for any $\omega \neq 0$, and use the Jacquet integral to construct an explicit basis for this space. Moreover, from (3.17) and (3.18) we see that $\dim W_{l,q}(\Upsilon_{\nu,p}, \omega) = 2$ always. We shall use the Goodman-Wallach operator to construct an element of exponential growth in $W_{l,q}(\Upsilon_{\nu,p}, \omega)$, $\omega \neq 0$.

Next we introduce the notion of a cusp form.

Definition 3.2.4. *Let $\kappa \in \mathcal{C}_\chi$ be a cusp for Γ. For a function $f \in L^2(\Gamma\backslash G)$, the function on $N\backslash G$ given by*

$$a_\kappa(g) = \frac{1}{|\Lambda_\kappa|} \int_{\mathcal{R}_\kappa} f(g_\kappa n[z] g)\, d_+z, \tag{3.20}$$

is called the Fourier term of order zero of f at κ. Here $|\Lambda_\kappa|$ is the area of the fundamental domain $\mathcal{R}_\kappa$ used in (1.22).

Definition 3.2.5. *An automorphic function whose zeroth order Fourier term a_κ is identically equal to zero for all cusps κ is called a cusp form.*

We denote by $\mathcal{A}_\chi^0(\Upsilon_{\nu,p}; l, q)$ the space of all χ-automorphic cusp forms of type (l, q) and character $\Upsilon_{\nu,p}$. The next lemma is an immediate corollary of Lemma 5.2.1 in Section 5.2.

Lemma 3.2.6. *All cusp forms f are real-analytic and of exponential decay at each cusp $\kappa \in \mathcal{C}_\chi$:*

$$f(g_\kappa n\mathrm{a}[r]k) = O(e^{-\alpha r}), \qquad \text{as} \quad r \to \infty$$

uniformly over N and K, for some $\alpha > 0$.

3.2.2 Spectral parameter

Let $f \in \mathcal{A}_\chi(\Upsilon; l, q)$ be a non-zero χ-automorphic function of type (l, q) with character Υ. It has a Fourier expansion $f = \sum_{\omega \in \mathcal{O}'} F_\omega f$. We assume that $f \neq F_0 f$. Since $f \not\equiv 0$, there exists $\omega \in \mathcal{O}'$ such that $0 \not\equiv F_\omega f \in W_{l,q}(\Upsilon, \omega)$. According to Lemma 3.2.2, there are $\nu \in \mathbb{C}$ and $p \in \frac{1}{2}\mathbb{Z}$, $|p| \leqslant l$ uniquely determined modulo $(\nu, p) \mapsto (-\nu, -p)$ such that $\Upsilon = \Upsilon_{\nu,p}$. This immediately gives

Lemma 3.2.7. *If $\mathcal{A}_\chi(\Upsilon; l, q) \neq \{0\}$ then $\Upsilon = \Upsilon_{\nu,p}$, for certain $\nu \in \mathbb{C}$ and $p \in \frac{1}{2}\mathbb{Z}$, $|p| \leqslant l$, which are uniquely determined modulo $(\nu, p) \mapsto (-\nu, -p)$.*

The pair of numbers (ν, p) is called the *spectral parameter* of the automorphic function $f \in \mathcal{A}_\chi(\Upsilon_{\nu,p}; l, q)$.

The numbers (ν, p) appear in the parameterization of the irreducible unitary representations of G, so they might also be referred to as the spectral parameter of a representation. The unitary principal series is indexed by pairs (ν, p) with $\nu \in i\mathbb{R}$, and the complementary series is indexed by $(\nu, 0)$ with $\nu \in (-1, 1)\backslash\{0\}$. The pairs (ν, p) and $(-\nu, -p)$ correspond to the same representation. Hence, we may assume that

$$(\nu, p) \in i[0, \infty) \times \tfrac{1}{2}\mathbb{Z} \qquad \text{or} \qquad (\nu, p) \in (0, 1) \times \{0\}. \tag{3.21}$$

For some congruence subgroups of $\mathrm{SL}_2(\mathbb{C})$, like $\mathrm{SL}_2(\mathbb{Z}[i])$ or $\mathrm{SL}_2(\mathbb{Z}[\sqrt{-2}])$ for example, it is known that there are no complementary series due to the absence of exceptional eigenvalues of the Laplacian on $\Gamma\backslash\mathbb{H}^3$. (See [11], Proposition 7.6.2.) Generalization of Selberg's conjecture states that the smallest positive eigenvalue of the Laplacian is $\geqslant 1$ for all $\Gamma_0(I) \subset \mathrm{SL}_2(\mathbb{C})$.

3.3 Eisenstein series

Eisenstein series form a very important example of automorphic functions which have polynomial growth, but are not cusp forms. The functions $\varphi_{l,q}(\nu, p)$ defined in (2.43) are going to be the building blocks for the Eisenstein series. Some of their properties are given in (2.44)–(2.45) and the rest of Section 2.3.

We need to investigate the behavior of the functions $\varphi_{l,q}(\nu, p)$ under the left action of $P = NH$. For that purpose, it suffices to consider elements $\mathrm{h}[u] \in H$, since the functions $\varphi_{l,q}(\nu, p)$ are N-invariant.

For $\mathrm{h}[t] \in M$ (that is $|t| = 1$), we have

$$\begin{aligned} \Phi^l_{p,q}\big(\mathrm{h}[t]k\big) &= \Phi^l_{p,q}\big(\mathrm{h}[e^{i(2\arg t + \varphi)/2}]\mathrm{v}[i\theta]\mathrm{h}[e^{i\psi/2}]\big) \\ &= e^{-ip(2\arg t + \varphi) - iq\psi}\Phi^l_{p,q}(\mathrm{v}[i\theta]) \\ &= e^{-2ip\arg t}\Phi^l_{p,q}\big(\mathrm{h}[e^{i\varphi/2}]\mathrm{v}[i\theta]\mathrm{h}[e^{i\psi/2}]\big) = t^{-2p}\Phi^l_{p,q}(k). \end{aligned} \tag{3.22}$$

For $u, z \in \mathbb{C}$, $u \neq 0$, $r > 0$, the identity

$$\mathrm{h}[u]\mathrm{n}[z]\mathrm{a}[r] = \mathrm{n}[zu^2]\mathrm{a}[r|u|^2]\mathrm{h}[u/|u|] \tag{3.23}$$

holds. Therefore, if $g = \mathrm{n}[z]\mathrm{a}[r]k \in G$, we have for $\mathrm{h}[u] \in H$:

$$\begin{aligned} \varphi_{l,q}(\nu,p)(\mathrm{h}[u]g) &= \varphi_{l,q}(\nu,p)(\mathrm{n}[zu^2]\mathrm{a}[r|u|^2]\mathrm{h}[u/|u|]k) \\ &= (r|u|^2)^{1+\nu}\Phi^l_{p,q}(\mathrm{h}[u/|u|]k) \overset{(3.22)}{=} |u|^{2(1+\nu)}\,(u/|u|)^{-2p}\,\varphi_{l,q}(\nu,p)(g). \end{aligned} \tag{3.24}$$

Hence $\varphi_{l,q}(\nu,p)$ is a function on $N\backslash G$ that satisfies

$$\varphi_{l,q}(\nu,p)(n\mathrm{h}[u]g) = |u|^{2(1+\nu)}\,(u/|u|)^{-2p}\,\varphi_{l,q}(\nu,p)(g) \tag{3.25}$$

for all $n\mathrm{h}[u] \in P$. It makes sense to form the sum $\sum_{\gamma\in\Gamma_P\backslash\Gamma}\chi(\gamma)^{-1}\varphi_{l,q}(\nu,p)(\gamma g)$, provided that the summands are invariant under the left action of Γ_P. That is equivalent to the following condition:

$$\chi(\varepsilon) = \varepsilon^{2p}, \qquad \forall\varepsilon\in\mathcal{O}^*. \tag{3.26}$$

Definition 3.3.1. *Let $\nu \in \mathbb{C}$, $l,p,q \in \frac{1}{2}\mathbb{Z}$, such that $l \equiv p \equiv q$ (mod 1), and $|p|,|q| \leqslant l$. Let χ be a character of $(\mathcal{O}/I)^*$. We define the Eisenstein series of type (l,q) at the cusp ∞ by*

$$E_{l,q}(\nu,p;\chi)(g) := \sum_{\gamma\in\Gamma_P\backslash\Gamma}\chi(\gamma)^{-1}\varphi_{l,q}(\nu,p)(\gamma g). \tag{3.27}$$

Because of (3.25) the Eisenstein series $E_{l,q}(\nu,p;\chi)$, whenever it converges absolutely, will have a χ-automorphic transformation behavior with respect to the discrete subgroup Γ.

We now generalize the Definition 3.3.1 by introducing an Eisenstein series at any cusp $\kappa \in \mathcal{C}_\chi$. Let $g_\kappa \in G$ is such that $\kappa = g_\kappa\cdot\infty$. We define a function

$$h_\kappa(g) := \varphi_{l,q}(\nu,p)(g_\kappa^{-1}g).$$

For $\gamma \in \Gamma_\kappa$, we have that $\gamma = g_\kappa\left(\begin{smallmatrix} u & * \\ 0 & u^{-1}\end{smallmatrix}\right)g_\kappa^{-1}$ for some $\left(\begin{smallmatrix} u & * \\ 0 & u^{-1}\end{smallmatrix}\right) \in P$, and

$$\begin{aligned} h_\kappa(\gamma g) &= \varphi_{l,q}(\nu,p)(g_\kappa^{-1}\gamma g) = \varphi_{l,q}(\nu,p)\left(\left(\begin{smallmatrix} u & * \\ 0 & u^{-1}\end{smallmatrix}\right)g_\kappa^{-1}g\right) \\ &\overset{(3.24)}{=} |u|^{2(1+\nu)}(u/|u|)^{-2p}\varphi_{l,q}(\nu,p)(g_\kappa^{-1}g) = |u|^{2(1+\nu)}\,(u/|u|)^{-2p}\,h_\kappa(g). \end{aligned} \tag{3.28}$$

Again, we want to consider the sum $\sum_{\gamma\in\Gamma_\kappa\backslash\Gamma}\chi(\gamma)^{-1}h_\kappa(\gamma g)$. In order to make the terms in this sum invariant under Γ_κ, we impose the following compatibility condition on Γ_κ:

$$\chi\left(g_\kappa\begin{pmatrix} u & * \\ 0 & u^{-1}\end{pmatrix}g_\kappa^{-1}\right) = |u|^{2(1+\nu)}\,(u/|u|)^{-2p}, \tag{3.29}$$

for all $\gamma = g_\kappa \left(\begin{smallmatrix} u & * \\ 0 & u^{-1} \end{smallmatrix}\right) g_\kappa^{-1} \in \Gamma_\kappa$. At the cusp ∞, the condition (3.29) coincides with (3.26).

We note that, by Theorem 2.1.8 in [11], $g_\kappa \left(\begin{smallmatrix} u & * \\ 0 & u^{-1} \end{smallmatrix}\right) g_\kappa^{-1} \in \Gamma$ implies that $|u| = 1$, which means that the condition (3.29) simplifies to

$$\chi\left(g_\kappa \begin{pmatrix} u & * \\ 0 & u^{-1} \end{pmatrix} g_\kappa^{-1}\right) = u^{-2p}. \tag{3.30}$$

Definition 3.3.2. *Let $\nu \in \mathbb{C}$, $l, q, p \in \frac{1}{2}\mathbb{Z}$, such that $l \equiv p \equiv q$ mod 1, and $|p|, |q| \leqslant l$. Let $\kappa \in \mathcal{C}_\chi$ with χ a character of $(\mathcal{O}/I)^*$ that satisfies the condition (3.29). We define the Eisenstein series of type (l, q) at the cusp κ by*

$$E^\kappa_{l,q}(\nu, p; \chi)(g) := \sum_{\gamma \in \Gamma_\kappa \backslash \Gamma} \chi(\gamma)^{-1} \varphi_{l,q}(\nu, p)(g_\kappa^{-1}\gamma g). \tag{3.31}$$

Because of (3.28), the Eisenstein series $E^\kappa_{l,q}(\nu, p; \chi)$, whenever it converges absolutely, will have a χ-automorphic transformation behavior with respect to the discrete subgroup Γ.

We have the isomorphism between G/K and $\mathbb{H}^3$ given by (1.8). Therefore we can identify the right K-invariant functions on G with the functions on the upper half space. In this way, using the convention $r(g) = r$ for $g = \mathrm{n}[z]\mathrm{a}[r]k \in G$, our Eisenstein series

$$E^\kappa_{0,0}(\nu, 0; 1)(g) = \sum_{\gamma \in \Gamma_\kappa \backslash \Gamma} r(g_\kappa^{-1}\gamma g)^{1+\nu}$$

corresponds to the Eisenstein series $\frac{1}{[\Gamma_\kappa : \Gamma'_\kappa]} E_{g_\kappa^{-1}}(z + rj, \nu)$, with $E_A(z + rj, \nu)$ defined in [11], §3.3.2.

The absolute convergence of the Eisenstein series $E^\kappa_{l,q}(\nu, p; \chi)$, follows then from the absolute convergence of $E_{g_\kappa^{-1}}(z + rj, \mathrm{Re}\,\nu)$. Namely,

$$\begin{aligned} |E^\kappa_{l,q}(\nu, p; \chi)(g)| &\leqslant \sum_{\gamma \in \Gamma_\kappa \backslash \Gamma} |\varphi_{l,q}(\nu, p)(g_\kappa^{-1}\gamma g)| \leqslant \|\Phi^l_{p,q}\|_\infty \sum_{\gamma \in \Gamma_\kappa \backslash \Gamma} |r(g_\kappa^{-1}\gamma g)^{1+\nu}| \\ &= \frac{\|\Phi^l_{p,q}\|_\infty}{[\Gamma_\kappa : \Gamma'_\kappa]} \sum_{\gamma \in \Gamma'_\kappa \backslash \Gamma} r(g_\kappa^{-1}\gamma g)^{1+\mathrm{Re}\,\nu} = \frac{\|\Phi^l_{p,q}\|_\infty}{[\Gamma_\kappa : \Gamma'_\kappa]} E_{g_\kappa^{-1}}(z + rj, \mathrm{Re}\,\nu), \end{aligned} \tag{3.32}$$

where $\|\Phi^l_{p,q}\|_\infty = \max_{k \in K}\{|\Phi^l_{p,q}(k)|\}$. The series $E_{g_\kappa^{-1}}(z + rj, s)$ converges absolutely and uniformly on compact subsets of $\mathbb{H}^3 \times \{s \mid \mathrm{Re}\,s > 1\}$. (See [11], Propositions 3.2.1 and 3.1.3).

Hence, for $\mathrm{Re}\,\nu > 1$, the Eisenstein series $E^\kappa_{l,q}(\nu, p; \chi)$ converges absolutely and uniformly on compact subsets of G.

3.4 Automorphic representations

Let $C^K(\Gamma\backslash G;\chi)$ be the space of K-finite elements in the space $C^\infty(\Gamma\backslash G;\chi)$ of smooth χ-automorphic functions on G with respect to Γ. The action of the real Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ on $C^\infty(\Gamma\backslash G;\chi)$ by right differentiation extends $\mathbb{C}$-linearly to $\mathfrak{g}$, and hence to the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$.

An intertwining operator $T : H(\nu,p) \to C^K(\Gamma\backslash G;\chi)$ for the action of the Lie algebra $\mathfrak{g}$ is called automorphic representation of $H(\nu,p)$. By $\mathcal{AR}(\nu,p)$ we denote the linear space of all automorphic representations of $H(\nu,p)$.

If the spectral parameter (ν,p) is such that $H(\nu,p)$ is irreducible, we know that $H(\nu,p)$ and $H(-\nu,-p)$ are isomorphic. This gives a linear bijection $T \mapsto T\circ\iota(-\nu,-p)$ between the spaces $\mathcal{AR}(\nu,p)$ and $\mathcal{AR}(-\nu,-p)$.

Let $T \in \mathcal{AR}(\nu,p)$. For all integers or half integers $l \geqslant |p|$, $p \equiv l \pmod 1$ and $q \equiv l \pmod 1$, $|q| \leqslant l$, the function $f = T\varphi_{l,q}(\nu,p) \in C^K(\Gamma\backslash G;\chi)$ satisfies:

$$\begin{aligned} &f(\gamma g \mathrm{h}[e^{it}]) = \chi(\gamma) f(g) e^{-2iqt}, \qquad \text{for } \gamma\in\Gamma,\ g\in G,\ t\in\mathbb{R}\\ &\Omega_{\mathfrak{k}} f = -\frac{l^2+l}{2} f,\\ &\Omega_\pm f = \frac{(\nu\mp p)^2-1}{8} f. \end{aligned}$$

This follows immediately from (2.44), (2.45), and the intertwining property of T. It shows that there are linear maps from automorphic representations to automorphic functions:

$$\mathcal{AR}(\nu,p) \to \mathcal{A}_\chi(\Upsilon_{\nu,p}; l,q) \ : \ T \mapsto T\varphi_{l,q}(\nu,p), \tag{3.33}$$

with $\Upsilon_{\nu,p}$ determined by its values $\frac{(\nu\mp p)^2-1}{8}$ on $\Omega_\pm$.

We define $\mathcal{AR}^{\mathrm{pol}}(\nu,p)$, respectively $\mathcal{AR}^2(\nu,p)$ as the subspaces of automorphic representations $T\in\mathcal{AR}(\nu,p)$ for which $T\varphi_{l,q}(\nu,p)\in\mathcal{A}^{\mathrm{pol}}_\chi(\Upsilon_{\nu,p};l,q)$, respectively $T\varphi_{l,q}(\nu,p)\in\mathcal{A}^2_\chi(\Upsilon_{\nu,p};l,q)$ for all $l\geqslant|p|$ and all $q\equiv l \pmod 1$, $|q|\leqslant l$. We call the elements of $\mathcal{AR}^{\mathrm{pol}}(\nu,p)$ automorphic representations of polynomial growth, and the elements of $\mathcal{AR}^2(\nu,p)$ square-integrable automorphic representations.

The maps from $\mathcal{AR}^2(\nu,p)$ to $\mathcal{A}^2_\chi(\Upsilon_{\nu,p};l,q)$ given by (3.33) are surjective for all $l\equiv q\equiv p \pmod 1$, $l\geqslant|p|$, $|q|\leqslant l$.

Chapter 4

N-equivariant eigenfunctions

The Whittaker functions for $\mathrm{SL}_2(\mathbb{C})$ will be of interest in the derivation of the sum formula. More precisely, we shall be interested in the spaces of eigenfunctions of the Casimir operators $\Omega_\pm$ that are N-equivariant, that is, they transform via a certain character under the left action of N.

4.1 Jacquet integral

One way of constructing Whittaker functions is the use of the Jacquet integral. Also, the Jacquet integral turns up in the computation of the Fourier expansion of automorphic functions and automorphic representations.

Definition 4.1.1. *For $\omega \in \mathbb{C}$, we define an integral operator on the space of functions $f \in C^\infty(G)$ satisfying an estimate of the form*

$$f(n\mathrm{a}[r]k) = O(r^{1+\sigma}) \qquad \text{as} \quad r \downarrow 0 \tag{4.1}$$

for some $\sigma > 0$, uniformly for $n \in N$, $k \in K$, by

$$\mathbf{J}_\omega f(g) := \int_N \chi_\omega(n)^{-1} f(\mathrm{w}ng)dn. \tag{4.2}$$

The integral (4.2) is called the Jacquet integral. It was studied by Jacquet [19], for more general groups than $\mathrm{SL}_2(\mathbb{C})$.

Remark 2. The definition (4.2) of $\mathbf{J}_\omega$ is the same as the definition of $\mathcal{A}_\omega$ in [9], (5.7) except that the character ψ_ω on N defined in [9] equation (4.5) satisfies $\psi_{2\omega}(n) = \chi_\omega(n)$, due to the use of $\mathrm{Re}(\omega z)$ in the definition, instead of $\mathrm{Tr}(\omega z) = \mathrm{Tr}_{F/\mathbb{Q}}(\omega z)$. This implies that $\mathbf{J}_\omega = \mathcal{A}_{2\omega}$.

We note the following relation in G:

$$\mathrm{wn}[z]\mathrm{a}[r] = \mathrm{n}\left[\frac{-\bar{z}}{r^2+|z|^2}\right]\mathrm{a}\left[\frac{r}{r^2+|z|^2}\right]\mathrm{k}\left[\frac{\bar{z}}{\sqrt{r^2+|z|^2}}, \frac{-r}{\sqrt{r^2+|z|^2}}\right]. \tag{4.3}$$

With the help of this relation, one concludes that the integral in (4.2) converges absolutely and uniformly on compact subsets of G.

It is obvious that

$$\mathbf{J}_\omega f(ng) = \chi_\omega(n)\mathbf{J}_\omega f(g), \qquad \text{for all } n \in N \tag{4.4}$$

whenever the integral converges absolutely.

Let $X \in \mathfrak{sl}_2(\mathbb{C})$, and suppose that Xf also satisfies (4.1). Then

$$X\mathbf{J}_\omega f = \mathbf{J}_\omega X f, \tag{4.5}$$

that is, the integral $\mathbf{J}_\omega$ is an intertwining operator for the action of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$.

If l_t is the left translation given by

$$l_t f(g) = f(\mathrm{h}[t]g), \tag{4.6}$$

and f satisfies the condition in Definition 4.1.1, then $\mathbf{J}_\omega$ has the following property

$$l_t\mathbf{J}_\omega l_t = |t|^4\mathbf{J}_{t^2\omega} \qquad \text{for any} \quad t \in \mathbb{C}^*. \tag{4.7}$$

Indeed,

$$\begin{aligned} l_t\mathbf{J}_\omega l_t f(g) &= \int_N \chi_\omega(n)^{-1} f(\mathrm{h}[t]\mathrm{w}n\mathrm{h}[t]g)dn \\ &= \int_{\mathbb{C}} e^{-2\pi i \mathrm{Tr}(\omega z)} f(\mathrm{wn}[zt^{-2}]g)d_+z \\ &\overset{(z\mapsto t^2 z)}{=} \int_{\mathbb{C}} e^{-2\pi i \mathrm{Tr}(t^2\omega z)} f(\mathrm{wn}[z]g)t^2\bar{t}^2 d_+z \\ &= |t|^4 \int_N \chi_{t^2\omega}(n)^{-1} f(\mathrm{w}ng)dn \;=\; |t|^4\mathbf{J}_{t^2\omega}f(g). \end{aligned}$$

This means that actually we may restrict our attention to $\mathbf{J}_0$ and $\mathbf{J}_1$.

Condition (4.1) holds for $\varphi_{l,q}(\nu,p)$ provided that $\mathrm{Re}\,\nu > 0$. Therefore the Jacquet integral (4.2) with $f = \varphi_{l,q}(\nu,p)$ converges absolutely for $\mathrm{Re}\,\nu > 0$. Since $H(\nu,p)$, defined in (2.46), is a $\mathcal{U}(\mathfrak{g})$-module, (4.5) implies the smoothness of the function $\mathbf{J}_\omega\varphi_{l,q}(\nu,p)(g)$, and (4.4) implies that $\mathbf{J}_\omega\varphi_{l,q}(\nu,p)$ has the right transformation behavior under the action of N. Hence, $\mathbf{J}_\omega\varphi_{l,q}(\nu,p) \in C^\infty(N\backslash G,\omega)$. The property (4.5) implies in particular that $\mathbf{J}_\omega\varphi_{l,q}(\nu,p)(g)$ is of type (l,q) and character $\Upsilon_{\nu,p}$. Hence $\mathbf{J}_\omega\varphi_{l,q}(\nu,p) \in W_{l,q}(\Upsilon_{\nu,p},\omega)$.

We want to compute $\mathbf{J}_\omega \varphi_{l,q}(\nu,p)$ explicitly. We have

$$\mathbf{J}_\omega \varphi_{l,q}(\nu,p)(\mathrm{n}[z]\mathrm{a}[r]k) = \int_N \chi_\omega(n)^{-1} \varphi_{l,q}(\nu,p)(\mathrm{w}n\mathrm{n}[z]\mathrm{a}[r]k)dn$$

$$\overset{(4.3)}{=} \int_{\mathbb{C}} e^{-2\pi i \mathrm{Tr}(\omega u)} \varphi_{l,q}(\nu,p) \Big(\mathrm{n}\left[\tfrac{-\bar{z}-\bar{u}}{r^2+|z+u|^2} \right] \cdot$$

$$\cdot\, \mathrm{a}\left[\tfrac{r}{r^2+|z+u|^2} \right] \mathrm{k}\left[\tfrac{\bar{z}+\bar{u}}{\sqrt{r^2+|z+u|^2}}, \tfrac{-r}{\sqrt{r^2+|z+u|^2}} \right] k \Big) \, d_+u$$

(change : $u \mapsto ru - z$)

$$= \int_{\mathbb{C}} e^{-2\pi i \mathrm{Tr}(\omega(ru-z))} \varphi_{l,q}(\nu,p) \Big(\mathrm{n}\left[\tfrac{-\bar{u}}{r(1+|u|^2)} \right] \cdot$$

$$\cdot\, \mathrm{a}\left[\tfrac{1}{r(1+|u|^2)} \right] \mathrm{k}\left[\tfrac{\bar{u}}{\sqrt{1+|u|^2}}, \tfrac{-1}{\sqrt{1+|u|^2}} \right] k \Big) \, r^2 d_+u$$

$$= \chi_\omega(\mathrm{n}[z]) \; r^{1-\nu} \int_{\mathbb{C}} \frac{e^{-2\pi i \mathrm{Tr}(\omega r u)}}{(1+|u|^2)^{1+\nu}} \, \Phi^l_{p,q} \left(\mathrm{k}\left[\tfrac{\bar{u}}{\sqrt{1+|u|^2}}, \tfrac{-1}{\sqrt{1+|u|^2}} \right] k \right) d_+u.$$

Because of (2.30) we have that

$$\Phi^l_{p,q}\left(\mathrm{k}\left[\tfrac{\bar{u}}{\sqrt{1+|u|^2}}, \tfrac{-1}{\sqrt{1+|u|^2}} \right] k \right) = \sum_{|m|\leqslant l} \Phi^l_{p,m}\left(\mathrm{k}\left[\tfrac{\bar{u}}{\sqrt{1+|u|^2}}, \tfrac{-1}{\sqrt{1+|u|^2}} \right] \right) \Phi^l_{m,q}(k),$$

which implies

$$\mathbf{J}_\omega \varphi_{l,q}(\nu,p)(n\mathrm{a}[r]k) = \chi_\omega(n) \sum_{|m|\leqslant l} v^l_m(r,\omega) \Phi^l_{m,q}(k) \tag{4.8}$$

with

$$v^l_m(r,\omega) := r^{1-\nu} \int_{\mathbb{C}} \frac{e^{-2\pi i \mathrm{Tr}(\omega r z)}}{(1+|z|^2)^{1+\nu}} \Phi^l_{p,m}\left(\mathrm{k}\left[\tfrac{\bar{z}}{\sqrt{1+|z|^2}}, \tfrac{-1}{\sqrt{1+|z|^2}} \right] \right) d_+z. \tag{4.9}$$

After a change of variables $z = ue^{i\phi}$ in the integral in (4.9), we have

$$v^l_m(r,\omega) = r^{1-\nu} \int_0^\infty \int_{-\pi}^{\pi} \frac{e^{-2\pi i \mathrm{Tr}(\omega r u e^{i\phi})}}{(1+u^2)^{1+\nu}} \Phi^l_{p,m}\left(\mathrm{k}\left[\tfrac{ue^{-i\phi}}{\sqrt{1+u^2}}, \tfrac{-1}{\sqrt{1+u^2}} \right] \right) u \, d\phi \, du.$$

Now, we use the relation

$$\mathrm{k}[e^{-i\phi}\alpha, \beta] = \mathrm{h}[e^{-i\phi/2}]\mathrm{k}[\alpha,\beta]\mathrm{h}[e^{-i\phi/2}]$$

and the property (P3) on page 15, to obtain

$$v^l_m(r,\omega) = r^{1-\nu} \int_0^\infty \frac{u}{(1+u^2)^{1+\nu}} \Phi^l_{p,m}\left(\mathrm{k}\left[\tfrac{u}{\sqrt{1+u^2}}, \tfrac{-1}{\sqrt{1+u^2}} \right] \right) I(\omega) \, du \tag{4.10}$$

where

$$I(\omega) := \int_{-\pi}^{\pi} e^{-2\pi i \operatorname{Tr}(\omega r u e^{i\phi}) + i(p+m)\phi} d\phi \tag{4.11}$$

with $p + m \in \mathbb{Z}$. We consider the cases $\omega = 0$ and $\omega \neq 0$ separately.

For $\omega = 0$, the fact that

$$I(0) = \int_{-\pi}^{\pi} e^{i(p+m)\phi} d\phi = \begin{cases} 2\pi & \text{if } m = -p, \\ 0 & \text{if } m \neq -p, \end{cases}$$

implies the equality

$$v_m^l(r, 0) = \delta_{m,-p}\, 2\pi r^{1-\nu} \int_0^\infty \frac{u}{(1+u^2)^{1+\nu}} \Phi_{p,m}^l \left(\mathrm{k} \left[\tfrac{u}{\sqrt{1+u^2}}, \tfrac{-1}{\sqrt{1+u^2}} \right] \right) du. \tag{4.12}$$

By definition of the function $\Phi_{p,-p}^l$ we have

$$\Phi_{p,-p}^l \left(\mathrm{k} \left[\tfrac{u}{\sqrt{1+u^2}}, \tfrac{-1}{\sqrt{1+u^2}} \right] \right) = (-1)^{p-l} \sum_{i=0}^{l-|p|} (-1)^i \binom{l-|p|}{i} \binom{l+|p|}{i} \frac{u^{2i}}{(1+u^2)^l},$$

and the integral in (4.12), for $m = -p$, is equal to

$$\begin{aligned}
&= (-1)^{p-l} \sum_{i=0}^{l-|p|} (-1)^i \binom{l-|p|}{i} \binom{l+|p|}{i} \int_0^\infty \frac{u^{2i+1} du}{(1+u^2)^{1+l+\nu}} \\
&= (-1)^{p-l} \sum_{i=0}^{l-|p|} (-1)^i \binom{l-|p|}{i} \binom{l+|p|}{i} \frac{\Gamma(i+1)\Gamma(l+\nu-i)}{2\Gamma(1+l+\nu)} \\
&= (-1)^{p-l} \sum_{i=0}^{l-|p|} (-1)^i \binom{l-|p|}{i} \frac{(l+|p|)!\Gamma(l+\nu-i)}{2(l+|p|-i)!\Gamma(1+l+\nu)} \\
&= (-1)^{p-l} \frac{\Gamma(l+|p|+1)}{2\Gamma(1+l+\nu)} \sum_{j=0}^{l-|p|} (-1)^{l-|p|-j} \binom{l-|p|}{j} \frac{\Gamma(|p|+\nu+j)}{\Gamma(2|p|+1+j)} \\
&= (-1)^{p-|p|} \frac{\Gamma(l+|p|+1)\Gamma(|p|+\nu)}{2\Gamma(1+l+\nu)\Gamma(2|p|+1)} \sum_{j=0}^{l-|p|} (-1)^j \binom{l-|p|}{j} \frac{(|p|+\nu)_j}{(2|p|+1)_j}. \qquad (4.13)
\end{aligned}$$

Here $(\alpha)_j = \alpha(\alpha+1)\dots(\alpha+j-1)$ is Pochhammer's symbol. The identity

$$\sum_{j=0}^{k} (-1)^j \binom{k}{j} \frac{(\alpha)_j}{(\beta)_j} = \frac{(\beta-\alpha)_k}{(\beta)_k} \tag{4.14}$$

can be proved by induction. We continue in (4.13):

$$
\begin{aligned}
&= (-1)^{p-|p|} \frac{\Gamma(l+|p|+1)\Gamma(|p|+\nu)}{2\Gamma(1+l+\nu)\Gamma(2|p|+1)} \frac{(|p|+1-\nu)_{l-|p|}}{(2|p|+1)_{l-|p|}} \\
&= (-1)^{p-|p|} \frac{\Gamma(l+1-\nu)\Gamma(|p|+\nu)}{2\Gamma(l+1+\nu)\Gamma(1+|p|-\nu)}.
\end{aligned}
$$

So, we get

$$
v_m^l(r,0) = \delta_{m,-p}(-1)^{p-|p|}\pi \frac{\Gamma(1+l-\nu)\Gamma(|p|+\nu)}{\Gamma(1+l+\nu)\Gamma(|p|+1-\nu)}\, r^{1-\nu}. \tag{4.15}
$$

For $\omega \neq 0$, we use the integral representation for the Bessel function of integer order

$$
J_n(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{iz\cos t} e^{in\left(t-\frac{\pi}{2}\right)} dt, \qquad n \geqslant 0
$$

(see [32], p. 79). The above formula and the fact (1.26) yield for $p+m \geqslant 0$

$$
\begin{aligned}
I(\omega) &= \int_{-\pi}^{\pi} e^{-4\pi i|\omega|ru\cos(\phi+\arg\omega)+i(p+m)\phi} d\phi \\
&= e^{-i(p+m)\left(\arg\omega-\frac{\pi}{2}\right)} \cdot 2\pi J_{p+m}(-4\pi|\omega|ru) \\
&= 2\pi\,(i\omega/|\omega|)^{-p-m}\; J_{p+m}(4\pi|\omega|ru),
\end{aligned}
$$

and similarly, for $p+m<0$

$$
\begin{aligned}
I(\omega) &= e^{-i(p+m)\left(\arg\omega+\frac{\pi}{2}\right)} \cdot 2\pi J_{-p-m}(-4\pi|\omega|ru) \\
&= 2\pi(-1)^{p+m}\,(i\omega/|\omega|)^{-p-m}\; J_{-p-m}(4\pi|\omega|ru).
\end{aligned}
$$

We combine the last two results in:

$$
I(\omega) = 2\pi i^{-|p+m|}\,(\omega/|\omega|)^{-p-m}\; J_{|p+m|}(4\pi|\omega|ru). \tag{4.16}
$$

According to $\operatorname{sgn}(p+m) = \pm 1$, the definition of $\Phi_{p,m}^l$ gives for $\alpha, \beta \in \mathbb{R}$

$$
\begin{aligned}
&\Phi_{p,m}^l(\mathrm{k}[\alpha,\beta]) = \\
&\quad = (-1)^x \sum_{i=0}^{\min\{l\mp m, l\mp p\}} (-1)^i \binom{l\mp m}{i} \binom{l\pm m}{l\mp p-i}\, \alpha^{2i+|p+m|}\beta^{2l-2i-|p+m|},
\end{aligned}
$$

where $x = l - \frac{1}{2}|m+p| - \frac{1}{2}(m-p)$. Thus,

$$\Phi^l_{p,m}\left(\mathrm{k}\left[\tfrac{u}{\sqrt{1+u^2}}, \tfrac{-1}{\sqrt{1+u^2}}\right]\right) =$$

$$= (-1)^{x-2l+|m+p|}\frac{u^{|p+m|}}{(1+u^2)^l}\sum_{i=0}^{\min\{l\mp m, l\mp p\}}(-1)^i u^{2i}\binom{l\mp m}{i}\binom{l\pm m}{l\mp p-i}$$

$$= (-1)^{-l+\frac{1}{2}|m+p|-\frac{1}{2}(m-p)}\frac{u^{|p+m|}}{(1+u^2)^l}\cdot$$

$$\cdot\sum_{i=0}^{\min\{l\mp m, l\mp p\}}\sum_{j=0}^{i}\binom{i}{j}(-1)^j(1+u^2)^j\binom{l\mp m}{i}\binom{l\pm m}{l\mp p-i}.$$

Let $A = l\mp m$, $B = l\mp p$, $c = |m+p|$. Changing the order of summation we get

$$\Phi^l_{p,m}\left(\mathrm{k}\left[\tfrac{u}{\sqrt{1+u^2}}, \tfrac{-1}{\sqrt{1+u^2}}\right]\right) =$$

$$= (-1)^{-l+\frac{c}{2}-\frac{1}{2}(m-p)}\frac{u^c}{(1+u^2)^l}\sum_{j=0}^{\min\{A,B\}}(-1)^j(1+u^2)^j\xi^l_p(m,j), \tag{4.17}$$

where $\xi^l_p(m,j) = \sum_{i=j}^{\min\{A,B\}}\binom{i}{j}\binom{A}{i}\binom{A\pm 2m}{B-i}$. We rewrite $\xi^l_p(m,j)$ in the following way

$$\xi^l_p(m,j) = \frac{A!(A\pm 2m)!}{j!}\sum_{i=j}^{\min\{A,B\}}\frac{1}{(i-j)!(A-i)!(B-i)!(c+i)!}.$$

The sum is symmetric in A and B, so without loss of generality we may assume that $A \leqslant B$. Then we have

$$\sum_{i=j}^{A}\frac{1}{(i-j)!(A-i)!(B-i)!(c+i)!} =$$

$$= \frac{(1}{(A-j)!(B-j)!(c+j)!}\sum_{k=0}^{A-j}(-1)^k\binom{A-j}{k}\frac{(j-B)_k}{(c+j+1)_k}$$

$$\overset{(4.14)}{=} \frac{(A+B+c-j)!}{(A-j)!(B-j)!(c+A)!(c+B)!},$$

and thus

$$\begin{aligned}\xi^l_p(m,j) &= \frac{A!(A\pm 2m)!}{j!}\frac{(A+B+c-j)!}{(A-j)!(B-j)!(c+A)!(c+B)!}\\ &= \frac{j!(2l-j)!}{(l-p)!(l+p)!}\binom{A}{j}\binom{B}{j}.\end{aligned}$$

Noting that $l-\frac{1}{2}(|m+p|+|m-p|)=\min\{A,B\}$ and $l-\frac{1}{2}(|m+p|-|m-p|)=\max\{A,B\}$ we get

$$\xi_p^l(m,j)=\frac{j!(2l-j)!}{(l-p)!(l+p)!}\cdot \\ \cdot\binom{l-\frac{1}{2}(|m+p|+|m-p|)}{j}\binom{l-\frac{1}{2}(|m+p|-|m-p|)}{j}. \qquad (4.18)$$

Substitution of (4.16) and (4.17) into (4.10) gives

$$v_m^l(r,\omega)=2\pi(-1)^{l-p}\left(i\omega/|\omega|\right)^{-p-m}\cdot \\ \cdot r^{1-\nu}\sum_{j=0}^{l-\frac{1}{2}(|m+p|+|m-p|)}(-1)^j\xi_p^l(m,j)Y(j), \qquad (4.19)$$

with

$$Y(j):=\int_0^\infty\frac{u^{1+|p+m|}}{(1+u^2)^{l+1+\nu-j}}J_{|p+m|}(4\pi|\omega|ru)du. \qquad (4.20)$$

The formula

$$\int_0^\infty\frac{u^{\tau+1}}{(1+u^2)^{\mu+1}}J_\tau(au)du=\frac{(a/2)^\mu}{\Gamma(1+\mu)}K_{\tau-\mu}(a)\qquad(a>0),$$

holds for $-1<\operatorname{Re}\tau<2\operatorname{Re}\mu+\frac{3}{2}$. (See [32], p.105, line 12). Applying it to the integral (4.20) gives

$$Y(j)=\frac{(2\pi|\omega|r)^{\nu+l-j}}{\Gamma(l+1+\nu-j)}K_{\nu+l-|m+p|-j}(4\pi|\omega|r). \qquad (4.21)$$

Now, substituting (4.21) into (4.19) yields

$$v_m^l(r,\omega)=(-1)^{l-p}(2\pi)^\nu|\omega|^{\nu-1}\left(i\omega/|\omega|\right)^{-p-m}\sum_{j=0}^{l-\frac{1}{2}(|m+p|+|m-p|)}(-1)^j\cdot \\ \cdot\xi_p^l(m,j)\frac{(2\pi|\omega|r)^{l+1-j}}{\Gamma(l+1+\nu-j)}K_{\nu+l-|m+p|-j}(4\pi|\omega|r). \qquad (4.22)$$

From the discussion above we have proved

Lemma 4.1.2. *We have for* $\omega=0$

$$\mathbf{J}_0\varphi_{l,q}(\nu,p)=(-1)^{p-|p|}\pi\frac{\Gamma(1+l-\nu)\Gamma(|p|+\nu)}{\Gamma(1+l+\nu)\Gamma(|p|+1-\nu)}\varphi_{l,q}(-\nu,-p), \qquad (4.23)$$

and for $\omega \neq 0$

$$\mathbf{J}_\omega \varphi_{l,q}(\nu,p)(n\mathrm{a}[r]k) = (-1)^{l-p}(2\pi)^\nu |\omega|^{\nu-1} \cdot \\ \cdot\, \chi_\omega(n) \sum_{|m|\leqslant l} \left(\frac{i\omega}{|\omega|}\right)^{-p-m} w_m^l(\nu,p;|\omega|r)\Phi^l_{m,q}(k), \tag{4.24}$$

where

$$w_m^l(\nu,p;r) := \sum_{j=0}^{l-\frac{1}{2}(|m+p|+|m-p|)} (-1)^j \cdot \\ \cdot\, \xi_p^l(m,j) \frac{(2\pi r)^{l+1-j}}{\Gamma(l+1+\nu-j)} K_{\nu+l-|m+p|-j}(4\pi r), \tag{4.25}$$

and $\xi_p^l(m,j)$ *is given by (4.18).*

For $\mathrm{Re}\,\nu > 0$, the function $\mathbf{J}_\omega\varphi_{l,q}(\nu,p)$ is well defined and equal to the expression in the right hand-sides of (4.23) and (4.24). These expressions are meromorphic in ν if $\omega = 0$, and holomorphic in ν if $\omega \neq 0$. Thus, (4.23) and (4.24) give a meromorphic, respectively holomorphic, extension of $\mathbf{J}_\omega\varphi_{l,q}(\nu,p)$ to $\nu \in \mathbb{C}$.

In the case $\omega \neq 0$, the exponential decay of $K_\mu(r)$ as $r \to \infty$ (see (1.35)) implies that $w_m^l(\nu,p;r)$, and hence $\mathbf{J}_\omega\varphi_{l,q}(\nu,p)(n\mathrm{a}[r]k)$, is of polynomial growth. Hence, $\mathbf{J}_\omega\varphi_{l,q}(\nu,p) \in W^{\mathrm{pol}}_{l,q}(\Upsilon_{\nu,p},\omega)$. Moreover, Lemma 3.2.3 tells us that $\mathbf{J}_\omega\varphi_{l,q}(\nu,p)$ spans the space $W^{\mathrm{pol}}_{l,q}(\Upsilon_{\nu,p},\omega)$, for all values of $\nu \in \mathbb{C}$, $p \in \frac{1}{2}\mathbb{Z}$.

In this way, (4.24) defines a linear operator, called the Jacquet operator, from $H(\nu,p) \to W^{\mathrm{pol}}(\Upsilon_{\nu,p},\omega)$, where $W^{\mathrm{pol}}(\Upsilon_{\nu,p},\omega)$ is the space spanned by all subspaces $W^{\mathrm{pol}}_{l,q}(\Upsilon_{\nu,p},\omega)$, $|p|,|q| \leqslant l$. It is an intertwining operator for the action of $\mathcal{U}(\mathfrak{g})$, see (4.5). We note that the term Jacquet operator is limited to its application to the space $H(\nu,p)$, whereas the term Jacquet integral will be used wherever it is defined.

In particular, since the space $W^{\mathrm{pol}}_{l,q}(\Upsilon_{-\nu,-p},\omega)$ is identical to $W^{\mathrm{pol}}_{l,q}(\Upsilon_{\nu,p},\omega)$, the function $\mathbf{J}_\omega\varphi_{l,q}(-\nu,-p)$ is a multiple of $\mathbf{J}_\omega\varphi_{l,q}(\nu,p)$. Checking the coefficients of $\Phi^l_{l,q}$ in these functions, we find the functional equation

$$(2\pi|\omega|)^{-\nu}\,(-i\omega/|\omega|)^{p+\xi}\,\Gamma(l+1+\nu)\,\mathbf{J}_\omega\varphi_{l,q}(\nu,p) = \\ = (2\pi|\omega|)^{\nu}\,(-i\omega/|\omega|)^{-p+\xi}\,\Gamma(l+1-\nu)\mathbf{J}_\omega\,\varphi_{l,q}(-\nu,-p), \tag{4.26}$$

where

$$\xi = \begin{cases} 0 & , \quad p \in \mathbb{Z} \\ \frac{1}{2} & , \quad p \in \frac{1}{2}+\mathbb{Z} \end{cases} \tag{4.27}$$

Note that the number ξ above actually parameterizes the central character $\mathrm{h}[\pm 1] \mapsto (\pm 1)^{2\xi}$, and because of the relation (3.1) we have $\chi(-1) = (-1)^{2\xi}$.

Remark 3. For $l \in \mathbb{Z}$ (which implies that p, q are also integers), Lemma 4.1.2 reduces to the Lemma 5.1 in [9], where $w^l_m(\nu, p; r) = \alpha^l_m(\nu, p; 2r)$. Also the functional equation (4.26) coincides with (5.29) in [9].

Lemma 4.1.3. *Let $\sigma > 0$. For $\omega \neq 0$, $\nu \in \mathbb{C}$, and $l \in \frac{1}{2}\mathbb{N}$, $p, q \equiv l$ (mod 1), $|p|, |q| \leqslant l$, the following estimates of the Jacquet integral are satisfied:*

$$\mathbf{J}_\omega \varphi_{l,q}(\nu, p)(n\mathrm{a}[r]k) \ll_{\omega,\varepsilon} r^{1-|\operatorname{Re}\nu|-\varepsilon}(1+|\operatorname{Im}\nu|)^{2|\operatorname{Re}\nu|-1}, \tag{4.28}$$

uniformly for $|\operatorname{Re}\nu| \leqslant \sigma$, $r \leqslant r_0$, $r_0 > 0$, $n \in N$, $k \in K$, for each $\varepsilon > 0$, and

$$\mathbf{J}_\omega \varphi_{l,q}(\nu, p)(g) \ll_{\omega,g} (1+|\operatorname{Im}\nu|)^{2|\operatorname{Re}\nu|-1}, \tag{4.29}$$

for fixed $g = n\mathrm{a}[r]k \in G$.

Proof. The expressions (4.24) and (4.25) in Lemma 4.1.2 and the estimate (1.33) imply that

$$\begin{aligned}
&\mathbf{J}_\omega \varphi_{l,q}(\nu, p)(n\mathrm{a}[r]k) \ll \\
&\quad \ll_\omega \max_{m,j} \left| \Gamma(l+1+\nu-j)^{-1} r^{l+1-j} K_{\nu+l-|m+p|-j}(4\pi|\omega|r) \right| \\
&\quad \ll_\omega \max_{m,j} \Big\{ r^{l+1-j-\left|\operatorname{Re}\nu+l-|m+p|-j\right|-\varepsilon} \cdot \\
&\qquad\qquad \cdot (1+|\operatorname{Im}\nu|)^{-\operatorname{Re}\nu-l-1+j+\frac{1}{2}+\left|\operatorname{Re}\nu+l-|m+p|-j\right|-\frac{1}{2}} \Big\},
\end{aligned} \tag{4.30}$$

where $|m| \leqslant l$, $0 \leqslant j \leqslant l - \frac{1}{2}|m+p| - \frac{1}{2}|m-p|$. On one side,

$$\operatorname{Re}\nu + l - |m+p| - j \leqslant \operatorname{Re}\nu + l - j \leqslant |\operatorname{Re}\nu| + l - j,$$

and on the other side, since $2j \leqslant 2l - |m+p| - |m-p| \leqslant 2l - |m+p|$,

$$\operatorname{Re}\nu + l - |m+p| - j \geqslant \operatorname{Re}\nu - l + j \geqslant -|\operatorname{Re}\nu| - l + j.$$

Hence $\left|\operatorname{Re}\nu + l - |m+p| - j\right| \leqslant |\operatorname{Re}\nu| + l - j$, which implies

$$l + 1 - j - \left|\operatorname{Re}\nu + l - |m+p| - j\right| - \varepsilon \geqslant 1 - |\operatorname{Re}\nu| - \varepsilon,$$

and

$$-\operatorname{Re}\nu - l - 1 + j + \left|\operatorname{Re}\nu + l - |m+p| - j\right| \leqslant -1 + 2|\operatorname{Re}\nu|.$$

Estimate (4.28) follows from (4.30) and the last two inequalities. For fixed $g = n\mathrm{a}[r]k \in G$, the power of r in (4.30) is not present, and we get (4.29). ∎

4.2 Goodman-Wallach operator

As we saw in the previous section, with the help of the Jacquet integral we constructed an explicit element in $W_{l,q}(\Upsilon_{\nu,p},\omega)$ of at most polynomial growth. In this section we shall use the Goodman-Wallach operator to construct an explicit element in $W_{l,q}(\Upsilon_{\nu,p},\omega)$ of exponential growth for all ν that are not integral and strictly smaller than $-|p|$. In that way we shall show that also for $\omega \neq 0$, $\dim W_{l,q}(\Upsilon_{\nu,p},\omega) = 2$. (See (3.17), Section 3.2.1).

In [13], Goodman and Wallach construct operators that in the context of $\mathrm{SL}_2(\mathbb{C})$ are $\mathfrak{g}$-morphisms $\mathbf{M}_\omega : H(\nu,p) \to C^\infty(G)$ of the form

$$\mathbf{M}_\omega\varphi(g) = \sum_{m,n\geqslant 0} a(m,n;\omega)\ \partial_z^m \partial_{\bar z}^n \varphi(\mathrm{wn}[z]\mathrm{w}^{-1}g)\big|_{z=0}\,. \tag{4.31}$$

The coefficients $a(m,n;\omega)$ are to be determined, depending only on ν,p,ω, so that

$$\mathbf{M}_\omega\varphi(ng) = \chi_\omega(n)\mathbf{M}_\omega\varphi(g), \qquad \forall n \in N \tag{4.32}$$

is satisfied. This is equivalent to

$$\begin{cases} \partial_t \mathbf{M}_\omega\varphi(\mathrm{n}[t]g)|_{t=0} = 2\pi i\omega\mathbf{M}_\omega\varphi(g) \\ \partial_{\bar t} \mathbf{M}_\omega\varphi(\mathrm{n}[t]g)|_{t=0} = 2\pi i\bar\omega\mathbf{M}_\omega\varphi(g) \end{cases} \quad \text{with} \quad t \in \mathbb{C}. \tag{4.33}$$

We still do not know the growth of the coefficients $a(m,n;\omega)$, so we work formally. From the definition (4.31), we have

$$\partial_t \mathbf{M}_\omega\varphi(\mathrm{n}[t]g)|_{t=0} = \sum_{m,n\geqslant 0} a(m,n;\omega)\ \partial_t\partial_z^m \partial_{\bar z}^n \varphi(\mathrm{wn}[z]\mathrm{w}^{-1}\mathrm{n}[t]g)\big|_{\substack{z=0\\t=0}}\,, \tag{4.34}$$

$$\partial_{\bar t} \mathbf{M}_\omega\varphi(\mathrm{n}[t]g)|_{t=0} = \sum_{m,n\geqslant 0} a(m,n;\omega)\ \partial_{\bar t}\partial_z^m \partial_{\bar z}^n \varphi(\mathrm{wn}[z]\mathrm{w}^{-1}\mathrm{n}[t]g)\big|_{\substack{z=0\\t=0}}\,. \tag{4.35}$$

Simple calculation shows that the following equalities hold:

$$\mathrm{wn}[z]\mathrm{w}^{-1} = \begin{pmatrix} 1 & 0 \\ -z & 1 \end{pmatrix} = \exp\left(-z\begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}\right),$$
$$\mathrm{wn}[z]\mathrm{w}^{-1}\begin{pmatrix} 0 & t \\ 0 & 0\end{pmatrix}(\mathrm{wn}[z]\mathrm{w}^{-1})^{-1} = t\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix} + tz\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix} - tz^2\begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}.$$

The exponential of the right side of the latter equality, as $t \to 0$, is

$$\begin{aligned} &\exp\left(t\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix} + tz\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix} - tz^2\begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}\right) \\ &\quad = \exp\left(t\begin{pmatrix} 0 & 1 \\ 0 & 0\end{pmatrix}\right)\exp\left(tz\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix}\right)\exp\left(-tz^2\begin{pmatrix} 0 & 0 \\ 1 & 0\end{pmatrix}\right) + O(t^2) \\ &\quad = \mathrm{n}[t]\mathrm{h}[e^{tz}]\mathrm{wn}[tz^2]\mathrm{w}^{-1} + O(t^2). \end{aligned} \tag{4.36}$$

On the other hand, the matrix identity $\exp(BAB^{-1}) = B(\exp A)B^{-1}$, for $A = \left(\begin{smallmatrix} 0 & t \\ 0 & 0 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} 1 & 0 \\ -z & 1 \end{smallmatrix}\right)$ yields for the exponential of the left side:

$$\exp\left(\mathrm{wn}[z]\mathrm{w}^{-1}\begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix}(\mathrm{wn}[z]\mathrm{w}^{-1})^{-1}\right) = \mathrm{wn}[z]\mathrm{w}^{-1}\mathrm{n}[t](\mathrm{wn}[z]\mathrm{w}^{-1})^{-1}. \quad (4.37)$$

Comparing (4.36) and (4.37) we get for t in the neighborhood of 0:

$$\mathrm{wn}[z]\mathrm{w}^{-1}\mathrm{n}[t] = \mathrm{n}[t]\mathrm{h}[e^{tz}]\mathrm{wn}[tz^2 + z]\mathrm{w}^{-1} + O(t^2). \quad (4.38)$$

Thus

$$\begin{aligned} \partial_t \varphi(\mathrm{wn}[z]\mathrm{w}^{-1}\mathrm{n}[t]g)\big|_{t=0} &= \partial_t \varphi(\mathrm{h}[e^{tz}]\mathrm{wn}[tz^2 + z]\mathrm{w}^{-1}g)\big|_{t=0} \\ &= \partial_t \left(e^{(1+\nu-p)tz+(1+\nu+p)\bar{t}\bar{z}}\varphi(\mathrm{wn}[tz^2 + z]\mathrm{w}^{-1}g)\right)\Big|_{t=0} \\ &= (1+\nu-p)z\varphi(\mathrm{wn}[z]\mathrm{w}^{-1}g) + z^2\partial_z\varphi(\mathrm{wn}[z]\mathrm{w}^{-1}g), \end{aligned} \quad (4.39)$$

and similarly

$$\begin{aligned} &\partial_{\bar{t}}\varphi(\mathrm{wn}[z]\mathrm{w}^{-1}\mathrm{n}[t]g)\big|_{t=0} = \\ &\quad = (1+\nu+p)\bar{z}\varphi(\mathrm{wn}[z]\mathrm{w}^{-1}g) + \bar{z}^2\partial_{\bar{z}}\varphi(\mathrm{wn}[z]\mathrm{w}^{-1}g). \end{aligned} \quad (4.40)$$

By induction, from (4.39) and (4.40), we obtain

$$\begin{aligned} &\partial_t\partial_z^m\partial_{\bar{z}}^n\varphi(\mathrm{wn}[z]\mathrm{w}^{-1}\mathrm{n}[t]g)\big|_{\substack{z=0\\t=0}} = \\ &\qquad = m(\nu-p+m)\ \partial_z^{m-1}\partial_{\bar{z}}^n\varphi(\mathrm{wn}[z]\mathrm{w}^{-1}g)\big|_{z=0}, \quad (4.41) \\ &\partial_{\bar{t}}\partial_z^m\partial_{\bar{z}}^n\varphi(\mathrm{wn}[z]\mathrm{w}^{-1}\mathrm{n}[t]g)\big|_{\substack{z=0\\t=0}} = \\ &\qquad = n(\nu+p+n)\ \partial_z^m\partial_{\bar{z}}^{n-1}\varphi(\mathrm{wn}[z]\mathrm{w}^{-1}g)\big|_{z=0}. \quad (4.42) \end{aligned}$$

Substituting (4.41) and (4.42) respectively into (4.34) and (4.35), we get that (4.33) is equivalent to

$$\begin{cases} 2\pi i\omega\, a(m,n;\omega) = (m+1)(\nu-p+1+m)\, a(m+1,n;\omega), \\ 2\pi i\bar{\omega}\, a(m,n;\omega) = (n+1)(\nu+p+1+n)\, a(m,n+1;\omega). \end{cases} \quad (4.43)$$

Hence

$$a(m,n;\omega) = \frac{(2\pi i\omega)^m(2\pi i\bar{\omega})^n}{m!\,n!\,(\nu-p+1)_m(\nu+p+1)_n}\, a(0,0;\omega).$$

Choosing $a(0,0;\omega) = \{\Gamma(\nu-p+1)\Gamma(\nu+p+1)\}^{-1}$, we obtain

$$a(m,n;\omega) = \frac{(2\pi i\omega)^m(2\pi i\bar{\omega})^n}{m!\,n!\,\Gamma(\nu-p+1+m)\Gamma(\nu+p+1+n)}. \quad (4.44)$$

With this choice of the vector $\{a(m,n;\omega) \mid m,n \geqslant 0\}$, the sum in (4.31) converges absolutely for any element $\varphi \in H(\nu,p)$. Indeed, the fact that $z \mapsto \varphi(\mathrm{wn}[z]\mathrm{w}^{-1}g)$ is an analytic function provides us with the necessary bound of the derivatives.

For any $X \in \mathcal{U}(\mathfrak{g})$, we have

$$\begin{aligned} X\mathbf{M}_\omega\varphi(g) &= \partial_t\left(\sum_{m,n\geqslant 0} a(m,n;\omega)\ \partial_z^m\partial_{\bar z}^n\varphi(\mathrm{wn}[z]\mathrm{w}^{-1}ge^{tX})\big|_{z=0}\right)\Bigg|_{t=0} \\ &= \sum_{m,n\geqslant 0} a(m,n;\omega)\ \partial_z^m\partial_{\bar z}^n\left(\partial_t\varphi(\mathrm{wn}[z]\mathrm{w}^{-1}ge^{tX})|_{t=0}\right)\big|_{z=0} \\ &= \sum_{m,n\geqslant 0} a(m,n;\omega)\ \partial_z^m\partial_{\bar z}^n X\varphi(\mathrm{wn}[z]\mathrm{w}^{-1}g)\big|_{z=0} = \mathbf{M}_\omega X\varphi(g). \end{aligned} \tag{4.45}$$

Because of (4.32) and (4.45), the intertwining operator $\mathbf{M}_\omega$ maps $\varphi_{l,q}(\nu,p)$ into $W_{l,q}(\Upsilon_{\nu,p},\omega)$. Hence, there should be an expansion $\mathbf{M}_\omega\varphi_{l,q}(\nu,p)$ in terms of $\Phi^l_{m,q}$, $|m| \leqslant l$. It is described explicitly in the following

Lemma 4.2.1. *For any $\omega \neq 0$, we have*

$$\begin{aligned} &\mathbf{M}_\omega\varphi_{l,q}(\nu,p)(n\mathrm{a}[r]k) = \\ &\quad = (2\pi|\omega|)^{-\nu-1}\chi_\omega(n)\sum_{|m|\leqslant l}\left(\frac{-i\omega}{|\omega|}\right)^{p-m}\mu^l_m(\nu,p;|\omega|r)\Phi^l_{m,q}(k), \end{aligned} \tag{4.46}$$

where

$$\mu^l_m(\nu,p;r) = \sum_{j=0}^{l-\frac{1}{2}(|m+p|+|m-p|)} \xi^l_p(m,j)\frac{(2\pi r)^{l+1-j}}{\Gamma(l+1+\nu-j)}\, I_{\nu+l-|m+p|-j}(4\pi r). \tag{4.47}$$

We also have the functional equation

$$\begin{aligned} &\pi^{-2}(2\pi|\omega|)^{-\nu}\,(-i\omega/|\omega|)^{p-\xi}\,\Gamma(1+l+\nu)\mathbf{J}_\omega\varphi_{l,q}(\nu,p) = \\ &\quad = -\frac{(-1)^{p-\xi}}{\sin\pi(\nu-p)}(2\pi|\omega|)^{\nu}\,(i\omega/|\omega|)^{-p-\xi}\,\Gamma(1+l+\nu)\mathbf{M}_\omega\varphi_{l,q}(\nu,p) + \\ &\quad + \frac{(-1)^{-p-\xi}}{\sin\pi(\nu-p)}(2\pi|\omega|)^{-\nu}\,(i\omega/|\omega|)^{p-\xi}\,\Gamma(1+l-\nu)\mathbf{M}_\omega\varphi_{l,q}(-\nu,-p), \end{aligned} \tag{4.48}$$

with ξ given by (4.27).

Remark 4. For $l,p,q \in \mathbb{Z}$, we have $\mu^l_m(\nu,p;r) = \beta^l_m(\nu,p;2r)$, where $\beta^l_m(\nu,p;r)$ is defined with (6.14) in [9]. Comparing our (4.46) with (6.13) in [9] we see that $\mathbf{M}_\omega = \mathcal{B}_{2\omega}$. Using this, we also see that the functional equation (4.48) in this case simplifies to (6.15) in [9].

Proof. Let us suppose that $\nu \notin \mathbb{Z}$. In the definition (4.25) of $w^l_m(\nu,p;r)$ we replace the K-Bessel function by its defining expression (1.30). Then $w^l_m(\nu,p;r)$ is a difference of two parts; one is $r^{-\nu}$ times a power series in r, and the other one is r^ν times another power series in r. Since $w^l_m(\nu,p;r)$ satisfies the system (3.10) and the terms of both parts are not mixed under differentiation, we conclude that each part satisfies (3.10). The part which contains r^ν is equal to $\frac{(-1)^{1+|m+p|}\pi}{2\sin\pi(\nu+l)}\mu^l_m(\nu,p;r)$, whence the right side of (4.46) belongs to $W_{l,q}(\Upsilon_{\nu,p},\omega)$. The other part yields another member of $W_{l,q}(\Upsilon_{\nu,p},\omega)$, and they are linearly independent. In Section 3.2.1, equation (3.18), we have seen that $\dim W_{l,q}(\Upsilon_{\nu,p},\omega) \leqslant 2$. Therefore we find that

$$\dim W_{l,q}(\Upsilon_{\nu,p},\omega) = 2. \tag{4.49}$$

On the other hand, there is a power series $P(r)$ such that

$$\mathbf{M}_1\varphi_{l,p}(\nu,p)(\mathrm{a}[r]) = a(0,0;1)r^{1+\nu}P(r). \tag{4.50}$$

Indeed, from the fact that

$$\mathrm{wn}[z]\mathrm{w}^{-1}\mathrm{a}[r] = \mathrm{n}\left[\frac{-r^2\bar z}{1+r^2|z|^2}\right]\mathrm{a}\left[\frac{r}{1+r^2|z|^2}\right]\mathrm{k}\left[\frac{1}{\sqrt{1+r^2|z|^2}},\frac{r\bar z}{\sqrt{1+r^2|z|^2}}\right],$$

it follows that

$$\begin{aligned}\mathbf{M}_1\varphi_{l,q}(\nu,p)(\mathrm{a}[r]) = {}& \sum_{m,n\geqslant 0} a(m,n;1)\cdot \\ &\cdot \partial_z^m\partial_{\bar z}^n\varphi_{l,q}(\nu,p)\left(\mathrm{a}\left[\tfrac{r}{1+r^2|z|^2}\right]\mathrm{k}\left[\tfrac{1}{\sqrt{1+r^2|z|^2}},\tfrac{r\bar z}{\sqrt{1+r^2|z|^2}}\right]\right)\Bigg|_{z=0} \\ = {}& a(0,0;1)r^{1+\nu}P(r),\end{aligned}$$

where

$$\begin{aligned}P(r) := {}& \sum_{m,n\geqslant 0}\frac{a(m,n;1)}{a(0,0;1)}\cdot \\ &\cdot\partial_z^m\partial_{\bar z}^n\left\{(1+r^2|z|^2)^{-\nu-1}\,\Phi^l_{p,q}\left(\mathrm{k}\left[\tfrac{1}{\sqrt{1+r^2|z|^2}},\tfrac{r\bar z}{\sqrt{1+r^2|z|^2}}\right]\right)\right\}\Bigg|_{z=0}.\end{aligned}$$

The term with $m=n=0$ in $P(r)$ is equal to $\Phi^l_{p,q}(1)$, which is 1 if $p=q$, and 0 otherwise. In the other terms, after differentiation we get an expression which has powers of r^2, $r^2 z$ or $r^2\bar z$ in the numerator, and powers of $1+r^2|z|^2$ in the denominator. When $z=0$, we are only left with the powers of r. Hence $P(r)$ is indeed a power series in r with $P(0)=1$ if $p=q$.

Now, $\mathbf{M}_1\varphi_{l,p}(\nu,p) \in W_{l,p}(\Upsilon_{\nu,p},1)$ and (4.50) imply that $\mathbf{M}_1\varphi_{l,p}(\nu,p)$ must be a constant multiple of the right hand side of (4.46). Entering the defining sum of $I_{\nu+l-|m+p|-j}(z)$ into the right hand side of (4.46) for $\omega = 1$, we see that lowest power of r, for $p = q$, is equal to

$$(2\pi)^{-\nu-1}\xi_p^l(p,l-|p|)\frac{(2\pi r)^{\nu+1}}{\Gamma(\nu+1+|p|)\Gamma(\nu+1-|p|)} = a(0,0;1)r^{\nu+1},$$

which immediately implies that the constant is 1. Hence (4.46) is true for $\omega = 1$.

Next we observe that

$$l_t\mathbf{M}_\omega l_t^{-1} = \mathbf{M}_{t^2\omega} \qquad \text{for any} \quad t \in \mathbb{C}^*. \tag{4.51}$$

Indeed, for any $\varphi \in H(\nu,p)$, we have

$$\begin{aligned} l_t\mathbf{M}_\omega l_t^{-1}\varphi(g) &= \sum_{m,n\geqslant 0} a(m,n;\omega)\ \partial_z^m\partial_{\bar z}^n\varphi\left(\mathrm{h}[1/t]\mathrm{wn}[z]\mathrm{w}^{-1}\mathrm{h}[t]g\right)\big|_{z=0} \\ &= \sum_{m,n\geqslant 0} a(m,n;\omega)\ \partial_z^m\partial_{\bar z}^n\varphi\left(\mathrm{wn}[t^2z]\mathrm{w}^{-1}g\right)\big|_{z=0} \\ &= \sum_{m,n\geqslant 0} a(m,n;\omega)t^{2m}\bar t^{2n}\ \partial_z^m\partial_{\bar z}^n\varphi\left(\mathrm{wn}[z]\mathrm{w}^{-1}g\right)\big|_{z=0} \\ &= \sum_{m,n\geqslant 0} a(m,n;t^2\omega)\ \partial_z^m\partial_{\bar z}^n\varphi\left(\mathrm{wn}[z]\mathrm{w}^{-1}g\right)\big|_{z=0} = \mathbf{M}_{t^2\omega}\varphi(g) \end{aligned}$$

Since (4.46) is true for $\omega = 1$, relation (4.51) implies that (4.46) holds for general non-zero ω.

As for equation (4.48), we note that $\mathbf{M}_\omega\varphi_{l,q}(\nu,p)$ and $\mathbf{M}_\omega\varphi_{l,q}(-\nu,-p)$ are linearly independent elements of $W_{l,q}\Upsilon_{\nu,p},\omega)$ because of (4.46)–(4.47) and the appearance of $I_\nu(4\pi r)$, respectively $I_{-\nu}(4\pi r)$ in the expressions for $\mathbf{M}_\omega\varphi_{l,q}(\nu,p)$, respectively $\mathbf{M}_\omega\varphi_{l,q}(-\nu,-p)$. Thus $\mathbf{J}_\omega\varphi_{l,q}(\nu,p) \in W_{l,q}(\Upsilon_{\nu,p},\omega)$ must be a linear combination of them. Comparing the coefficients of $\Phi_{l,q}^l$ in these three elements we obtain (4.48).

The case $\nu \in \mathbb{Z}$ is settled by analytic continuation, since both sides of (4.46) are entire in ν. Similarly for (4.48). ■

Another property of the Goodman-Wallach operator which we shall use later is the following:

$$l_\tau\mathbf{M}_\omega\varphi_{l,q}(\nu,p) = |\tau|^{2(1+\nu)}(\tau/|\tau|)^{-2p}\mathbf{M}_{\tau^2\omega}\varphi_{l,q}(\nu,p), \tag{4.52}$$

for any $\tau \in \mathbb{C}^*$.

Indeed, direct computation gives for any $g \in G$

$$\begin{aligned}
&l_\tau \mathbf{M}_{\omega_2}\varphi_{l,q}(\nu,p)(g) = \\
&\quad = \sum_{m,n\geqslant 0} a(m,n;\omega_2)\partial_z^m\partial_{\bar z}^n\varphi_{l,q}(\nu,p)(\mathrm{wn}[z]\mathrm{w}^{-1}\mathrm{h}[\tau]g)|_{z=0} \\
&\quad = \sum_{m,n\geqslant 0} a(m,n;\omega_2)\partial_z^m\partial_{\bar z}^n\varphi_{l,q}(\nu,p)(\mathrm{h}[\tau]\mathrm{wn}[\tau^2 z]\mathrm{w}^{-1}g)|_{z=0} \\
&\quad \overset{(3.24)}{=} |\tau|^{2(1+\nu)}\left(\tau/|\tau|\right)^{-2p}\sum_{m,n\geqslant 0} a(m,n;\omega_2)\tau^{2m}\bar\tau^{2n}\cdot \\
&\qquad\qquad\qquad \cdot\partial_z^m\partial_{\bar z}^n\varphi_{l,q}(\nu,p)(\mathrm{wn}[z]\mathrm{w}^{-1}g)|_{z=0} \\
&\quad = |\tau|^{2(1+\nu)}\left(\tau/|\tau|\right)^{-2p}\mathbf{M}_{\tau^2\omega_2}\varphi_{l,q}(\nu,p)(g).
\end{aligned}$$

We now investigate the behavior of $r \mapsto \mathbf{M}_\omega(n\mathrm{a}[r]k)$ as the argument approaches 0 or ∞. Note that

$$\begin{aligned}
&\mathbf{M}_\omega\varphi_{l,q}(\nu,p)(\mathrm{a}[r]g) = \\
&\quad = \sum_{m,n\geqslant 0} a(m,n;\omega)r^{1+\nu+m+n}\,\partial_z^m\partial_{\bar z}^n\varphi_{l,q}(\nu,p)(\mathrm{wn}[z]\mathrm{w}^{-1}g)\big|_{z=0} \\
&\quad = \varphi_{l,q}(\nu,p)(\mathrm{a}[r]g)\left(a(0,0;\omega)+O(r)\right), \qquad \text{as} \quad r\downarrow 0. \qquad (4.53)
\end{aligned}$$

Since for $\operatorname{Re}\nu > 0$, $\varphi_{l,q}(\nu,p)$ satisfies condition (4.1) with $\sigma = \operatorname{Re}\nu$, the estimate (4.53) implies

$$\mathbf{M}_\omega\varphi_{l,q}(\nu,p)(n\mathrm{a}[r]k) = O(r^{1+\operatorname{Re}\nu}), \qquad \text{as} \quad r\downarrow 0 \qquad (4.54)$$

uniformly for $k \in K$, $n \in N$.

Later we shall need a refined version of the estimate above. Namely, Lemma 4.2.1 and the estimate (1.32) give

$$\begin{aligned}
&\mathbf{M}_\omega\varphi_{l,q}(\nu,p)(n\mathrm{a}[r]k) \ll \\
&\quad \ll \max_{p,m,j}\left|\Gamma(l+1+\nu-j)^{-1}r^{l+1-j}I_{\nu+l-|m+p|-j}(4\pi|\omega|r)\right| \\
&\quad \ll \max_{p,m,j}\left\{r^{1+\operatorname{Re}\nu+2l-2j-|m+p|}(1+|\operatorname{Im}\nu|)^{-2\operatorname{Re}\nu-1-2l+2j+|m+p|}\right\}e^{\pi|\operatorname{Im}\nu|} \\
&\quad \ll r^{1+\operatorname{Re}\nu}(1+|\operatorname{Im}\nu|)^{-2\operatorname{Re}\nu-1}e^{\pi|\operatorname{Im}\nu|}, \qquad (4.55)
\end{aligned}$$

uniformly for $|\operatorname{Re}\nu| \leqslant \sigma$, $0 < r \leqslant r_0$ with $\sigma > 0$, $r_0 > 0$, and $n \in N$, $k \in K$.

As to the behavior of $\mathbf{M}_\omega(n\mathrm{a}[r]k)$ as $r \to \infty$, we see from Lemma 4.2.1 and the estimate (1.34) that $r \mapsto \mathbf{M}_\omega(n\mathrm{a}[r]k)$ increases exponentially.

Because of (4.54) it makes sense to apply the Jacquet integral $\mathbf{J}_{\omega_1}$ to the function $\mathbf{M}_{\omega_2}\varphi_{l,q}(\nu,p)$ and the integral will be absolutely convergent.

Lemma 4.2.2. *Let* $\omega_2 \neq 0$ *and* $\operatorname{Re}\nu > 0$. *Then*

$$\mathbf{J}_0\mathbf{M}_{\omega_2}\varphi_{l,q}(\nu,p) = \frac{\sin\pi(\nu-p)}{\nu^2-p^2}\frac{\Gamma(1+l-\nu)}{\Gamma(1+l+\nu)}\,\varphi_{l,q}(-\nu,-p), \tag{4.56}$$

and for $\omega_1 \neq 0$

$$\mathbf{J}_{\omega_1}\mathbf{M}_{\omega_2}\varphi_{l,q}(\nu,p) = \mathcal{J}^*_{\nu,p}(4\pi\sqrt{\omega_1\omega_2})\mathbf{J}_{\omega_1}\varphi_{l,q}(\nu,p), \tag{4.57}$$

with

$$\mathcal{J}^*_{\nu,p}(z) = J^*_{\nu-p}(z)J^*_{\nu+p}(\bar{z}), \tag{4.58}$$

where $J^*_\nu(z)$ *is the even entire function of* z *which is equal to* $J_\nu(z)(z/2)^{-\nu}$ *for* $z > 0$.

Remark 5. In the case when $p \in \mathbb{Z}$, $\mathcal{J}^*_{\nu,p}(z) = |z/2|^{-2\nu}(z/|z|)^{2p}\mathcal{J}_{\nu,p}(z)$ with $\mathcal{J}_{\nu,p}$ the Bessel function given by [9], (6.21). Obviously, the function $\mathcal{J}_{\nu,p}(\sqrt{z})$ is of importance to us, and for $p \in \frac{1}{2}+\mathbb{Z}$ this function is no longer continuous in z. This discontinuity is actually neutralized by the discontinuity of the factor $(z/|z|)^p$ in the expression $|z/\sqrt{2}|^{-\nu}(z/|z|)^p\mathcal{J}_{\nu,p}(\sqrt{z})$, but to avoid the complications arising from the choices of square roots, we have decided to introduce the new notation $\mathcal{J}^*_{\nu,p}(\sqrt{z})$ for this expression. In this way, the function $\mathcal{J}^*_{\nu,p}$ is continuous for all p, integer or half-integer. The use of the letter $\mathcal{J}$ in this notation is to indicate the relation with the Bessel functions, although in general $\mathcal{J}^*_{\nu,p}$ is not a Bessel function.

Proof. For $\operatorname{Re}\nu > 0$, $\mathbf{J}_{\omega_1}\mathbf{M}_{\omega_2}\varphi_{l,q}(\nu,p)$ is given by the integral (4.2) applied to the sum (4.31). By absolute convergence, we may change the order of integration and summation:

$$\begin{aligned}
\mathbf{J}_{\omega_1}\mathbf{M}_{\omega_2}\varphi_{l,q}(\nu,p)(g) &= \int_{\mathbb{C}} e^{-2\pi i\operatorname{Tr}(\omega_1 u)}\mathbf{M}_{\omega_2}\varphi_{l,q}(\nu,p)(\mathrm{wn}[u]g)\,d_+u\\
&= \sum_{m,n\geqslant 0} a(m,n;\omega_2)\int_{\mathbb{C}} e^{-2\pi i\operatorname{Tr}(\omega_1 u)}\;\partial_z^m\partial_{\bar z}^n\varphi_{l,q}(\nu,p)\,(\mathrm{wn}[z+u]g)\big|_{z=0}\,d_+u\\
&\quad(\text{change}: z\mapsto z-u)\\
&= \sum_{m,n\geqslant 0} a(m,n;\omega_2)\int_{\mathbb{C}} e^{-2\pi i\operatorname{Tr}(\omega_1 u)}\;\partial_z^m\partial_{\bar z}^n\varphi_{l,q}(\nu,p)\,(\mathrm{wn}[z]g)\big|_{z=u}\,d_+u\\
&= \sum_{m,n\geqslant 0} a(m,n;\omega_2)\int_{\mathbb{C}} e^{-2\pi i\operatorname{Tr}(\omega_1 u)}\partial_u^m\partial_{\bar u}^n\varphi_{l,q}(\nu,p)\,(\mathrm{wn}[u]g)\,d_+u.
\end{aligned} \tag{4.59}$$

We transform the last integral by deforming the integration area and using integration by parts. If $S_R = \{z\in\mathbb{C} : |z| \leqslant R\}$ is a closed disk in $\mathbb{R}^2$ with radius R, then $\lim_{R\to\infty}\int_{S_R} = \int_{\mathbb{C}}$ holds. Integration by parts gives

$$\int_{S_R} e^{-2\pi i\operatorname{Tr}(\omega_1 u)}\partial_u\varphi_{l,q}(\nu,p)\,(\mathrm{wn}[u]g)\,d_+u =$$

$$= \int_{\partial S_R} e^{-2\pi i \mathrm{Tr}(\omega_1 u)} \varphi_{l,q}(\nu,p)\,(\mathrm{wn}[u]g)\,d\bar{u} - \\ - \int_{S_R} (-2\pi i\omega_1) e^{-2\pi i \mathrm{Tr}(\omega_1 u)} \varphi_{l,q}(\nu,p)\,(\mathrm{wn}[u]g)\,d_+u,$$

and by letting $R \to \infty$, we obtain

$$\int_{\mathbb{C}} e^{-2\pi i \mathrm{Tr}(\omega_1 u)} \partial_u \varphi_{l,q}(\nu,p)\,(\mathrm{wn}[u]g)\,d_+u = \\ = 2\pi i\omega_1 \int_{\mathbb{C}} e^{-2\pi i \mathrm{Tr}(\omega_1 u)} \varphi_{l,q}(\nu,p)\,(\mathrm{wn}[u]g)\,d_+u. \tag{4.60}$$

Similarly

$$\int_{\mathbb{C}} e^{-2\pi i \mathrm{Tr}(\omega_1 u)} \partial_{\bar{u}} \varphi_{l,q}(\nu,p)\,(\mathrm{wn}[u]g)\,d_+u = \\ = 2\pi i\bar{\omega}_1 \int_{\mathbb{C}} e^{-2\pi i \mathrm{Tr}(\omega_1 u)} \varphi_{l,q}(\nu,p)\,(\mathrm{wn}[u]g)\,d_+u. \tag{4.61}$$

By induction, from (4.60) and (4.61), it follows

$$\int_{\mathbb{C}} e^{-2\pi i \mathrm{Tr}(\omega_1 u)} \partial_u^m \partial_{\bar{u}}^n \varphi_{l,q}(\nu,p)\,(\mathrm{wn}[u]g)\,d_+u = \\ = (2\pi i\omega_1)^m (2\pi i\bar{\omega}_1)^n \int_{\mathbb{C}} e^{-2\pi i \mathrm{Tr}(\omega_1 u)} \varphi_{l,q}(\nu,p)\,(\mathrm{wn}[u]g)\,d_+u. \tag{4.62}$$

We insert (4.62) in (4.59), and get

$$\mathbf{J}_{\omega_1}\mathbf{M}_{\omega_2}\varphi_{l,q}(\nu,p)(g) = \\ = \sum_{m,n\geqslant 0} a(m,n;\omega_2)(2\pi i\omega_1)^m(2\pi i\bar{\omega}_1)^n \int_N \chi_{\omega_1}(n)\varphi_{l,q}(\nu,p)\,(\mathrm{wn}g)\,dn \\ = \sum_{m,n\geqslant 0} a(m,n;\omega_2)(2\pi i\omega_1)^m(2\pi i\bar{\omega}_1)^n\ \mathbf{J}_{\omega_1}\varphi_{l,q}(\nu,p)(g). \tag{4.63}$$

For $\omega_1 = 0$, all the summands in (4.63) are zero, except for the term with $m = n = 0$, which is equal to $a(0,0;\omega_2)$. Using (4.23) we get

$$\mathbf{J}_0\mathbf{M}_{\omega_2}\varphi_{l,q}(\nu,p)(g) = a(0,0;\omega_2)\mathbf{J}_0\varphi_{l,q}(\nu,p)(g) \\ \overset{(4.23)}{=} a(0,0;\omega_2)(-1)^{p-|p|}\pi\frac{\Gamma(1+l-\nu)\Gamma(|p|+\nu)}{\Gamma(1+l+\nu)\Gamma(|p|+1-\nu)}\varphi_{l,q}(-\nu,-p)(g) \\ = (-1)^{p-|p|}\frac{\sin\pi(\nu-|p|)}{\nu^2-p^2}\frac{\Gamma(1+l-\nu)}{\Gamma(1+l+\nu)}\varphi_{l,q}(-\nu,-p)(g),$$

which proves (4.56), since $\sin\pi(\nu-|p|) = (-1)^{p-|p|}\sin\pi(\nu-p)$.

For $\omega_1 \neq 0$, we transform the sum (4.63) as follows:

$$\sum_{m,n\geqslant 0} a(m,n;\omega_2)(2\pi i\omega_1)^m(2\pi i\bar{\omega}_1)^n =$$
$$= \sum_{m,n\geqslant 0} \frac{(-4\pi^2\omega_1\omega_2)^m(-4\pi^2\bar{\omega}_1\bar{\omega}_2)^n}{m!n!\Gamma(\nu-p+m+1)\Gamma(\nu+p+n+1)}$$
$$= \sum_{m\geqslant 0} \frac{(-1)^m(2\pi\sqrt{\omega_1\omega_2})^{2m}}{m!\Gamma(\nu-p+m+1)} \sum_{n\geqslant 0} \frac{(-1)^n(2\pi\sqrt{\bar{\omega}_1\bar{\omega}_2})^n}{n!\Gamma(\nu+p+n+1)}$$
$$= J^*_{\nu-p}(4\pi\sqrt{\omega_1\omega_2})J^*_{\nu+p}(4\pi\sqrt{\bar{\omega}_1\bar{\omega}_2}) = \mathcal{J}^*_{\nu,p}(4\pi\sqrt{\omega_1\omega_2}).$$

This proves (4.57). ■

Chapter 5

Fourier coefficients

5.1 Fourier expansion of Eisenstein series

Let $\kappa, \eta \in \mathcal{C}_\chi$ be two cusps of Γ (not necessarily distinct) with corresponding $g_\kappa, g_\eta \in \mathrm{SL}_2(\mathbb{C})$ as in §1.3. Direct generalization of Section 3.4 in [11] to non-trivial K-types gives the explicit Fourier expansions of the Eisenstein series $E^\kappa_{l,q}(\nu, p; \chi)$ at the cusp η.

Let $\Lambda_\eta \subset \mathbb{C}$ be the lattice that corresponds to the discrete subgroup ${g_\eta}^{-1}\Gamma'_\eta g_\eta$, see (1.21). Let $|\Lambda_\eta|$ be the Euclidean area of a fundamental domain for Λ_η and let $\Lambda'_\eta = \{z \in \mathbb{C} \mid \mathrm{Tr}(z\lambda) \in \mathbb{Z},\ \forall \lambda \in \Lambda_\eta\}$ be its dual lattice.

For $\mathrm{Re}\,\nu > 1$ the Λ_η-invariant function $g \mapsto E^\kappa_{l,q}(\nu, p; \chi)(g_\eta g)$, that is

$$E^\kappa_{l,q}(\nu, p; \chi)(g_\eta \mathrm{n}[\lambda] g) = E^\kappa_{l,q}(\nu, p; \chi)(g_\eta g), \qquad \text{for all } \lambda \in \Lambda_\eta, \tag{5.1}$$

has Fourier expansion

$$\begin{aligned} E^\kappa_{l,q}(\nu, p; \chi)(g_\eta g) &= \delta_{\kappa,\eta}\, \varphi_{l,q}(\nu, p)(g) \\ &+ \frac{(-1)^{p-|p|}\,\pi}{|\Lambda_\eta|[\Gamma_\kappa : \Gamma'_\kappa]} \frac{\Gamma(l+1-\nu)\Gamma(|p|+\nu)}{\Gamma(l+1+\nu)\Gamma(|p|+1-\nu)} D^{\kappa,\eta}_\chi(0; \nu, p)\varphi_{l,q}(-\nu, -p)(g) \\ &+ \frac{1}{|\Lambda_\eta|[\Gamma_\kappa : \Gamma'_\kappa]} \sum_{0 \neq \omega \in \Lambda'_\eta} D^{\kappa,\eta}_\chi(\omega; \nu, p)\mathbf{J}_\omega \varphi_{l,q}(\nu, p)(g). \end{aligned} \tag{5.2}$$

where the Fourier coefficient of order $\omega \neq 0$ is given by

$$\begin{aligned} D^{\kappa,\eta}_\chi(\omega; \nu, p) &:= \\ &= \sum_{\left(\begin{smallmatrix} * & * \\ c & d \end{smallmatrix}\right) \in \mathcal{R}} \chi\left(g_\kappa \begin{pmatrix} * & * \\ c & d \end{pmatrix} {g_\eta}^{-1}\right)^{-1} |c|^{-2(1+\nu)} \left(c/|c|\right)^{2p} e^{2\pi i \mathrm{Tr}(\omega d/c)}, \end{aligned} \tag{5.3}$$

and $\mathcal{R}$ is a system of representatives of the double cosets in

$$g_\kappa{}^{-1}\Gamma'_\kappa g_\kappa \backslash g_\kappa{}^{-1}\Gamma g_\eta / g_\eta{}^{-1}\Gamma'_\eta g_\eta$$

such that $c \neq 0$.

The Eisenstein series $E^\kappa_{l,q}(\nu, p; \chi)$, $p \in \frac{1}{2}\mathbb{Z}$, have a meromorphic continuation in ν to the whole complex plane, and moreover, this continuation is holomorphic on the line $\operatorname{Re}\nu = 0$. The same holds for the Fourier coefficients $D^{\kappa,\infty}_\chi(\omega; \nu, p)$.

5.2 Fourier expansion of automorphic representations

In this section, we restrict ourselves to the Fourier expansion of an automorphic representation at the cusp ∞, since that is sufficient for our purpose of deriving a spectral sum formula.

One can also develop a sum formula using the Fourier expansion of automorphic representations at other cusps. The Fourier coefficients will then depend on two data: their order and the cusp at which the expansion is done. The ideas stay the same as in the case restricted to the cusp ∞, only the necessary book keeping is more complicated. In the case of $\mathrm{SL}_2(\mathbb{R})$, such a formula is derived by Proskurin, [38], and by Bruggeman, [3], with more emphasis on representation theory.

Let $\omega \in \mathcal{O}$ and F_ω be the operator giving the Fourier term of order ω defined in (3.5). If $T \in \mathcal{AR}(\nu, p)$ is an automorphic representation of $H(\nu, p)$, then $F_\omega T$ gives an intertwining operator from $H(\nu, p)$ to $C^\infty(N\backslash G, \omega)$. Let $\mathcal{W}(\nu, p; \omega)$ be the linear space of intertwining operators from $H(\nu, p)$ to $C^\infty(N\backslash G, \omega)$. So if $S \in \mathcal{W}(\nu, p; \omega)$, then $S\varphi_{l,q}(\nu, p) \in W_{l,q}(\Upsilon_{\nu,p}, \omega)$ for each type (l, q).

Let $\omega = 0$. We recall that we have defined the space $H(\nu, p)$ as a subspace of $C^\infty(N\backslash G, 0)$ (see (2.46)), and therefore the identity map $\mathrm{Id}(\nu, p) : H(\nu, p) \to H(\nu, p)$ is an element in $\mathcal{W}(\nu, p; 0)$. If the pair (ν, p) is such that $H(\nu, p)$ is irreducible, then $\iota(\nu, p)$ given by (2.47) gives an other element in $\mathcal{W}(\nu, p; 0)$. These two elements are linearly independent for $(\nu, p) \neq (0, 0)$, which together with the fact that $\dim \mathcal{W}(\nu, p; 0) \leqslant \dim W_{l,q}(\Upsilon_{\nu,p}, 0) = 2$ (see (3.17)), implies that $\mathrm{Id}(\nu, p)$ and $\iota(\nu, p)$ form a basis for $\mathcal{W}(\nu, p; 0)$ in this case. If $(\nu, p) = (0, 0)$, then a basis is given by $\mathrm{Id}(0, 0)$ and an other linear operator which acting on the elements of $H(0, 0)$ gives a function with logarithmic term in r.

Let $T \in \mathcal{AR}^{\mathrm{pol}}(\nu, p)$, and (ν, p) such that $H(\nu, p)$ is irreducible. For each $\omega \in \mathcal{O}$, the intertwining operator $F_\omega T$ is an element in $\mathcal{W}(\nu, p; \omega)$. If $\omega = 0$, the Fourier term $F_0 T$ is a linear combination of the bases discussed above.

Next we consider $\omega \neq 0$. The operators $\mathbf{J}_\omega$ and $\mathbf{M}_\omega$ give an explicit basis for the space $\mathcal{W}(\nu, p; \omega)$, since $\mathbf{J}_\omega \varphi_{l,q}(\nu, p)$ and $\mathbf{M}_\omega \varphi_{l,q}(\nu, p)$ form a basis of $W_{l,q}(\Upsilon_{\nu,p}, \omega)$ for each type (l, q) satisfying $l \geqslant |p|$, $|q| \leqslant l$. One sees this from (4.49) and the behavior of the functions $\mathbf{J}_\omega \varphi_{l,q}(\nu, p)(n\mathrm{a}[r]k)$ and $\mathbf{M}_\omega \varphi_{l,q}(\nu, p)(n\mathrm{a}[r]k)$ as $r \to \infty$.

(See §4.1 and §4.2.) So, $F_\omega T$ must be a linear combination of $\mathbf{J}_\omega$ and $\mathbf{M}_\omega$. Since $T \in \mathcal{AR}^{\mathrm{pol}}(\nu,p)$, the function $T\varphi_{l,q}(\nu,p) \in \mathcal{A}^{\mathrm{pol}}_\chi(\Upsilon_{\nu,p};l,q)$ for all types (l,q), and its Fourier term of order ω belongs to the space $W_{l,q}(\Upsilon_{\nu,p},\omega)$. Moreover, $F_\omega T\varphi_{l,q}(\nu,p)$ has polynomial growth inherited from $T\varphi_{l,q}(\nu,p)$. Therefore, we cannot have a contribution from $\mathbf{M}_\omega$ in $F_\omega T$. Hence, $F_\omega T$ must be a multiple of the Jacquet operator:

$$F_\omega T = c_T(\omega)\mathbf{J}_\omega. \tag{5.4}$$

The coefficients $c_T(\omega)$ depend only on the order ω and the automorphic representation T, since both T and $\mathbf{J}_\omega$ commute with the action of $\mathfrak{g}$.

The automorphic representations T such that $F_0T = 0$ are called cuspidal. The Fourier expansion of a cuspidal automorphic representation of an irreducible $H(\nu,p)$ is given by

$$T = \sum_{0\neq\omega\in\mathcal{O}'} c_T(\omega)\mathbf{J}_\omega, \tag{5.5}$$

In particular, if T is a cuspidal representation, then $T\varphi_{l,q}(\nu,p) \in \mathcal{A}^0_\chi(\Upsilon_{\nu,p};l,q)$, and we have

$$T\varphi_{l,q}(\nu,p) = \sum_{0\neq\omega\in\mathcal{O}'} c_T(\omega)\mathbf{J}_\omega\varphi_{l,q}(\nu,p). \tag{5.6}$$

From this we see that the Fourier coefficients $c_T(\omega)$ of a cuspidal automorphic representation T with spectral parameter (ν_T,p_T) also appear as Fourier coefficients of the cups forms with the same spectral parameter and type (l,q), for all $l \geqslant |p_T|$, $|q| \leqslant l$.

Since $l_\varepsilon \mathbf{J}_\omega\varphi_{l,q}(\nu,p) = \varepsilon^{2p}\mathbf{J}_{\varepsilon^2\omega}\varphi_{l,q}(\nu,p)$ for a unit $\varepsilon \in \mathcal{O}^*$, we conclude that the Fourier coefficients must satisfy

$$c_T(\varepsilon^2\omega) = \varepsilon^{2p}c_T(\omega), \qquad \forall\varepsilon\in\mathcal{O}^*. \tag{5.7}$$

Let now $\kappa \in \mathcal{C}_\chi$ be a cusp for Γ. The following lemma describes more closely the behavior near the cusps of any Γ'_κ-invariant function with an expansion of the type (5.6).

Lemma 5.2.1. *Let $\kappa \in \mathcal{C}_\chi$ be a cusp, $p \in \frac{1}{2}\mathbb{Z}$, and $l \in \frac{1}{2}\mathbb{N}$ such that $l \equiv p$ (mod 1), $l \geqslant |p|$. Let $\Lambda_\kappa \subset \mathbb{C}$ be the lattice corresponding to the discrete subgroup ${g_\kappa}^{-1}\Gamma'_\kappa g_\kappa$ with dual Λ'_κ and set $\omega_0 := \min\{|\omega| : \omega \in \Lambda'_\kappa\backslash\{0\}\}$.*

(i) Any Γ'_κ-invariant function f on G of type (l,q) ($q \equiv l$ (mod 1), $|q| \leqslant l$) and spectral parameter (ν,p) which has an expansion

$$f(g_\kappa n\mathrm{a}[r]k) = \sum_{0\neq\omega\in\Lambda'_\kappa} c(\omega)\mathbf{J}_\omega\varphi_{l,q}(\nu,p)(n\mathrm{a}[r]k), \tag{5.8}$$

satisfies

$$f(g_\kappa n\mathrm{a}[r]k) = O(r^{\,l+\frac{1}{2}} e^{-2\pi\omega_0 r}) \qquad \text{as} \quad r \to \infty, \tag{5.9}$$

with the implicit constant depending on l, ν, and p.

(ii) Any Γ'_κ-invariant family of functions $f(\nu) \in L^2(\Gamma\backslash G)$ of type (l,q) and spectral parameter (ν,p), where ν runs through a compact set $\mathcal{N} \subset \mathbb{C}$, with an expansion of type (5.8) satisfies

$$f(\nu; g_\kappa n\mathrm{a}[r]k) = O(r^{\,l+\frac{1}{2}} e^{-2\pi\omega_0 r}) \qquad \text{as} \quad r \to \infty, \tag{5.10}$$

uniformly for $\nu \in \mathcal{N}$.

Proof. The analysis on the compact group K, see Section 2.2, implies that for fixed $n \in N$ and $a \in A$ the function $k \mapsto f(g_\kappa nak)$ is of type (l,q). The space of such functions is finite-dimensional with basis $\Phi^l_{m,q}$, $|m| \leqslant l$. Thus, using Lemma 4.1.2, we have

$$f(g_\kappa na[r]k) = \sum_{0\neq\omega\in\Lambda'_\kappa} \sum_{|m|\leqslant l} c(\omega) j^\omega_m(\nu;r)\chi_\omega(n)\Phi^l_{m,q}(k), \tag{5.11}$$

with

$$j^\omega_m(\nu;r) = (-1)^{l-p}(2\pi)^\nu |\omega|^{\nu-1}\,(i\omega/|\omega|)^{-p-m}\, w^l_m(\nu,p;|\omega|r),$$

and $w^l_m(\nu,p;r)$ as given in (4.25), where the series

$$\sum_{0\neq\omega\in\Lambda'_\kappa} c(\omega) j^\omega_m(\nu;r)\chi_\omega(n) \tag{5.12}$$

converges absolutely.

In order to estimate the double sum (5.11), we use the asymptotic estimate of the K-Bessel function for large argument, see (1.35),

$$K_{\nu+l-|m+p|-j}(4\pi|\omega|r) \sim (8|\omega|r)^{-\frac{1}{2}} e^{-4\pi|\omega|r} \qquad \text{as} \quad r \to \infty, \tag{5.13}$$

for all $|m| \leqslant l$ and $0 \leqslant j \leqslant l - \frac{1}{2}(|m+p| + |m-p|)$ appearing in $w^l_m(\nu,p;r)$. This gives

$$\begin{aligned}
&\lim_{r\to\infty} (2\pi|\omega|r)^{-l-1}(8\pi|\omega|r)^{\frac{1}{2}} e^{4\pi|\omega|r} w^l_m(\nu,p;|\omega|r) = \\
&\quad = \lim_{r\to\infty} \Bigg\{ \frac{\xi^l_p(m,0)}{\Gamma(l+1+\nu)} (8\pi|\omega|r)^{\frac{1}{2}} e^{4\pi|\omega|r} K_{\nu+l-|m+p|}(4\pi|\omega|r) \\
&\qquad + \sum_{j=1}^{l-\frac{1}{2}(|m+p|+|m-p|)} \xi^l_p(m,j) \frac{(-2\pi|\omega|r)^{-j}}{\Gamma(l+1+\nu-j)} (8\pi|\omega|r)^{\frac{1}{2}} \cdot \\
&\qquad\qquad \cdot e^{4\pi|\omega|r} K_{\nu+l-|m+p|-j}(4\pi|\omega|r) \Bigg\} \\
&\quad = \binom{2l}{l-|p|} \Gamma(l+1+\nu)^{-1} + \lim_{r\to\infty} O(r^{-1}) = \binom{2l}{l-|p|} \Gamma(l+1+\nu)^{-1}.
\end{aligned}$$

That is,

$$w_m^l(\nu, p; |\omega|r) \sim \frac{1}{2} \binom{2l}{l-|p|} \frac{(2\pi|\omega|r)^{l+\frac{1}{2}}}{\Gamma(l+1+\nu)} e^{-4\pi|\omega|r} \qquad \text{as} \quad r \to \infty. \tag{5.14}$$

If ν is fixed, then the last estimate and the absolute convergence of the series (5.12) at any point $(0, t)$ with $t > \frac{1+(l+|\nu|)^2}{4\pi\omega_0}$, imply that the coefficients $c(\omega)$ must satisfy the following estimate

$$c(\omega) = O\left(|\omega|^{-\operatorname{Re}\nu - l + \frac{1}{2}} e^{4\pi|\omega|t}\right). \tag{5.15}$$

So if $r > 2t$,

$$\begin{aligned} f(g_\kappa n a[r]k) &< \sum_{\omega \in \Lambda'_\kappa} \sum_{|m| \leqslant l} \left|c(\omega)\right|\left|j_m^\omega(\nu; r)\right|\left|\Phi_{m,q}^l(k)\right| \\ &\ll \sum_{0 \neq \omega \in \Lambda'_\kappa} \left|c(\nu;\omega)\right|\left|j_m^\omega(\nu; r)\right| \ll_{l,\nu,p} r^{\,l+\frac{1}{2}} \sum_{0 \neq \omega \in \Lambda'_\kappa} e^{-4\pi|\omega|(r-t)} \\ &\ll_{l,\nu,p} r^{\,l+\frac{1}{2}} e^{-2\pi\omega_0 r} \qquad \text{as} \quad r \to \infty, \end{aligned}$$

which proves (5.9).

Let now ν run over a compact set $\mathcal{N} \subset \mathbb{C}$. Suppose that each $f = f(\nu)$ is square integrable on $\Gamma\backslash G$, and that its L^2- norm $\|f(\nu)\|_{\Gamma\backslash G}$ is bounded uniformly for $\nu \in \mathcal{N}$. Then this bound holds also for the L^2-norm of each term in the expansion (5.11) when integrated over a cusp sector at κ:

$$\int_{\mathcal{R}_\kappa \times (r_0, \infty)} \int_{K_{1/2}} \left|c(\nu;\omega) j_m^\omega(\nu; r) e^{2\pi i \operatorname{Tr}(\omega z)} \Phi_{m,q}^l(k)\right|^2 dk\, d_+ z\, \frac{dr}{r^3} < C_1, \tag{5.16}$$

where $C_1 = \max_{\nu \in \mathcal{N}} \|f(\nu)\|^2_{\Gamma\backslash G}$ and $c(\nu;\omega) = c(\omega)$. This inequality remains valid when we shrink the cusp sector by increasing r_0. The left side of (5.16) is equal to

$$\begin{aligned} &\int_{r_0}^\infty \int_{\mathcal{R}_\kappa} \left|c(\nu;\omega)\right|^2 \left|j_m^\omega(\nu; r)\right|^2 \int_{K_{1/2}} \left|\Phi_{m,q}^l(k)\right|^2 dk\, d_+ z\, \frac{dr}{r^3} = \\ &= \frac{1}{2} \|\Phi_{m,q}^l\|_K^2 |\mathcal{R}_\kappa| \left|c(\nu;\omega)\right|^2 \int_{r_0}^\infty \left|j_m^\omega(\nu; r)\right|^2 \frac{dr}{r^3} \\ &= \frac{1}{2} \|\Phi_{p,q}^l\|_K^2 |\mathcal{R}_\kappa| \left|c(\nu;\omega)\right|^2 (2\pi)^{2\operatorname{Re}\nu} |\omega|^{2(\operatorname{Re}\nu - 1)} \int_{r_0}^\infty \left|w_m^l(\nu, p; |\omega|r)\right|^2 \frac{dr}{r^3}. \end{aligned}$$

If $\mu = \min_{|m| \leqslant l} \left\{\|\Phi_{p,q}^l\|_K\right\}$, then (5.16) implies that

$$\left|c(\nu;\omega)\right|^2 < C_2 (2\pi)^{-2\operatorname{Re}\nu} |\omega|^{-2(\operatorname{Re}\nu - 1)} \left(\int_{r_1}^\infty \left|w_m^l(\nu, p; |\omega|r)\right|^2 \frac{dr}{r^3}\right)^{-1}, \tag{5.17}$$

for all $r_1 \geqslant r_0$, where $C_2 = 2C_1|\mathcal{R}_\kappa|^{-1}\mu^{-2}$. It is essential that the asymptotic equality (5.13), and hence (5.14), is uniform for $\nu \in \mathcal{N}$. Since (5.14) implies

$$\left|w_m^l(\nu, p; |\omega|r)\right| > \frac{1}{4}\begin{pmatrix} 2l \\ l-|p| \end{pmatrix}\frac{(2\pi|\omega|r)^{l+\frac{1}{2}}}{\Gamma(l+1+\nu)}e^{-4\pi|\omega|r}$$

for large enough r, we have for suitably large $r_1 \geqslant r_0$

$$\int_{r_1}^{\infty}\left|w_m^l(\nu, p; |\omega|r)\right|^2\frac{dr}{r^3} > \frac{4^{1-2l}}{16}\begin{pmatrix} 2l \\ l-|p| \end{pmatrix}^2\frac{(2\pi|\omega|)^2}{\Gamma(l+1+\nu)^2}\Gamma(2l-1, 8\pi|\omega|r_1).$$

For any $a \in \mathbb{Z}_{\geqslant 1}$, the incomplete gamma function has an expansion of the form $\Gamma(a,z) = e^{-z}z^{a-1}\big(1+O(z^{-1})\big)$ as $z \to \infty$, which implies that $\Gamma(a,z) > \frac{1}{2}e^{-z}z^{a-1}$ for suitably large z. So, for all $r_2 \geqslant r_1$ suitably large, we get

$$\int_{r_2}^{\infty}\left|w_m^l(\nu, p; |\omega|r)\right|^2\frac{dr}{r^3} > \frac{1}{128}\begin{pmatrix} 2l \\ l-|p| \end{pmatrix}^2\frac{(2\pi|\omega|)^{2l}}{\Gamma(l+1+\nu)^2}{r_2}^{2l-2}e^{-8\pi|\omega|r_2}. \tag{5.18}$$

We substitute (5.18) into (5.17) with r_1 replaced by r_2 and obtain

$$\left|c(\nu;\omega)\right| < 8\sqrt{2}\,C_2\begin{pmatrix} 2l \\ l-|p| \end{pmatrix}^{-1}\frac{|\Gamma(l+1+\nu)|}{(2\pi)^{l+\operatorname{Re}\nu}}\,{r_2}^{1-l}|\omega|^{-l-\operatorname{Re}\nu+1}e^{4\pi|\omega|r_2}. \tag{5.19}$$

From (5.14) we obtain the asymptotic estimate for $j_m^\omega(\nu;r)$, which for r large enough yields

$$\left|j_m^\omega(\nu;r)\right| < \begin{pmatrix} 2l \\ l-|p| \end{pmatrix}\frac{(2\pi)^{l+1+\operatorname{Re}\nu}}{|\Gamma(l+1+\nu)|}\,r^{\,l+\frac{1}{2}}|\omega|^{\operatorname{Re}\nu+l-\frac{1}{2}}e^{-4\pi|\omega|r}. \tag{5.20}$$

For $r > 2r_2$, inequalities (5.20) and (5.19) give

$$\begin{aligned}&\sum_{0\neq\omega\in\Lambda'_\kappa}\left|c(\nu;\omega)\right|\left|j_m^\omega(\nu;r)\right| < \\ &\qquad < 16\pi\sqrt{2}\,C_2\,{r_2}^{1-l}\,r^{\,l+\frac{1}{2}}\sum_{0\neq\omega\in\Lambda'_\kappa}|\omega|^{\frac{1}{2}}e^{-4\pi|\omega|(r-r_2)} \\ &\qquad \ll r^{\,l+\frac{1}{2}}\sum_{0\neq\omega\in\Lambda'_\kappa}|\omega|^{\frac{1}{2}}e^{-2\pi|\omega|r} \ll r^{\,l+\frac{1}{2}}e^{-2\pi\omega_0 r}, \qquad \text{for all} \quad |m| \leqslant l.\end{aligned} \tag{5.21}$$

Expansion (5.11) and estimate (5.21) above imply, for all $\nu \in \mathcal{N}$,

$$\begin{aligned}\left|f(\nu; g_\kappa n\mathrm{a}[r]k)\right| &< \sum_{\omega\in\Lambda'_\kappa}\sum_{|m|\leqslant l}\left|c(\nu;\omega)\right|\left|j_m^\omega(\nu;r)\right|\left|\Phi_{m,q}^l(k)\right| \\ &\ll \sum_{0\neq\omega\in\Lambda'_\kappa}\left|c(\nu;\omega)\right|\left|j_m^\omega(\nu;r)\right| \ll_{\mathcal{N}} r^{\,l+\frac{1}{2}}e^{-2\pi\omega_0 r} \qquad \text{as} \quad r \to \infty,\end{aligned}$$

which proves (5.10). ∎

Chapter 6

Kloosterman sums

The spectral sum formula connects the Fourier coefficients of cuspidal automorphic representations with Kloosterman sums.

6.1 Definition and properties

Definition 6.1.1. *Let F be an imaginary number field, and $\mathcal{O}$ its ring of integers. For $c \in \mathcal{O} \setminus \{0\}$, $\omega, \omega' \in \mathcal{O}' \setminus \{0\}$, and a character χ of $(\mathcal{O}/(c))^*$, we define a Kloosterman sum associated to F by*

$$S_\chi(\omega, \omega'; c) = \sum_{d \bmod (c)}^{*} \chi(d)^{-1}\, e^{2\pi i \mathrm{Tr}((d\omega + \tilde{d}\omega')/c)}, \tag{6.1}$$

Here $\sum^*$ means that d runs over representatives of $(\mathcal{O}/(c))^*$ and $d\tilde{d} \equiv 1 \bmod (c)$.

Definition 6.1.1 generalizes the classical Kloosterman sum over $\mathbb{Q}$:

$$S(m, n; c) = \sum_{\substack{d \bmod c \\ d\tilde{d} \equiv 1 \bmod c}} e^{2\pi i (dm + \tilde{d}n)/c}, \qquad \text{for} \quad m, n, c \in \mathbb{Z} \backslash \{0\}. \tag{6.2}$$

Kloosterman sums over $\mathbb{Q}$ were introduced by Kloosterman, [21], in the study of quadratic forms.

Using the symmetries $d \mapsto -d$ and $d \mapsto \tilde{d}$ respectively, we obtain some simple properties for a Kloosterman sum (6.1):

$$S_\chi(\omega, \omega'; -c) = \chi(-1) S_\chi(\omega, \omega'; c), \tag{6.3}$$

$$S_\chi(\omega', \omega; c) = S_{\chi^{-1}}(\omega, \omega'; c), \tag{6.4}$$

$$\overline{S_\chi(\omega, \omega'; c)} = \chi(-1) S_{\chi^{-1}}(\omega, \omega'; c) = \chi(-1) S_\chi(\omega', \omega; c). \tag{6.5}$$

6.2 Estimates

We have the following trivial estimate for the Kloosterman sums

$$|S_\chi(\omega,\omega';c)| \leqslant |N(c)|, \tag{6.6}$$

where $N(c) = |\mathcal{O}/(c)| = |c|^2$.

To derive the sum formula, we do not need an estimate for Kloosterman sums. However, it will turn out in §11.2 that a nontrivial estimate for a Kloosterman sum allows us to enlarge the class of test functions.

The classical Kloosterman sum (6.2) satisfies the Salié-Weil type estimate

$$S(m,n;c) \ll_{m,n,\delta} c^{\frac{1}{2}+\delta}$$

for each $\delta > 0$. See [39], [44], [12]. In [4], §5, the authors worked out the generalization to Kloosterman sums $S_1(\omega,\omega';c)$ over an arbitrary algebraic number field. Following closely the approach of Estermann and using the results in [4], we shall prove similar estimate for $S_\chi(\omega,\omega';c)$, with χ not necessarily trivial.

For each $c \in I\backslash\{0\}$, we have $(c) = \prod_j \mathfrak{p}_j^{m_j}$, with $\mathfrak{p}_j$ running through different prime ideals of $\mathcal{O}$. Let $I_c = \prod_{j:\mathfrak{p}_j|I} \mathfrak{p}_j^{m_j}$. Then $(c) = I_c \cdot ((c)/I_c)$ is a decomposition of (c) in relatively prime ideals.

Proposition 6.2.1. *Let $\omega,\omega' \in \mathcal{O}'\backslash\{0\}$, and let χ be a character of $(\mathcal{O}/I)^*$. For $c \in I\backslash\{0\}$ and $\delta > 0$ we have*

$$S_\chi(\omega,\omega';c) = O\left(N(I_c)|N((c)/I_c)|^{\frac{1}{2}+\delta}\right). \tag{6.7}$$

Proof. For an ideal $J \subset I$ and characters φ and ψ of the additive group $\mathcal{O}/J$, there is the generalized Kloosterman sum

$$S_\chi(\varphi,\psi;J) = \sum_{d\in(\mathcal{O}/J)^*} \chi(d)^{-1}\varphi(d)\psi(\tilde{d}), \tag{6.8}$$

with $d\tilde{d} \equiv 1 \bmod J$.

If $J = \prod_{j=1}^r \mathfrak{p}_j^{m_j}$ is expanded as a product of prime ideals $\mathfrak{p}_j$ in $\mathcal{O}$, then $\mathcal{O}/J \equiv \prod_{j=1}^r \mathcal{O}/\mathfrak{p}_j^{m_j}$, and correspondingly $\varphi = \otimes_{j=1}^r \varphi_j$, $\psi = \otimes_{j=1}^r \psi_j$, and $\chi = \otimes_{j=1}^r \chi_j$, where φ_j and ψ_j are characters of the additive group $\mathcal{O}/\mathfrak{p}_j^{m_j}$, and χ_j are characters of $(\mathcal{O}/\mathfrak{p}_j^{m_j})^*$. The assumption that χ is a character modulo I implies that $\chi_j = 1$ if $\mathfrak{p}_j$ does not divide I. Thus we have the multiplicative property

$$S_\chi(\varphi,\psi;J) = \prod_{j=1}^r S_{\chi_j}(\varphi_j,\psi_j;\mathfrak{p}_j^{m_j}). \tag{6.9}$$

We estimate the factors indexed by j such that $\mathfrak{p}_j|I$ with the trivial bound (6.6). For the other factors, Proposition 9 in [4] gives:

$$|S_1(\varphi_j, \psi_j; \mathfrak{p}_j^{m_j})| \leqslant c(\mathfrak{p}_j) N(\mathfrak{p}_j)^{m_j - N_j/2} \tag{6.10}$$

with N_j minimal such that $\mathfrak{p}_j^{m_j}$ is contained in both $\ker(\varphi_j)$ and $\ker(\psi_j)$. The constants $c(\mathfrak{p}_j) = 2$ if $2 \nmid N(\mathfrak{p}_j)$ and $c(\mathfrak{p}_j) = 2N(\mathfrak{p}_j)^{v_j(2)+\frac{1}{2}}$ if $v_j(2) \geqslant 1$, where v_j is the valuation at the prime $\mathfrak{p}_j$.

We now take $J = (c)$, $\varphi(d) = e^{2\pi i \operatorname{Tr}(\omega d)}$, $\psi(d) = e^{2\pi i \operatorname{Tr}(\omega' d)}$, and obtain $S_\chi(\varphi, \psi; J) = S_\chi(\omega, \omega'; c)$. In [4], §5.2, it is shown that

$$N_j = \max\{0, -v_j(\omega) - d_j + v_j(c), -v_j(\omega') - d_j + v_j(c)\},$$

where d_j is the order at $\mathfrak{p}_j$ of the different of $\mathcal{O}$. This gives

$$\begin{aligned} |S_\chi(\omega, \omega'; c)| &\leqslant \prod_{j:\mathfrak{p}_j|I} N(\mathfrak{p}_j)^{m_j} \cdot \prod_{j:\mathfrak{p}_j\nmid I} c(\mathfrak{p}_j) N(\mathfrak{p}_j)^{v_j(c) - N_j/2} \leqslant \\ &\leqslant |d_F| \prod_{j:\mathfrak{p}_j|I} N(\mathfrak{p}_j)^{m_j} \cdot \prod_{j:\mathfrak{p}_j\nmid I} c(\mathfrak{p}_j) N(\mathfrak{p}_j)^{\frac{1}{2}v_j(c) + \frac{1}{2}\max\{v_j(\omega), v_j(\omega')\}} \ll \\ &\ll_{F,I,\omega,\omega'} N(I_c)|N((c)/I_c)|^{1/2} 2^r, \end{aligned} \tag{6.11}$$

where r is the number of prime ideals dividing (c). By the last remark (iv) in §5.2, [4], we have $2^r = O(|N(c)|^\delta)$ for each $\delta > 0$, which proves the proposition. ■

A consequence of Proposition 6.2.1 is the following non-trivial estimate for a Kloosterman sum:

$$S_\chi(\omega, \omega'; c) \ll_{\omega,\omega',\epsilon} |N(c)|^{\frac{1}{2}+\epsilon}, \tag{6.12}$$

for each $\epsilon > 0$.

Chapter 7

Poincaré series

7.1 Definition and properties

Let χ denote a character of $(\mathcal{O}/I)^*$ and also the corresponding unitary character on Γ of the form (1.14). Let $L^2(\Gamma\backslash G, \chi)$, be the space of all square-integrable χ-automorphic functions on G.

Definition 7.1.1. *Let $f \in C^\infty(N\backslash G, \omega)$, $\omega \in \mathcal{O}'$. The Poincaré series $P_\chi f$ generated by f, is defined by*

$$P_\chi f(g) = \frac{1}{[\Gamma_P : \Gamma_N]} \sum_{\gamma \in \Gamma_N \backslash \Gamma} \chi(\gamma)^{-1} f(\gamma g). \tag{7.1}$$

Since f is a Γ_N-invariant function, the series $P_\chi f$ has χ-automorphic behavior with respect to Γ, provided that the sum converges absolutely.

The convergence is determined by the behavior of the function $f(n\mathrm{a}[r]k)$ as $r \downarrow 0$. Let us impose the following condition:

$$f(n\mathrm{a}[r]k) \ll r^{1+\sigma_0}\ , \qquad \text{as } r \downarrow 0 \quad \text{for some } \sigma_0 > 0, \tag{7.2}$$

uniformly for $n \in N$, $k \in K$.

On the basis of the Bruhat decomposition $\Gamma \cap G = \Gamma \cap (P \sqcup P\mathrm{w}N)$, we can decompose the series $P_\chi f$ into two sub-sums

$$P_\chi f(g) = \Sigma_1(g) + \Sigma_2(g). \tag{7.3}$$

The first sum $\Sigma_1(g) = \frac{1}{[\Gamma_P:\Gamma_N]} \sum_{\gamma \in \Gamma_N \backslash \Gamma_P} \chi(\gamma)^{-1} f(\gamma g)$ is finite, hence convergent, and the second sum $\Sigma_2(g) = \frac{1}{[\Gamma_P:\Gamma_N]} \sum_{\gamma \in \Gamma_N \backslash (\Gamma \cap P\mathrm{w}N)} \chi(\gamma)^{-1} f(\gamma g)$ is estimated by the corresponding sum in the Eisenstein series $E_{0,0}(\sigma_0, 0; 1)$. Specifically, the elements γg that appear in the sum belong to a region $\mathcal{G}(r_0(g)) := N\{\mathrm{a}[r] : r \leqslant r_0(g)\}K$. In this region $|f(\gamma g)| \ll r^{1+\sigma_0}$, and thus

$$|\Sigma_2(g)| \leqslant \frac{1}{[\Gamma_P : \Gamma_N]} \sum_{\gamma \in \Gamma_N \backslash (\Gamma \cap P\mathrm{w}N)} |f(\gamma g)| = O(r^{1-\sigma_0}) \tag{7.4}$$

is convergent. Therefore the Poincaré series $P_\chi f$ converges absolutely if the function f satisfies (7.2) with $\sigma_0 > 1$.

We now examine the square integrability of the function $P_\chi f$ on $\Gamma\backslash G$. We assume that f has polynomial decay near infinity, that is,

$$f(n\mathrm{a}[r]k) \ll r^{1-\sigma_\infty}\ , \quad \text{as } r \to \infty \quad \text{for some } \sigma_\infty > 0. \tag{7.5}$$

We saw in §1.3 that the fundamental domain of $\Gamma\backslash\mathbb{H}^3$ is of the form (1.23), see also Proposition 2.3.9 in [11]. Near the cusp ∞, we estimate the two sums in (7.3) as follows:

$$\begin{aligned} |\Sigma_1(n\mathrm{a}[r]k)| &\leqslant \frac{1}{[\Gamma_P : \Gamma_N]} \sum_{\varepsilon \in \mathcal{O}^*} |f(\mathrm{h}[\varepsilon]n\mathrm{a}[r]k)| \\ &= \frac{1}{[\Gamma_P : \Gamma_N]} \sum_{\varepsilon \in \mathcal{O}^*} |f(n'\mathrm{a}[r]k')| \overset{(7.5)}{=} O(r^{1-\sigma_\infty}), \qquad \text{as} \quad r \to \infty, \end{aligned} \tag{7.6}$$

which is square-integrable on the interval $[1, \infty)$ with respect to the measure $r^{-3}dr$. If $r \geqslant c$, all $\gamma n\mathrm{a}[r]k$ that appear in the sum $\Sigma_2(n\mathrm{a}[r]k)$ belong to a region $\mathcal{G}(r_c)$. Because of (7.5), we have in this region $|f(\gamma n\mathrm{a}[r]k)| \ll {r_c}^{1+\sigma_\infty}$, and we estimate the second sub-sum by the corresponding sum in the Eisenstein series $E_{0,0}(\sigma_\infty, 0; 1)(n\mathrm{a}[\mathrm{r}]k)$:

$$|\Sigma_2(n\mathrm{a}[r]k)| \leqslant \frac{1}{[\Gamma_P : \Gamma_N]} \sum_{\gamma \in \Gamma_N \backslash (\Gamma \cap P\mathrm{w}N)} |f(\gamma n\mathrm{a}[r]k)| = O(r^{1-\sigma_\infty}), \tag{7.7}$$

which is square-integrable on $[1, \infty)$. The estimates (7.6) and (7.7) show that $P_\chi f(g)$ is square-integrable on the cusp sector at ∞.

Near a cusp $\kappa \neq \infty$, $\Sigma_1(g_\kappa g)$ disappears since $\Gamma g_\kappa \cap P = \emptyset$. Therefore, for $g = n\mathrm{a}[r]k$ with $r \geqslant r'$, all $\gamma g_\kappa g$ belong to a region $\mathcal{G}(r(r', \kappa))$, and we estimate the Poincaré series $P_\chi f(g_\kappa g)$ by the whole Eisenstein series $E_{0,0}(\sigma_\infty, 0; 1)(g_\kappa g)$:

$$|P_\chi f(g_\kappa g)| \leqslant \frac{1}{[\Gamma_P : \Gamma_N]} \sum_{\gamma \in \Gamma_N \backslash \Gamma} |f(\gamma g_\kappa g)| = O(r^{1-\sigma_\infty}), \qquad \text{as} \quad r \to \infty.$$

Hence $P_\chi f$ is also square-integrable on the cusp sectors at $\kappa \neq \infty$. We summarize the discussion above into

Proposition 7.1.2. *If the function $f \in C^\infty(N\backslash G, \omega)$, $\omega \in \mathcal{O}'$, satisfies the following growth conditions* *(i)* $f(\mathrm{n}a[r]k) \ll r^{1+\sigma_0}$, *as* $r \downarrow 0$ *for some* $\sigma_0 > 1$,

(ii) $f(\mathrm{n}a[r]k) \ll r^{1-\sigma_\infty}$, *as* $r \to \infty$ *for some* $\sigma_\infty > 0$,

uniformly for $n \in N$, $k \in K$, *then* $P_\chi f \in L^2(\Gamma\backslash G, \chi)$.

7.2 Fourier expansion

Let $f \in C^\infty(N\backslash G, \omega)$ with some $\omega \in \mathcal{O}'$, satisfies conditions (i) and (ii) in Proposition 7.1.2. Since the function $z \mapsto P_\chi f(\mathrm{n}[z]g)$ is periodic on $\mathbb{C}$ for the lattice $\mathcal{O}$, the Poincaré series $P_\chi f$ has a Fourier expansion at the cusp ∞

$$P_\chi f(g) = \sum_{\omega' \in \mathcal{O}'} F_{\omega'} P_\chi f(g), \tag{7.8}$$

with a Fourier term given by

$$F_{\omega'} P_\chi f(g) = \frac{2}{\sqrt{|d_F|}} \int_{\Gamma_N\backslash N} \chi_{\omega'}(n)^{-1} P_\chi f(ng)dn. \tag{7.9}$$

We want to obtain an explicit expression for the Fourier term of order ω'. Toward this goal, we substitute (7.1) into (7.9), and use the absolute convergence of the series to interchange the order of summation and integration:

$$F_{\omega'} P_\chi f(g) = \frac{2}{[\Gamma_P : \Gamma_N]\sqrt{|d_F|}} \sum_{\gamma \in \Gamma_N\backslash\Gamma} \chi(\gamma)^{-1} \int_{\Gamma_N\backslash N} \chi_{\omega'}(n)^{-1} f(\gamma ng)dn. \tag{7.10}$$

From the Bruhat decomposition $G = P \sqcup P\mathrm{w}N$, we have

$$\Gamma = \Gamma_P \sqcup \left(\bigsqcup_{\substack{c \in \mathcal{O} \\ c \neq 0}} \bigsqcup_{\substack{d \bmod (c) \\ \langle c,d\rangle = \mathcal{O}}} \Gamma_N \begin{pmatrix} * & * \\ c & d \end{pmatrix} \Gamma_N \right) \tag{7.11}$$

where we use the notation $\langle c, d\rangle = c \cdot \mathcal{O} + d \cdot \mathcal{O}$. The sum in (7.10) splits into two sub-sums:

$$F_{\omega'} P_\chi f(g) = \frac{2}{[\Gamma_P : \Gamma_N]\sqrt{|d_F|}} \left\{ \sum_{\gamma \in \Gamma_N\backslash\Gamma_P} + \sum_{\gamma \in \Gamma_N\backslash(\Gamma \cap P\mathrm{w}N)} \right\}. \tag{7.12}$$

Using the fact that Γ_P has a structure of a semi-direct product, we take elements $\mathrm{h}[1/\varepsilon]$ with $\varepsilon \in \mathcal{O}^*$ as representatives for $\Gamma_N\backslash\Gamma_P$, and the first sub-sum then equals

$$\sum_{\varepsilon\in\mathcal{O}^*}\chi(\varepsilon)^{-1}\int_{\Gamma_N\backslash N}\chi_{\omega'}(n)^{-1}f(\mathrm{h}[1/\varepsilon]ng)dn =$$

$$= \sum_{\varepsilon\in\mathcal{O}^*}\chi(\varepsilon)^{-1}\int_{\mathbb{C}\,\mathrm{mod}\,\mathcal{O}} e^{-2\pi i\mathrm{Tr}(\omega' z)}f(\mathrm{n}[z\varepsilon^{-2}]\mathrm{h}[1/\varepsilon]g)d_+z$$

$$= \sum_{\varepsilon\in\mathcal{O}^*}\chi(\varepsilon)^{-1}f(\mathrm{h}[1/\varepsilon]g)\int_{\mathbb{C}\,\mathrm{mod}\,\mathcal{O}} e^{2\pi i\mathrm{Tr}\left(z\omega\varepsilon^{-2}-z\omega'\right)}d_+z$$

$$= \sum_{\varepsilon\in\mathcal{O}^*}\chi(\varepsilon)^{-1}\delta_{\varepsilon^2\omega',\omega}l_{1/\varepsilon}f(g)\cdot\mathrm{vol}(\mathbb{C}\,\mathrm{mod}\,\mathcal{O})$$

$$= \frac{\sqrt{|d_F|}}{2}\sum_{\varepsilon\in\mathcal{O}^*}\delta_{\varepsilon^2\omega',\omega}\,\chi(\varepsilon)^{-1}l_{1/\varepsilon}f(g), \tag{7.13}$$

where l_t is the left translation given by (4.6).

Let us denote the big cell in the Bruhat decomposition by $C = P\mathrm{w}N$. The second sub-sum is equal to

$$\sum_{\gamma\in\Gamma_N\backslash(\Gamma\cap C)}\chi(\gamma)^{-1}\int_{\Gamma_N\backslash N}\chi_{\omega'}(n)^{-1}f(\gamma ng)dn =$$

$$= \sum_{\gamma\in\Gamma_N\backslash(\Gamma\cap C)/\Gamma_N}\sum_{\delta\in\Gamma_N}\chi(\gamma\delta)^{-1}\int_{\Gamma_N\backslash N}\chi_{\omega'}(n)^{-1}f(\gamma\delta ng)dn$$

$$= \sum_{\gamma\in\Gamma_N\backslash(\Gamma\cap C)/\Gamma_N}\chi(\gamma)^{-1}\sum_{\delta\in\Gamma_N}\int_{\Gamma_N\backslash N}\chi_{\omega'}(\delta n)^{-1}f(\gamma\delta ng)dn$$

$$= \sum_{\gamma\in\Gamma_N\backslash(\Gamma\cap C)/\Gamma_N}\chi(\gamma)^{-1}\int_{N}\chi_{\omega'}(n)^{-1}f(\gamma ng)dn. \tag{7.14}$$

Using (7.11), for $\gamma\in\Gamma_N\backslash\left(\Gamma\cap C\right)/\Gamma_N$, we get $\gamma = \mathrm{n}[a/c]\mathrm{h}[1/c]\mathrm{wn}[d/c]$, where c runs over the non-zero elements in I, d runs over representatives of $(\mathcal{O}/(c))^*$, and a is such that $ad\equiv 1\,\mathrm{mod}\,(c)$. As χ is a character of $(\mathcal{O}/(c))^*$, we continue with (7.14)

$$= \sum_{c\in I}{}' \sum_{d\,\mathrm{mod}\,(c)}{}^{*}\chi(d)^{-1}\int_N\chi_{\omega'}(n)^{-1}f(\mathrm{n}[a/c]\mathrm{h}[1/c]\mathrm{wn}[d/c]ng)dn$$

$$= \sum_{c\in I}{}' \sum_{d\,\mathrm{mod}\,(c)}{}^{*}\chi(d)^{-1}\int_{\mathbb{C}} e^{-2\pi i\mathrm{Tr}(\omega' z)+2\pi i\mathrm{Tr}(\omega a/c)}f(\mathrm{h}[1/c]\mathrm{wn}[d/c+z]g)d_+z$$

(change: $z\mapsto z-d/c$)

$$= \sum_{c\in I}{}' \sum_{d\,\mathrm{mod}\,(c)}{}^{*}\chi(d)^{-1}e^{2\pi i\,\mathrm{Tr}(a\omega/c)}\int_{\mathbb{C}} e^{-2\pi i\mathrm{Tr}\left(\omega'(z-d/c)\right)}f(\mathrm{h}[1/c]\mathrm{wn}[z]g)d_+z$$

$$= \sum_{c \in I}{}' \sum_{d \bmod (c)}{}^{*} \chi(d)^{-1} \, e^{2\pi i \operatorname{Tr}\left((d\omega'+a\omega)/c\right)} \int_N \chi_{\omega'}(n)^{-1} f(\mathrm{h}[1/c]\mathrm{w}ng)dn$$
$$= \sum_{c \in I}{}' S_\chi(\omega', \omega; c) \mathbf{J}_{\omega'} l_{1/c} f(g). \tag{7.15}$$

Here $S_\chi(\omega', \omega; c)$ is the Kloosterman sum defined by (6.1), and $\mathbf{J}_{\omega'} f$ is the Jacquet integral described in §4.1.

Finally, substituting (7.13) and (7.15) into (7.12), we obtain the following explicit form of the Fourier term of order ω' of the Poincaré series $P_\chi f$:

$$F_{\omega'} P_\chi f = \frac{1}{[\Gamma_P : \Gamma_N]} \sum_{\varepsilon \in \mathcal{O}^*} \delta_{\varepsilon^2 \omega', \omega} \, \chi(\varepsilon)^{-1} l_{1/\varepsilon} f + \\ + \frac{2}{[\Gamma_P : \Gamma_N]\sqrt{|d_F|}} \sum_{c \in I}{}' S_\chi(\omega', \omega; c) \mathbf{J}_{\omega'} l_{1/c} f. \tag{7.16}$$

7.3 Scalar product of Poincaré series

Under the conditions of Proposition 7.1.2, the Poincaré series are square-integrable functions on $\Gamma\backslash G$. So, it makes sense to consider the inner product of such Poincaré series with a square-integrable function on $\Gamma\backslash G$. We may also consider the inner product of such Poincaré series with a smooth χ-automorphic function on $\Gamma\backslash G$ that is not square-integrable, provided that its product with the function that generates the Poincaré series is integrable on $\Gamma\backslash G$.

Let $\omega \neq 0$. Denote by $P_{l,q}(N\backslash G, \omega)$ the space of functions $f \in C^\infty(N\backslash G, \omega)$ that have type (l, q) and satisfy the following growth conditions:

$$f(n\mathrm{a}[r]k) = \begin{cases} O(r^{1+\sigma_0}) & \text{as} \quad r \downarrow 0 \quad \text{for some } \sigma_0 > 0, \\ O(r^{1-\sigma_\infty}) & \text{as} \quad r \to \infty \quad \text{for some } \sigma_\infty > 0. \end{cases} \tag{7.17}$$

The numbers σ_0 and σ_∞ may depend on the function f.

The following lemma is a well known result for absolutely convergent Poincaré series:

Lemma 7.3.1. *Let $\omega \in \mathcal{O}' \setminus \{0\}$ and $l, q \in \frac{1}{2}\mathbb{Z}$, $l \equiv q$ (mod 1), $|q| \leqslant l$. Let $f \in P_{l,q}(N\backslash G, \omega)$ satisfy conditions (7.17) with $\sigma_0 > 1$, and suppose that $\phi \in C^\infty(\Gamma\backslash G, \chi; l, q)$ is such that $P_\chi f \cdot \phi \in L^1(\Gamma\backslash G)$. Then*

$$\langle P_\chi f, \phi \rangle_{\Gamma\backslash G} = \frac{\sqrt{|d_F|}}{2[\Gamma_P : \Gamma_N]} \langle f, F_\omega \phi \rangle_{N\backslash G}.$$

Proof. A simple computation yields

$$[\Gamma_P : \Gamma_N] \langle P_\chi f, \phi \rangle_{\Gamma\backslash G} = [\Gamma_P : \Gamma_N] \int_{\Gamma\backslash G} P_\chi f(g) \overline{\phi(g)} dg$$

$$= \int_{\Gamma\backslash G} \sum_{\gamma\in\Gamma_N\backslash\Gamma} \chi(\gamma)^{-1} f(\gamma g)\overline{\phi(g)}dg = \sum_{\gamma\in\Gamma_N\backslash\Gamma} \int_{\Gamma\backslash G} f(\gamma g)\overline{\phi(\gamma g)}dg$$
$$= \int_{\Gamma_N\backslash G} f(g)\overline{\phi(g)}dg = \int_{N\backslash G}\int_{\Gamma_N\backslash N} f(ng)\overline{\phi(ng)}dn\, d\dot{g}$$
$$= \int_{N\backslash G} f(g) \int_{\Gamma_N\backslash N} \chi_\omega(n)\overline{\phi(ng)}dn\, d\dot{g}$$
$$= \mathrm{vol}(\Gamma_N\backslash N) \int_{N\backslash G} f(g)\overline{F_\omega\phi(g)}d\dot{g} = \frac{\sqrt{|d_F|}}{2}\langle f, F_\omega \phi\rangle_{N\backslash G}\ . \qquad \blacksquare$$

Using Lemma 7.3.1 and the expression (7.16) for the Fourier coefficient of a Poincaré series, we can obtain an explicit expression for the scalar product of two square-integrable Poincaré series, which explains how and why Kloosterman sums appear.

Lemma 7.3.2. *Let $\omega_1, \omega_2 \in \mathcal{O}' \setminus \{0\}$ and $f_i \in P_{l,q}(N\backslash G, \omega_i)$, for $i = 1, 2$, be two functions that satisfy the conditions (7.17) with $\sigma_{0,i} > 1$.*

We have the following expression for the scalar product of the square-integrable Poincaré series $P_\chi f_1$ and $P_\chi f_2$:

$$\langle P_\chi f_1, P_\chi f_2\rangle_{\Gamma\backslash G} = \frac{\sqrt{|d_F|}}{2[\Gamma_P:\Gamma_N]^2} \sum_{\varepsilon\in\mathcal{O}^*} \delta_{\varepsilon^2\omega_1,\omega_2}\,\chi(\varepsilon)\langle f_1, l_{1/\varepsilon} f_2\rangle_{N\backslash G}$$
$$+ \frac{\chi(-1)}{[\Gamma_P:\Gamma_N]^2} {\sum_{c\in I}}' S_\chi(\omega_2,\omega_1;c)\langle f_1, \mathbf{J}_{\omega_1} l_{1/c} f_2\rangle_{N\backslash G}. \qquad (7.18)$$

Proof. First, Lemma 7.3.1 gives the relation between the Poincaré series and Fourier terms, and then (7.16) gives the explicit result:

$$\langle P_\chi f_1, P_\chi f_2\rangle_{\Gamma\backslash G} = \frac{\sqrt{|d_F|}}{2[\Gamma_P:\Gamma_N]}\langle f_1, F_{\omega_1} P_\chi f_2\rangle_{N\backslash G}$$
$$= \frac{\sqrt{|d_F|}}{2[\Gamma_P:\Gamma_N]^2} \sum_{\varepsilon\in\mathcal{O}^*} \delta_{\varepsilon^2\omega_1,\omega_2}\,\langle f_1, \chi(\varepsilon)^{-1} l_{1/\varepsilon} f_2\rangle_{N\backslash G}$$
$$+ \frac{1}{[\Gamma_P:\Gamma_N]^2} {\sum_{c\in I}}' \langle f_1, S_\chi(\omega_1,\omega_2;c)\mathbf{J}_{\omega_1} l_{1/c} f_2\rangle_{N\backslash G}$$
$$= \frac{\sqrt{|d_F|}}{2[\Gamma_P:\Gamma_N]^2} \sum_{\varepsilon\in\mathcal{O}^*} \delta_{\varepsilon^2\omega_1,\omega_2}\,\chi(\varepsilon)\langle f_1, l_{1/\varepsilon} f_2\rangle_{N\backslash G}$$
$$+ \frac{\chi(-1)}{[\Gamma_P:\Gamma_N]^2} {\sum_{c\in I}}' S_\chi(\omega_2,\omega_1;c)\langle f_1, \mathbf{J}_{\omega_1} l_{1/c} f_2\rangle_{N\backslash G}.$$

In the last line, we used the property (6.5) of a Kloosterman sum.

To see the convergence of the inner products in (7.18) it suffices to note the following: If we replace the functions f_1, f_2 by $|f_1|$, $|f_2|$, and work with $\chi = 1$ and $\omega_1 = \omega_2 = 0$, then the conditions in Proposition 7.1.2 are satisfied by $|f_1|, |f_2|$, the Poincaré series $P_1|f_1|$ and $P_1|f_2|$ are square-integrable over $\Gamma\backslash G$, so their scalar product $\langle P_1|f_1|, P_1|f_2|\rangle_{\Gamma\backslash G}$ is finite. By Fubini's theorem, the scalar products $\langle |f_1|, l_{1/\varepsilon}|f_2|\rangle_{N\backslash G}$ and $\langle |f_1|, \mathbf{J}_0 l_{1/c}|f_2|\rangle_{N\backslash G}$ are finite as well, and they provide a majorant for the right side of (7.18). This implies the absolute convergence without the necessity of providing any estimates for the Kloosterman sums. ■

Chapter 8

Spectral decomposition of the space $L^2(\Gamma\backslash G, \chi)$

Let $L^2(\Gamma\backslash G, \chi)$ be the Hilbert space of χ-automorphic functions with respect to the subgroup Γ which are square-integrable on $\Gamma\backslash G$ with respect to the measure induced by dg. We have the following orthogonal decomposition:

$$L^2(\Gamma\backslash G, \chi) = \mathbb{C} \oplus L^{2,\mathrm{cusp}}(\Gamma\backslash G, \chi) \oplus L^{2,\mathrm{cont}}(\Gamma\backslash G, \chi). \tag{8.1}$$

Here $\mathbb{C}$ stands for the one-dimensional subspace of constant functions on $\Gamma\backslash G$, $L^{2,\mathrm{cusp}}(\Gamma\backslash G, \chi)$ for the cuspidal subspace, and $L^{2,\mathrm{cont}}(\Gamma\backslash G, \chi)$ for the orthogonal complement of $\mathbb{C} \oplus L^{2,\mathrm{cusp}}(\Gamma\backslash G, \chi)$. The spectral decomposition (8.1) is a consequence of the general theory of Eisenstein series due to Langlands [28]. In our arithmetical situation, it can be shown that the residues of the Eisenstein series contribute only the constant functions in the spectral decomposition.

The closed subspace $L^{2,\mathrm{cont}}(\Gamma\backslash G, \chi)$ is generated by integrals of Eisenstein series.

The cuspidal subspace $L^{2,\mathrm{cusp}}(\Gamma\backslash G, \chi)$ is spanned by the cusp forms, functions on $L^2(\Gamma\backslash G, \chi)$ with vanishing constant terms in their Fourier expansions at all cusps. It can be decomposed into at most countably many cuspidal subspaces V irreducible with respect to the action of G:

$$L^{2,\mathrm{cusp}}(\Gamma\backslash G, \chi) = \overline{\bigoplus V}. \tag{8.2}$$

The two Casimir elements $\Omega_\pm$ given by (2.16) and (2.17) act as multiplication by a constant in each V. There are numbers ν_V, p_V, specified in (3.21), such that

$$\Omega_\pm|_V = \Upsilon_{\nu_V, p_V}(\Omega_\pm) \cdot 1,$$

with the character Υ_{ν_V, p_V} on $\mathfrak{g}$ as in Lemma 3.2.2. Therefore, each subspace V is characterized by the spectral parameter (ν_V, p_V).

According to the right action of the group K, the space V decomposes into K-irreducible subspaces

$$V = \overline{\bigoplus_{l \geqslant |p_V|, |q| \leqslant l} V_{l,q}}, \qquad V_{l,q} = \mathbb{C}\, T_V \varphi_{l,q}(\nu_V, p_V), \tag{8.3}$$

where $V_{l,q}$ has dimension one. It consists of cuspidal χ-automorphic forms of type (l, q) and spectral parameter (ν_V, p_V). One can choose a unitary isomorphism of $\mathfrak{g}$-modules $T_V : H(\nu_V, p_V) \to V$. The image consists of the K-finite vectors in V, and it is dense in V. For each (l, q) in (8.3) the automorphic function $T_V \varphi_{l,q}(\nu_V, p_V)$ spans $V_{l,q}$. (The space $H(\nu_V, p_V)$ has been defined in (2.46), see also Section 2.3).

If $p_V = 0$, then V contains a one-dimensional space of K-trivial vectors in $L^{2,\mathrm{cusp}}(\Gamma\backslash G, \chi)$. The set $\{T_V \varphi_{0,0}(\nu_V, p_V) \mid p_V = 0\}$ corresponds to an orthogonal system in $L^2(\Gamma\backslash\mathbb{H}^3, \chi)$, where $\chi(-1) = 1$. If $p_V \neq 0$, then in V occur only the K-types with $l \geqslant |p_V|$. The unitarity of T_V implies that

$$\|T_V \varphi_{l,q}(\nu_V, p_V)\| = \begin{cases} \|\varphi_{l,q}(\nu_V, p_V)\|_{\mathrm{ps}} & \text{if } \nu_V \in i[0, \infty) \\ \|\varphi_{l,q}(\nu_V, 0)\|_{\mathrm{cs}} & \text{if } \nu_V \in (0, 1), \end{cases} \tag{8.4}$$

where (ν_V, p_V) are as in (3.21) and the norms $\|\cdot\|_{\mathrm{ps}}$ in the unitary principal series and $\|\cdot\|_{\mathrm{cs}}$ in the complementary series are given by (2.49) and (2.51), respectively.

We restrict the decomposition (8.1) to the subspace $L^2(\Gamma\backslash G, \chi; l, q)$ spanned by all square-integrable χ-automorphic functions of type (l, q).

Theorem 8.1. *Let $f_1, f_2 \in L^2(\Gamma\backslash G, \chi; l, q)$ be represented by bounded functions in $C^\infty(\Gamma\backslash G)$. We denote their inner product by*

$$\langle f_1, f_2 \rangle_{\Gamma\backslash G} = \int_{\Gamma\backslash G} f_1(g)\overline{f_2(g)}dg.$$

Then, the inner products $\langle f_j, E^\kappa_{l,q}(\nu, p; \chi)\rangle_{\Gamma\backslash G}$ for $j = 1, 2$, are defined and square-integrable as functions of $\nu \in i\mathbb{R}$, and the following equality holds

$$\begin{aligned}\langle f_1, f_2 \rangle_{\Gamma\backslash G} &= \delta_{l,0} \frac{1}{\mathrm{vol}(\Gamma\backslash G)} \langle f_1, 1\rangle_{\Gamma\backslash G} \langle 1, f_2\rangle_{\Gamma\backslash G} \\ &+ \sum_V \frac{1}{\|\Phi^l_{p_V,q}\|^2_K} \langle f_1, T_V\varphi_{l,q}(\nu_V, p_V)\rangle_{\Gamma\backslash G} \langle T_V\varphi_{l,q}(\nu_V, p_V), f_2\rangle_{\Gamma\backslash G}\end{aligned}$$

$$+\sum_{V'}\frac{\Gamma(l+1+\nu_{V'})}{\Gamma(l+1-\nu_{V'})}\frac{1}{\|\Phi^l_{0,q}\|^2_K}\langle f_1, T_{V'}\varphi_{l,q}(\nu_{V'},0)\rangle_{\Gamma\backslash G}\cdot$$
$$\cdot\langle T_{V'}\varphi_{l,q}(\nu_{V'},0), f_2\rangle_{\Gamma\backslash G}$$
$$+\frac{1}{4\pi i}\sum_{\kappa\in\mathcal{C}_\chi}\frac{[\Gamma_\kappa:\Gamma'_\kappa]}{|\Lambda_\kappa|}\sum_{|p|\leqslant l}^{\chi}\frac{1}{\|\Phi^l_{p,q}\|^2_K}\int_{(0)}\langle f_1, E^\kappa_{l,q}(\nu,p;\chi)\rangle_{\Gamma\backslash G}\cdot$$
$$\cdot\langle E^\kappa_{l,q}(\nu,p;\chi), f_2\rangle_{\Gamma\backslash G}\,d\nu. \tag{8.5}$$

Here V, respectively V', runs through the subset of unitary principal series, respectively complementary series, in the orthogonal system chosen above, $\sum^{\chi}_{|p|\leqslant l}$ means that the sum runs through all $p\in\frac{1}{2}\mathbb{Z}$ such that $|p|\leqslant l$ with the condition $\chi(\varepsilon)=\varepsilon^{2p}$ satisfied for all $\varepsilon\in\mathcal{O}^*$, and $|\Lambda_\kappa|$ is the Euclidean area of a period parallelogram for the lattice $\Lambda_\kappa\in\mathbb{C}$ corresponding to the discrete subgroup ${g_\kappa}^{-1}\Gamma'_\kappa g_\kappa$.

Remark 6. The normalization factors $\frac{1}{4\pi i}[\Gamma_\kappa:\Gamma'_\kappa]|\Lambda_\kappa|^{-1}$ in the last term of (8.5) are taken from [11], Theorem 6.3.4, (3.12). Although the authors there work with trivial character χ and trivial K-type, their results are applicable in our case too, since the normalization factors depend only on the geometry of the part of the fundamental domain near the cusp κ, and not on the character χ or on the parameter $p\in\frac{1}{2}\mathbb{Z}$.

Chapter 9

Auxiliary test functions

In order to build the Poincaré series that will be used in the proof of the sum formula, we shall employ auxiliary test functions. More precisely, we shall consider the Lebedev transforms of a certain class of functions as building blocks for the Poincaré series.

The first section of this chapter is devoted to the introduction of the Lebedev transformation $\mathbf{L}^{\omega}_{l,q}$, its one-sided inverse $\tilde{\mathbf{L}}^{\omega}_{l,q}$, and some of their properties. In the second section we shall choose the auxiliary test functions.

9.1 Lebedev transformation

Definition 9.1.1. *Let $f \in P_{l,q}(N\backslash G,\omega)$, and $\xi \in \{0,\frac{1}{2}\}$ as in (4.27). Let $\sigma_0 > 0$ such that (7.17) is satisfied for f. We define the Lebedev transform $\mathbf{L}^{\omega}_{l,q} f(\nu,p)$ of f, with $|\operatorname{Re}\nu| \leqslant \sigma_0$ and $p \in \xi + \mathbb{Z}$, by*

$$\mathbf{L}^{\omega}_{l,q} f(\nu,p) := \frac{(-i\omega/|\omega|)^{-p+\xi}}{\pi^2 \|\Phi^l_{p,q}\|_K} (2\pi|\omega|)^{\nu} \cdot \\ \cdot \Gamma(l+1-\nu) \int_{N\backslash G} f(g) \overline{\mathbf{J}_{\omega}\varphi_{l,q}(-\bar{\nu},p)(g)} d\dot{g}, \tag{9.1}$$

where $d\dot{g}$ is the quotient measure on $N\backslash G$, which corresponds to the measure $r^{-3}drdk$ under the isomorphism $N\backslash G \cong AK$.

From (4.24) and (4.25) we see that the absolute convergence of the integral in (9.1) is no problem for $r > 1$, as K_ν is exponentially decreasing. For $r \in (0,1]$, using the estimate (4.28), we see that the contributions to the integral are: $r^{1+\sigma_0}$ from the function f, $r^{1-\sigma_0-\varepsilon}$ from the Jacquet integral, and r^{-3} from the measure. This implies the convergence of the integral on the interval $(0,1]$. The function $\nu \mapsto \mathbf{L}^{\omega}_{l,q} f(\nu,p)$ is holomorphic on the strip $|\operatorname{Re}\nu| \leqslant \sigma_0$.

Remark 7. In the case $l \in \mathbb{Z}$, the transform (9.1) is a multiple of the Lebedev transform in [9] given by (7.4). Namely, $\mathbf{L}^{\omega}_{l,q} = \sqrt{2}\mathcal{L}^{2\omega}_{l,q}$. The factor $\sqrt{2}$ comes from the different normalizations of the Haar measure on K here and in [9]; see (2.33) and [9], (3.23).

We transform the integral in (9.1) in terms of $\mathbf{J}_0 f$ as follows:

$$\begin{aligned}
&\int_{N\backslash G} f(g)\overline{\mathbf{J}_\omega \varphi_{l,q}(-\bar{\nu},p)(g)}d\dot{g} = \\
&\quad = \int_{N\backslash G}\int_N f(g)\chi_\omega(n)\overline{\varphi_{l,q}(-\bar{\nu},p)(\mathrm{w}ng)}dn\,d\dot{g} \\
&\quad = \int_G f(g)\overline{\varphi_{l,q}(-\bar{\nu},p)(\mathrm{w}g)}dg \\
&\quad \overset{(g\mapsto \mathrm{w}^{-1}g)}{=} \int_G f(\mathrm{w}^{-1}g)\overline{\varphi_{l,q}(-\bar{\nu},p)(g)}dg. \qquad (9.2)
\end{aligned}$$

Note that $f(\mathrm{w}^{-1}g) = f(\mathrm{w}g\mathrm{h}[-1]) = \chi(-1)f(\mathrm{w}g)$, since f has K-type l and the consistency relation (3.1) holds. Thus, we continue in (9.2):

$$\begin{aligned}
&= \chi(-1)\int_{N\backslash G}\int_N f(\mathrm{w}ng)\,\overline{\varphi_{l,q}(-\bar{\nu},p)(g)}dn\,d\dot{g} \\
&= \chi(-1)\int_{N\backslash G}\mathbf{J}_0 f(g)\overline{\varphi_{l,q}(-\bar{\nu},p)(g)}d\dot{g}. \qquad (9.3)
\end{aligned}$$

The behavior $f(n\mathrm{a}[r]k) = O(r^{1+\sigma_0})$ as $r \downarrow 0$ implies that $\mathbf{J}_0 f(g)$ converges absolutely, and as $r \to \infty$ it satisfies

$$\begin{aligned}
\mathbf{J}_0 f(\mathrm{a}[r]k) &= \int_{\mathbb{C}} f(\mathrm{wn}[z]\mathrm{a}[r]k)d_+z \\
&\overset{(4.3)}{=} \int_{\mathbb{C}} f\left(\mathrm{n}\left[\tfrac{-\bar{z}}{r^2+|z|^2}\right]\mathrm{a}\left[\tfrac{r}{r^2+|z|^2}\right]k'\right)d_+z \\
&\overset{(z=\rho e^{i\phi})}{=} 2\pi\int_0^\infty f\left(\mathrm{n}\left[\tfrac{-\rho e^{-i\phi}}{r^2+\rho^2}\right]\mathrm{a}\left[\tfrac{r}{r^2+\rho^2}\right]k'\right)\rho\,d\rho \\
&\ll \int_0^\infty r^{1+\sigma_0}(r^2+\rho^2)^{-1-\sigma_0}\rho\,d\rho = r^{1+\sigma_0}\frac{r^{-2\sigma_0}}{2\sigma_0} = \frac{1}{2\sigma_0}r^{1-\sigma_0}. \qquad (9.4)
\end{aligned}$$

As $r \downarrow 0$, the function $\mathbf{J}_0 f$ with f satisfying (7.17) is estimated as follows:

$$\begin{aligned}
\mathbf{J}_0 f(\mathrm{a}[r]k) &= \int_{\mathbb{C}} f(\mathrm{wn}[z]\mathrm{a}[r]k)d_+z \\
&\overset{(z\mapsto rz)}{=} r^2\int_{\mathbb{C}} f(\mathrm{wn}[rz]\mathrm{a}[r]k)d_+z \\
&\overset{(4.3)}{=} r^2\int_{\mathbb{C}} f\left(\mathrm{n}\left[\tfrac{-\bar{z}}{r(1+|z|^2)}\right]\mathrm{a}\left[\tfrac{1}{r(1+|z|^2)}\right]k'\right)d_+z
\end{aligned}$$

$$\overset{(z=\rho e^{i\phi})}{=} 2\pi r^2 \int_0^\infty f\left(\mathrm{n}\left[\tfrac{-\rho e^{-i\phi}}{r(1+\rho^2)}\right] \mathrm{a}\left[\tfrac{1}{r(1+\rho^2)}\right] k'\right) \rho\, d\rho$$
$$\ll r^2 \int_0^\infty \min\left\{(r(1+\rho^2))^{-1+\sigma_\infty}, (r(1+\rho^2))^{-1-\sigma_0}\right\} \rho\, d\rho$$
$$= r^2 \int_0^{\sqrt{1/r-1}} r^{-1+\sigma_\infty}(1+\rho^2)^{-1+\sigma_\infty} \rho\, d\rho$$
$$+ r^2 \int_{\sqrt{1/r-1}}^\infty r^{-1-\sigma_0}(1+\rho^2)^{-1-\sigma_0} \rho\, d\rho \leqslant \frac{1}{2\sigma_\infty} r + \frac{1}{2\sigma_0} r \ll r. \tag{9.5}$$

The properties of the intertwining operator $\mathbf{J}_0$ together with estimates (9.4) and (9.5) show that $\mathbf{J}_0 f$ is a continuous function satisfying the growth conditions

$$\mathbf{J}_0 f(n\mathrm{a}[r]k) = \begin{cases} O(r) & \text{as} \quad r \downarrow 0, \\ O(r^{1-\sigma_0}) & \text{as} \quad r \to \infty. \end{cases} \tag{9.6}$$

Any such function has an expansion in terms of $\Phi^l_{m,q}$, $|m| \leqslant l$. Thus, we may write

$$\mathbf{J}_0 f(n\mathrm{a}[r]k) = \sum_{|m|\leqslant l} u_m(r) \Phi^l_{m,q}(k), \tag{9.7}$$

where the functions $u_m(r)$ are continuous and satisfy

$$u_m(r) \ll \begin{cases} r & , \quad r \downarrow 0 \\ r^{1-\sigma_0} & , \quad r \to \infty \end{cases} \qquad \text{for all } m. \tag{9.8}$$

The integral (9.3) then equals

$$= \chi(-1) \int_0^\infty \sum_{|m|\leqslant l} u_m(r) r^{-2-\nu} \int_K \Phi^l_{m,q}(k) \overline{\Phi^l_{p,q}(k)} dk\, dr$$
$$= \chi(-1) \|\Phi^l_{p,q}\|^2_K \int_0^\infty u_p(r) r^{-\nu-2} dr = \chi(-1)\|\Phi^l_{p,q}\|^2_K \mathcal{M}u_p(-\nu-1). \tag{9.9}$$

Here $\mathcal{M}\phi$ is the Mellin transform of a function ϕ given by

$$\mathcal{M}\phi(s) = \int_0^\infty \phi(r) r^{s-1} dr. \tag{9.10}$$

(See e.g. [41], p.125). We recall that the inversion formula for this transformation is given by

$$\phi(r) = \frac{1}{2\pi i} \int_{\mathrm{Re}\, s = \sigma} \mathcal{M}\phi(s) r^{-s} ds. \tag{9.11}$$

Applying the Mellin transformation to the function u_p imposes a condition on $\mathrm{Re}\, s$. Namely, from the estimates (9.8) we see that $\mathcal{M}u_p(s)$ exists on the strip $-1 < \mathrm{Re}\, s < \sigma_0 - 1$. Thus, we have proved

Lemma 9.1.2. *Let $f \in P_{l,q}(N\backslash G, \omega)$ and let $\mathbf{J}_0 f$ be expanded as in (9.7). Then, for ν such that $-\sigma_0 < \operatorname{Re}\nu < 0$, the Lebedev transform (9.1) of f is linked to the Mellin transform in the following way:*

$$\mathbf{L}^{\omega}_{l,q} f(\nu,p) = \chi(-1)\pi^{-2}\|\Phi^l_{p,q}\|_K \,(-i\omega/|\omega|)^{-p+\xi} \cdot \\ \cdot \Gamma(l+1-\nu)(2\pi|\omega|)^{\nu}\, \mathcal{M}u_p(-\nu-1). \qquad (9.12)$$

Next, we shall see that the Lebedev transformation is invertible on a suitable space of functions.

Definition 9.1.3. *Let $\sigma > 0$ and $l \in \frac{1}{2}\mathbb{N}$. We denote by $\mathcal{T}^l_\sigma$ the linear space of functions η defined on the set*

$$\left\{(\nu,p) \in \mathbb{C} \times \tfrac{1}{2}\mathbb{Z} \;:\; |\operatorname{Re}\nu| \leqslant \sigma,\; p \equiv l \,(\operatorname{mod} 1),\; |p| \leqslant l\right\} \qquad (9.13)$$

such that

(i) *$\eta(\nu,p)$ is holomorphic on a neighborhood of the strip $|\operatorname{Re}\nu| \leqslant \sigma$,*

(ii) *$\eta(\nu,p) \ll e^{-\frac{\pi}{2}|\operatorname{Im}\nu|}(1+|\operatorname{Im}\nu|)^{-A}$ for any $A > 0$,*

(iii) *$\eta(\nu,p) = \eta(-\nu,-p)$.*

Theorem 9.1.4. *Let $\sigma \in \left(1, \frac{3}{2}\right)$ and $\xi \in \{0, \frac{1}{2}\}$ as in (4.27). For $\eta \in \mathcal{T}^l_\sigma$ we define the following transform:*

$$\tilde{\mathbf{L}}^{\omega}_{l,q}\eta(g) := \frac{1}{2\pi^3 i} \sum_{|p| \leqslant l} \frac{(i\omega/|\omega|)^{p-\xi}}{\|\Phi^l_{p,q}\|_K} \int_{(0)} \eta(\nu,p)(2\pi|\omega|)^{-\nu} \cdot \\ \cdot \Gamma(l+1+\nu)\mathbf{J}_\omega \varphi_{l,q}(\nu,p)(g)\nu^{\epsilon(p)} \sin\pi(\nu-p)d\nu, \qquad (9.14)$$

with $\epsilon(0) = 1$, $\epsilon(p) = -1$ for $p \in \mathbb{Z}\setminus\{0\}$, and $\epsilon(p) = 0$ for $p \in \frac{1}{2}+\mathbb{Z}$.

Then $\tilde{\mathbf{L}}^{\omega}_{l,q}\eta(g) \in P_{l,q}(N\backslash G,\omega)$, and we have

$$\mathbf{L}^{\omega}_{l,q}\tilde{\mathbf{L}}^{\omega}_{l,q}\eta(\nu,p) = \\ = -\frac{2}{\pi^2}(-1)^{p-\xi}\Gamma(l+1-\nu)\Gamma(l+1+\nu)\frac{\sin\pi(\nu-p)}{\nu^2-p^2}\,\nu^{\epsilon(p)}\eta(\nu,p) \qquad (9.15)$$

on any strip $|\operatorname{Re}\nu| < \alpha$ with $0 < \alpha < 1$.

Remark 8. For $l \in \mathbb{Z}$, $\tilde{\mathbf{L}}^{\omega}_{l,q}\eta = \sqrt{2}\mathcal{M}^{2\omega}_{l,q}\eta$, where $\mathcal{M}^{\omega}_{l,q}$ is the transform given by (7.9) in [9]. Also the equation (9.15) reduces to [9], (7.10) with the different normalizations of dk in mind.

Proof. First we prove that $\tilde{\mathbf{L}}^{\omega}_{l,q}\eta(g) \in P_{l,q}(N\backslash G,\omega)$. The relations (4.24)–(4.25) express $\mathbf{J}_\omega\varphi_{l,q}(\nu,p)(na[r]k)$ in terms of K-Bessel functions. For the convergence of the integral, we use estimate (1.37). This estimate also shows that $\tilde{\mathbf{L}}^{\omega}_{l,q}\eta(na[r]k)$ is of rapid decay with respect to r as $r \to \infty$. To investigate the behavior as $r \downarrow 0$ we observe that by (1.37) the contour in (9.14) can be shifted to $\operatorname{Re}\nu = \alpha$ with $0 < \alpha < 1$. Then the functional equation (4.48) and condition (iii) give

$$\begin{aligned}
\tilde{\mathbf{L}}^{\omega}_{l,q}\eta(g) &= \frac{1}{2\pi i}\sum_{|p|\leqslant l}\frac{1}{\|\Phi^l_{p,q}\|_K}\int_{(\alpha)}\eta(\nu,p)\,\cdot \\
&\cdot\left\{-(2\pi|\omega|)^{\nu}\left(\frac{i\omega}{|\omega|}\right)^{-p-\xi}\Gamma(l+1+\nu)\mathbf{M}_\omega\varphi_{l,q}(\nu,p)(g)\right.\\
&\left.+(-1)^{2\xi}(2\pi|\omega|)^{-\nu}\left(\frac{i\omega}{|\omega|}\right)^{p-\xi}\Gamma(l+1-\nu)\mathbf{M}_\omega\varphi_{l,q}(-\nu,-p)(g)\right\}\nu^{\epsilon(p)}d\nu\\
&= \frac{-1}{2\pi i}\sum_{|p|\leqslant l}\frac{(i\omega/|\omega|)^{-p-\xi}}{\|\Phi^l_{p,q}\|_K}\left\{\int_{(\alpha)}+\int_{(-\alpha)}\right\}\eta(\nu,p)\,\cdot\\
&\qquad\cdot(2\pi|\omega|)^{\nu}\Gamma(l+1+\nu)\mathbf{M}_\omega\varphi_{l,q}(\nu,p)(g)\nu^{\epsilon(p)}d\nu\\
&= \frac{i}{\pi}\sum_{|p|\leqslant l}\frac{(i\omega/|\omega|)^{-p-\xi}}{\|\Phi^l_{p,q}\|_K}\int_{(\alpha)}\eta(\nu,p)(2\pi|\omega|)^{\nu}\,\cdot\\
&\qquad\cdot\Gamma(l+1+\nu)\mathbf{M}_\omega\varphi_{l,q}(\nu,p)(g)\nu^{\epsilon(p)}d\nu\\
&\quad+l!\sum_{\substack{p\in\mathbb{Z}\\1\leqslant|p|\leqslant l}}\frac{(i\omega/|\omega|)^{-p}}{\|\Phi^l_{p,q}\|_K}\eta(0,p)\mathbf{M}_\omega\varphi_{l,q}(0,p)(g).
\end{aligned}\tag{9.16}$$

We note that the second term after the last equality sign in (9.16) is only present if l is integral and $l \geqslant 1$.

The estimate (4.54) implies that, as $r \downarrow 0$, the first sum after the last equality sign in (9.16) is $O(r^{1+\alpha})$. Using the estimate (7.15) in [9] we get

$$\begin{aligned}
l!\sum_{\substack{p\in\mathbb{Z}\\1\leqslant|p|\leqslant l}}&\frac{(i\omega/|\omega|)^{-p}}{\|\Phi^l_{p,q}\|_K}\eta(0,p)\mathbf{M}_\omega\varphi_{l,q}(0,p)(g) =\\
&= B(\eta)\chi_\omega(n)r^2\Phi^l_{0,q}(k)+O(r^3) = B(\eta)\mathbf{M}_\omega\varphi_{l,q}(1,0)(g)+O(r^3),
\end{aligned}\tag{9.17}$$

where $l \in \mathbb{Z}$, $l \geqslant 1$, and $B(\eta) = 2\pi l \cdot l!\,\eta(0,1)|\omega|\|\Phi^l_{1,q}\|_K^{-1}$. Collecting these, we have the following estimates

$$\tilde{\mathbf{L}}^{\omega}_{l,q}\eta(na[r]k) = \begin{cases} O(r^{1+\alpha}) & \text{as} \quad r\downarrow 0 \quad \text{for } 0<\alpha<1,\\ O(r^{-k}) & \text{as} \quad r\to\infty \quad \text{for all } k\geqslant 1,\end{cases}\tag{9.18}$$

and we conclude that $\tilde{\mathbf{L}}^{\omega}_{l,q}\eta \in P_{l,q}(N\backslash G, \omega)$.

In order to use Lemma 9.1.2 in computing $\mathbf{L}^{\omega}_{l,q}\tilde{\mathbf{L}}^{\omega}_{l,q}\eta$, we need to express $\mathbf{J}_0\tilde{\mathbf{L}}^{\omega}_{l,q}\eta$ in the form (9.7). We apply $\mathbf{J}_0$ to the last expression in (9.16) and by absolute convergence (obtained using the estimate (4.54)) we have:

$$\mathbf{J}_0\tilde{\mathbf{L}}^{\omega}_{l,q}\eta(n\mathrm{a}[r]k) = \frac{i}{\pi}\sum_{|p|\leqslant l}\frac{(i\omega/|\omega|)^{-p-\xi}}{\|\Phi^l_{p,q}\|_K}\int_{(\alpha)}\eta(\nu,p)(2\pi|\omega|)^{\nu}\cdot$$
$$\cdot\,\Gamma(l+1+\nu)\mathbf{J}_0\mathbf{M}_{\omega}\varphi_{l,q}(\nu,p)(g)\nu^{\epsilon(p)}d\nu$$
$$+\,l!\sum_{\substack{p\in\mathbb{Z}\\1\leqslant|p|\leqslant l}}\frac{(i\omega/|\omega|)^{-p}}{\|\Phi^l_{p,q}\|_K}\eta(0,p)\mathbf{J}_0\mathbf{M}_{\omega}\varphi_{l,q}(0,p)(g).$$

Note that (4.56) implies $\mathbf{J}_0\mathbf{M}_{\omega}\varphi_{l,q}(0,p) = 0$ for $p \in \mathbb{Z}\backslash\{0\}$. So, the second term in the above expression vanishes, and furthermore by (4.56) we have:

$$= \frac{i}{\pi}\sum_{|p|\leqslant l}\frac{(i\omega/|\omega|)^{-p-\xi}}{\|\Phi^l_{p,q}\|_K}\int_{(\alpha)}\eta(\nu,p)(2\pi|\omega|)^{\nu}\Gamma(l+1-\nu)\frac{\sin\pi(\nu-p)}{\nu^2-p^2}\cdot$$
$$\cdot\,\nu^{\epsilon(p)}r^{1-\nu}\Phi^l_{-p,q}(k)d\nu$$
$$\overset{(p\mapsto -p)}{=}\frac{i}{\pi}\sum_{|p|\leqslant l}\frac{(i\omega/|\omega|)^{p-\xi}}{\|\Phi^l_{p,q}\|_K}\int_{(\alpha)}\eta(-\nu,p)(2\pi|\omega|)^{\nu}\Gamma(l+1-\nu)\frac{\sin\pi(\nu+p)}{\nu^2-p^2}\cdot$$
$$\cdot\,\nu^{\epsilon(p)}r^{1-\nu}d\nu\cdot\Phi^l_{p,q}(k)$$
$$= \sum_{|p|\leqslant l}u_p(r)\Phi^l_{p,q}(k),$$

where

$$u_p(r) := \frac{i}{\pi\|\Phi^l_{p,q}\|_K}(i\omega/|\omega|)^{p-\xi}\int_{(\alpha)}\eta(-\nu,p)\cdot$$
$$\cdot\,(2\pi|\omega|)^{\nu}\Gamma(l+1-\nu)\frac{\sin\pi(\nu+p)}{\nu^2-p^2}\,\nu^{\epsilon(p)}r^{1-\nu}d\nu. \tag{9.19}$$

The equality (9.19) can be written in the following form:

$$r^{-1}u_p(r) = \frac{1}{2\pi i}\int_{(\alpha)}g(\nu)r^{-\nu}d\nu,$$

with

$$g(\nu) := -2\frac{(i\omega/|\omega|)^{p-\xi}}{\|\Phi^l_{p,q}\|_K}\eta(-\nu,p)(2\pi|\omega|)^{\nu}\Gamma(l+1-\nu)\frac{\sin\pi(\nu+p)}{\nu^2-p^2}\nu^{\epsilon(p)}. \tag{9.20}$$

From (9.18) and (9.6) we get

$$\mathbf{J}_0\tilde{\mathbf{L}}^{\omega}_{l,q}\eta(n\mathrm{a}[r]k) = \begin{cases} O(r) & , \quad \text{as} \quad r \downarrow 0 \\ O(r^{1-\alpha}) & , \quad \text{as} \quad r \to \infty \quad \text{for } 0<\alpha<1, \end{cases} \tag{9.21}$$

The functions $u_p(r)$ given by (9.19) inherit these growth conditions, and therefore the Mellin inversion formula (9.11) gives for $0 < \operatorname{Re}\nu < \alpha$

$$g(\nu) = \int_0^\infty r^{-1}u_p(r)r^{\nu-1}dr = \mathcal{M}u_p(\nu-1).$$

Hence, for $-\alpha < \operatorname{Re}\nu < 0$, we have

$$\begin{aligned}\mathbf{L}^{\omega}_{l,q}\tilde{\mathbf{L}}^{\omega}_{l,q}\eta(\nu,p) &= \\ &\overset{(9.12)}{=} \frac{\chi(-1)}{\pi^2}\|\Phi^l_{p,q}\|_K\,(-i\omega/|\omega|)^{-p+\xi}\,\Gamma(l+1-\nu)(2\pi|\omega|)^\nu\,g(-\nu) \\ &\overset{(9.20)}{=} -\frac{2}{\pi^2}(-1)^{p-\xi}\Gamma(l+1-\nu)\Gamma(l+1+\nu)\frac{\sin\pi(\nu-p)}{\nu^2-p^2}\,\nu^{\epsilon(p)}\eta(\nu,p).\end{aligned} \tag{9.22}$$

Estimate (9.18) and Definition 9.1.1, imply that the integral defining $\mathbf{L}^{\omega}_{l,q}\tilde{\mathbf{L}}^{\omega}_{l,q}\eta(\nu,p)$ is absolutely convergent and holomorphic as a function in ν on $|\operatorname{Re}\nu| < \alpha$, for all $\alpha \in (0,1)$. The right side of (9.22) is holomorphic as a function in ν on the wider strip $|\operatorname{Re}\nu| < \alpha$ for all $\alpha \in [0,1]$. The equality (9.22) holds on $-\alpha < \operatorname{Re}\nu < 0$. By holomorphic continuation, the equality stays valid on the strip $|\operatorname{Re}\nu| < \alpha$, with $\alpha \in (0,1)$. ■

The fact that $\tilde{\mathbf{L}}^{\omega}_{l,q}\eta \in P_{l,q}(N\backslash G,\omega)$ implies in particular that $\tilde{\mathbf{L}}^{\omega}_{l,q}\eta$ is square-integrable on $N\backslash G$. Related to this we have a Parseval property of the transformation $\tilde{\mathbf{L}}^{\omega}_{l,q}$:

Lemma 9.1.5. *Let $\sigma \in \left(1,\frac{3}{2}\right)$, and $\eta,\theta \in \mathcal{T}^l_\sigma$. Then we have*

$$\begin{aligned}\langle\tilde{\mathbf{L}}^{\omega}_{l,q}\eta,\tilde{\mathbf{L}}^{\omega}_{l,q}\theta\rangle_{N\backslash G} = \frac{1}{\pi^3 i}\sum_{|p|\leqslant l}\int_{(0)}\eta(\nu,p)\overline{\theta(\nu,p)}\,\cdot \\ \cdot\,\Gamma(l+1-\nu)\Gamma(l+1+\nu)\frac{\sin^2\pi(\nu-p)}{p^2-\nu^2}\,\nu^{2\epsilon(p)}d\nu.\end{aligned} \tag{9.23}$$

Remark 9. In case $l \in \mathbb{Z}$, this lemma reduces to Lemma 7.1 in [9].

Proof. We replace $\tilde{\mathbf{L}}^{\omega}_{l,q}\theta(g)$ by its defining expression (9.14). The resulting double integral over $N\backslash G \times i\mathbb{R}$ is absolutely convergent (as $r \downarrow 0$ use estimates (4.28) and (9.18) for $\operatorname{Re}\nu = 0$, and as $r \to \infty$ use (1.35), (4.24)–(4.25), and the second part of (9.18)), and we have

$$\begin{aligned}\int_{N\backslash G}\tilde{\mathbf{L}}^{\omega}_{l,q}\eta(g)\overline{\tilde{\mathbf{L}}^{\omega}_{l,q}\theta(g)}d\dot g = \frac{i}{2\pi^3}\sum_{|p|\leqslant l}\frac{(i\omega/|\omega|)^{-p+\xi}}{\|\Phi^l_{p,q}\|_K}\int_{(0)}\overline{\theta(\nu,p)}(2\pi|\omega|)^\nu\,\cdot \\ \cdot\,\Gamma(l+1-\nu)\int_{N\backslash G}\tilde{\mathbf{L}}^{\omega}_{l,q}\eta(g)\overline{\mathbf{J}_\omega\varphi_{l,q}(\nu,p)(g)}d\dot g\,\sin\pi(\nu-p)\nu^{\epsilon(p)}d\nu.\end{aligned} \tag{9.24}$$

Here we have used the fact that, for $\operatorname{Re}\nu = 0$,

$$\begin{aligned}\overline{\sin\pi(\nu-p)\nu^{\epsilon(p)}} &= \sin\pi(-\nu-p)(-\nu)^{\epsilon(p)} \\ &= -\sin\pi(\nu-p+2p)(-1)^{\epsilon(p)}\nu^{\epsilon(p)} \\ &= -(-1)^{\epsilon(p)+2p}\sin\pi(\nu-p)\nu^{\epsilon(p)} = \sin\pi(\nu-p)\nu^{\epsilon(p)}. \end{aligned} \tag{9.25}$$

Using the definition (9.1) of the Lebedev transform, and then the property (9.15), we continue in (9.24):

$$\begin{aligned} &= \frac{i}{2\pi}\sum_{|p|\leqslant l}(-1)^{p-\xi}\int_{(0)}\overline{\theta(\nu,p)}\,\mathbf{L}^{\omega}_{l,q}\tilde{\mathbf{L}}^{\omega}_{l,q}\eta(\nu,p)\sin\pi(\nu-p)\nu^{\epsilon(p)}d\nu \\ &= \frac{1}{\pi^3 i}\sum_{|p|\leqslant l}\int_{(0)}\eta(\nu,p)\overline{\theta(\nu,p)}\Gamma(l+1-\nu)\Gamma(l+1+\nu)\frac{\sin^2\pi(\nu-p)}{p^2-\nu^2}\nu^{2\epsilon(p)}d\nu, \end{aligned}$$

which ends the proof. ∎

We now define a function that shall later appear as the kernel of the Bessel transformation (11.1).

Definition 9.1.6. *For $\nu\in\mathbb{C}$, $p\in\frac{1}{2}\mathbb{Z}$, and $z\in\mathbb{C}^*$, we define*

$$\begin{aligned}\mathcal{K}^*_{\nu,p}(z) := \frac{1}{\sin\pi(\nu-p)}\Big\{&|z/2|^{-2\nu}\,(iz/|z|)^{2p-2\xi}\,\mathcal{J}^*_{-\nu,-p}(z)- \\ &-|z/2|^{2\nu}\,(iz/|z|)^{-2p-2\xi}\,\mathcal{J}^*_{\nu,p}(z)\Big\}, \end{aligned} \tag{9.26}$$

with $\mathcal{J}^*_{\nu,p}$ as given in (4.58) and $\xi\in\{0,\frac{1}{2}\}$ as in (4.27).

Directly from the definition we obtain:

(K1) $\mathcal{K}^*_{-\nu,-p}(z) = \mathcal{K}^*_{\nu,p}(z)$,

(K2) $\mathcal{K}^*_{\nu,p}(-z) = \mathcal{K}^*_{\nu,p}(z)$,

(K3) $\overline{\mathcal{K}^*_{\nu,p}(z)} = (z/|z|)^{4\xi}\mathcal{K}^*_{\bar\nu,-p}(z) = (z/|z|)^{4\xi}\mathcal{K}^*_{-\bar\nu,p}(z)$.

Some further properties are given in the following

Lemma 9.1.7. *Let $\sigma\in\left(1,\frac{3}{2}\right)$, $l\in\frac{1}{2}\mathbb{Z}$, $l\geqslant|p|$, $z\in\mathbb{C}^*$. The function $\mathcal{K}^*_{\nu,p}(z)$ is holomorphic in $\nu\in\mathbb{C}$ and satisfies*

$$\mathcal{K}^*_{\nu,p}(z) \ll_W (1+|\operatorname{Im}\nu|)^{2\sigma-1} \tag{9.27}$$

on the strip $|\operatorname{Re}\nu|<\sigma$. The estimate (9.27) is uniform in $z\in W$, for an arbitrary compact subset $W\subset\mathbb{C}^*$.

Proof. We have $|z/2|^{2\nu}(z/|z|)^{-2p}\mathcal{J}^*_{\nu,p}(z) = J_{\nu-p}(z)J_{\nu+p}(\bar{z})$ if we choose the branches suitably. Thus we see that the zeros of the difference

$$|z/2|^{-2\nu}(iz/|z|)^{2p-2\xi}\,\mathcal{J}^*_{-\nu,-p}(z) - |z/2|^{2\nu}(iz/|z|)^{-2p-2\xi}\,\mathcal{J}^*_{\nu,p}(z)$$

appear exactly when $\nu - p \in \mathbb{Z}$. So, the poles of $\frac{1}{\sin\pi(\nu-p)}$ are cancelled by these zeros, which for fixed z makes the function $\mathcal{K}^*_{\nu,p}(z)$ holomorphic in $\nu \in \mathbb{C}$.

Assuming that $\mu \in \mathbb{C}\setminus\mathbb{Z}_{\leqslant -1}$, from the power series expansion of $J^*_\mu(z)$, we have for fixed z

$$|J^*_\mu(z)| \leqslant e^{|z|^2/4}\max_{m\geqslant 0}\left\{|\Gamma(\mu+1+m)|^{-1}\right\},$$

and thus $J^*_\mu(z) \ll_z |\Gamma(\mu+1)|^{-1}$ uniformly for $\operatorname{Re}\mu > -\epsilon$, $\epsilon > 0$.

If $\operatorname{Re}\mu < 0$, then

$$|\Gamma(\mu+1+m)| = |\Gamma(\mu+1)|\prod_{j=0}^{m-1}|\mu+1+j| \geqslant |\Gamma(\mu+1)|\prod_{j=0}^{m-1}|\operatorname{Re}\mu+1+j|,$$

where only two factors $|\operatorname{Re}\mu+1+j|$ are in the neighborhood of the origin, and all the others satisfy $|\operatorname{Re}\mu+1+j| \geqslant 1$. This means that for all $m \geqslant 0$,

$$|\Gamma(\mu+1+m)| \geqslant |\Gamma(\mu+1)|\epsilon(1-\epsilon),$$

for some $\epsilon > 0$. Hence $J^*_\mu(z) \ll_{z,\epsilon} |\Gamma(\mu+1)|^{-1}$ uniformly for $\operatorname{Re}\mu < 0$ with distance greater than ϵ from elements in $\mathbb{Z}_{\leqslant -1}$.

We now look at the functions $J^*_{\nu\pm p}(z)$, $p \in \frac{1}{2}\mathbb{Z}$, for ν in a vertical strip with distance greater than ϵ from elements in $\frac{1}{2}\mathbb{Z}_{\leqslant -1}$ that belong to that strip. Then,

$$J^*_{\nu\pm p}(z) \ll_{z,\epsilon} |\Gamma(\nu\pm|p|+1)|^{-1}. \tag{9.28}$$

We note that

$$|\Gamma(\nu+|p|+1)|^{-1} \ll (1+|\operatorname{Im}\nu|)^{-\operatorname{Re}\nu-|p|-\frac{1}{2}}e^{\frac{\pi}{2}|\operatorname{Im}\nu|}, \tag{9.29}$$

$$\begin{aligned}\Gamma(\nu-|p|+1)|^{-1} &= \frac{|\sin\pi(\nu-|p|)||\Gamma(|p|-\nu)|}{\pi}\\ &\ll (1+|\operatorname{Im}\nu|)^{-\operatorname{Re}\nu+|p|-\frac{1}{2}}e^{\frac{\pi}{2}|\operatorname{Im}\nu|}.\end{aligned} \tag{9.30}$$

Thus, for fixed z, we have

$$J^*_{\nu\pm p}(z) \ll_\epsilon (1+|\operatorname{Im}\nu|)^{-\operatorname{Re}\nu\mp|p|-\frac{1}{2}}e^{\frac{\pi}{2}|\operatorname{Im}\nu|},$$

which implies the estimate

$$\mathcal{J}^*_{\nu,p}(z) \ll_W (1+|\operatorname{Im}\nu|)^{-2\operatorname{Re}\nu-1}e^{\pi|\operatorname{Im}\nu|} \tag{9.31}$$

uniformly for $p \in \frac{1}{2}\mathbb{Z}$, $z \in W$ for any compact subset $W \subset \mathbb{C}^*$, and $|\operatorname{Re}\nu| < \sigma$, $\sigma \in (1, \frac{3}{2})$, such that $|\nu - k| > \epsilon$ for $k \in \{0, -\frac{1}{2}, -1\}$. Definition 9.1.6 now gives

$$\begin{aligned}\mathcal{K}^*_{\nu,p}(z) &\ll_W (1 + |\operatorname{Im}\nu|)^{-1}\Big((1 + |\operatorname{Im}\nu|)^{2\operatorname{Re}\nu} + (1 + |\operatorname{Im}\nu|)^{-2\operatorname{Re}\nu}\Big) \\ &\ll_W (1 + |\operatorname{Im}\nu|)^{2|\operatorname{Re}\nu|-1}, \end{aligned} \tag{9.32}$$

which implies (9.27). ■

Lemma 9.1.8. *For any non-zero $\omega_1, \omega_2, \tau \in \mathbb{C}$, we define the map*

$$\kappa^*(\omega_1, \omega_2, \tau) : \eta \mapsto \mathcal{K}^*_{\nu,p}\left(4\pi\tau\sqrt{\omega_1\omega_2}\right)\eta, \tag{9.33}$$

for $\eta \in \mathcal{T}^l_\sigma$ and $\mathcal{K}^*_{\nu,p}(z)$ given by (9.26).

Then, $\kappa^*(\omega_1, \omega_2, \tau)$ is a linear operator in the space of functions $\mathcal{T}^l_\sigma$, and we have

$$\mathbf{J}_{\omega_1} l_\tau \tilde{\mathbf{L}}^{\omega_2}_{l,q}\eta = |\pi\tau|^2\left(\frac{i\tau\omega_1}{|\tau\omega_1|}\right)^{2\xi}\tilde{\mathbf{L}}^{\omega_1}_{l,q}\big(\kappa^*(\omega_1, \omega_2, \tau)\eta\big). \tag{9.34}$$

We note that the choice of the square root $\sqrt{\omega_1\omega_2}$ in the definition of the operator $\kappa^*(\omega_1, \omega_2, \tau)$ does not matter since $\mathcal{K}^*_{\nu,p}$ is even function on $\mathbb{C}^*$.

Remark 10. For integer values of p, $\mathcal{K}^*_{\nu,p}(z) = \mathcal{K}_{\nu,p}(z)$ and $\kappa^*(\omega_1, \omega_2, \tau) = \kappa(2\omega_1, 2\omega_2, \tau) = \kappa(\omega_1, \omega_2, 2\tau)$, where $\mathcal{K}_{\nu,p}$ and $\kappa(\omega_1, \omega_2, \tau)$ are the functions defined in [9], (7.21) and (7.20), respectively. In that sense, this lemma generalizes Lemma 7.2 in [9].

Proof. Lemma 9.1.7 shows that $\kappa^*(\omega_1, \omega_2, \tau)$ is indeed an operator on the set $\mathcal{T}^l_\sigma$, given in Definition 9.1.3, which acts by multiplication with the function $\mathcal{K}^*_{\nu,p}\left(4\pi\tau\sqrt{\omega_1\omega_2}\right)$. Hence the right side of (9.34) is well-defined. Because of the property (4.52), we transform the left side of (9.34) using (9.16), where the exchange of the order of integrals is allowed because of the estimate (4.55). So, for $\alpha \in (1, \frac{3}{2})$, we obtain:

$$\begin{aligned}\mathbf{J}_{\omega_1} l_\tau \tilde{\mathbf{L}}^{\omega_2}_{l,q}\eta(g) = \frac{i}{\pi}|\tau|^2\left(\frac{\tau}{|\tau|}\right)^{2\xi}\sum_{|p|\leqslant l}\frac{\big(i\tau^2\omega_2/|\tau^2\omega_2|\big)^{-p-\xi}}{\|\Phi^l_{p,q}\|_K}\int_{(\alpha)}\eta(\nu,p)\,\cdot \\ \cdot\,(2\pi|\tau^2\omega_2|)^\nu\Gamma(l+1+\nu)\mathbf{J}_{\omega_1}\mathbf{M}_{\tau^2\omega_2}\varphi_{l,q}(\nu,p)(g)\nu^{\epsilon(p)}d\nu \\ + l!\,|\tau|^2\sum_{\substack{p\in\mathbb{Z}\\ 1\leqslant|p|\leqslant l}}\frac{\big(i\tau^2\omega_2/|\tau^2\omega_2|\big)^{-p}}{\|\Phi^l_{p,q}\|_K}\eta(0,p)\mathbf{J}_{\omega_1}\mathbf{M}_{\tau^2\omega_2}\varphi_{l,q}(0,p)(g).\end{aligned}$$

By (4.57) we further have

$$\begin{aligned}= \frac{i}{\pi}|\tau|^2\left(\frac{\tau}{|\tau|}\right)^{2\xi}\sum_{|p|\leqslant l}\frac{\big(i\tau^2\omega_2/|\tau^2\omega_2|\big)^{-p-\xi}}{\|\Phi^l_{p,q}\|_K}\int_{(\alpha)}\eta(\nu,p)(2\pi|\tau^2\omega_2|)^\nu\,\cdot \\ \cdot\,\Gamma(l+1+\nu)\mathcal{J}^*_{\nu,p}(4\pi\tau\sqrt{\omega_1\omega_2})\mathbf{J}_{\omega_1}\varphi_{l,q}(\nu,p)(g)\nu^{\epsilon(p)}d\nu\end{aligned}$$

$$+l!\,|\tau|^2\left(\frac{\tau}{|\tau|}\right)^{2\xi}\sum_{\substack{l,p\in\mathbb{Z}\\1\leqslant|p|\leqslant l}}\frac{\left(i\tau^2\omega_2/|\tau^2\omega_2|\right)^{-p}}{\|\Phi^l_{p,q}\|_K}\eta(0,p)\cdot$$
$$\cdot\mathcal{J}^*_{0,p}(4\pi\tau\sqrt{\omega_1\omega_2})\mathbf{J}_{\omega_1}\varphi_{l,q}(0,p)(g). \tag{9.35}$$

The estimates (4.29) and (9.31) allow us to shift the contour (α) of one half of the integral to $(-\alpha)$; then the last sum vanishes. In the integral over $(-\alpha)$, we make the change $(\nu,p)\mapsto(-\nu,-p)$ and apply the functional equation (4.26). We continue in (9.35) with some rearrangement:

$$= \frac{i}{2\pi}|\tau|^2\left(\frac{\tau}{|\tau|}\right)^{2\xi}\sum_{|p|\leqslant l}\frac{\left(i\tau^2\omega_2/|\tau^2\omega_2|\right)^{-p-\xi}}{\|\Phi^l_{p,q}\|_K}\left\{\int_{(\alpha)}+\int_{(-\alpha)}\right\}\eta(\nu,p)\cdot$$
$$\cdot(2\pi|\tau^2\omega_2|)^{\nu}\Gamma(l+1+\nu)\mathcal{J}^*_{\nu,p}(4\pi\tau\sqrt{\omega_1\omega_2})\mathbf{J}_{\omega_1}\varphi_{l,q}(\nu,p)(g)\nu^{\epsilon(p)}d\nu$$
$$= \frac{i}{2\pi}|\tau|^2\left(\frac{\tau}{|\tau|}\right)^{2\xi}\sum_{|p|\leqslant l}\frac{(i\omega_1/|\omega_1|)^{p+\xi}}{\|\Phi^l_{p,q}\|_K}\int_{(\alpha)}\eta(\nu,p)\Gamma(l+1+\nu)(2\pi|\omega_1|)^{-\nu}\cdot$$
$$\cdot\left\{(4\pi^2|\tau^2\omega_1\omega_2|)^{\nu}\left(\frac{-\tau^2\omega_1\omega_2}{|\tau^2\omega_1\omega_2|}\right)^{-p-\xi}\mathcal{J}^*_{\nu,p}(4\pi\tau\sqrt{\omega_1\omega_2})-(4\pi^2|\tau^2\omega_1\omega_2|)^{-\nu}\cdot\right.$$
$$\left.\cdot\left(\frac{-\tau^2\omega_1\omega_2}{|\tau^2\omega_1\omega_2|}\right)^{p-\xi}\mathcal{J}^*_{-\nu,-p}(4\pi\tau\sqrt{\omega_1\omega_2})\right\}\mathbf{J}_{\omega_1}\varphi_{l,q}(\nu,p)(g)\nu^{\epsilon(p)}d\nu$$
$$= \frac{|\tau|^2}{2\pi i}\left(\frac{\tau}{|\tau|}\right)^{2\xi}\left(\frac{i\omega_1}{|\omega_1|}\right)^{2\xi}\sum_{|p|\leqslant l}\frac{(i\omega_1/|\omega_1|)^{p-\xi}}{\|\Phi^l_{p,q}\|_K}\int_{(0)}\mathcal{K}^*_{\nu,p}(4\pi\tau\sqrt{\omega_1\omega_2})\eta(\nu,p)\cdot$$
$$\cdot(2\pi|\omega_1|)^{-\nu}\Gamma(l+1+\nu)\mathbf{J}_{\omega_1}\varphi_{l,q}(\nu,p)(g)\nu^{\epsilon(p)}\sin\pi(\nu-p)d\nu$$
$$= |\pi\tau|^2\left(\frac{i\tau\omega_1}{|\tau\omega_1|}\right)^{2\xi}\tilde{\mathbf{L}}^{\omega_1}_{l,q}\big(\kappa^*(\omega_1,\omega_2,\tau)\eta\big)(g),$$

and the lemma has been proved. ■

9.2 Choice of Poincaré series

We shall use the Lebedev transform $\tilde{\mathbf{L}}^{\omega}_{l,q}\eta$ of functions $\eta\in\mathcal{T}^l_\sigma$, to generate a Poincaré series, which will later be used for deriving the preliminary sum formula.

Let us consider the Poincaré series

$$P_\chi\tilde{\mathbf{L}}^{\omega}_{l,q}\eta(g) := \frac{1}{[\Gamma_P:\Gamma_N]}\sum_{\gamma\in\Gamma_N\backslash\Gamma}\chi(\gamma)^{-1}\tilde{\mathbf{L}}^{\omega}_{l,q}\eta(\gamma g) \tag{9.36}$$

with non-zero $\omega\in\mathcal{O}'$.

The behavior of $\tilde{\mathbf{L}}^{\omega}_{l,q}\eta$ near 0 is important for the absolute convergence of the sum in (9.36). We use (9.16), where the line of integration is allowed to be moved to any $\alpha \in (0, \sigma]$ with $\sigma > 0$, because of the estimate (4.55). The contribution of the first term in (9.16) to $\tilde{\mathbf{L}}^{\omega}_{l,q}\eta(n\mathrm{a}[r]k)$ is then $O(r^{1+\sigma})$. This term is determining the behavior of $\tilde{\mathbf{L}}^{\omega}_{l,q}\eta(n\mathrm{a}[r]k)$ as $r \downarrow 0$ if $\sigma < 1$. However, for the absolute convergence of the Poincaré series (9.36) we need $\sigma > 1$. If $\sigma > 1$, the first term in (9.16) has the right behavior, but the second term causes problems with convergence. Namely, we see from (9.17) that it has a contribution $O(r^2)$, and it therefore determines the behavior of $\tilde{\mathbf{L}}^{\omega}_{l,q}\eta(n\mathrm{a}[r]k)$ as $r \downarrow 0$, which is not enough for the absolute convergence of $P_\chi \tilde{\mathbf{L}}^{\omega}_{l,q}\eta$.

To solve this problem, we shall use results of Miatello and Wallach in [34] concerning meromorphic continuation of Poincaré series. Their results are derived for the trivial character $\chi = 1$, but for any cofinite discrete subgroup. In order to apply those results we need to work with Poincaré series for $\Gamma_1(I)$. So, we shall write the Poincaré series (9.36) as a linear combination of (shifted) Poincaré series over $\Gamma_1(I)$ with trivial character $\chi = 1$, use results in [34] about the meromorphic continuation of those series, and obtain a meromorphic continuation of the originally defined series. Let

$$Pf(g) := \sum_{\gamma \in \Gamma_N \backslash \Gamma_1(I)} f(\gamma g),$$

denote a Poincaré series generated by the function f over the group $\Gamma_1(I)$ and trivial character.

Each $\gamma \in \Gamma_N \backslash \Gamma$ can be written as a product $\gamma = \zeta\delta$, with ζ running over a set of representatives for $\Gamma_N \backslash \Gamma_1(I)$ and δ over a set of representatives for $\Gamma_1(I) \backslash \Gamma$. Then, we may rewrite the Poincaré series $P_\chi f$ in the following way

$$\begin{aligned} P_\chi f(g) &:= \frac{1}{[\Gamma_P : \Gamma_N]} \sum_{\gamma \in \Gamma_N \backslash \Gamma} \chi(\gamma)^{-1} f(\gamma g) \\ &= \frac{1}{[\Gamma_P : \Gamma_N]} \sum_{\zeta \in \Gamma_N \backslash \Gamma_1(I)} \sum_{\delta \in \Gamma_1(I) \backslash \Gamma} \chi(\zeta\delta)^{-1} f(\zeta\delta g) \\ &= \frac{1}{[\Gamma_P : \Gamma_N]} \sum_{\delta \in \Gamma_1(I) \backslash \Gamma} \chi(\delta)^{-1} Pf(\delta g). \end{aligned} \tag{9.37}$$

Note that the character χ is trivial on $\Gamma_1(I)$.

If $T > 0$ is fixed, we introduce a so called cut off function $\rho \in C^\infty(G)$ defined as a left N-invariant, right K-invariant function such that $\rho(\mathrm{a}[r]) = 1$ if $r \in (0, T]$, $\rho(\mathrm{a}[r]) = 0$ if $r \in (T + 1, \infty)$, and $0 \leqslant \rho \leqslant 1$. For $\omega \in \mathcal{O}' \backslash \{0\}$, we consider the following Poincaré series

$$P_\chi \mathbf{M}_\omega \varphi_{l,q}(\nu, p)(g) = \frac{1}{[\Gamma_P : \Gamma_N]} \sum_{\gamma \in \Gamma_N \backslash \Gamma} \chi(\gamma)^{-1} \mathbf{M}_\omega \varphi_{l,q}(\nu, p)(\gamma g), \tag{9.38}$$

and

$$P_\chi \rho \mathbf{M}_\omega \varphi_{l,q}(\nu,p)(g) = \frac{1}{[\Gamma_P : \Gamma_N]} \sum_{\gamma \in \Gamma_N \backslash \Gamma} \chi(\gamma)^{-1} \rho(\gamma g) \mathbf{M}_\omega \varphi_{l,q}(\nu,p)(\gamma g), \qquad (9.39)$$

for $\mathrm{Re}\,\nu > 1$. From (9.39) we see that $P_\chi \rho \mathbf{M}_\omega \varphi_{l,q}(\nu,p)$ can be estimated at infinity by a finite sum over the units in $\mathcal{O}$ plus a part of the Eisenstein series $E_{0,0}(\mathrm{Re}\,\nu, 0; 1)$ corresponding to the big cell in the Bruhat decomposition. This gives

$$P_\chi \rho \mathbf{M}_\omega \varphi_{l,q}(\nu,p)(n\mathrm{a}[r]k) = O(r^{1-\mathrm{Re}\,\nu}), \qquad \text{as} \quad r \to \infty, \qquad (9.40)$$

for $\mathrm{Re}\,\nu > 1$. At a cusp $\kappa \in \mathcal{C}_\chi$ which is not Γ-equivalent to ∞, we estimate the series (9.39) by the whole Eisenstein series $E_{0,0}(\mathrm{Re}\,\nu, 0; 1)(g_\kappa g)$, and get

$$P_\chi \rho \mathbf{M}_\omega \varphi_{l,q}(\nu,p)(g_\kappa n\mathrm{a}[r]k) = O(r^{1-\mathrm{Re}\,\nu}), \qquad \text{as} \quad r \to \infty, \qquad (9.41)$$

for $\mathrm{Re}\,\nu > 1$.

Because of (9.37) we have

$$P_\chi \mathbf{M}_\omega \varphi_{l,q}(\nu,p)(g) = \frac{1}{[\Gamma_P : \Gamma_N]} \sum_{\delta \in \Gamma_1(I) \backslash \Gamma} \chi(\delta)^{-1} P \mathbf{M}_\omega \varphi_{l,q}(\nu,p)(\delta g), \qquad (9.42)$$

and

$$P_\chi \rho \mathbf{M}_\omega \varphi_{l,q}(\nu,p)(g) = \frac{1}{[\Gamma_P : \Gamma_N]} \sum_{\delta \in \Gamma_1(I) \backslash \Gamma} \chi(\delta)^{-1} P \rho \mathbf{M}_\omega \varphi_{l,q}(\nu,p)(\delta g). \qquad (9.43)$$

Here the Poincaré series $P\mathbf{M}_\omega \varphi_{l,q}(\nu,p)$ and $P\rho \mathbf{M}_\omega \varphi_{l,q}(\nu,p)$ correspond respectively to the Poincaré series $\mathbf{M}(\xi_p, \nu, g, \Phi^l_{p,q})$ and $\tilde{\mathbf{M}}(\xi_p, \nu, g, \Phi^l_{p,q})$ for $\Gamma_1(I)$ in [34], §2, with $\xi_p : \mathrm{h}[e^{it}] \mapsto e^{-2pit}$ a character of $M = H \cap K$.

Returning to (9.16) and (9.17), we define $\tilde{\mathbf{L}}^{\omega,*}_{l,q}\eta$ by

$$\tilde{\mathbf{L}}^{\omega}_{l,q}\eta = \tilde{\mathbf{L}}^{\omega,*}_{l,q}\eta + B(\eta)\rho \mathbf{M}_\omega \varphi_{l,q}(1,0). \qquad (9.44)$$

By construction the function $\tilde{\mathbf{L}}^{\omega,*}_{l,q}\eta$ satisfies $\tilde{\mathbf{L}}^{\omega,*}_{l,q}\eta(n\mathrm{a}[r]k) = O(r^{1+\sigma})$ as $r \downarrow 0$ with $\sigma \in \left(1, \frac{3}{2}\right)$. This implies absolute convergence of the Poincaré series $P_\chi \tilde{\mathbf{L}}^{\omega,*}_{l,q}\eta$. We shall also want this series to be square-integrable, which is determined by the behavior of the function $\tilde{\mathbf{L}}^{\omega,*}_{l,q}\eta$ near infinity. The estimate (9.18) implies that $\tilde{\mathbf{L}}^{\omega,*}_{l,q}\eta(n\mathrm{a}[r]k) = O(r^{-k})$ as $r \to \infty$ for all $k > 0$, and by Proposition 7.1.2 we have that $P_\chi \tilde{\mathbf{L}}^{\omega,*}_{l,q}\eta \in L^2(\Gamma\backslash G; l, q)$.

Theorem 2.5 in [34] implies that for any Γ the function $\nu \mapsto P\rho \mathbf{M}_\omega \varphi_{l,q}(\nu, 0)$ can be analytically continued as meromorphic function in $\nu \in \mathbb{C}$, where the singularities in the region $\mathrm{Re}\,\nu > 0$ occur only at values of ν for which $(\nu, 0)$ is a spectral parameter. The pole at $\nu = 1$, which might occur if $l = 0$, does not concern us since in that case the problematic term in (9.16) is not present. The spectral parameters ν, where $(\nu, 0)$ characterize the complementary series, are called exceptional and they form a discrete subset of the interval $(0, 1)$. So, there exists $\epsilon > 0$ such that the neighborhood $|\nu - 1| \leqslant \epsilon$ of 1 does not contain exceptional spectral parameters. At present, it is known that exceptional spectral parameters, if present, are concentrated in a small subinterval of $(0, 1)$. Hence the family of functions $\nu \mapsto P\rho\mathbf{M}_\omega\varphi_{l,q}(\nu, 0)$ has an analytic continuation as a holomorphic function in ν for $\operatorname{Re}\nu \geqslant 1 - \epsilon$. Because of (9.43) the same holds for $\nu \mapsto P_\chi\rho\mathbf{M}_\omega\varphi_{l,q}(\nu, 0)$.

Remark 11. This ϵ can be almost 1. The best results up until now known to me are those of Kim and Shahidi, [20]. However, for our purposes it will be sufficient to consider a small $\epsilon > 0$.

For the derivation of the sum formula (see next chapter) we shall also need that $P_\chi\rho\mathbf{M}_\omega\varphi_{l,q}(\nu, 0)$ is an element of $L^2(\Gamma\backslash G, \chi; l, q)$ for ν in a neighborhood of 1. We derive this conclusion using the results of Miatello and Wallach in [34]. For reasons of convenience let us write $\psi_\omega(\nu) := \rho\mathbf{M}_\omega\varphi_{l,q}(\nu, p)$ and assume for the time being that $l \geqslant 1$ is integral. (In the case $l = 0$ or $l \in \frac{1}{2} + \mathbb{Z}$, the offending term in the behavior of $\tilde{\mathbf{L}}^\omega_{l,q}\eta$ does not occur.) We consider general p although we will need the results only for $p = 0$. The function $\psi_\omega(\nu)$ has the following growth behavior, for $\operatorname{Re}\nu > 0$:

$$\psi_\omega(\nu)(n\mathrm{a}[r]k) = \begin{cases} O(r^{1+\operatorname{Re}\nu}) & \text{as} \quad r \downarrow 0 \\ 0 & \text{if } r \text{ is sufficiently large,} \end{cases} \tag{9.45}$$

uniformly in $n \in N$, $k \in K$. For $\operatorname{Re}\nu > 1$, these estimates and Proposition 7.1.2 imply that the Poincaré series $P_\chi\psi_\omega(\nu)$ is square-integrable on $\Gamma\backslash G$.

To see that $\nu \mapsto P_\chi\psi_\omega(\nu)$ extends to an L^2-holomorphic map with values in $L^2(\Gamma\backslash G, \chi; l, q)$ also for $\operatorname{Re}\nu \geqslant 1 - \epsilon$, we may restrict our reasoning to $P\psi_\omega(\nu)$ because of (9.37). We then obtain the desired conclusion from the results in [34]. We consider the function $\big(4(\Omega_+ + \Omega_-) - \nu^2 - p^2 + 1\big)\, P\psi_\omega(\nu)$. For $\operatorname{Re}\nu > 1$, we may differentiate within the Poincaré series, and find

$$\big(4(\Omega_+ + \Omega_-) - \nu^2 - p^2 + 1\big)\, P\psi_\omega(\nu) = P\check{\psi}_\omega(\nu),$$

where the function $\check{\psi}_\omega(\nu) := \big(4(\Omega_+ + \Omega_-) - \nu^2 - p^2 + 1\big)\, \psi_\omega(\nu)$ is holomorphic pointwise in g for at least $\operatorname{Re}\nu > 0$. The Poincaré series $P\check{\psi}_\omega(\nu)$ corresponds to the series $\check{\mathbf{M}}(\xi_p, \nu, g, \Phi^l_{p,q})$ in [34] for $\Gamma_1(I)$. The fact that the linear operator $\big(4(\Omega_+ + \Omega_-) - \nu^2 - p^2 + 1\big)$ applied to the function $\mathbf{M}_\omega\varphi_{l,q}(\nu, p)$ yields zero, implies that $\check{\psi}_\omega(\nu)(n\mathrm{a}[r]k)$ is non-zero only for $T \leqslant r \leqslant T + 1$. Thus, the Poincaré series $P\check{\psi}_\omega(\nu)$ is locally given by a finite sum. This sum converges absolutely for $\operatorname{Re}\nu > 0$, uniformly for g in a fundamental domain for $\Gamma\backslash G$. Hence, $P\check{\psi}_\omega(\nu)$ converges absolutely for all ν with $\operatorname{Re}\nu > 0$. Since $\big|\psi_\omega(\nu)(n\mathrm{a}[r]k)\big| = O(1)$ for

$T \leqslant r \leqslant T+1$ uniformly for $|\nu - 1| \leqslant \epsilon$, the Poincaré series $P\check{\psi}_\omega(\nu)$ can be analytically continued to a C^∞–function in ν and g with compact support in $\Gamma_1(I)\backslash G$ independent of ν and it is square-integrable on $\Gamma_1(I)\backslash G$ uniformly for $|\nu - 1| \leqslant \epsilon$. For $\mathrm{Re}\,\nu > 1$, the series $P\psi_\omega(\nu)$ has a spectral expansion, see p. 424 in [34], and the spectral expansions of $P\psi_\omega(\nu)$ and $P\check{\psi}_\omega(\nu)$ are linked, see relations on p. 428–429 in [34]. Miatello and Wallach use these relations between the spectral coefficients to show that the Poincaré series $P\psi_\omega(\nu)$ is square-integrable over $\Gamma_1(I)\backslash G$ also for $|\nu - 1| \leqslant \epsilon$, with a uniformly bounded norm. Hence, we conclude that the Poincaré series $P_\chi\psi_\omega(\nu) = P_\chi\rho\mathbf{M}_\omega\varphi_{l,q}(\nu,p)$ is square-integrable on $\Gamma\backslash G$ for $\mathrm{Re}\,\nu \geqslant 1-\epsilon$.

The estimates (9.40) and (9.41) can be extended to $|\nu - 1| \leqslant \epsilon$ in order to describe the behavior of $P_\chi\rho\mathbf{M}_\omega\varphi_{l,q}(\nu,p)$ at the cusps also for ν in a neighborhood of 1 when the series (9.39) does not converge absolutely. For that purpose we observe that if we choose the truncation parameter T large enough, then for $g = n\mathrm{a}[r]k \in G$ with $r > T$, we have

$$P_\chi\rho\mathbf{M}_\omega\varphi_{l,q}(\nu,p)(g_\kappa g) = P_\chi\mathbf{M}_\omega\varphi_{l,q}(\nu,p)(g_\kappa g) + \frac{\delta_{\kappa,\infty}}{[\Gamma_P : \Gamma_N]} \sum_{\varepsilon\in\mathcal{O}^*} \chi(\varepsilon)^{-1}\varepsilon^{2p}\big(\rho(g)-1\big)\mathbf{M}_{\omega/\varepsilon^2}\varphi_{l,q}(\nu,p)(g). \tag{9.46}$$

Obviously, the series $P_\chi\rho\mathbf{M}_\omega\varphi_{l,q}(\nu,p)$ and $P_\chi\mathbf{M}_\omega\varphi_{l,q}(\nu,p)$ are equal near all the cusps κ that are not Γ-equivalent to ∞. From the Fourier expansion of the Poincaré series $P_\chi\mathbf{M}_\omega\varphi_{l,q}(\nu,p)$ at a cusp $\kappa \in \mathcal{C}_\chi$

$$\begin{aligned} P_\chi\mathbf{M}_\omega\varphi_{l,q}(\nu,p)(g_\kappa g) = \\ = a_0^\kappa(\nu,p)\varphi_{l,q}(-\nu,-p)(g) + \sum_{0\neq\omega'\in\Lambda'_\kappa} a_{\omega'}^\kappa(\nu,p)\mathbf{J}_{\omega'}\varphi_{l,q}(\nu,p)(g) \\ + \frac{\delta_{\kappa,\infty}}{[\Gamma_P : \Gamma_N]} \sum_{\substack{\varepsilon\in\mathcal{O}^* \\ \omega'\varepsilon^2=\omega}} \chi(\varepsilon)^{-1}\varepsilon^{2p}\mathbf{M}_{\omega/\varepsilon^2}\varphi_{l,q}(\nu,p)(g), \end{aligned} \tag{9.47}$$

and (9.46), we see that for $g = n\mathrm{a}[r]k$ with $r \to \infty$

$$\begin{aligned} P_\chi\rho\mathbf{M}_\omega\varphi_{l,q}(\nu,p)(g_\kappa g) = \\ = a_0^\kappa(\nu,p)\varphi_{l,q}(-\nu,-p)(g) + \sum_{0\neq\omega'\in\Lambda'_\kappa} a_{\omega'}^\kappa(\nu,p)\mathbf{J}_{\omega'}\varphi_{l,q}(\nu,p)(g) \\ + \frac{\delta_{\kappa,\infty}}{[\Gamma_P : \Gamma_N]} \sum_{\substack{\varepsilon\in\mathcal{O}^* \\ \omega'\varepsilon^2=\omega}} \chi(\varepsilon)^{-1}\varepsilon^{2p}\rho(g)\mathbf{M}_{\omega/\varepsilon^2}\varphi_{l,q}(\nu,p)(g). \end{aligned} \tag{9.48}$$

If κ is not Γ-equivalent to ∞ then $\delta_{\kappa,\infty} = 0$, while if $\kappa = \infty$ then $\rho(n\mathrm{a}[r]k) = 0$ as $r \to \infty$, so the last term in (9.48) vanishes at all cusps. For $|\nu - 1| \leqslant \epsilon$, the sum over non-zero $\omega' \in \Lambda'_\kappa$ is absolutely convergent and square-integrable over $\Gamma\backslash G$, so according to Lemma 5.2.1, (ii) it has exponential decay in r as $r \to \infty$ uniformly in ν. This means that $\varphi_{l,q}(-\nu,-p)(n\mathrm{a}[r]k) = r^{1-\nu}\Phi^{l}_{-p,q}(k)$ determines the growth of $P_\chi\rho\mathbf{M}_\omega\varphi_{l,q}(\nu,p)$ at each cusp. Hence

$$P_\chi\rho\mathbf{M}_\omega\varphi_{l,q}(\nu,p)(g_\kappa n\mathrm{a}[r]k) = O(r^{1-\mathrm{Re}\,\nu}), \qquad \text{as} \quad r \to \infty, \tag{9.49}$$

uniformly for $\mathrm{Re}\,\nu \geqslant 1-\epsilon$, at each cusp $\kappa \in \mathcal{C}_\chi$.

Since the Poincaré series $P_\chi\rho\mathbf{M}_\omega\varphi_{l,q}(\nu,0)$ is square-integrable on $\Gamma\backslash G$ for $\mathrm{Re}\,\nu \geqslant 1-\epsilon$, we denote by $P_\chi\rho\mathbf{M}_\omega\varphi_{l,q}(1,0)$ its value at $\nu = 1$, and define a Poincaré series $P_\chi\tilde{\mathbf{L}}^{\omega}_{l,q}\eta$ by

$$P_\chi\tilde{\mathbf{L}}^{\omega}_{l,q}\eta = P_\chi\tilde{\mathbf{L}}^{\omega,*}_{l,q}\eta + B(\eta)P_\chi\rho\mathbf{M}_\omega\varphi_{l,q}(1,0). \tag{9.50}$$

The discussion above implies square-integrability of the Poincaré series $P_\chi\tilde{\mathbf{L}}^{\omega}_{l,q}\eta$ on $\Gamma\backslash G$. Moreover, it is a bounded function on $\Gamma\backslash G$.

Indeed, for $\nu = 1$ and $p = 0$, estimate (9.49) gives

$$P_\chi\rho\mathbf{M}_\omega\varphi_{l,q}(1,0)(g_\kappa n\mathrm{a}[r]k) \ll 1 \qquad \text{as} \quad r \to \infty, \tag{9.51}$$

for all $\kappa \in \mathcal{C}_\chi$. Hence

$$P_\chi\rho\mathbf{M}_\omega\varphi_{l,q}(1,0)(g) \ll 1, \qquad \text{for all} \quad g \in \Gamma\backslash G. \tag{9.52}$$

On the other hand, since $\tilde{\mathbf{L}}^{\omega,*}_{l,q}\eta(n\mathrm{a}[r]k)$ is $O(r^{-b})$ as $r \to \infty$ for all $b \geqslant 0$, we have $P_\chi\tilde{\mathbf{L}}^{\omega,*}_{l,q}\eta(g_\kappa n\mathrm{a}[r]k) \ll r^{-b}$ as $r \to \infty$ at each cusp $\kappa \in \mathcal{C}_\chi$. This implies boundedness of the Poincaré series $\tilde{\mathbf{L}}^{\omega,*}_{l,q}\eta$ on $\Gamma\backslash G$. Hence, we conclude

$$P_\chi\tilde{\mathbf{L}}^{\omega}_{l,q}\eta(g) = O(1), \qquad \text{for all} \quad g \in \Gamma\backslash G. \tag{9.53}$$

Chapter 10

Preliminary sum formula

In this chapter we shall derive the preliminary sum formula via spectral and geometric computations of the inner product of two Poincaré series as discussed in Section 9.2. We shall first carry out some preparations.

Lemma 10.1. *Let f be a continuous χ-automorphic function with respect to Γ which is integrable over $\Gamma\backslash G$, $f(\mathrm{n a}[r]k)$ is at least $O(1)$ as $r \downarrow 0$, and $f(g_\kappa \mathrm{n a}[r]k)$ is at most $O(r)$ as $r \to \infty$ at each cusp κ. Then, the following equality holds:*

$$\langle P_\chi \tilde{\mathbf{L}}^\omega_{l,q}\eta, f\rangle_{\Gamma\backslash G} = \frac{\sqrt{|d_F|}}{2[\Gamma_P : \Gamma_N]}\langle \tilde{\mathbf{L}}^\omega_{l,q}\eta, F_\omega f\rangle_{N\backslash G}, \tag{10.1}$$

Proof. From the definition (9.50), we see that the function $P_\chi \tilde{\mathbf{L}}^\omega_{l,q}\eta$ is a sum of two functions. The function $\tilde{\mathbf{L}}^{\omega,*}_{l,q}\eta$ satisfies the growth conditions from Proposition 7.1.2 and therefore, by Lemma 7.3.1, we have

$$\langle P_\chi \tilde{\mathbf{L}}^{\omega,*}_{l,q}\eta, f\rangle_{\Gamma\backslash G} = \frac{\sqrt{|d_F|}}{2[\Gamma_P : \Gamma_N]}\langle \tilde{\mathbf{L}}^{\omega,*}_{l,q}\eta, F_\omega f\rangle_{N\backslash G}. \tag{10.2}$$

For $\mathrm{Re}\,\nu > 1$, the equality

$$\langle P_\chi \rho \mathbf{M}_\omega \varphi_{l,q}(\nu,0), f\rangle_{\Gamma\backslash G} = \frac{\sqrt{|d_F|}}{2[\Gamma_P : \Gamma_N]}\langle \rho \mathbf{M}_\omega \varphi_{l,q}(\nu,0), F_\omega f\rangle_{N\backslash G} \tag{10.3}$$

holds by Lemma 7.3.1. Because of the growth conditions imposed on f, the right side of (10.3) converges absolutely to a holomorphic function for $\mathrm{Re}\,\nu > 0$. Also the left side of (10.3) is holomorphic for $\mathrm{Re}\,\nu \geqslant 1-\epsilon$ with $\epsilon > 0$; it follows from estimate (9.49) and the growth of f on cusp sectors. Note that f is not necessarily square-integrable; so the holomorphy does not follow from the L^2-holomorphy of $P_\chi \rho \mathbf{M}_\omega \varphi_{l,q}(\nu,0)$. On compact subsets of $\Gamma\backslash G$ the integrability of

$P_\chi \rho \mathbf{M}_\omega \varphi_{l,q}(\nu,0)\cdot f$ is not a problem. Estimate (9.49) implies that we also have integrability of $P_\chi \rho \mathbf{M}_\omega \varphi_{l,q}(\nu,0)\cdot f$ on cusp sectors. Hence, the equality (10.3) extends holomorphically to $\mathrm{Re}\,\nu \geqslant 1-\epsilon$. In particular, it holds for $\nu=1$.

Combining (9.50), (10.2), and (10.3) for $\nu=1$, we obtain(10.1). ∎

10.1 Scalar product of Poincaré series, spectral description

The cuspidal automorphic functions $T_V\varphi_{l,q}(\nu_V,p_V)$ are continuous and integrable over $\Gamma\backslash G$ with exponential decay at the cusps and appropriate growth on compact subsets of $\Gamma\backslash G$. So, we may apply Lemma 10.1 with $f=T_V\varphi_{l,q}(\nu_V,p_V)$.

By (5.6) we have the Fourier expansion

$$T_V\varphi_{l,q}(\nu_V,p_V)=\sum_{0\neq\omega\in\mathcal{O}'} c_{T_V}(\omega)\mathbf{J}_\omega\varphi_{l,q}(\nu_V,p_V). \tag{10.4}$$

It will be convenient to use the normalization of the Fourier coefficients $c_{T_V}(\omega)$ given by

$$C_V(\omega;\nu_V,p_V):=(2\pi|\omega|)^{\nu_V}\,(\omega/|\omega|)^{-p_V+\xi}\,c_{T_V}(\omega), \tag{10.5}$$

Equality (10.1) yields

$$\langle P_\chi\tilde{\mathbf{L}}^\omega_{l,q}\eta, T_V\varphi_{l,q}(\nu_V,p_V)\rangle_{\Gamma\backslash G}=\frac{\overline{c_{T_V(\omega)}}\sqrt{|d_F|}}{2[\Gamma_P:\Gamma_N]}\int_{N\backslash G}\tilde{\mathbf{L}}^\omega_{l,q}\eta(g)\overline{\mathbf{J}_\omega\varphi_{l,q}(\nu_V,p_V)(g)}d\dot{g},$$

which by definition of the Lebedev transform (9.1) is further equal to

$$\begin{aligned}
&=\frac{\pi^2\sqrt{|d_F|}}{2[\Gamma_P:\Gamma_N]}\frac{\|\Phi^l_{p_V,q}\|_K\,i^{-p_V+\xi}}{\Gamma(l+1+\overline{\nu_V})}\overline{C_V(\omega;\nu_V,p_V)}\,\mathbf{L}^\omega_{l,q}\tilde{\mathbf{L}}^\omega_{l,q}\eta(-\overline{\nu_V},p_V)\\
&\overset{(9.15)}{=}-\frac{\sqrt{|d_F|}}{[\Gamma_P:\Gamma_N]}\,i^{\,p_V-\xi}\|\Phi^l_{p_V,q}\|_K\,\overline{C_V(\omega;\nu_V,p_V)}\,\cdot\\
&\qquad\cdot\Gamma(l+1-\overline{\nu_V})\frac{\sin\pi(\overline{\nu_V}-p_V)}{\overline{\nu_V}^2-p_V^2}\,\overline{\nu_V}^{\,\epsilon(p_V)}\eta(-\overline{\nu_V},p_V).
\end{aligned}\tag{10.6}$$

Taking the complex conjugate of the last expression yields

$$\begin{aligned}
\langle T_V\varphi_{l,q}(\nu_V,p_V),P_\chi\tilde{\mathbf{L}}^\omega_{l,q}\eta\rangle_{\Gamma\backslash G}&=-\frac{\sqrt{|d_F|}}{[\Gamma_P:\Gamma_N]}\,i^{\,-p_V+\xi}\|\Phi^l_{p_V,q}\|_K\,\cdot\\
&\cdot C_V(\omega;\nu_V,p_V)\Gamma(l+1-\nu_V)\frac{\sin\pi(\nu_V-p_V)}{\nu_V^2-p_V^2}\nu_V^{\,\epsilon(p_V)}\overline{\eta(-\overline{\nu_V},p_V)}.
\end{aligned}\tag{10.7}$$

Let $\theta \in \mathcal{T}^l_\sigma$ be another function with the same properties as η.

If V is in the unitary principal series, that is $\nu_V \in i[0,\infty)$, then we recall (9.25), and equalities (10.6)–(10.7) give

$$\langle P_\chi \tilde{\mathbf{L}}^{\omega_1}_{l,q}\eta, T_V\varphi_{l,q}(\nu_V,p_V)\rangle_{\Gamma\backslash G}\langle T_V\varphi_{l,q}(\nu_V,p_V), P_\chi\tilde{\mathbf{L}}^{\omega_2}_{l,q}\theta\rangle_{\Gamma\backslash G} = \\ = \frac{|d_F|}{[\Gamma_P:\Gamma_N]^2}\|\Phi^l_{p_V,q}\|^2_K \overline{C_V(\omega_1;\nu_V,p_V)}\cdot \\ \cdot C_V(\omega_2;\nu_V,p_V)\lambda_l(\nu_V,p_V)\eta(\nu_V,p_V)\overline{\theta(\nu_V,p_V)}, \tag{10.8}$$

where

$$\lambda_l(\nu,p) := \Gamma(l+1+\nu)\Gamma(l+1-\nu)\frac{\sin^2\pi(\nu-p)}{(\nu^2-p^2)^2}\nu^{\,2\epsilon(p)}. \tag{10.9}$$

If V is in the complementary series, i.e. $\nu_V \in (0,1) \subset \mathbb{R}$, then $p_V = 0$, and from (10.6)–(10.7) we get

$$\langle P_\chi \tilde{\mathbf{L}}^{\omega_1}_{l,q}\eta, T_V\varphi_{l,q}(\nu_V,0)\rangle_{\Gamma\backslash G}\langle T_V\varphi_{l,q}(\nu_V,0), P_\chi\tilde{\mathbf{L}}^{\omega_2}_{l,q}\theta\rangle_{\Gamma\backslash G} = \\ = \frac{|d_F|}{[\Gamma_P:\Gamma_N]^2}\frac{\Gamma(l+1-\nu_V)}{\Gamma(l+1+\nu_V)}\|\Phi^l_{0,q}\|^2_K \overline{C_V(\omega_1;\nu_V,0)}\cdot \\ \cdot C_V(\omega_2;\nu_V,0)\lambda_l(\nu_V,0)\eta(\nu_V,0)\overline{\theta(\nu_V,0)}. \tag{10.10}$$

We now turn to the Eisenstein series. The fact that the functions $E^\kappa_{l,q}(\nu,p;\chi)$ with $\operatorname{Re}\nu = 0$ are continuous and integrable functions over $\Gamma\backslash G$, and $O(r)$ on cusp sectors as well as on compact subsets of $\Gamma\backslash G$, allow us to apply Lemma 10.1 with $f = E^\kappa_{l,q}(\nu,p;\chi)$.

The expression (5.2) gives, in particular, the Fourier expansion of $E^\kappa_{l,q}(\nu,p;\chi)$ at infinity. Again, for convenience, we normalize the Fourier coefficients $D^{\kappa,\infty}_\chi(\omega;\nu,p)$ of the Eisenstein series as follows

$$B_{\kappa,\chi}(\omega;\lambda,p) := (2\pi|\omega|)^\lambda\,(\omega/|\omega|)^{-p+\xi}\,D^{\kappa,\infty}_\chi(\omega;\lambda,p). \tag{10.11}$$

We mentioned at the end of Section 5.1 that the Fourier coefficients $D^{\kappa,\infty}_\chi(\omega;\nu,p)$, and hence $B_{\kappa,\chi}(\omega;\nu,p)$, of the Eisenstein series are meromorphic over $\mathbb{C}$ with respect to ν, and holomorphic on the line $\operatorname{Re}\nu = 0$. So, on the line $\operatorname{Re}\nu = 0$, we have by (10.1)

$$\langle P_\chi \tilde{\mathbf{L}}^{\omega}_{l,q}\eta, E^\kappa_{l,q}(\nu,p;\chi)\rangle_{\Gamma\backslash G} = \\ = \frac{\sqrt{|d_F|}}{2\,[\Gamma_P:\Gamma_N]}\frac{\overline{D^{\kappa,\infty}_\chi(\omega;\nu,p)}}{[\Gamma_\kappa:\Gamma'_\kappa]}\int_{N\backslash G}\tilde{\mathbf{L}}^\omega_{l,q}\eta(g)\overline{\mathbf{J}_\omega\varphi_{l,q}(\nu,p)(g)}\dot{d}g \\ \overset{(9.1)}{=} \frac{\pi^2\sqrt{|d_F|}}{2[\Gamma_P:\Gamma_N]}\frac{\|\Phi^l_{p,q}\|_K}{[\Gamma_\kappa:\Gamma'_\kappa]}\frac{\overline{D^{\kappa,\infty}_\chi(\omega;\nu,p)}}{\Gamma(l+1-\nu)}\frac{(-i\omega/|\omega|)^{p-\xi}}{(2\pi|\omega|)^\nu}\mathbf{L}^\omega_{l,q}\tilde{\mathbf{L}}^\omega_{l,q}\eta(\nu,p).$$

Equality (9.15) and normalization (10.11) yield

$$\langle P_\chi \tilde{\mathbf{L}}^\omega_{l,q}\eta, E^\kappa_{l,q}(\nu,p;\chi)\rangle_{\Gamma\backslash G} = -\frac{\sqrt{|d_F|}}{[\Gamma_P:\Gamma_N]}\frac{i^{\,p-\xi}\|\Phi^l_{p,q}\|_K}{[\Gamma_\kappa:\Gamma'_\kappa]}\cdot \\ \cdot\overline{B_{\kappa,\chi}(\omega;\nu,p)}\,\Gamma(l+1+\nu)\frac{\sin\pi(\nu-p)}{\nu^2-p^2}\nu^{\epsilon(p)}\eta(\nu,p). \tag{10.12}$$

Taking the complex conjugate of the last expression, using $\bar{\nu}=-\nu$ and (9.25), gives

$$\langle E^\kappa_{l,q}(\nu,p;\chi), P_\chi \tilde{\mathbf{L}}^\omega_{l,q}\eta\rangle_{\Gamma\backslash G} = -\frac{\sqrt{|d_F|}}{[\Gamma_P:\Gamma_N]}\frac{i^{\,-p+\xi}\|\Phi^l_{p,q}\|_K}{[\Gamma_\kappa:\Gamma'_\kappa]}\cdot \\ \cdot B_{\kappa,\chi}(\omega;\nu,p)\Gamma(l+1-\nu)\frac{\sin\pi(\nu-p)}{\nu^2-p^2}\nu^{\epsilon(p)}\overline{\eta(\nu,p)}. \tag{10.13}$$

Then, from (10.12) and (10.13) we get

$$\langle P_\chi \tilde{\mathbf{L}}^{\omega_1}_{l,q}\eta, E^\kappa_{l,q}(\nu,p;\chi)\rangle_{\Gamma\backslash G}\langle E^\kappa_{l,q}(\nu,p;\chi), P_\chi \tilde{\mathbf{L}}^{\omega_2}_{l,q}\theta\rangle_{\Gamma\backslash G} = \\ = \frac{|d_F|}{[\Gamma_P:\Gamma_N]^2}\frac{\|\Phi^l_{p,q}\|^2_K}{[\Gamma_\kappa:\Gamma'_\kappa]^2}\overline{B_{\kappa,\chi}(\omega_1;\nu,p)}\cdot \\ \cdot B_{\kappa,\chi}(\omega_2;\nu,p)\lambda_l(\nu,p)\eta(\nu,p)\overline{\theta(\nu,p)}, \tag{10.14}$$

with $\lambda_l(\nu,p)$ as in (10.9).

For the constant function $f\equiv 1$, the fact that $\omega\neq 0$ implies $F_\omega 1\equiv 0$, and therefore we have $\langle P_\chi \tilde{\mathbf{L}}^\omega_{l,q}\eta, 1\rangle_{\Gamma\backslash G}=0$.

Since the square-integrable Poincaré series $P_\chi \tilde{\mathbf{L}}^\omega_{l,q}\eta$ is bounded, see (9.53), we may take $f_1 = P_\chi \tilde{\mathbf{L}}^{\omega_1}_{l,q}\eta$ and $f_2 = P_\chi \tilde{\mathbf{L}}^{\omega_2}_{l,q}\theta$ in Theorem 8.1. Using (10.8), (10.10), and (10.14) we obtain the spectral side of the sum formula:

$$\langle P_\chi \tilde{\mathbf{L}}^{\omega_1}_{l,q}\eta, P_\chi \tilde{\mathbf{L}}^{\omega_2}_{l,q}\theta\rangle_{\Gamma\backslash G} = \frac{|d_F|}{[\Gamma_P:\Gamma_N]^2}\sum_V \overline{C_V(\omega_1;\nu_V,p_V)}\cdot \\ \cdot C_V(\omega_2;\nu_V,p_V)\lambda_l(\nu_V,p_V)\eta(\nu_V,p_V)\overline{\theta(\nu_V,p_V)} \\ +\frac{|d_F|}{[\Gamma_P:\Gamma_N]^2}\sum_{\kappa\in\mathcal{C}_\chi}\frac{1}{4\pi i\,[\Gamma_\kappa:\Gamma'_\kappa]|\Lambda_\kappa|}\sum_{|p|\leqslant l}^{\chi}\int_{(0)}\overline{B_{\kappa,\chi}(\omega_1;\nu,p)}\cdot \\ \cdot B_{\kappa,\chi}(\omega_2;\nu,p)\lambda_l(\nu,p)\eta(\nu,p)\overline{\theta(\nu,p)}\,d\nu. \tag{10.15}$$

Here V runs over an orthogonal system of irreducible cuspidal subspaces of the space $L^2(\Gamma\backslash G,\chi)$ that intersect $L^2(\Gamma\backslash G,\chi;l,q)$ non-trivially.

10.2 Scalar product of Poincaré series, geometric description

We now proceed with the geometric computation of the same inner product as in the previous section.

For $\omega_1 \in \mathcal{O}'$, $\omega_1 \neq 0$, we observe that (9.50) implies

$$F_{\omega_1} P_\chi \tilde{\mathbf{L}}^{\omega_2}_{l,q} \eta = F_{\omega_1} P_\chi \tilde{\mathbf{L}}^{\omega_2,*}_{l,q} \eta + B(\eta) F_{\omega_1} P_\chi \rho \mathbf{M}_{\omega_2} \varphi_{l,q}(1,0). \tag{10.16}$$

The functions $\tilde{\mathbf{L}}^{\omega_2,*}_{l,q}\eta$ and $\rho\mathbf{M}_{\omega_2}\varphi_{l,q}(\nu,0)$, $\operatorname{Re}\nu > 1$, satisfy the growth conditions in Proposition 7.1.2. (See the discussion just after (9.44), and the estimate (9.45), respectively.) Thus, we have by (7.16):

$$\begin{aligned} F_{\omega_1} P_\chi \tilde{\mathbf{L}}^{\omega_2,*}_{l,q} \eta = {}& \frac{1}{[\Gamma_P : \Gamma_N]} \sum_{\varepsilon \in \mathcal{O}^*} \delta_{\varepsilon^2\omega_1,\omega_2}\, \chi(\varepsilon)^{-1} l_{1/\varepsilon} \tilde{\mathbf{L}}^{\omega_2,*}_{l,q} \eta \\ & + \frac{2}{[\Gamma_P : \Gamma_N]\sqrt{|d_F|}} \sideset{}{'}\sum_{c \in I} S_\chi(\omega_1, \omega_2; c) \mathbf{J}_{\omega_1} l_{1/c} \tilde{\mathbf{L}}^{\omega_2,*}_{l,q} \eta \end{aligned} \tag{10.17}$$

as well as

$$\begin{aligned} F_{\omega_1} P_\chi \rho \mathbf{M}_{\omega_2} \varphi_{l,q}(\nu,0) = {}& \frac{1}{[\Gamma_P : \Gamma_N]} \sum_{\varepsilon \in \mathcal{O}^*} \delta_{\varepsilon^2\omega_1,\omega_2}\, \chi(\varepsilon)^{-1} l_{1/\varepsilon} \rho \mathbf{M}_{\omega_2} \varphi_{l,q}(\nu,0) \\ & + \frac{2}{[\Gamma_P : \Gamma_N]\sqrt{|d_F|}} \sideset{}{'}\sum_{c \in I} S_\chi(\omega_1, \omega_2; c) \mathbf{J}_{\omega_1} l_{1/c} \rho \mathbf{M}_{\omega_2} \varphi_{l,q}(\nu,0), \end{aligned} \tag{10.18}$$

for $\operatorname{Re}\nu > 1$. Using again the notation $\psi_\omega(\nu) = \rho\mathbf{M}_\omega \varphi_{l,q}(\nu,0)$ (here with $p = 0$), by the definition of the Jacquet operator, we have

$$\begin{aligned} \mathbf{J}_{\omega_1} l_{1/c} \psi_{\omega_2}(\nu)(g) &= \int_N \chi_{\omega_1}(n)^{-1} \psi_{\omega_2}(\nu)(g)(\mathrm{h}[1/c]\mathrm{w} n g)\, dn \\ &< \int_{N^*} \left| \psi_{\omega_2}(\nu) \left(\mathrm{n}\left[\tfrac{\bar{z}}{c^2(r^2+|z|^2)} \right] \mathrm{a}\left[\tfrac{r}{|c|^2(r^2+|z|^2)} \right] k' \right) \right| d_+ z, \end{aligned} \tag{10.19}$$

where $N^* := \{ z \in \mathbb{C} : r \leqslant (T+1)|c|^2(r^2+|z|^2)\}$ with $T > 0$ and $g = \mathrm{a}[r]k$. For the last inequality we used the relation (4.3) and the fact that the cut off function ρ is non-zero if its argument is in $(0, T+1]$. Estimate (9.45), for $z \in N^*$, yields

$$\psi_{\omega_2}(\nu) \left(\mathrm{n}\left[\tfrac{\bar{z}}{c^2(r^2+|z|^2)} \right] \mathrm{a}\left[\tfrac{r}{|c|^2(r^2+|z|^2)} \right] k' \right) \ll \left(\frac{r}{|c|^2(r^2+|z|^2)} \right)^{1+\operatorname{Re}\nu}, \tag{10.20}$$

uniformly for $\operatorname{Re}\nu > 0$ and $r \geqslant 0$. Here the implicit constant does not depend on either c or r. Thus, for $\operatorname{Re}\nu \geqslant 1 - \epsilon$ we get the estimate

$$\mathbf{J}_{\omega_1} l_{1/c} \rho \mathbf{M}_{\omega_2} \varphi_{l,q}(\nu,0)(\mathrm{a}[r]k) \ll r^{-1-\operatorname{Re}\nu} |c|^{-2(1+\operatorname{Re}\nu)} \ll r^{\epsilon} |c|^{-4+2\epsilon}, \tag{10.21}$$

uniformly for $r \geqslant 0$ and $c \in I \setminus \{0\}$. This means that the second term in the right side of (10.18) converges absolutely to a holomorphic function for $\mathrm{Re}\,\nu \geqslant 1-\epsilon$. Hence, the expression (10.18) extends holomorphically to $\mathrm{Re}\,\nu \geqslant 1-\epsilon$. In particular, it may be evaluated at $\nu = 1$.

Substitution of (10.17) and (10.18) with $\nu = 1$ into (10.16) gives

$$F_{\omega_1} P_\chi \tilde{\mathbf{L}}^{\omega_2}_{l,q}\eta = \frac{1}{[\Gamma_P:\Gamma_N]} \sum_{\varepsilon\in\mathcal{O}^*} \delta_{\varepsilon^2\omega_1,\omega_2}\,\chi(\varepsilon)^{-1} l_{1/\varepsilon}\tilde{\mathbf{L}}^{\omega_2}_{l,q}\eta + \frac{2}{[\Gamma_P:\Gamma_N]\sqrt{|d_F|}} \sideset{}{'}\sum_{c\in I} S_\chi(\omega_1,\omega_2;c)\mathbf{J}_{\omega_1} l_{1/c}\tilde{\mathbf{L}}^{\omega_2}_{l,q}\eta. \tag{10.22}$$

From (9.16) and (4.52) we get that

$$l_\tau \tilde{\mathbf{L}}^{\omega_2}_{l,q}\eta = |\tau|^2(\tau/|\tau|)^{2\xi}\,\tilde{\mathbf{L}}^{\tau^2\omega_2}_{l,q}\eta \qquad \text{for any} \quad \tau\in\mathbb{C}^*,$$

which together with Lemma 9.1.8 implies

$$F_{\omega_1} P_\chi \tilde{\mathbf{L}}^{\omega_2}_{l,q}\eta = \frac{1}{[\Gamma_P:\Gamma_N]} \sum_{\varepsilon\in\mathcal{O}^*} \delta_{\varepsilon^2\omega_1,\omega_2}\,\chi(\varepsilon)^{-1}\varepsilon^{-2\xi}\tilde{\mathbf{L}}^{\omega_1}_{l,q}\eta + \frac{2\pi^2\,(i\omega_1/|\omega_1|)^{2\xi}}{[\Gamma_P:\Gamma_N]\sqrt{|d_F|}} \sideset{}{'}\sum_{c\in I}\left(\frac{c}{|c|}\right)^{-2\xi}\frac{S_\chi(\omega_1,\omega_2;c)}{|c|^2}\tilde{\mathbf{L}}^{\omega_1}_{l,q}\big(\kappa^*(\omega_1,\omega_2,1/c)\eta\big). \tag{10.23}$$

On the other hand, since the square-integrable function $P_\chi\tilde{\mathbf{L}}^{\omega_2}_{l,q}\theta$ is also bounded on $\Gamma\backslash G$, see (9.53), we may apply Lemma 10.1 with $f = P_\chi\tilde{\mathbf{L}}^{\omega_2}_{l,q}\theta$ and obtain

$$\langle P_\chi\tilde{\mathbf{L}}^{\omega_1}_{l,q}\eta, P_\chi\tilde{\mathbf{L}}^{\omega_2}_{l,q}\theta\rangle_{\Gamma\backslash G} = \frac{\sqrt{|d_F|}}{2[\Gamma_P:\Gamma_N]}\langle\tilde{\mathbf{L}}^{\omega_1}_{l,q}\eta, F_{\omega_1}P_\chi\tilde{\mathbf{L}}^{\omega_2}_{l,q}\theta\rangle_{N\backslash G}. \tag{10.24}$$

We now insert (10.23) into (10.24), change the order of summation and integration, and get:

$$\begin{aligned}\langle P_\chi\tilde{\mathbf{L}}^{\omega_1}_{l,q}\eta, P_\chi\tilde{\mathbf{L}}^{\omega_2}_{l,q}\theta\rangle_{\Gamma\backslash G} &= \\ &= \frac{\sqrt{|d_F|}}{2[\Gamma_P:\Gamma_N]^2}\sum_{\varepsilon\in\mathcal{O}^*}\delta_{\varepsilon^2\omega_1,\omega_2}\,\chi(\varepsilon)\varepsilon^{2\xi}\langle\tilde{\mathbf{L}}^{\omega_1}_{l,q}\eta,\tilde{\mathbf{L}}^{\omega_1}_{l,q}\theta\rangle_{N\backslash G} \\ &\quad + \chi(-1)\frac{\pi^2(i\omega_1/|\omega_1|)^{-2\xi}}{[\Gamma_P:\Gamma_N]^2}\sideset{}{'}\sum_{c\in I}\left(\frac{c}{|c|}\right)^{2\xi}\frac{S_\chi(\omega_2,\omega_1;c)}{|c|^2}\,\cdot \\ &\qquad \cdot\langle\tilde{\mathbf{L}}^{\omega_1}_{l,q}\eta,\tilde{\mathbf{L}}^{\omega_1}_{l,q}\big(\kappa^*(\omega_1,\omega_2,1/c)\eta\big)\rangle_{N\backslash G},\end{aligned} \tag{10.25}$$

In the last line we used property (6.5) of a Kloosterman sum.

Finally, an application of Lemma 9.1.5 gives the geometric side of the sum formula

$$\begin{aligned}\langle P_\chi \tilde{\mathbf{L}}^{\omega_1}_{l,q}\eta, P_\chi \tilde{\mathbf{L}}^{\omega_2}_{l,q}\theta\rangle_{\Gamma\backslash G} &= \frac{\sqrt{|d_F|}}{2\pi^3 i\,[\Gamma_P:\Gamma_N]^2}\sum_{\varepsilon\in\mathcal{O}^*}\delta_{\varepsilon^2\omega_1,\omega_2}\,\chi(\varepsilon)\varepsilon^{2\xi}\cdot\\ &\quad\cdot\sum_{|p|\leqslant l}\int_{(0)}\eta(\nu,p)\overline{\theta(\nu,p)}\lambda_l(\nu,p)(p^2-\nu^2)d\nu\\ &\quad+\frac{(i\omega_2/|\omega_2|)^{2\xi}}{\pi i\,[\Gamma_P:\Gamma_N]^2}\sideset{}{'}\sum_{c\in I}\left(\frac{c}{|c|}\right)^{-2\xi}\frac{S_\chi(\omega_2,\omega_1;c)}{|c|^2}\cdot\\ &\quad\cdot\sum_{|p|\leqslant l}\int_{(0)}\mathcal{K}^*_{\nu,p}\left(\tfrac{4\pi}{c}\sqrt{\omega_1\omega_2}\right)\eta(\nu,p)\overline{\theta(\nu,p)}\lambda_l(\nu,p)(p^2-\nu^2)d\nu, \qquad (10.26)\end{aligned}$$

where we used properties (K1) and (K3) of the function $\mathcal{K}^*_{\nu,p}$ with $\operatorname{Re}\nu=0$, see page 80.

10.3 Preliminary sum formula

The equality of the expressions in (10.15) and (10.26) is the basis for the sum formula. Equating their right sides proves:

Proposition 10.3.1. *Let $\eta,\theta\in\mathcal{T}^l_\sigma$ with fixed $\sigma\in(1,\frac{3}{2})$. Then, for any $\omega_1,\omega_2\in\mathcal{O}'\setminus\{0\}$ we have:*

$$\begin{aligned}&\sum_V\overline{C_V(\omega_1;\nu_V,p_V)}C_V(\omega_2;\nu_V,p_V)\eta(\nu_V,p_V)\overline{\theta(\nu_V,p_V)}\lambda_l(\nu_V,p_V)\\ &\quad+\frac{1}{4\pi i}\sum_{\kappa\in\mathcal{C}_\chi}\frac{1}{[\Gamma_\kappa:\Gamma'_\kappa]|\Lambda_\kappa|}\sum_{|p|\leqslant l}^{\chi}\int_{(0)}\overline{B_{\kappa,\chi}(\omega_1;\nu,p)}\cdot\\ &\qquad\cdot B_{\kappa,\chi}(\omega_2;\nu,p)\eta(\nu,p)\overline{\theta(\nu,p)}\lambda_l(\nu,p)\,d\nu=\\ &=\frac{1}{2\pi^3 i\sqrt{|d_F|}}\sum_{\varepsilon\in\mathcal{O}^*}\delta_{\varepsilon^2\omega_1,\omega_2}\,\chi(\varepsilon)\varepsilon^{2\xi}\cdot\\ &\qquad\cdot\sum_{|p|\leqslant l}\int_{(0)}\eta(\nu,p)\overline{\theta(\nu,p)}\lambda_l(\nu,p)(p^2-\nu^2)d\nu\\ &\quad+\frac{(i\omega_2/|\omega_2|)^{2\xi}}{|d_F|\pi i}\sideset{}{'}\sum_{c\in I}\left(\frac{c}{|c|}\right)^{-2\xi}\frac{S_\chi(\omega_2,\omega_1;c)}{|c|^2}\sum_{|p|\leqslant l}\int_{(0)}\mathcal{K}^*_{\nu,p}\left(\tfrac{4\pi}{c}\sqrt{\omega_1\omega_2}\right)\cdot\\ &\qquad\cdot\eta(\nu,p)\overline{\theta(\nu,p)}\lambda_l(\nu,p)(p^2-\nu^2)d\nu, \qquad (10.27)\end{aligned}$$

where $|\Lambda_\kappa|$ is the Euclidean area of a period parallelogram for the lattice Λ_κ in $\mathbb{C}$ corresponding to the discrete subgroup ${g_\kappa}^{-1}\Gamma'_\kappa g_\kappa$, V runs over a maximal orthogo-

nal system of irreducible cuspidal subspaces of $L^2(\Gamma\backslash G;\chi)$ *such that the type* (l,q) *occurs in* V *for one (hence for all)* $q \in \frac{1}{2}\mathbb{Z}$ *satisfying* $q \equiv l$ *(mod 1),* $|q| \leqslant l$*, and* $\sum_{|p|\leqslant l}^{\chi}$ *means that the sum runs through all* $p \in \frac{1}{2}\mathbb{Z}$ *such that* $|p| \leqslant l$ *with the condition* $\chi(\varepsilon) = \varepsilon^{2p}$ *satisfied for all* $\varepsilon \in \mathcal{O}^*$.

Chapter 11

Spectral sum formula

Based on the preliminary version (10.27), we shall extend the validity of the sum formula to a wider class of test functions. (See Theorem 11.3.3.) Before we enlarge the class of test functions, we have to know more about the Bessel transformation appearing in the geometric side of the formula.

11.1 Bessel transformation

For $\sigma \in (0, \frac{3}{2})$, $\sigma \neq 1$, and $a, b \in \mathbb{R}$, we define $\mathcal{H}^\sigma(a,b)$ to be the space of functions $h : \{\nu \in \mathbb{C} : |\operatorname{Re}\nu| \leqslant \sigma\} \times \frac{1}{2}\mathbb{Z} \longrightarrow \mathbb{C}$ that satisfy the following conditions

(i) $h(\nu, p) = h(-\nu, -p)$,

(ii) $\nu \mapsto h(\nu, p)$ is holomorphic on a neighborhood of the strip $|\operatorname{Re}\nu| \leqslant \sigma$,

(iii) $h(\nu, p) \ll (1 + |\operatorname{Im}\nu|)^{-a}(1 + |p|)^{-b}$,

(iv) If $\sigma > \frac{1}{2}$ and $p \in \frac{1}{2} + \mathbb{Z}$, then the function $\nu \mapsto h(\nu, p)$ has at $\nu = \pm\frac{1}{2}$ zeros of at least order 2, and if $\sigma > 1$ and $p \in \mathbb{Z} \setminus \{0\}$, then the function $\nu \mapsto h(\nu, p)$ has at $\nu = \pm 1$ zeros of at least order 2.

The functions h built from $\eta, \theta \in \mathcal{T}_\sigma^l$ in (11.8) are elements of $\mathcal{H}^\sigma(a,b)$ for all $a, b \in \mathbb{R}$.

Lemma 11.1.1. *Let $\sigma \in (0, \frac{3}{2})$, $\sigma \neq 1$ and $h \in \mathcal{H}^\sigma(a,b)$ with $a, b \in \mathbb{R}$. We define the Bessel transform $\mathbf{B}h$ by*

$$\mathbf{B}h(u) := \frac{1}{2\pi i} \sum_{p \in \frac{1}{2}\mathbb{Z}} \int_{(0)} \mathcal{K}^*_{\nu,p}(u) h(\nu, p)(p^2 - \nu^2)\, d\nu, \tag{11.1}$$

with $\mathcal{K}^*_{\nu,p}$ defined in (9.26).

If $a > 2$ and $b > 3$, then the sum and the integral in (11.1) converge absolutely.

Proof. For $\operatorname{Re}\nu = 0$, we have

$$|p^2 - \nu^2| \leqslant (|p| + |\operatorname{Im}\nu|)^2 \leqslant (1 + |p| + |\operatorname{Im}\nu|)^2 \leqslant (1 + |\operatorname{Im}\nu|)^2(1 + |p|)^2.$$

This and the estimate (9.32), for $\operatorname{Re}\nu = 0$ and fixed $u \in \mathbb{C}^*$, gives

$$\mathcal{K}^*_{\nu,p}(u)h(\nu,p)(p^2 - \nu^2) \ll (1 + |\operatorname{Im}\nu|)^{1-a}(1 + |p|)^{2-b}. \tag{11.2}$$

This estimate further implies

$$\mathbf{B}h(u) \ll \sum_{p \in \frac{1}{2}\mathbb{Z}} (1 + |p|)^{2-b} \int_{(0)} (1 + |\operatorname{Im}\nu|)^{1-a} d\nu, \tag{11.3}$$

from which we clearly see that the integral converges absolutely if $a > 2$, and the sum is absolutely convergent if $b > 3$. ■

Remark 12. For integer values of p, and in case of the Gaussian number field $\mathbb{Q}(i)$, the Bessel transform defined in (11.1) is four times the Bessel transform $\mathrm{B}h$ defined in [9], (10.2).

Let $a > 2$, $b > 3$. The estimate (9.32) allows us to shift the line of integration to the line $\operatorname{Re}\nu = \alpha_1$ for any $|\alpha_1| \leqslant \sigma < \frac{a}{2} - 1$:

$$\mathbf{B}h(u) = \frac{1}{2\pi i} \sum_{p \in \frac{1}{2}\mathbb{Z}} \int_{(\alpha_1)} \mathcal{K}^*_{\nu,p}(u)h(\nu,p)(p^2 - \nu^2)d\nu.$$

We then use (9.26) to split this integral into sum of two integrals with $\mathcal{J}^*_{\nu,p}(u)$ and $\mathcal{J}^*_{-\nu,-p}(u)$. This is possible because of the estimate (9.31). The same estimate makes the resulting integrals absolutely convergent, and also the sum over $p \in \frac{1}{2}\mathbb{Z}$. Condition (iv) in the definition of $\mathcal{H}^\sigma(a,b)$ ensures that the integrands are holomorphic inside the strip $|\operatorname{Re}\nu| < \sigma$ except for singularities at $\nu = 0$ in the terms with $p \in \mathbb{Z}\backslash\{0\}$. We now rearrange the sum, shift one of the lines of integration, and pick up the residues, to obtain:

$$\mathbf{B}h(u) = \frac{1}{\pi i} \sum_{p \in \frac{1}{2}\mathbb{Z}} \int_{(\alpha_1)} \frac{|u/2|^{2\nu}(iu/|u|)^{-2p-2\xi}}{\sin \pi(\nu - p)} \mathcal{J}^*_{\nu,p}(u)h(\nu,p)(\nu^2 - p^2)d\nu$$
$$+ \frac{1}{\pi} \sum_{p \in \mathbb{Z}\backslash\{0\}} p^2 \, (u/|u|)^{-2p} \, \mathcal{J}^*_{0,p}(u)h(0,p). \tag{11.4}$$

For $\sigma \in (0, \frac{3}{2})$, $\sigma \neq 1$, we take $\alpha_1 = \sigma$ in (11.4), and estimate both terms separately in order to obtain an estimate for the Bessel transform $\mathbf{B}h(u)$ as $|u| \downarrow 0$.

In case $p \in \mathbb{Z}$, we know that $(u/|u|)^{-2p}\mathcal{J}^*_{0,p}(u) = \mathcal{J}_{0,p}(u)$. (See Remark 5 on page 48). Using the estimate $\mathcal{J}_{0,p}(u) \ll (|u|/2)^{2|p|}/(|p|!)^2$, see (10.8) in [9], we conclude that

$$(u/|u|)^{-2p}\mathcal{J}^*_{0,p}(u) \ll_u \frac{(|u|/2)^{2|p|}}{(|p|!)^2}, \qquad \text{as} \quad |u| \downarrow 0. \tag{11.5}$$

Therefore the second term in the right side of (11.4) is $O(|u|^2)$, as $|u| \downarrow 0$.

For $\operatorname{Re}\nu = \sigma$, we have uniformly for $p \in \frac{1}{2}\mathbb{Z}$

$$\begin{aligned}
\left| \frac{|u/2|^{2\nu}(iu/|u|)^{-2p-2\xi} \mathcal{J}^*_{\nu,p}(u)}{\sin \pi(\nu - p)} \right| &\leqslant \frac{|u|^{2\sigma} |J^*_{\nu-p}(u)||J^*_{\nu,p}(\bar{u})|}{|\sin \pi(\nu - p)|} \\
&\overset{(9.28)}{\ll} \frac{|u|^{2\sigma}}{|\sin \pi(\nu - p)|} |\Gamma(\nu + |p| + 1)|^{-1} |\Gamma(\nu - |p| + 1)|^{-1} \\
&= \frac{|u|^{2\sigma}}{\pi} \frac{|\Gamma(|p| - \nu)|}{|\Gamma(\nu + |p| + 1)|} \ll_\sigma |u|^{2\sigma} |\nu|^{-2\sigma - 1}, \qquad \text{as} \qquad |u| \downarrow 0. \qquad (11.6)
\end{aligned}$$

In the case $p \in \mathbb{Z}$, the last estimate reduces to (10.9) in [9], and in the case $p \in \frac{1}{2} + \mathbb{Z}$ it is obtained as follows: We write $|p| = \frac{1}{2} + n$ with $n \in \mathbb{N}$ and get

$$\begin{aligned}
\frac{|\Gamma(|p| - \nu)|}{|\Gamma(\nu + |p| + 1)|} &= \frac{|\Gamma(-\nu - 1/2)|}{|\Gamma(\nu + 1/2)|} \prod_{j=0}^{n} \left| \frac{j - \nu - 1/2}{j + \nu + 1/2} \right| \\
&\ll |\Gamma(-\nu - 1/2)||\Gamma(\nu + 1/2)|^{-1} \ll |\nu + 1/2|^{-2\sigma - 1} e^{2\sigma + 1} \ll_\sigma |\nu|^{-2\sigma - 1}.
\end{aligned}$$

The estimate (11.6) implies that the first term in the right side of (11.4) is $O(|u|^{2\sigma})$, as $|u| \downarrow 0$.

Collecting these results we have proved

Lemma 11.1.2. *For $\sigma \in (0, \frac{3}{2})$, $\sigma \neq 1$, $a > 2$, $b > 3$. For each $h \in \mathcal{H}^\sigma(a, b)$ we have*

$$\mathbf{B}h(u) \ll_\sigma |u|^{2 \min\{\sigma, 1\}}, \qquad \text{as} \qquad |u| \downarrow 0.$$

We need the following lemma in the proof of the sum formula in Section 11.3.

Lemma 11.1.3. *For $\sigma \in (0, \frac{3}{2})$, $\sigma \neq 1$, $a > 2$, $b > 3$. Take $\alpha \in (0, \sigma]$. Let (f_n) be a sequence of elements in $\mathcal{H}^\sigma(a, b)$ converging pointwise to $f \in \mathcal{H}^\sigma(a, b)$ for all (ν, p) with $\operatorname{Re}\nu \in (0, \alpha]$, $p \in \frac{1}{2}\mathbb{Z}$, and moreover*

$$\sup_{0 \leqslant \operatorname{Re}\nu \leqslant \alpha, p \in \frac{1}{2}\mathbb{Z}, n \in \mathbb{N}} (1 + |\operatorname{Im}\nu|)^a (1 + |p|)^b \big| f_n(\nu, p) - f(\nu, p) \big| < \infty.$$

Then, the integral defining $\mathbf{B}f(u)$ converges absolutely, and

$$\lim_{n \to \infty} |u|^{-2 \min\{\sigma, 1\}} \mathbf{B} f_n(u) = |u|^{-2 \min\{\sigma, 1\}} \mathbf{B} f(u) \qquad (11.7)$$

uniformly on each set $\{u \in \mathbb{C}^* : |u| \leqslant r_0\}$ with $r_0 > 0$.

Proof. Using the integral representation (11.5) and (11.6) we get

$$\Big||u|^{-2\min\{\sigma,1\}}\mathbf{B}f_n(u) - |u|^{-2\min\{\sigma,1\}}\mathbf{B}f(u)\Big| \leqslant$$
$$\ll |u|^{-2\min\{\sigma,1\}} \sum_{p\in\frac{1}{2}\mathbb{Z}} \int_{(\alpha)} |u|^{2\sigma}|\nu|^{-2\sigma-1}|h(\nu,p)||\nu^2-p^2|d\nu$$
$$+ |u|^{-2\min\{\sigma,1\}} \sum_{p\in\mathbb{Z}\setminus\{0\}} p^2|u|^{2|p|}(|p|!)^{-2}|h(0,p)|.$$

If we denote $S := \sup_{\nu,p,n}(1+|\operatorname{Im}\nu|)^a(1+|p|)^b\big|f_n(\nu,p)-f(\nu,p)\big|$, the difference can be further estimated as follows

$$\ll S|u|^{-2\min\{\sigma,1\}}\left\{|u|^{2\sigma}\sum_{p\in\frac{1}{2}\mathbb{Z}}(1+|p|)^{2-b}\int_{(\alpha)}(1+|\operatorname{Im}\nu|)^{-2\sigma+1-a}d\nu\right.$$
$$\left.+|u|^2\sum_{p\in\mathbb{Z}\setminus\{0\}}(1+|p|)^{2-b}\right\} \ll |u|^{-2\min\{\sigma,1\}}\cdot|u|^{\min\{2\sigma,2\}} \ll 1.$$

Applying dominated convergence gives the result. ■

11.2 Extension method

In the preliminary sum formula, Proposition 10.3.1, the test functions η and θ do not occur separately, but always as a product $\eta\cdot\bar{\theta}$. We also have the presence of the function λ_l containing Γ-factors. We want our test functions to be as simple as possible, and certainly not built as a product of different functions. We also want to eliminate the dependence of l in the test functions. The latter is easily handled by defining functions on the set $\{\nu\in\mathbb{C} : |\operatorname{Re}\nu|\leqslant\sigma\}\times\frac{1}{2}\mathbb{Z}$ vanishing for $|p|>l$.

We define a function h built from $\eta,\theta\in\mathcal{T}^l_\sigma$, with $\sigma\in(1,\frac{3}{2})$, and λ_l as in (10.9), in the following way:

$$h(\nu,p)=\begin{cases}\eta(\nu,p)\bar{\theta}(\nu,p)\lambda_l(\nu,p) & , \quad |p|\leqslant l\\ 0 & , \quad |p|>l,\end{cases} \tag{11.8}$$

where $\bar{\theta}(\nu,p) := \overline{\theta(-\bar{\nu},p)}$. For each such function h, we obtain from (10.27):

$$\sum_V \overline{C_V(\omega_1;\nu_V,p_V)}C_V(\omega_2;\nu_V,p_V)h(\nu_V,p_V)$$
$$+\frac{1}{4\pi i}\sum_{\kappa\in\mathcal{C}_\chi}\frac{1}{[\Gamma_\kappa:\Gamma'_\kappa]|\Lambda_\kappa|}\sum_{p\in\frac{1}{2}\mathbb{Z}}^{\chi}\int_{(0)}\overline{B_{\kappa,\chi}(\omega_1;\nu,p)}B_{\kappa,\chi}(\omega_2;\nu,p)h(\nu,p)d\nu=$$

$$= \frac{1}{2\pi^3 i\sqrt{|d_F|}} \sum_{\varepsilon\in\mathcal{O}^*} \delta_{\varepsilon^2\omega_1,\omega_2}\,\chi(\varepsilon)\varepsilon^{2\xi} \sum_{p\in\frac12\mathbb{Z}} \int_{(0)} h(\nu,p)(p^2-\nu^2)d\nu$$
$$+ \frac{2}{|d_F|}\left(\frac{i\omega_2}{|\omega_2|}\right)^{2\xi} {\sum_{c\in I}}' \left(\frac{c}{|c|}\right)^{-2\xi} \frac{S_\chi(\omega_2,\omega_1;c)}{|c|^2}\cdot$$
$$\cdot\frac{1}{2\pi i}\sum_{p\in\frac12\mathbb{Z}} \int_{(0)} \mathcal{K}^*_{\nu,p}\left(\tfrac{4\pi}{c}\sqrt{\omega_1\omega_2}\right) h(\nu,p)(p^2-\nu^2)d\nu, \tag{11.9}$$

where all sums and integrals in the relation above converge absolutely.

It is convenient to introduce notation for the various terms in the sum formula. On the spectral side, the choice of the spectral parameter (ν_V, p_V) is as indicated in (3.21), and we introduce a measure $d\sigma_{\omega,\omega'}$ on the set

$$Y := \Big(\big(i[0,\infty)\cup(0,1)\big)\times\{0\}\Big)\bigcup\Big(i[0,\infty)\times\left(\tfrac12\mathbb{Z}\backslash\{0\}\right)\Big)\subset\mathbb{C}\times\tfrac12\mathbb{Z} \tag{11.10}$$

by

$$\int_Y f(\nu,p)d\sigma_{\omega,\omega'}(\nu,p) := \sum_V \overline{C_V(\omega;\nu_V,p_V)}C_V(\omega';\nu_V,p_V)f(\nu_V,p_V)$$
$$+\frac{1}{2\pi i}\sum_{\kappa\in\mathcal{C}_\chi}\frac{1}{[\Gamma_\kappa:\Gamma'_\kappa]|\Lambda_\kappa|}\sum_{p\in\frac12\mathbb{Z}}^{\chi}\int_{i[0,\infty)}\overline{B_{\kappa,\chi}(\omega;\nu,p)}\cdot$$
$$\cdot B_{\kappa,\chi}(\omega';\nu,p)f(\nu,p)d\nu. \tag{11.11}$$

It is clear that for $\omega=\omega'$ the measures $d\sigma_{\omega,\omega}$ are non-negative.

If the function f is integrable for $d\sigma_{\omega,\omega}$ and for $d\sigma_{\omega',\omega'}$, then it is also integrable for $d\sigma_{\omega,\omega'}$, and

$$\left|\int_Y f\,d\sigma_{\omega,\omega'}\right| \leqslant \left(\int_Y |f|\,d\sigma_{\omega,\omega}\right)^{1/2}\left(\int_Y |f|\,d\sigma_{\omega',\omega'}\right)^{1/2}. \tag{11.12}$$

On the geometric side, the delta term can be described by introducing another measure $d\delta_{\omega,\omega'}$ on the same set Y:

$$\int_Y f(\nu,p)d\delta_{\omega,\omega'}(\nu,p) := \frac{\alpha(\chi,\xi;\omega,\omega')}{\pi^3 i\sqrt{|d_F|}}\sum_{p\in\frac12\mathbb{Z}}\int_{i[0,\infty)} f(\nu,p)(p^2-\nu^2)d\nu, \tag{11.13}$$

where

$$\alpha(\chi,\xi;\omega,\omega') := \begin{cases} 2\chi(\varepsilon)\varepsilon^{2\xi} & , \text{ if } \omega'=\varepsilon^2\omega \text{ with } \varepsilon\in\mathcal{O}^*/\{\pm1\},\\ 0 & , \text{ otherwise.}\end{cases} \tag{11.14}$$

Actually, the support of $d\delta_{\omega,\omega'}$ is the subset $i[0,\infty)\times\frac12\mathbb{Z}\subset Y$. This measure is positive if $\omega'=\varepsilon^2\omega$ with some $\varepsilon\in\mathcal{O}^*/\{\pm1\}$.

If f is an even function on $i\mathbb{R}$, then the integrals over $i[0,\infty)$ can be replaced by one half of the integrals over the imaginary axis.

To describe the Kloosterman term, we note that for a test function $h(\nu,p)$ discussed above, the expression defining the Bessel transform $\mathbf{B}h$ converges absolutely.

Now, for any even function $f : \mathbb{C}^* \longrightarrow \mathbb{C}$ we define the following sum of Kloosterman sums:

$$\mathrm{Kl}(\omega,\omega';f) := \frac{2}{|d_F|}\left(\frac{i\omega'}{|\omega'|}\right)^{2\xi} \sum_{c\in I}{}' \left(\frac{c}{|c|}\right)^{-2\xi} \frac{S_\chi(\omega',\omega;c)}{|c|^2} f\left(\tfrac{4\pi}{c}\sqrt{\omega\omega'}\right). \tag{11.15}$$

Absolute convergence of $\mathrm{Kl}(\omega,\omega';f)$ follows from the estimate (6.12) as soon as the function f satisfies $f(u) \ll |u|^{2\alpha}$ with $\alpha > \frac{1}{2}$. Indeed,

$$|\mathrm{Kl}(\omega,\omega';f)| \leqslant \frac{2}{|d_F|}\sum_{c\in I}{}' \frac{|S_\chi(\omega',\omega;c)|}{|c|^2}\left|f\left(\tfrac{4\pi}{c}\sqrt{\omega\omega'}\right)\right| \ll_{\omega,\omega',\epsilon} \sum_{c\in I}{}' |c|^{-1+\epsilon-2\alpha}.$$

Since $I \subset \mathcal{O}$ is a lattice in $\mathbb{C}$, the last sum may be estimated by an integral, and the convergence will follow for $-1+\epsilon-2\alpha < -2$.

For $h(\nu,p)$ as above, the absolute convergence of $\mathrm{Kl}(\omega,\omega';\mathbf{B}h)$ is clear because of Lemma 11.1.2.

Each function h defined as in (11.8), based on $\eta,\theta \in \mathcal{T}^l_\sigma$ with $\sigma \in (1,\frac{3}{2})$, is well-defined on Y and the support of the measure $d\sigma_{\omega,\omega'}$ is contained in the domain in h, so we may reformulate the result (11.9) with the new notations:

Lemma 11.2.1. *Let $\sigma \in (1,\frac{3}{2})$ and $\omega,\omega' \in \mathcal{O}'\backslash\{0\}$. Let $l \in \frac{1}{2}\mathbb{N}$ and suppose that h is defined as in (11.8), based on $\eta,\theta \in \mathcal{T}^l_\sigma$.*

Then h is integrable for the measures $d\sigma_{\omega,\omega'}$ and $d\delta_{\omega,\omega'}$, the Bessel transformation as defined in (11.1) converges absolutely, and

$$\int_Y h\,d\sigma_{\omega,\omega'} = \int_Y h\,d\delta_{\omega,\omega'} + \mathrm{Kl}(\omega,\omega';\mathbf{B}h). \tag{11.16}$$

By the phrase *the sum formula with (ω,ω') holds for f* we mean that all sums and integrals involved in (11.16) converge absolutely and the equality (11.16) holds for f.

It is clear that any function $f \in \mathcal{H}^\sigma(a,b)$ with $\sigma \in (1,\frac{3}{2})$ is well-defined on Y, since Y is contained in the domain of f, and the integral $\int_Y f\,d\sigma_{\omega,\omega'}$ makes sense. We would like to extend the sum formula to functions $f \in \mathcal{H}^\sigma(a,b)$ with $\sigma \in (\frac{1}{2},1)$, but in this case the domain of f does not contain the support Y of the measure $d\sigma_{\omega,\omega'}$, and the integral $\int_Y f d\sigma_{\omega,\omega'}$ does not make sense. To "fix" this, for $\sigma \in (\frac{1}{2},1)$, we define

$$Y^\sigma := \Big(\big(i[0,\infty)\cup(0,\sigma]\big)\times\{0\}\Big)\bigcup\Big(i[0,\infty)\times\big(\tfrac{1}{2}\mathbb{Z}\backslash\{0\}\big)\Big), \tag{11.17}$$

and recall that the results of Kim and Shahidi, [20], imply that there are no exceptional spectral parameters in the interval $[\sigma, 1) \subset (\frac{1}{2}, 1)$; see Section 9.2. This means that $\operatorname{supp} d\sigma_{\omega,\omega'} \subset Y^\sigma$, and hence the integral $\int_{Y^\sigma} f d\sigma_{\omega,\omega'}$ makes sense. We extend the definition of the set Y^σ for $\sigma \in (1, \frac{3}{2})$ by setting $Y^\sigma := Y$.

We shall now give two lemmas which will be used in the extension of the sum formula.

Lemma 11.2.2. *Let* $\sigma > \frac{1}{2}$ *and* $\omega \in \mathcal{O}'\backslash\{0\}$. *Suppose the function* f *and the sequence of functions* (f_n) *on* Y^σ *satisfy the following conditions:*

(i) The sum formula with (ω, ω) *holds for each* f_n.

(ii) The integral defining $\mathbf{B}f(u)$ *converges absolutely, and*

$$\lim_{n\to\infty} |u|^{-2\min\{\sigma,1\}} \mathbf{B}f_n(u) = |u|^{-2\min\{\sigma,1\}} \mathbf{B}f(u)$$

uniformly on each set $\{u \in \mathbb{C}^* : |u| \leqslant 4\pi|\omega|\}$.

(iii) f *is integrable for* $d\delta_{\omega,\omega}$.

(iv) $f_n \geqslant 0$ *on* Y^σ, *and* $\lim_{n\to\infty} f_n = f$ *pointwise on* Y^σ.

Then the sum formula with (ω, ω) *holds for* f, *and*

$$\lim_{n\to\infty} \int_{Y^\sigma} f_n \, d\sigma_{\omega,\omega} = \int_{Y^\sigma} f \, d\sigma_{\omega,\omega}, \tag{11.18}$$

$$\lim_{n\to\infty} \int_{Y^\sigma} f_n \, d\delta_{\omega,\omega} = \int_{Y^\sigma} f \, d\delta_{\omega,\omega}, \tag{11.19}$$

$$\lim_{n\to\infty} \mathrm{Kl}(\omega, \omega; \mathbf{B}f_n) = \mathrm{Kl}(\omega, \omega; \mathbf{B}f). \tag{11.20}$$

Proof. We consider the Kloosterman term. The arguments $u_c := \frac{4\pi}{c}|\omega|$ occurring in (11.15) all satisfy $|u_c| \leqslant 4\pi|\omega|$, since $|c| \geqslant 1$ for all $c \in I\backslash\{0\}$. If $\epsilon > 0$ then, from the condition (ii), we have for all sufficiently large n:

$$\left|\mathbf{B}f_n\left(\tfrac{4\pi}{c}|\omega|\right) - \mathbf{B}f\left(\tfrac{4\pi}{c}|\omega|\right)\right| < \epsilon \left(\tfrac{4\pi|\omega|}{|c|}\right)^{2\min\{\sigma,1\}}.$$

Using this, Lemma 11.1.2, and the non-trivial estimate of a Kloosterman sum (6.12), we get

$$\begin{aligned}
|\mathrm{Kl}(\omega, \omega; f)| &\leqslant |\mathrm{Kl}(\omega, \omega; \mathbf{B}f_n)| + |\mathrm{Kl}(\omega, \omega; \mathbf{B}f_n) - \mathrm{Kl}(\omega, \omega; \mathbf{B}f)| \\
&\leqslant \frac{2}{|d_F|} \sum_{c\in I}{}' \frac{|S_\chi(\omega, \omega; c)|}{|c|^2} \Big\{ \left|\mathbf{B}f_n\left(\tfrac{4\pi}{c}|\omega|\right)\right| + \left|\mathbf{B}f_n\left(\tfrac{4\pi}{c}|\omega|\right) - \mathbf{B}f\left(\tfrac{4\pi}{c}|\omega|\right)\right| \Big\} \\
&\ll_{\omega,\epsilon} \sum_{c\in I}{}' |c|^{-1+\epsilon-2\min\{\sigma,1\}}.
\end{aligned}$$

The expression $\mathrm{Kl}(\omega,\omega;\mathbf{B}f)$ converges absolutely since $\min\{\sigma,1\} > \frac{1}{2}$, and (11.20) holds.

Conditions (iii) and (iv) allow us to apply dominated convergence to see that $\lim_{n\to\infty}\int_{Y^\sigma} f_n\, d\delta_{\omega,\omega} = \int_{Y^\sigma} f\, d\delta_{\omega,\omega}$. As the sum formula with $\omega=\omega'$ holds for each f_n, we have a sequence (f_n) of $d\sigma_{\omega,\omega}$-integrable functions on Y^σ for which the integrals tend to a limit equal to $\int_{Y^\sigma} f\, d\delta_{\omega,\omega} + \mathrm{Kl}(\omega,\omega;\mathbf{B}f)$. So, by Fatou's lemma (see e.g. [1], p. 105, [27], p. 141), the limit function f is also integrable for $d\sigma_{\omega,\omega}$. Again by dominated convergence we see that $\lim_{n\to\infty}\int_{Y^\sigma} f_n\, d\sigma_{\omega,\omega} = \int_{Y^\sigma} f\, d\sigma_{\omega,\omega}$, and the sum formula with $\omega=\omega'$ must hold for f. ■

Lemma 11.2.3. *Let $\sigma > \frac{1}{2}$ and $\omega,\omega' \in \mathcal{O}'\backslash\{0\}$. Suppose the functions f, h and the sequence of functions (f_n) on Y^σ satisfy the following conditions:*

(i) The sum formula with (ω,ω') holds for each f_n.

(ii) The sum formula holds for h with (ω,ω) as well as with (ω',ω').

(iii) The integral defining $\mathbf{B}f(u)$ converges absolutely, and

$$\lim_{n\to\infty} |u|^{-2\min\{\sigma,1\}}\mathbf{B}f_n(u) = |u|^{-2\min\{\sigma,1\}}\mathbf{B}f(u)$$

uniformly on each set $\{u\in\mathbb{C}^* : |u| \leqslant r_0\}$ with $r_0 > 0$.

(iv) $\lim_{n\to\infty} f_n = f$ pointwise on Y^σ.

(v) $|f_n| \leqslant h$ on Y^σ.

Then the sum formula with (ω,ω') holds for f, and

$$\lim_{n\to\infty}\int_{Y^\sigma} f_n\, d\sigma_{\omega,\omega'} = \int_{Y^\sigma} f\, d\sigma_{\omega,\omega'}, \tag{11.21}$$

$$\lim_{n\to\infty}\int_{Y^\sigma} f_n\, d\delta_{\omega,\omega'} = \int_{Y^\sigma} f\, d\delta_{\omega,\omega'}, \tag{11.22}$$

$$\lim_{n\to\infty} \mathrm{Kl}(\omega,\omega';\mathbf{B}f_n) = \mathrm{Kl}(\omega,\omega';\mathbf{B}f). \tag{11.23}$$

Proof. The absolute convergence of the sum of Kloosterman sums and the corresponding limit formula follow as in the proof of the previous lemma.

The definition (11.14) gives that $\alpha(\chi,\xi;\omega,\omega)\neq 0$, and for $\omega'=\varepsilon^2\omega$ with $\varepsilon\in\mathcal{O}^*$, we have $\alpha(\chi,\xi;\omega,\omega') = \chi(\varepsilon)\varepsilon^{2\xi}\alpha(\chi,\xi;\omega,\omega)$. This implies that h is integrable for $d\delta_{\omega,\omega'}$. By dominated convergence, using h as a majorant, we see that (11.19) is satisfied.

From (11.12) we see that h is integrable for the measure $d\sigma_{\omega,\omega'}$ as well. Therefore it can be used as majorant in the application of dominated convergence for the spectral side, which implies that (11.18) holds. ■

11.3 Extension of the sum formula

From Lemma 11.2.1 we know that the sum formula holds for functions h of the form (11.8). We shall approximate the elements $f \in \mathcal{H}^\sigma(a,b)$ with $\sigma \in (1, \frac{3}{2})$ by such functions and then conclude that the sum formula holds also for f.

Lemma 11.3.1. *Let $\sigma \in (1, \frac{3}{2})$, $\omega, \omega' \in \mathcal{O}' \backslash \{0\}$, $a, b \in \mathbb{R}$, and $f \in \mathcal{H}^\sigma(a,b)$. The sequence of functions f_n, $n \geqslant 1$, defined by*

$$f_n(\nu,p) := \begin{cases} f(\nu,p)e^{2\nu^2/n} & , \ \text{if } |p| \leqslant n, \\ 0 & , \ \text{if } |p| > n \end{cases} \tag{11.24}$$

converges to the function f pointwise on Y (see (11.10)), and the sum formula with (ω, ω') holds for each f_n.

Proof. The pointwise convergence on Y is clear from the definition (11.24) of f_n. To prove the latter statement, we put

$$\Lambda_n(\nu,p) := \begin{cases} \lambda_l(\nu,p) & , \ \text{if } n \leqslant l, \\ 0 & , \ \text{if } n > l, \end{cases}$$

$$\eta_n(\nu,p) := \begin{cases} \Lambda_n(\nu,p)^{-1} f(\nu,p) e^{\nu^2/n} & , \ \text{if } |p| \leqslant n, \\ 0 & , \ \text{if } |p| > n, \end{cases}$$

$$\theta_n(\nu,p) := \begin{cases} e^{\nu^2/n}, & \text{if } |p| \leqslant n, \\ 0 & , \ \text{if } |p| > n. \end{cases}$$

For $n \leqslant l$, it is obvious that $\theta_n \in \mathcal{T}^l_\sigma$. To conclude the same for η_n, we note that the function Λ_n^{-1} is meromorphic on the strip $|\operatorname{Re}\nu| \leqslant \sigma$ with double poles at $\nu = \pm\frac{1}{2}$ if $p \in \frac{1}{2} + \mathbb{Z}$. Those poles are cancelled by the double zeros of the function $f(\nu,p)$ at $\nu = \pm\frac{1}{2}$ if $p \in \frac{1}{2} + \mathbb{Z}$, prescribed in the definition of f. This means that the product $\Lambda_n^{-1} f$, and hence η_n, are holomorphic on the strip $|\operatorname{Re}\nu| \leqslant \sigma$. Directly from the definition of $\lambda_l(\nu,p)$ we get that $\Lambda_n^{-1}(-\nu,-p) = \Lambda_n^{-1}(\nu,p)$, which implies that also η_n remains unchanged for $(-\nu,-p) \mapsto (\nu,p)$. Some work shows that we have the estimate $\Lambda_n(\nu,p)^{-1} \ll (1 + |\operatorname{Im}\nu|)^{-2l+1} e^{-\pi|\operatorname{Im}\nu|}$, which implies that η_n has indeed the necessary growth behavior. So, $\eta_n \in \mathcal{T}^l_\sigma$.

In addition, the constructions above ensure that $\overline{\theta_n} = \theta_n$, and $\eta_n \overline{\theta_n} \Lambda_n = f_n$. So, the functions f_n are of the form (11.8), and by Lemma 11.2.1 the sum formula with (ω, ω') holds for each f_n, $n \leqslant l$. ■

We shall now conclude that the sum formula holds for elements of $\mathcal{H}^\sigma(a,b)$, $\sigma \in (1, \frac{3}{2})$ in the following "step by step" way:

Define a function $f_{a,b}$, for some $a, b \in \mathbb{R}$, with

$$f_{a,b}(\nu,p) := \left((1-\xi)^2 - \nu^2\right)^2 \left(4 - \nu^2\right)^{-2-a/2} \left(1 + p^2\right)^{-b/2}. \tag{11.25}$$

Here ξ depends on p and it is defined in (4.27). The conditions that the numbers a, b should satisfy will be determined later. By construction, the function $f_{a,b}$ is an element of $\mathcal{H}^\sigma(a,b)$.

1. We take $f = f_{a,b}$ and construct a sequence of functions (f_n) as in Lemma 11.3.1. The lemma then tells us that each f_n satisfies the condition (i) both in Lemma 11.2.2 and Lemma 11.2.3.

2. Using Lemma 11.1.3 we see that the conditions (ii) in Lemma 11.2.2 and (iii) in Lemma 11.2.3 are satisfied provided that $a > 2$ and $b > 3$.

3. We have $f_{a,b}(\nu, p)(p^2 - \nu^2) \ll (1 + |\operatorname{Im}\nu|)^{2-a}(1 + |p|)^{2-b}$, which implies integrability of the function $f_{a,b}$ for the measure $d\delta_{\omega,\omega'}$ under the conditions $a > 3$ and $b > 3$.

4. A check in (11.25) convinces us that $f_{a,b}(\nu, p)$ is chosen in such a way that is non-negative on the set Y defined in (11.10). The factor $e^{2\nu^2/n}$ is also positive on Y. Hence, $f_n \geqslant 0$, and $f_n(\nu, p) \to f_{a,b}(\nu, p)$ on Y for $n \to \infty$, as required in condition (iv) in Lemma 11.2.2.

5. Let $\omega \in \mathcal{O}\backslash\{0\}$. Steps 1–4 tell us that all conditions in Lemma 11.2.2 are satisfied. Therefore we conclude that the sum formula with (ω, ω) holds for $f_{a,b}$ provided that $a > 3$ and $b > 3$.

6. We take a suitable multiple of $f_{a,b}$ as a majorant in Lemma 11.2.3. Let $h = Cf_{a,b}$ and $\omega, \omega' \in \mathcal{O}\backslash\{0\}$. Step 5 implies that if $a > 3$, $b > 3$, the sum formula holds for h with (ω, ω) as well as for $(\omega'\omega')$. This gives condition (ii) in Lemma 11.2.3.

7. If we choose the constant C in h such that $C > e^{2\sigma^2}$, where $|\operatorname{Re}\nu| \leqslant \sigma$, then $|f_n| \leqslant Cf_{a,b} = h$ on Y. This is condition (v) in Lemma 11.2.3.

8. Steps 1, 2, 4, 6, and 7 imply that all the conditions in Lemma 11.2.3 are satisfied, which means that the sum formula with (ω, ω') holds for $f_{a,b}$ provided that $a > 3$ and $b > 3$.

The discussion above proves

Proposition 11.3.2. *Let $\omega, \omega' \in \mathcal{O}\backslash\{0\}$, $\sigma \in (1, \frac{3}{2})$, and $a, b \in \mathbb{R}$ such that $a > 3$, $b > 3$. Each function $h \in \mathcal{H}^\sigma(a,b)$ is integrable for the measures $d\sigma_{\omega,\omega'}$ and $d\delta_{\omega,\omega'}$, the Bessel transformation $\mathbf{B}h$ in (11.1) converges absolutely, and the sum $\mathrm{Kl}(\omega, \omega'; \mathbf{B}h)$ in (11.15) converges absolutely. Moreover*

$$\int_Y h\, d\sigma_{\omega,\omega'} = \int_Y h\, d\delta_{\omega,\omega'} + \mathrm{Kl}(\omega, \omega'; \mathbf{B}h). \tag{11.26}$$

Our final goal is to extend the class of test functions in Proposition 11.3.2 to $h \in \mathcal{H}^\sigma(a,b)$ with $\sigma \in (\frac{1}{2}, 1)$.

Let $\Sigma \in (1, \frac{3}{2})$ and $\sigma \in (\frac{1}{2}, 1)$. We consider a function $f \in \mathcal{H}^\sigma(a,b)$ with $a > 3$, $b > 3$. We shall prove that the sum formula holds for f approximating it by functions $f_n \in \mathcal{H}^\Sigma(a,b)$.

The function $f_{0,0}$ as defined in (11.25) has the minimal number of zeros prescribed in the definition of $\mathcal{H}^\sigma(a,b)$, see p. 97, and it is holomorphic on any strip $|\operatorname{Re}\nu| \leqslant \sigma$ with $\sigma < 2$. We can write the function f as a product $f = f_{0,0}\, g$, with a function g satisfying (i)–(iii) in the definition of the set $\mathcal{H}^\sigma(a,b)$.

We define g_n by convolution of g with a Gauss kernel:

$$g_n(\nu,p) := -i\sqrt{\frac{n}{\pi}} \int_{(0)} g(\mu,p) e^{n(\mu-\nu)^2} d\mu. \tag{11.27}$$

The integral converges absolutely for each $\nu \in \mathbb{C}$ and defines a holomorphic function $\nu \mapsto g_n(\nu,p)$ on $\mathbb{C}$ invariant under $(\nu,p) \mapsto (-\nu,-p)$. The line of integration in (11.27) may be moved to any line $\operatorname{Re}\nu = \alpha_1$, $\alpha_1 \in [-\sigma,\sigma]$.

Let $\nu = \alpha + ir$ with $\alpha \in [-\Sigma,\Sigma]$, $r \in \mathbb{R}$. To estimate the functions g_n, we first rewrite the integral (11.27), by putting $\mu = it$, $t \in \mathbb{R}$, and use the fact that g satisfies estimate (iii) in the definition of the set $\mathcal{H}^\sigma(a,b)$:

$$\begin{aligned} g_n(\nu,p) &= \sqrt{\frac{n}{\pi}} \int_{-\infty}^{\infty} g(it,p) e^{n\alpha^2 - n(r-t)^2 + 2n\alpha(r-t)i} dt \\ &\ll n^{1/2} e^{n\alpha^2} (1+|p|)^{-b} \int_{-\infty}^{\infty} (1+|t|)^{-a} e^{-n(r-t)^2} dt. \end{aligned} \tag{11.28}$$

For any $a \geqslant 0$ and $n \geqslant 1$, the integral is estimated as follows:

$$\begin{aligned} &\int_{-\infty}^{\infty} (1+|t|)^{-a} e^{-n(r-t)^2} dt \ll \\ &\quad \ll \int_{|t|\leqslant|r|/2} 1 \cdot e^{-n(t-r)^2} dt + (1+|r|)^{-a} \int_{|t|\geqslant|r|/2} e^{-n(t-r)^2} dt \\ &\quad \ll \int_{|r|/2}^{3|r|/2} e^{-nt^2} dt + (1+|r|)^{-a} \int_{-\infty}^{\infty} e^{-nt^2} dt \\ &\quad \ll n^{-1/2} \int_{|r|/2}^{\infty} e^{-t^2} dt + n^{1/2} (1+|r|)^{-a} \\ &\quad \ll n^{-1/2} \left\{ \begin{array}{ll} 1 & , |r| \leqslant 1 \\ e^{-|r|\sqrt{n}/2} & , |r| \geqslant 1 \end{array} \right\} + n^{1/2}(1+|r|)^{-a} \\ &\quad \ll n^{-1/2}(1+|r|)^{-a}. \end{aligned} \tag{11.29}$$

From (11.28) and (11.29) we get

$$g_n(\nu,p) \ll_n (1+|p|)^{-b} (1+|\operatorname{Im}\nu|)^{-a}. \tag{11.30}$$

This implies that all functions g_n, $n \geqslant 1$, satisfy the conditions (i)–(iii) in the definition of $\mathcal{H}^{\Sigma}(a,b)$. Hence, the product $f_n := f_{0,0}\, g_n$ is an element of $\mathcal{H}^{\Sigma}(a,b)$. By Proposition 11.3.2 the sum formula holds with (ω,ω') for all f_n, which means that the condition (i) in Lemma 11.2.3 is satisfied. We want to show that the sequence of functions (f_n) approximates f in such a way that Lemma 11.2.3 can be applied.

For all $|\alpha| \leqslant \sigma$, we can move the line of integration:

$$g_n(\nu,p) = -i\sqrt{\frac{n}{\pi}}\int_{(\alpha)} g(\mu,p)e^{n(\nu-\mu)^2}\,d\mu = \sqrt{\frac{n}{\pi}}\int_{-\infty}^{\infty} g(\alpha+it,p)e^{-n(t-r)^2}\,dt$$
$$\ll \sqrt{n}\int_{-\infty}^{\infty}(1+|p|)^{-b}(1+|t|)^{-a}e^{-n(t-r)^2}\,dt \overset{(11.29)}{\ll} (1+|p|)^{-b}(1+|r|)^{-a}.$$

This estimate is uniform in $n \geqslant 1$. It shows that it is possible to choose a constant $C \geqslant 0$ such that $|g_n(\nu,p)| \leqslant C\frac{|f_{a,b}(\nu,p)|}{|f_{0,0}(\nu,p)|}$ for all (ν,p) with $|\operatorname{Re}\nu| \leqslant \sigma$. Then the function $h := Cf_{a,b}$ is the necessary majorant for condition (v) in Lemma 11.2.3 to be satisfied.

On the other hand, $h \in \mathcal{H}^{\Sigma}(a,b)$ and Proposition 11.3.2 implies that condition (ii) in Lemma 11.2.3 is also satisfied.

To obtain the pointwise convergence of $g_n(\nu,p)$ for $|\operatorname{Re}\nu| \leqslant \sigma$, $(\operatorname{Re}\nu = \alpha)$, we carry out the following change of variables:

$$\begin{aligned} g_n(\nu,p) &= \sqrt{\frac{n}{\pi}}\int_{-\infty}^{\infty} g(\alpha+it,p)e^{-n(t-r)^2}\,dt \\ &\quad \big(\text{change} : t \mapsto r + t/\sqrt{n}\big) \\ &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} g\left(\nu + \frac{it}{\sqrt{n}}, p\right)e^{-t^2}\,dt. \end{aligned} \tag{11.31}$$

The function $\nu \mapsto g(\nu,p)$ is bounded on vertical lines, and thus we have an integrable majorant. We are allowed to take the limit as $n \to \infty$ inside the integral, which gives

$$\lim_{n\to\infty} g_n(\nu,p) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\lim_{n\to\infty} g\left(\nu + \frac{it}{\sqrt{n}}, p\right)e^{-t^2}\,dt = g(\nu,p).$$

This further implies pointwise convergence of f_n:

$$\lim_{n\to\infty} f_n(\nu,p) = f_{0,0}(\nu,p)\lim_{n\to\infty} g_n(\nu,p) = f_{0,0}(\nu,p)g(\nu,p) = f(\nu,p), \tag{11.32}$$

for all (ν,p) with $|\operatorname{Re}\nu| \leqslant \sigma$, and hence also for $(\nu,p) \in Y^{\sigma}$ as required in the condition (iv) in Lemma 11.2.3.

Finally, we use Lemma 11.1.3 to establish validity of condition (iii) in Lemma 11.2.3. For that purpose we need an estimate for the difference $g\left(\nu + \frac{it}{\sqrt{n}}, p\right) - g(\nu,p)$, which we shall obtain by estimating the partial derivative $\partial_\nu g$.

Let $\alpha \in (-\sigma, \sigma)$ and $\gamma = \{\mu \in \mathbb{C} : |\mu - \nu| = \sigma - |\alpha|\}$ be a small circle around the point ν. Using the Cauchy integral formula and differentiating under the integral sign yields

$$\begin{aligned}\partial_\nu g(\nu,p) &= \frac{1}{2\pi i}\int_\gamma \frac{g(\mu,p)}{(\mu-\nu)^2}d\mu \ll (1+|p|)^{-b}\int_\gamma (1+|\operatorname{Im}\mu|)^{-a}(\mu-\nu)^{-2}d\mu \\ &\ll (1+|p|)^{-b}(1+|\operatorname{Im}\nu|-\sigma+|\alpha|)^{-a}(\sigma+|\alpha|)^{-1} \\ &\ll_\alpha (1+|p|)^{-b}(1+|\operatorname{Im}\nu|)^{-a}. \end{aligned} \tag{11.33}$$

We use this estimate with $\alpha \in [0, \sigma]$ to obtain from (11.31):

$$\begin{aligned} g_n(\nu,p) - g(\nu,p) &= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}\left[g\left(\nu + \frac{it}{\sqrt{n}}, p\right) - g(\nu,p)\right]e^{-t^2}dt \\ &\overset{(11.33)}{\ll} \int_{-\infty}^{\infty}\left|\frac{it}{\sqrt{n}}\right|(1+|p|)^{-b}(1+|\operatorname{Im}\nu|)^{-a}e^{-t^2}dt \\ &\leqslant (1+|p|)^{-b}(1+|\operatorname{Im}\nu|)^{-a}\int_{-\infty}^{\infty}|t|e^{-t^2}dt = (1+|p|)^{-b}(1+|\operatorname{Im}\nu|)^{-a}. \end{aligned}$$

Multiplication by $f_{0,0}$ does not change this estimate. Therefore, we may apply Lemma 11.1.3 for f and f_n to see that the condition (ii) in Lemma 11.2.3 is also satisfied.

Hence, by Lemma 11.2.3, the sum formula holds for f. This proves

Theorem 11.3.3. (Spectral sum formula) *Let $\sigma \in (\frac{1}{2}, 1)$, and h be a function defined over the set $\{\nu \in \mathbb{C} : |\operatorname{Re}\nu| \leqslant \sigma\} \times \frac{1}{2}\mathbb{Z}$ such that*

(i) $h(\nu,p) = h(-\nu,-p)$,

(ii) $\nu \mapsto h(\nu,p)$ *is holomorphic on a neighborhood of the strip* $|\operatorname{Re}\nu| \leqslant \sigma$,

(iii) $h(\nu,p) \ll (1+|\operatorname{Im}\nu|)^{-a}(1+|p|)^{-b}$ *with* $a > 3$, $b > 3$,

(iv) the function $\nu \mapsto h(\nu,p)$ *has at least double zeros at* $\nu = \pm\frac{1}{2}$ *if* $p \in \frac{1}{2} + \mathbb{Z}$.

Then, for any non-zero $\omega, \omega' \in \mathcal{O}'$, *we have:*

$$\begin{aligned} &\sum_V \overline{C_V(\omega;\nu_V,p_V)}C_V(\omega';\nu_V,p_V)h(\nu_V,p_V) \\ &\quad + \frac{1}{4\pi i}\sum_{\kappa\in\mathcal{C}_\chi}\frac{1}{[\Gamma_\kappa:\Gamma'_\kappa]|\Lambda_\kappa|}\sum_{p\in\frac{1}{2}\mathbb{Z}}^{\chi}\int_{(0)}\overline{B_{\kappa,\chi}(\omega;\nu,p)}B_{\kappa,\chi}(\omega';\nu,p)h(\nu,p)\,d\nu = \\ &= \frac{1}{2\pi^3 i\sqrt{|d_F|}}\sum_{\varepsilon\in\mathcal{O}^*}\delta_{\varepsilon^2\omega,\omega'}\,\chi(\varepsilon)\varepsilon^{2\xi}\sum_{p\in\frac{1}{2}\mathbb{Z}}\int_{(0)}h(\nu,p)(p^2-\nu^2)d\nu \\ &\quad + \frac{2}{|d_F|}\left(\frac{i\omega'}{|\omega'|}\right)^{2\xi}\sum_{c\in I}{}'\left(\frac{c}{|c|}\right)^{-2\xi}\frac{S_\chi(\omega',\omega;c)}{|c|^2}\mathbf{B}h\left(\tfrac{4\pi}{c}\sqrt{\omega\omega'}\right), \end{aligned} \tag{11.34}$$

where $\sum^{\chi}_{p\in\frac{1}{2}\mathbb{Z}}$ *means that the sum runs through all* $p \in \frac{1}{2}\mathbb{Z}$ *such that the condition* $\chi(\varepsilon) = \varepsilon^{2p}$ *is satisfied for all* $\varepsilon \in \mathcal{O}^*$*,* V *runs over a maximal orthogonal system of irreducible cuspidal subspaces of* $L^2(\Gamma\backslash G;\chi)$*,* $|\Lambda_\kappa|$ *is the Euclidean area of a period parallelogram for the lattice* $\Lambda_\kappa \in \mathbb{C}$ *corresponding to* ${g_\kappa}^{-1}\Gamma'_\kappa g_\kappa$*, and* $\mathbf{B}h$ *is the Bessel transform in (11.1). Convergence of these expressions is absolute throughout.*

11.4 Discussion of the spectral sum formula

With condition (iv) in Theorem 11.3.3, we have one extra restriction in the class of test functions in the case of half-integer p, making this class smaller than the corresponding class in the case of integer p. Even if we somehow manage to prove that there are no exceptional eigenvalues for $\Gamma_0(I)$ in the interval $(0, \frac{1}{2})$ and choose $\sigma < \frac{1}{2}$, with σ determining the width of the strip where the spectral parameter ν belongs, the estimation of the sum of Kloosterman sums will cause a problem. More precisely, for such σ, we will not have the absolute convergence of $\mathrm{Kl}(\omega, \omega'; \mathbf{B}h)$.

As a consequence, (iv) is a strong condition which seems to be unavoidable, and it causes complications in some applications of the sum formula.

11.4.1 Comparison with the case of Gaussian number field

In [9], Bruggeman and Motohashi prove the spectral sum formula in the case of the Gaussian number field $F = \mathbb{Q}(i)$, trivial character $\chi = 1$ and $\Gamma = \mathrm{PSL}_2(\mathbb{Z}[i])$. In this case we have

$$d_F = -4, \qquad \xi = 0, \qquad l, p, q \in \mathbb{Z}, \qquad \mathcal{O}^* = \{\pm 1, \pm i\}.$$

Viewing Γ_P and Γ_N as subgroups of $\mathrm{PSL}_2(\mathbb{Z}[i])$ we have $[\Gamma_P : \Gamma_N] = 2$. There is only one cusp for Γ in this case, i.e. $\mathcal{C}_\chi = \{\infty\}$.

It is known (see [11], Proposition 7.6.2) that in this case there are no complementary series due to the absence of exceptional eigenvalues of the Laplacian. Hence the sum in the first term in the left side of (11.34) runs only over V's that are isomorphic to a principal series representation of G.

Recalling that the dual lattice to $\mathbb{Z}[i]$ is $\frac{1}{2}\mathbb{Z}[i]$, we return to (5.6) and see that

$$T_V \varphi_{l,q}(\nu_V, p_V) = \sum_{\substack{\omega\in\frac{1}{2}\mathbb{Z}[i]\\ \omega\neq 0}} c_{T_V}(\omega)\mathbf{J}_\omega\varphi_{l,q}(\nu_V, p_V) = \sum_{\substack{\omega\in\mathbb{Z}[i]\\ \omega\neq 0}} c_{T_V}\left(\tfrac{\omega}{2}\right)\mathcal{A}_\omega\varphi_{l,q}(\nu_V, p_V),$$

because of the relation $\mathbf{J}_\omega = \mathcal{A}_{2\omega}$. This mean that our Fourier coefficients c_{T_V} and the Fourier coefficients c_V in [9], (8.8) satisfy

$$c_{T_V}(\omega) = c_V(2\omega).$$

If $t_V(\omega) \in \mathbb{R}$ is the eigenvalue of the Hecke operator $\mathcal{T}_\omega$ given by [9], (8.12), the relation (8.17) in [9] gives

$$C_V(\omega, \nu_V, p_V) = \pi^{\nu_V} c_V(1) t_V(2\omega).$$

On noting that $\nu_V \in i\mathbb{R}$, we have that the first term in the left side of (11.34) equals:

$$\sum_V |c_V(1)|^2 t_V(2\omega) t_V(2\omega') h(\nu_V, p_V), \tag{11.35}$$

where V runs over all Hecke invariant right-irreducible cuspidal subspaces of $L^2(\Gamma\backslash G)$ that intersect the space $L^2(\Gamma\backslash G; l, q)$ non-trivially.

For the Fourier coefficients of the Eisenstein series $E_{l,q}(\nu, p; 1)$ we have:

$$B_{\infty,1}(\omega; \nu, p) = (2\pi|\omega|)^\nu (\omega/|\omega|)^{-p} D_1^{\infty,\infty}(\omega; \nu, p),$$

with

$$D_1^{\infty,\infty}(\omega; \nu, p) = \frac{1}{2} \sum_{c \neq 0} |c|^{-2(1+\nu)} \, (c/|c|)^{2p} \, S_F(2\omega, 0; c),$$

where $S_F(\omega, \omega'; c)$ is the Kloosterman sum defined in [9], (1.2). Here p must satisfy the condition $i^{2p} = 1$, i.e. $p \in 2\mathbb{Z}$. We note that in this case, $I = \mathbb{Z}[i]/\{\pm 1\}$, and hence the sum $\sum'_{c \in I} = \frac{1}{2} \sum_{c \neq 0}$. Using the Ramanujan identity for $\mathbb{Q}(i)$, see (2.18) in [9], we get

$$B_{\infty,1}(\omega; \nu, p) = 2(2\pi|\omega|)^\nu (\omega/|\omega|)^{-p} \frac{\sigma_{-\nu}(2\omega, p/2)}{\zeta_F(1+\nu, p/2)},$$

for $\omega \in \frac{1}{2}\mathbb{Z}[i]\backslash\{0\}$. Here $\zeta_F(s, p/2)$ is the Hecke L-function of $\mathbb{Q}(i)$ associated with the Grössencharakter $(\omega/|\omega|)^{2p}$, and $\sigma_\nu(\omega, p)$ is the divisor sum given by (2.4) in [9].

Since $|\Lambda_\infty| = 1$, the second term in the left side of (11.34) is equal to

$$\frac{1}{2\pi i} \sum_{p \in 2\mathbb{Z}} \left(\frac{\omega\omega'}{|\omega\omega'|} \right)^p \int_{(0)} \frac{\sigma_\nu(2\omega, -p/2) \sigma_\nu(2\omega', -p/2)}{|4\omega\omega'|^\nu |\zeta_F(1+\nu, p/2)|^2} h(\nu, p) \, d\nu. \tag{11.36}$$

The units in $\mathrm{PSL}_2(\mathbb{Z}[i])$ form the set $\mathcal{O}^* = \{1, i\}$, so

$$\sum_{\varepsilon \in \mathcal{O}^*} \delta_{\omega', \omega\varepsilon^2} \, \chi(\varepsilon) \varepsilon^{2\xi} = \delta_{\omega,\omega'} + \delta_{\omega,-\omega'},$$

and therefore the delta term in (11.34) equals

$$\frac{\delta_{\omega,\omega'} + \delta_{\omega,-\omega'}}{4\pi^3 i} \sum_{p \in \mathbb{Z}} \int_{(0)} h(\nu, p)(p^2 - \nu^2) d\nu. \tag{11.37}$$

We have $\mathcal{K}^*_{\nu,p}(z) = \mathcal{K}_{\nu,p}(z)$, where $\mathcal{K}_{\nu,p}(z)$ is the Bessel kernel in [9]. This implies for the Bessel transform

$$\mathbf{B}h(u) := \frac{1}{2\pi i}\sum_{p\in\mathbb{Z}}\int_{(0)}\mathcal{K}_{\nu,p}(u)h(\nu,p)(p^2-\nu^2)d\nu = 4\,\mathrm{B}h(u),$$

where $\mathrm{B}h(u)$ is the Bessel transform given by [9], (10.2). The Kloosterman sums $S_F(\omega,\omega';c)$ in [9] and $S_1(\omega,\omega';c)$ are related by

$$S_1(\omega,\omega';c) = S_F(2\omega,2\omega';c) = S_F(2\omega',2\omega;c).$$

Recall that $\sum'_{c\in I} = \frac{1}{2}\sum_{c\neq 0}$, so the Kloosterman term in (11.34) is equal to

$$\sum_{c\neq 0}\frac{S_F(2\omega,2\omega';c)}{|c|^2}\mathrm{B}h\left(\frac{2\pi}{c}\sqrt{4\omega\omega'}\right). \tag{11.38}$$

Collecting (11.35)–(11.38), replacing $(2\omega,2\omega')$ by (ω,ω'), and noting that (iv) in Theorem 11.3.3 is an empty condition, we conclude that our Theorem 11.3.3 reduces in the case of the Gaussian number field and trivial character to Theorem 10.1 in [9].

11.4.2 Remarks concerning general number fields

Comparing our sum formula given in Theorem 11.3.3 with the sum formula for SL_2 over a totally real number field given in [6], Theorem 2.7.1, as well as all the necessary ingredients (Fourier expansion of automorphic forms on the upper half-space, Kloosterman sums, test functions, Bessel transformations etc.), we see that they are completely analogous. We will not enter into detailed comparisons, but we would like to mention that our result might enable one to extend the validity of the sum formula to SL_2 over an arbitrary number field.

11.5 Application of the spectral sum formula

We shall use the spectral sum formula, as stated in Theorem 11.3.3 with fixed $\omega = \omega'$, to obtain a density result for the cuspidal automorphic representations by choosing a suitable test function depending on some $a > 0$ and then exploring the asymptotic behavior of all the terms as $a \downarrow 0$. We will show that the delta term produces the leading contribution.

Let $a > 0$. We fix a number $p \in \frac{1}{2}\mathbb{Z}$, integer or half-integer, and choose the following test function

$$h(\nu,q) := \begin{cases} m_p(\nu)e^{-a\lambda} & , \quad q = \pm p \\ 0 & , \quad q \neq \pm p \end{cases} \tag{11.39}$$

where $\lambda = 1 - \nu^2 - p^2$ is the eigenvalue of the real Casimir operator $-4(\Omega_+ + \Omega_-)$, and

$$m_p(\nu) := \begin{cases} \left(\frac{1}{4} - \nu^2\right)^2 & \text{if } p \in \frac{1}{2} + \mathbb{Z} \\ 1 & \text{if } p \in \mathbb{Z} \end{cases}$$

is the factor providing the necessary zeros of the test function in the odd case. With this function and $\omega = \omega'$, formula (11.34) becomes

$$\begin{aligned} &\sum_{V : p_V = \pm p} |C_V(\omega; \nu_V, p_V)|^2 m_{p_V}(\nu_V) e^{-a(1-\nu_V^2 - p_V^2)} = \\ &= \frac{i}{4\pi} \sum_{\kappa \in \mathcal{C}_\chi} \frac{1}{[\Gamma_\kappa : \Gamma'_\kappa]|\Lambda_\kappa|} \sum_{\pm p} \int_{(0)} |B_{\kappa,\chi}(\omega; \nu, p)|^2 m_p(\nu) e^{-a(1-\nu^2-p^2)} d\nu \\ &+ \frac{1}{\pi^3 i \sqrt{|d_F|}} \sum_{\pm p} \int_{(0)} m_p(\nu) e^{-a(1-\nu^2-p^2)} (p^2 - \nu^2) d\nu \\ &+ \frac{2}{|d_F|} \left(\frac{i\omega}{|\omega|}\right)^{2\xi} \sum_{c \in I}{}' \left(\frac{c}{|c|}\right)^{-2\xi} \frac{S_\chi(\omega, \omega; c)}{|c|^2} \mathbf{B}h\left(\tfrac{4\pi|\omega|}{c}\right), \end{aligned} \tag{11.40}$$

where

$$\mathbf{B}h\left(\tfrac{4\pi|\omega|}{c}\right) = \frac{1}{2\pi i} \sum_{\pm p} \int_{(0)} \mathcal{K}^*_{\nu,p}\left(\tfrac{4\pi|\omega|}{c}\right) m_p(\nu) e^{-a(1-\nu^2-p^2)} (p^2 - \nu^2) d\nu. \tag{11.41}$$

Note that the Eisenstein term in (11.40) appears only if the chosen p satisfies the condition $\chi(\varepsilon) = \varepsilon^{2p}$ for all units $\varepsilon \in \mathcal{O}^*$.

We now consider the terms on the right side of (11.40) separately.

Delta term. Let us define $\epsilon_p = 2$ if $p \neq 0$, and $\epsilon_0 = 1$. We have

$$\begin{aligned} &\frac{1}{\pi^3 i \sqrt{|d_F|}} \sum_{\pm p} \int_{(0)} m_p(\nu) e^{-a(1-\nu^2-p^2)} (p^2 - \nu^2) d\nu = \\ &\quad = \frac{2\epsilon_p}{\pi^3 \sqrt{|d_F|}} e^{-a(1-p^2)} \int_0^\infty m_p(it) e^{-at^2} (p^2 + t^2) dt. \end{aligned} \tag{11.42}$$

An easy computation gives for the integral

$$\int_0^\infty m_p(it) e^{-at^2} (p^2 + t^2) dt = \begin{cases} \frac{15\sqrt{\pi}}{16} a^{-7/2} + O_p(a^{-5/2}) & \text{if } p \in \frac{1}{2} + \mathbb{Z}, \\ \frac{\sqrt{\pi}}{4} a^{-3/2} + O_p(a^{-1/2}) & \text{if } p \in \mathbb{Z}. \end{cases}$$

The index p in the notations $O_p(a^{-5/2})$ and $O_p(a^{-1/2})$, means that the implicit constants depend on the fixed number p. Hence, as $a \downarrow 0$, the delta term is

$$\frac{1}{\pi^3 i\sqrt{|d_F|}} \sum_{\pm p} \int_{(0)} m_p(\nu) e^{-a(1-\nu^2-p^2)}(p^2-\nu^2)d\nu =$$

$$= \begin{cases} \frac{\epsilon_p}{2\pi^{5/2}\sqrt{|d_F|}} a^{-3/2} + O_p(a^{-1/2}) & \text{if} \quad p \in \mathbb{Z}, \\ \frac{15}{4\pi^{5/2}\sqrt{|d_F|}} a^{-7/2} + O_p(a^{-5/2}) & \text{if} \quad p \in \frac{1}{2}+\mathbb{Z}. \end{cases} \tag{11.43}$$

Kloosterman term. We start with estimation of the Bessel transformation. From (11.4), with $\alpha_1 = \sigma \in (\frac{1}{2}, 1)$ and h as in (11.39), we have

$$\mathbf{B}h(u) = \frac{2}{\pi} p^2 \,(u/|u|)^{-2p} \,\mathcal{J}^*_{0,p}(u) e^{-a(1-p^2)}$$
$$+ \frac{1}{\pi i} \sum_{\pm p} \int_{(\sigma)} \frac{|u/2|^{2\nu}(iu/|u|)^{-2p-2\xi}}{\sin \pi(\nu - p)} \mathcal{J}^*_{\nu,p}(u) m_p(\nu) e^{-a(1-\nu^2-p^2)}(\nu^2-p^2)d\nu,$$

where the first term comes from the sum of residues at $\nu = 0$, and it is present only if the chosen p is a non-zero integer. We have used that for $p \in \mathbb{Z}$, the relation $(u/|u|)^{-2p} \,\mathcal{J}^*_{0,p}(u) = (u/|u|)^{2p} \,\mathcal{J}^*_{0,-p}(u)$ holds.

For $p \in \mathbb{Z}\backslash\{0\}$, estimate (11.5) implies uniformly in p

$$\frac{2}{\pi} p^2 \,(u/|u|)^{-2p} \,\mathcal{J}^*_{0,p}(u) e^{-a(1-p^2)} \ll$$
$$\ll p^2 \frac{|u|^{2|p|}}{(|p|!)^2} e^{-a(1-p^2)} \leqslant |u|^{2|p|} e^{-a(1-p^2)} \ll_p |u|^{2|p|}, \qquad \text{as} \quad |u| \downarrow 0.$$

Estimate (11.6) implies, as $|u| \downarrow 0$,

$$\frac{1}{\pi i} \sum_{\pm p} \int_{(\sigma)} \frac{|u/2|^{2\nu}(iu/|u|)^{-2p-2\xi}}{\sin \pi(\nu - p)} \mathcal{J}^*_{\nu,p}(u) m_p(\nu) e^{-a(1-\nu^2-p^2)}(\nu^2-p^2)d\nu \ll$$
$$\ll \int_{(\sigma)} \frac{|u|^{2\sigma}}{\big|\nu+|p|\big|} |\nu|^{-2\sigma} |m_p(\nu)| \left| e^{-a(1-\nu^2-p^2)} \right| |\nu^2-p^2||d\nu|. \tag{11.44}$$

Since $\sigma > \frac{1}{2}$, both 2σ and $2\sigma - 1$ are positive, and using $|\nu| \geqslant |\operatorname{Re}\nu| = \sigma$ we obtain

$$\frac{\big|\nu^2-p^2\big|}{\big|\nu+|p|\big||\nu|^{2\sigma}} \leqslant |p||\nu|^{-2\sigma} + |\nu|^{-(2\sigma-1)} < |p|\sigma^{-2\sigma} + \sigma^{-(2\sigma-1)} = \sigma^{-2\sigma}(|p|+\sigma).$$

Thus, by writing $\nu = \sigma + it$, we continue in (11.44)

$$< 2\,|u|^{2\sigma} e^{-a(1-p^2-\sigma^2)} \sigma^{-2\sigma}(|p|+\sigma) \int_0^\infty |m_p(\sigma+it)| e^{-at^2} dt. \tag{11.45}$$

If $p \in \mathbb{Z}$, then $m_p(\sigma+it) = 1$, and the last integral in (11.45) is $\frac{\sqrt{\pi}}{2}a^{-1/2}$. If $p \in \frac{1}{2}+\mathbb{Z}$, then $\left|m_p(\sigma+it)\right| = \left(\frac{1}{4}-\sigma^2+t^2\right)^2 + 4\sigma^2t^2$, and the integral is

$$\int_0^\infty |m_p(\sigma+it)|e^{-at^2}\,dt = \frac{3\sqrt{\pi}}{8}a^{-5/2}\left(1+\frac{1+4\sigma^2}{3}a+\frac{1-4\sigma^2}{12}a^2\right).$$

Hence, as $a \downarrow 0$,

$$\mathbf{B}h(u) \ll_{p,\sigma} \begin{cases} \varsigma_p|u|^{2|p|}+|u|^{2\sigma}a^{-1/2} & \text{if} \quad p \in \mathbb{Z}, \\ |u|^{2\sigma}a^{-5/2} & \text{if} \quad p \in \frac{1}{2}+\mathbb{Z}. \end{cases} \tag{11.46}$$

with $\varsigma_p = 1$ if $p \in \mathbb{Z}\backslash\{0\}$, and $\varsigma_0 = 0$.

The estimate (6.12), for $\omega = \omega' \neq 0$ and any $c \in I\backslash\{0\}$, yields

$$S_\chi(\omega,\omega;c) \ll_{\omega,\delta} |c|^{1+\delta}, \tag{11.47}$$

for each $\delta > 0$. From (11.46) and (11.47), we obtain the following estimate for the Kloosterman term:

$$\begin{aligned}
&\left|\frac{2}{|d_F|}\left(\frac{i\omega}{|\omega|}\right)^{2\xi}\sum_{c\in I}{}' \left(\frac{c}{|c|}\right)^{-2\xi}\frac{S_\chi(\omega,\omega;c)}{|c|^2}\mathbf{B}h\left(\tfrac{4\pi|\omega|}{c}\right)\right| \ll \\
&\quad \ll_{F,\omega,\delta} \sum_{c\in I}{}' |c|^{\delta-1}\left|\mathbf{B}h\left(\tfrac{4\pi|\omega|}{c}\right)\right| \\
&\quad \ll_{F,\omega,\delta,\sigma,p} \begin{cases} \varsigma_p\sum'_{c\in I}|c|^{\delta-1-2|p|}+\left(\sum'_{c\in I}|c|^{\delta-1-2\sigma}\right)a^{-1/2} & \text{if } p\in\mathbb{Z}, \\ \left(\sum'_{c\in I}|c|^{\delta-1-2\sigma}\right)a^{-5/2} & \text{if } p \in \frac{1}{2}+\mathbb{Z}. \end{cases}
\end{aligned}$$

If $p = 0$ or $p \in \frac{1}{2}+\mathbb{Z}$, the sum $\sum'_{c\in I}|c|^{\delta-1-2|p|}$ is absent and the choice of δ such that $\delta < 2\alpha-1$ implies convergence of the sum $\sum'_{c\in I}|c|^{\delta-1-2\sigma}$. If $p \in \mathbb{Z}\backslash\{0\}$, then by choosing $\delta < \min\{2|p|-1, 2\alpha-1\}$, both sums converge. Hence, for any fixed $p \in \frac{1}{2}\mathbb{Z}$, the Kloosterman term, as $a \downarrow 0$, is estimated as follows,

$$\ll_{F,\omega,\delta,\sigma,p} \begin{cases} a^{-1/2} & \text{if } p \in \mathbb{Z}, \\ a^{-5/2} & \text{if } p \in \frac{1}{2}+\mathbb{Z}. \end{cases} \tag{11.48}$$

Eisenstein term. The Eisenstein series $E^\kappa_{l,q}(\nu,p;\chi)$ for $\Gamma = \Gamma_0(I)$ with a character χ are linear combinations of Eisenstein series for the principal congruence subgroup $\Gamma(I)$ with trivial character. Thus, also the Fourier coefficients of the Eisenstein series for Γ are linear combinations of the Fourier coefficients of Eisenstein series for $\Gamma(I)$. The Fourier coefficients of the Eisenstein series for the principal congruence subgroup, on the other hand, may be written as a sum of quotients of a bounded expressions and certain L-series. A lower bound for those L-series on the critical line will hence give an estimate for the Fourier coefficient $D_\chi^{\kappa,\infty}(\omega;\nu,p)$. We use the following claim:

For each cusp $\kappa \in \mathcal{C}_\chi$, $\operatorname{Re}\nu = 0$ and $\varepsilon > 0$, we have

$$D_\chi^{\kappa,\infty}(\omega;\nu,p) \ll |\omega|^\varepsilon \begin{cases} \log^7(2+|\operatorname{Im}\nu|) + \log^7|p| & \text{if } p \neq 0, \\ \log^7(2+|\operatorname{Im}\nu|) & \text{if } p = 0. \end{cases} \tag{11.49}$$

The implicit constants in the estimates depend on the field F, the ideal I, the choice of the elements g_κ describing cusps. We do not work out all details of the proof here. The reasoning to reach estimate (76) in [7] can be also carried out in the present situation.

REMARK 13. Lower powers of the logarithm can be obtained in a much more direct way. The technique in Sections 3.10–3.11 in [40] is applicable for all L-series with Euler products (this is the important point). Motohashi has done the calculations in the case of the Gaussian number field and the ideal $I = \mathbb{Z}[i]$. His idea is that an upper bound analogue to the first condition in [40], Theorem 3.10, is a consequence of the functional equation satisfied by the L-series in question and the Phragmén-Lindelöf convexity principle. Then, via Landau's lemmas one can obtain a lower bound for the L-series.

Estimate (11.49) implies that

$$B_{\kappa,\chi}(\omega;\nu,p) \ll_{F,I,\varepsilon,\kappa} |\omega|^{\varepsilon+\operatorname{Re}\nu}\Big\{\log^7(2+|\operatorname{Im}\nu|) + \varsigma_p \log^7|p|\Big\},$$

where $\varsigma_p = 1$ if $p \neq 0$, and $\varsigma_0 = 0$. Let $C := \frac{|\mathcal{C}_\chi|}{|d_F|\pi i}\big\{\min_{\kappa\in\mathcal{C}_\chi}[\Gamma_\kappa : \Gamma'_\kappa]|\Lambda_\kappa|\big\}^{-1}$. The Eisenstein term in (11.40) is then estimated as follows

$$\begin{aligned}
&\frac{1}{4\pi i}\sum_{\kappa\in\mathcal{C}_\chi}\frac{1}{[\Gamma_\kappa : \Gamma'_\kappa]|\Lambda_\kappa|}\sum_{\pm p}\int_{(0)}|B_{\kappa,\chi}(\omega;\nu,p)|^2 m_p(\nu) e^{-a(1-\nu^2-p^2)}d\nu \ll \\
&\ll 2C\int_{(0)}|\omega|^{2(\varepsilon+\operatorname{Re}\nu)}\Big\{\log^{14}(2+|\operatorname{Im}\nu|) + \varsigma_p(\log|p|)^{14}\Big\}\cdot \\
&\qquad\qquad\qquad \cdot\big|m_p(\nu)\big|\,\Big|e^{-a(1-\nu^2-p^2)}\Big|\,|d\nu| \\
&\ll |\omega|^{2\varepsilon}e^{-a(1-p^2)}\int_0^\infty\Big\{\log^{14}(2+t) + \varsigma_p(\log|p|)^{14}\Big\}\big|m_p(it)\big|e^{-at^2}dt \\
&\ll_{\omega,p} a^{-1/2}\int_0^\infty \log^{14}(2\sqrt{a}+t)\big|m_p(ia^{-1/2}t)\big|e^{-t^2}dt \\
&\quad + a^{-1/2}\Big\{|\log a|^{14} + \varsigma_p(\log|p|)^{14}\Big\}\int_0^\infty\big|m_p(ia^{-1/2}t)\big|e^{-t^2}dt.
\end{aligned} \tag{11.50}$$

To estimate the integrals, we consider separately the cases when p is an integer and when p is a half-integer.

If $p \in \mathbb{Z}$, then $m_p(ia^{-1/2}t) = 1$ and the Eisenstein term is estimated by

$$\ll_{\omega,p} a^{-1/2} \int_0^\infty \log^{14}(2\sqrt{a}+t)e^{-t^2}dt + \frac{\sqrt{\pi}}{2}a^{-1/2}\left\{|\log a|^{14} + \varsigma_p\left(\log|p|\right)^{14}\right\}.$$

For $0 < t < 1$, we have $\log^{14}(2\sqrt{a}+t)e^{-t^2} = |\log a|^{14} + O(t)$, and thus

$$\int_0^1 \log^{14}(2\sqrt{a}+t)e^{-t^2}dt \ll |\log a|^{14}. \tag{11.51}$$

For $t > 1$ and $0 < a < 1$,

$$\int_1^\infty \log^{14}(2\sqrt{a}+t)e^{-t^2}dt < \int_1^\infty (2+t)^{14}e^{-t^2}dt = O(1). \tag{11.52}$$

Hence, the Eisenstein term in this case is

$$\ll a^{-1/2}\left\{|\log a|^{14} + O(1) + \varsigma_p\left(\log|p|\right)^{14}\right\} \ll_{\omega,p} |\log a|^{14}a^{-1/2}, \tag{11.53}$$

as $a \downarrow 0$.

If $p \in \frac{1}{2}+\mathbb{Z}$, then $\left|m_p(ia^{-1/2}t)\right| = \frac{t^4}{a^2} + \frac{t^2}{2a} + \frac{1}{16}$, and the Eisenstein term is estimated by

$$\begin{aligned}
&\ll_{\omega,p} a^{-1/2}\int_0^\infty \log^{14}(2\sqrt{a}+t)\left|m_p(ia^{-1/2}t)\right|e^{-t^2}dt \\
&\qquad + a^{-1/2}\left\{|\log a|^{14} + \varsigma_p\left(\log|p|\right)^{14}\right\}\int_0^\infty\left(\frac{t^4}{a^2}+\frac{t^2}{2a}+\frac{1}{16}\right)e^{-t^2}dt \\
&\ll a^{-1/2}\left\{\int_0^1 \log^{14}(2\sqrt{a}+t)e^{-t^2}dt + \int_1^\infty \log^{14}(2\sqrt{a}+t)a^{-2}t^4e^{-t^2}dt\right\} \\
&\qquad + a^{-1/2}\left\{|\log a|^{14} + \varsigma_p\left(\log|p|\right)^{14}\right\}\frac{\sqrt{\pi}}{32}a^{-2}(12+4a+a^2) \\
&\overset{(11.51)}{\ll} a^{-1/2}|\log a|^{14} + a^{-5/2}\int_1^\infty \log^{14}(2\sqrt{a}+t)t^4e^{-t^2}dt \\
&\qquad + a^{-5/2}\left\{|\log a|^{14} + \varsigma_p\left(\log|p|\right)^{14}\right\}.
\end{aligned}$$

Similarly to (11.52), we have for $t > 1$ and $0 < a < 1$,

$$\int_1^\infty \log^{14}(2\sqrt{a}+t)t^4e^{-t^2}dt < \int_1^\infty (2+t)^{14}t^4e^{-t^2}dt = O(1),$$

and therefore conclude that the Eisenstein term in this case is

$$\ll_{\omega,p} |\log a|^{14}a^{-5/2}, \qquad \text{as} \quad a \downarrow 0. \tag{11.54}$$

Density results. Finally, collecting (11.43), (11.48), (11.53), and (11.54) into the formula (11.40), we obtain

$$\sum_{V:p_V=\pm p} |C_V(\omega;\nu_V,p_V)|^2 m_{p_V}(\nu_V) e^{-a(1-\nu_V^2-p_V^2)} = K_p\, a^{-s_p} + \mathrm{rest}(a),$$

where

$$K_p = \begin{cases} \frac{\epsilon_p}{2\pi^{5/2}\sqrt{|d_F|}} & \text{if} \quad p\in\mathbb{Z} \\ \frac{15}{4\pi^{5/2}\sqrt{|d_F|}} & \text{if} \quad p\in\frac{1}{2}+\mathbb{Z} \end{cases}, \qquad s_p = \begin{cases} \frac{3}{2}, & p\in\mathbb{Z} \\ \frac{7}{2}, & p\in\frac{1}{2}+\mathbb{Z}, \end{cases}, \tag{11.55}$$

and

$$|\mathrm{rest}(a)| = \begin{cases} O(|\log a|^{14} a^{-s_p+1}) & \text{if } p \text{ s.t. } \chi(\varepsilon)=\varepsilon^{2p},\ \forall\varepsilon\in\mathcal{O}^* \\ O(a^{-s_p+1}) & \text{otherwise.} \end{cases} \tag{11.56}$$

Since obviously, both $O(|\log a|^{14}a^{-s_p+1})$ and $O(a^{-s_p+1})$ are $o(a^{-s_p})$ as $a\downarrow 0$, this proves

Proposition 11.5.1. *Let $p\in\frac{1}{2}\mathbb{Z}$ and $\omega\in\mathcal{O}'\backslash\{0\}$ be fixed, χ a character of $\Gamma_0(I)$, and $\epsilon_p = 2$ if $p\in\mathbb{Z}\backslash\{0\}$ and $\epsilon_0 = 1$. Let V run through all the χ-automorphic representations with spectral parameter $(\nu_V,\pm p)$, let $C_V(\omega;\nu_V,p_V)$ be the Fourier coefficient of order ω in the expansion (10.4) normalized as in (10.5), and let λ_V be the eigenvalue of the real Casimir operator $-4(\Omega_+ + \Omega_-)$.*

The following estimates are true, for each $\varepsilon>0$, as $a\downarrow 0$:

(i) if $p\in\mathbb{Z}$, then

$$\sum_{V:p_V=\pm p} |C_V(\omega;\nu_V,p_V)|^2 e^{-a\lambda_V} = \frac{\epsilon_p}{2\pi^{5/2}\sqrt{|d_F|}}\, a^{-3/2} + O(a^{-1/2-\varepsilon}),$$

(ii) if $p\in\frac{1}{2}+\mathbb{Z}$, then

$$\sum_{V:p_V=\pm p} |C_V(\omega;\nu_V,p_V)|^2 \left(\tfrac{1}{4}-\nu_V^2\right)^2 e^{-a\lambda_V} = \frac{15}{4\pi^{5/2}\sqrt{|d_F|}}\, a^{-7/2} + O(a^{-5/2-\varepsilon}).$$

Applying a Tauberian argument to the result in the proposition above, we obtain the following density result:

Theorem 11.5.2. *Let $p\in\frac{1}{2}\mathbb{Z}$ and $\omega\in\mathcal{O}'\backslash\{0\}$ be fixed, χ a character of $\Gamma_0(I)$, and $\epsilon_p = 2$ if $p\in\mathbb{Z}\backslash\{0\}$ and $\epsilon_0 = 1$. Let V run through all the χ-automorphic representations with spectral parameter $(\nu_V,\pm p)$, let $C_V(\omega;\nu_V,p_V)$ be the Fourier coefficient of order ω in the expansion (10.4) normalized as in (10.5), and let λ_V be the eigenvalue of the real Casimir operator $-4(\Omega_+ + \Omega_-)$.*

The following estimates, concerning the cuspidal representations V with eigenvalue λ_V not exceeding X, hold:

(i) if $p \in \mathbb{Z}$, then

$$\sum_{\substack{V:p_V=\pm p\\ \lambda_V\leqslant X}} |C_V(\omega;\nu_V,p_V)|^2 \sim \frac{2\epsilon_p}{3\pi^3\sqrt{|d_F|}}X^{3/2}, \qquad \text{as} \quad X\to\infty,$$

(ii) if $p \in \frac{1}{2}+\mathbb{Z}$, then

$$\sum_{\substack{V:p_V=\pm p\\ \lambda_V\leqslant X}} |C_V(\omega;\nu_V,p_V)|^2 \left(\tfrac{1}{4}-\nu_V^2\right)^2 \sim \frac{4}{7\pi^3\sqrt{|d_F|}}X^{7/2}, \qquad \text{as} \quad X\to\infty.$$

Proof. We set the non-negative function $\alpha(t)$ to be equal to the step-function

$$\alpha(t) := \sum_{\substack{V:p_V=\pm p\\ \lambda_V\leqslant t}} |C_V(\omega;\nu_V,p_V)|^2 m_{p_V}(\nu_V), \qquad t>0,$$

where $\lambda_V = 1-\nu_V^2-p^2$. Then the Stieltjes integral $f(x) := \int_0^\infty e^{-xt}d\,\alpha(t)$ becomes

$$f(x) = \sum_{V:p_V=\pm p} |C_V(\omega;\nu_V,p_V)|^2 m_{p_V}(\nu_V)e^{-x\lambda_V},$$

which by Proposition 11.5.1 is approximated as follows: $f(x)\sim K_p\,x^{-s_p}$ as $x\downarrow 0$, with K_p and s_p as in (11.55). The Tauberian result, Theorem 4.3 on page 192 in [45], then gives

$$\alpha(X) \sim \frac{K_p}{\Gamma(s_p+1)}X^{s_p}, \qquad \text{as} \quad X\to\infty$$

which is (i) for $p\in\mathbb{Z}$, and (ii) for $p\in\frac{1}{2}+\mathbb{Z}$. ■

It would be nice to get rid of the factor $\left(\frac{1}{4}-\nu_V^2\right)^2$ in (ii), in order to get asymptotic for the same sum both for integer and half-integer p. For that purpose we shall use Theorem 11.5.2, (ii) and a partial summation formula. We could not find precisely the needed result in the literature so we give a proof here without claiming any originality.

Let us denote $\mu_V := \frac{1}{4}-\nu_V^2$. Since we consider the odd case, i.e. $p\in\frac{1}{2}+\mathbb{Z}$, we know that there are no complementary series, and hence $\nu_V\in i\mathbb{R}$. This implies that $\mu_V = \frac{1}{4}+|\nu_V|^2 \geqslant \frac{1}{4}$, and $\lambda_V\leqslant X$ is equivalent to $\mu_V\leqslant X+p^2-\frac{3}{4}$.

Let $A(t) := \sum_{\substack{V:p_V=\pm p\\ \mu_V\leqslant t}} |C_V(\omega;\nu_V,p_V)|^2\mu_V^2$. The result (ii) of Theorem 11.5.2 can be then rewritten as follows

$$A(Y)\sim \beta Y^\gamma, \qquad \text{as} \quad Y\to\infty, \tag{11.57}$$

where

$$\beta := \tfrac{4}{7\pi^3\sqrt{|d_F|}}, \qquad \gamma := \tfrac{7}{2} > 2, \qquad \text{and} \qquad Y := X+p^2-\tfrac{3}{4}. \tag{11.58}$$

We take $g(x) := x^{-2}$ in the partial summation formula (A.21) on page 489 in [18] and get

$$\sum_{\substack{V: p_V = \pm p \\ \mu_V \leqslant Y}} |C_V(\omega; \nu_V, p_V)|^2 = \frac{A(Y)}{Y^2} + 2 \int_{1/4}^{Y} \frac{A(t)}{t^3}\, dt. \tag{11.59}$$

From (11.57) we have that for any $\varepsilon > 0$, there is $n_\varepsilon > \frac{1}{4}$ such that for all $t \geqslant n_\varepsilon$ holds

$$\beta(1-\varepsilon)\, t^\gamma < A(t) < \beta(1+\varepsilon)\, t^\gamma. \tag{11.60}$$

Since the set $\{\nu_V\}$, and hence also $\{\mu_V\}$, is discrete in $\mathbb{R}$ with finite multiplicities, the function $A(t)$ is a step-function and the interval $[\frac{1}{4}, n_\varepsilon)$ is contained in a finite union $\bigcup_{i=0}^{k-1} [t_i, t_{i+1})$ for some $k \in \mathbb{N}$, $t_0 := \frac{1}{4}$ and $t_k \geqslant n_\varepsilon$, where A is constant on each subinterval. If we denote

$$M := \max_{i=0,\dots,k} \left| \frac{A(t_i)}{\beta {t_i}^\gamma} \right| \qquad \text{and} \qquad m := \min_{i=0,\dots,k} \left| \frac{A(t_i)}{\beta {t_{i+1}}^\gamma} \right|,$$

then, for all $t \in [\frac{1}{4}, n_\varepsilon)$, we have

$$\beta m\, t^\gamma \leqslant A(t) \leqslant \beta M\, t^\gamma. \tag{11.61}$$

From (11.60) and (11.61), for $Y \geqslant \max\{2, n_\varepsilon\}$, we get

$$\begin{aligned} &\frac{\beta}{\gamma-2} \left\{ 1 - \varepsilon + (m - 1 + \varepsilon) \left(\frac{n_\varepsilon}{Y}\right)^{\gamma-2} - \frac{m}{(4Y)^{\gamma-2}} \right\} \leqslant \\ &\qquad \leqslant \frac{1}{Y^{\gamma-2}} \int_{1/4}^{Y} \frac{A(t)}{t^3}\, dt \leqslant \frac{\beta}{\gamma-2} \left\{ 1 + \varepsilon + (M - 1 - \varepsilon) \left(\frac{n_\varepsilon}{Y}\right)^{\gamma-2} - \frac{M}{(4Y)^{\gamma-2}} \right\}. \end{aligned}$$

This means that both $\overline{\lim}$ and $\underline{\lim}$ as $Y \to \infty$ of the quantity $Y^{-\gamma+2} \int_{1/4}^{Y} \frac{A(t)}{t^3}\, dt$ are equal to the constant $\frac{\beta}{\gamma-2}$. Hence

$$\lim_{Y\to\infty} \frac{1}{Y^{\gamma-2}} \int_{1/4}^{Y} \frac{A(t)}{t^3}\, dt = \frac{\beta}{\gamma-2}. \tag{11.62}$$

Thus, (11.59) yields

$$\begin{aligned} &\lim_{Y\to\infty} \frac{1}{Y^{\gamma-2}} \sum_{\substack{V: p_V = \pm p \\ \mu_V \leqslant Y}} |C_V(\omega; \nu_V, p_V)|^2 = \\ &\qquad = \lim_{Y\to\infty} \frac{A(Y)}{Y^\gamma} + 2 \lim_{Y\to\infty} \frac{1}{Y^{\gamma-2}} \int_{1/4}^{Y} \frac{A(t)}{t^3}\, dt \overset{(11.57),(11.62)}{=} \beta + \frac{2\beta}{\gamma-2}, \end{aligned}$$

that is,

$$\sum_{\substack{V:p_V=\pm p\\ \mu_V\leqslant Y}} |C_V(\omega;\nu_V,p_V)|^2 \sim \frac{\beta\gamma}{\gamma-2}\,Y^{\gamma-2}, \qquad \text{as}\quad Y\to\infty.$$

Returning to the notations (11.58), this proves

Theorem 11.5.3. *Let $p\in\frac{1}{2}+\mathbb{Z}$ and $\omega\in\mathcal{O}'\backslash\{0\}$ be fixed, and χ a character of $\Gamma_0(I)$. Let V run through all the χ-automorphic representations with spectral parameter $(\nu_V,\pm p)$, let $C_V(\omega;\nu_V,p_V)$ be the Fourier coefficient of order ω in the expansion (10.4) normalized as in (10.5), and let λ_V be the eigenvalue of the real Casimir operator $-4(\Omega_+ + \Omega_-)$.*

The following estimate, concerning the odd cuspidal representations V with eigenvalue λ_V not exceeding X, holds:

$$\sum_{\substack{V:p_V=\pm p\\ \lambda_V\leqslant X}} |C_V(\omega;\nu_V,p_V)|^2 \sim \frac{4}{3\pi^3\sqrt{|d_F|}}\,X^{3/2}, \qquad \text{as}\quad X\to\infty.$$

Comparing Theorem 11.5.2, (i) with Theorem 11.5.3, we see that the same density result holds for all $p\neq 0$, integer or half-integer.

One of the symmetries of interest for automorphic forms on $\Gamma\backslash G$ is derived from the Galois action on the group G, $g\to\bar{g}$. This involution maps Γ onto itself and induces the involution on $L^2(\Gamma\backslash G)$ given by $f^*(g)=f(\bar{g})$. It sends even functions to even functions and odd functions to odd functions, it preserves the properties of polynomial growth, square-integrability, orthogonality, and it satisfies $(F_\omega f)^* = F_{\bar{\omega}} f^*$. For all $g=n\mathrm{a}[r]k\in G$, we have

$$\begin{aligned}\varphi_{l,q}(\nu,p)^*(g) &= \varphi_{l,q}(\nu,p)(\bar{n}\mathrm{a}[r]\bar{k}) = r^{1+\nu}\Phi^l_{p,q}(\bar{k})\\ &= (-1)^{p+q} r^{1+\nu}\Phi^l_{-p,-q}(k) = (-1)^{p+q}\varphi_{l,-q}(\nu,-p)(g). \qquad (11.63)\end{aligned}$$

Each $\varphi_{l,q}(\nu,p)$ generates the space of automorphic forms on $\Gamma\backslash G$ of type (l,q) with spectral parameter (ν,p), and therefore the equality (11.63) means that $\varphi_{l,q}(\nu,p)^*$ belongs to the space of automorphic forms on $\Gamma\backslash G$ of type $(l,-q)$ with spectral parameter $(\nu,-p)$.

The symmetry in p means that whenever a representation with $p_V=p\neq 0$ occurs in the sum $\sum_{\substack{V:p_V=\pm p\\ \lambda_V\leqslant X}}$, a representation with $p_V=-p$ occurs also. Having in mind that we only consider a Fourier coefficients of fixed order ω and the choice (3.21), this implies that the sum over representations with $p_V=p\neq 0$ is one half of the sum over representations with $p_V=\pm p$, provided that $\nu\neq 0$.

An immediate consequence of Theorem 11.5.2, (i) and Theorem 11.5.3, is

Corollary 11.5.4. *For each $p\in\frac{1}{2}\mathbb{Z}$, there is an infinite orthogonal system of cuspidal automorphic representations in $L^2(\mathrm{SL}_2(\mathcal{O})\backslash\mathrm{SL}_2(\mathbb{C}))$ with $p_V=p$.*

Moreover, all cuspidal automorphic representations in $L^2(\mathrm{SL}_2(\mathcal{O})\backslash\mathrm{SL}_2(\mathbb{C}))$ with $p_V=p$ are nicely spread with the same density for different p's.

Chapter 12

Bessel inversion

The sum formula (11.34) has the independent test function h on the spectral side and therefore it is useful to obtain information concerning spectral data: the spectral parameter (ν_V, p_V) and the coefficients $C_V(\omega; \nu_V, p_V)$ of the cuspidal representations. The Bessel transform $\mathbf{B}h$ in the sum of Kloosterman sums $\mathrm{Kl}(\omega, \omega'; \mathbf{B}h)$ on the geometric side of the formula depends on the test function h. If we can treat the function $\mathbf{B}h$ as an independent test function, then the sum formula becomes more useful: it can be used to investigate sums of Kloosterman sums $S_\chi(\omega, \omega'; c)$. To do that, we have to invert the Bessel transformation $\mathbf{B}$. The sum formula with the independent test function on the geometric side is called the Kloosterman sum formula.

12.1 Inverse Bessel transformation

For any function f on $\mathbb{C}^*$ with suitable growth behavior near 0 and ∞, we have the transforms:

$$\mathbf{J}f(\nu, p) := \int_{\mathbb{C}^*} f(u)|u/2|^{2\nu}(iu/|u|)^{-2p+2\xi}\mathcal{J}^*_{\nu,p}(u)d_*u, \tag{12.1}$$

$$\mathbf{K}f(\nu, p) := \frac{\pi}{2}\int_{\mathbb{C}^*} f(u)(u/|u|)^{4\xi}\mathcal{K}^*_{\nu,p}(u)d_*u, \tag{12.2}$$

with the measure $d_*u = |u|^{-2}d_+u$ on $\mathbb{C}^*$, $\xi \in \{0, \frac{1}{2}\}$ as in (4.27), and

$$\mathcal{J}^*_{\nu,p}(z) = J^*_{\nu-p}(z)J^*_{\nu+p}(\bar{z}),$$

$$\mathcal{K}^*_{\nu,p}(z) = \frac{(iz/|z|)^{-2\xi}}{\sin\pi(\nu-p)}\left\{\left|\frac{z}{2}\right|^{-2\nu}\left(\frac{iz}{|z|}\right)^{2p}\mathcal{J}^*_{-\nu,-p}(z) - \left|\frac{z}{2}\right|^{2\nu}\left(\frac{iz}{|z|}\right)^{-2p}\mathcal{J}^*_{\nu,p}(z)\right\}$$

as in (4.58) and (9.26), respectively.

We call the transformation $\mathbf{K}$ the inverse Bessel transformation simply because of the order of appearance. We started with the spectral sum formula, where the Bessel transformation $\mathbf{B}$ appeared, and then by inverting $\mathbf{B}$ we shall obtain the Kloosterman sum formula, where the inverse Bessel transformation $\mathbf{K}$ appears.

Some simple properties of these transforms are

$$\mathbf{K}f(\nu,p) = \frac{(-1)^{2\xi}\pi}{2\,\sin\pi(\nu-p)}\Big\{\mathbf{J}f(-\nu,-p) - \mathbf{J}f(\nu,p)\Big\}, \tag{12.3}$$

$$\mathbf{K}f(-\nu,-p) = \mathbf{K}f(\nu,p), \tag{12.4}$$

$$\overline{\mathbf{J}f(\nu,p)} = (-1)^{2\xi}\mathbf{J}\overline{f_{(\xi)}}(\bar\nu,-p), \tag{12.5}$$

$$\overline{\mathbf{K}f(\nu,p)} = (-1)^{2\xi}\mathbf{K}\overline{f_{(\xi)}}(\bar\nu,-p) = (-1)^{2\xi}\mathbf{K}\overline{f_{(\xi)}}(-\bar\nu,p). \tag{12.6}$$

Here $f_{(\xi)}(u) := (u/|u|)^{4\xi}f(u)$.

Remark 14. In the case $p \in \mathbb{Z}$, we have $\mathbf{J}f(\nu,p) = (-1)^p\mathrm{J}f(\nu,p)$ and $\mathbf{K}f(\nu,p) = \frac{\pi}{2}\,\mathrm{K}f(\nu,p)$, where $\mathrm{J}f(\nu,p)$ and $\mathrm{K}f(\nu,p)$ are the functions in [9] given by (11.6) and (11.1), respectively.

For most of this section, we shall consider these transforms for $f \in C^\infty_{c,\mathrm{ev}}(\mathbb{C}^*)$, the space of smooth, compactly supported, even functions on $\mathbb{C}^*$.

Each function $f \in C^\infty_{c,\mathrm{ev}}(\mathbb{C}^*)$ has a polar Fourier expansion

$$f(re^{i\varphi}) = \sum_{n\in\mathbb{Z}} e^{2in\varphi}f_n(r), \qquad f_n(r) = \frac{1}{2\pi}\int_0^{2\pi} f(re^{i\varphi})e^{-2in\varphi}d\varphi.$$

The Fourier coefficients $f_n(r)$ are smooth, compactly supported functions on the interval $(0,\infty)$; their support is contained in a compact set not depending on n. Moreover, a multiple application of integration by parts gives, for any integer $A > 0$,

$$f_n(r) \ll C_f^A(1+|n|)^{-A}, \tag{12.7}$$

where the constant C_f depends on the compact support of f, and the implicit constant only on A and f.

For a smooth, compactly supported function f on $(0,\infty)$, the Mellin transform $\mathcal{M}f$, see (9.10) and (9.11), is holomorphic on $\mathbb{C}$ and satisfies for each $a \in \mathbb{N}$:

$$\mathcal{M}f(s) \ll_{f,a} C_f^{|\operatorname{Re} s|}\frac{1}{|(s)_a|}.$$

The constant C_f depends on the compact support of f, and the implicit constant depends on the support of f and the supremum norm of f and its derivatives up to order a.

The complex Mellin transform of $f \in C^\infty_{c,\mathrm{ev}}(\mathbb{C}^*)$ is given by

$$\mathcal{M}_c f(\nu,q) := \int_{\mathbb{C}^*} f(u)|u|^{2\nu}(u/|u|)^{-2q} d_* u \tag{12.8}$$

for $\nu \in \mathbb{C}$ and $q \in \mathbb{Z}$. We carry out the change of variables $u = re^{i\varphi}$ with $r \in (0,\infty)$, $\varphi \in [0, 2\pi)$, and get

$$\mathcal{M}_c f(\nu,q) = \int_0^\infty \int_0^{2\pi} f(re^{i\varphi}) r^{2\nu} e^{-2qi\varphi} r^{-2} r \, d\varphi \, dr = 2\pi \mathcal{M} f_q(2\nu).$$

Again, by repeated integration by parts, we get for integers $A, B \geqslant 0$

$$\mathcal{M}_c f(\nu,q) \ll C_f^{2|\operatorname{Re}\nu|+A+B}(1+|q|)^{-A}\frac{1}{|(2\nu)_B|}. \tag{12.9}$$

The constant C_f depends on the compact support of f, and the implicit constant depends on the support of f and the supremum norms of derivatives of f up to order $A+B$.

The inversion formula for the complex Mellin transform is

$$f(u) = \frac{1}{2\pi^2 i}\sum_{q\in\mathbb{Z}} \int_{(\sigma)} \mathcal{M}_c f(\nu,q)|u|^{-2\nu}(u/|u|)^{2q} d\nu, \tag{12.10}$$

and the Parseval-Plancherel formula yields

$$\int_{\mathbb{C}^*} f(u)\overline{g(u)} d_* u = \frac{1}{2\pi^2 i}\sum_{q\in\mathbb{Z}} \int_{(\sigma)} \mathcal{M}_c f(\nu,q)\mathcal{M}_c \bar{g}(-\nu,-q) d\nu. \tag{12.11}$$

From the power series expansion of the function $\mathcal{J}^*_{\nu,p}$, we express the transform $\mathbf{J}f(\nu,p)$ as a linear combination of complex Mellin transforms:

$$\mathbf{J}f(\nu,p) = \sum_{m,n\geqslant 0} b_{m,n}(\nu,p)\mathcal{M}_c f(\nu+m+n, p-\xi-m+n), \tag{12.12}$$

with

$$b_{m,n}(\nu,p) := \frac{(-1)^{m+n+p-\xi}\, 4^{-\nu-m-n}}{m!n!\Gamma(\nu-p+m+1)\Gamma(\nu+p+n+1)}. \tag{12.13}$$

In order to get an estimate for $\mathbf{J}f(\nu,p)$, we estimate the series of absolute values corresponding to (12.12). We write $\operatorname{Re}\nu = \sigma$, $|\operatorname{Im}\nu| = t$, $P := \left[\frac{|p-\xi|}{2}\right]$, and assume that $t \geqslant 1$ and $p \geqslant \xi$. Estimate (12.9) implies

$$\begin{aligned}
&\sum_{m,n\geqslant 0} |b_{m,n}(\nu,p)||\mathcal{M}_c f(\nu+m+n, p-m+n)| \ll \\
&\quad \ll C_f^{2|\sigma|} \sum_{m,n\geqslant 0} \frac{(C_f^2/4)^{m+n}(1+|p-\xi-m+n|)^{-A}|(2(\nu+m+n))_B|^{-1}}{m!\,n!\,|\Gamma(\nu-p+m+1)||\Gamma(\nu+p+n+1)|} \\
&\quad = S_1 + S_2,
\end{aligned} \tag{12.14}$$

where S_1 is the sum over $m, n \geqslant 0$ such that $|p - \xi - m + n| > P$, and S_2 is the sum over $m, n \geqslant 0$ such that $|p - \xi - m + n| \leqslant P$.

For all $m, n \geqslant 0$ we have

$$|(2(\nu + m + n))_B| \geqslant |2\nu|^B \geqslant (2|\operatorname{Im}\nu|)^B \geqslant (1 + t)^B,$$

as well as

$$|\Gamma(\nu - p + m + 1)| \geqslant |\Gamma(\nu - p + 1)|, \qquad |\Gamma(\nu + p + n + 1)| \geqslant |\Gamma(\nu + p + 1)|.$$

If $|p - \xi - m + n| > P$, then $(1 + |p - \xi - m + n|)^{-A} < (1 + P)^{-A} \ll (1 + |p|)^{-A}$, and hence the first sub-sum in (12.14) is estimated by

$$\begin{aligned} S_1 &\ll C_f^{2|\sigma|} \sum_{\substack{m,n \geqslant 0 \\ |p-\xi-m+n|>P}} \frac{(C_f^2/4)^{m+n}(1 + |p|)^{-A}(1 + t)^{-B}}{m!\, n!\, |\Gamma(\nu - p + 1)||\Gamma(\nu + p + 1)|} \\ &\ll_f \frac{(1 + |p|)^{-A}(1 + t)^{-B}}{|\Gamma(\nu - p + 1)||\Gamma(\nu + p + 1)|}. \end{aligned} \tag{12.15}$$

If $|p - \xi - m + n| \leqslant P$, then trivially $(1 + |p - \xi - m + n|)^{-A} < 1$, and the second sub-sum in (12.14) is

$$\begin{aligned} S_2 &\ll C_f^{2|\sigma|} \sum_{n=0}^{\infty} \frac{(C_f^2/4)^n (1 + t)^{-B}}{n!\, |\Gamma(\nu + p + n + 1)|} \sum_{m=n+p-\xi-P}^{n+p-\xi+P} \frac{(C_f^2/4)^m}{m! |\Gamma(\nu - p + m + 1)|} \\ &\ll_f \sum_{n=0}^{\infty} \frac{(C_f^2/4)^n (1 + t)^{-B}}{n! |\Gamma(\nu + p + n + 1)|} \frac{(C_f^2/4)^{n+p-\xi-P}}{(n + p - \xi - P)!\, |\Gamma(\nu + n - \xi - P)|} \cdot \\ &\qquad \cdot \sum_{k=0}^{2P} \frac{(C_f^2/4)^k (n + p - \xi - P)!\, |\Gamma(\nu + n - \xi - P)|}{(n + p - \xi - P + k)!\, |\Gamma(\nu + n - \xi - P + k)|}. \end{aligned}$$

To estimate the sum over k, we use that $|\Gamma(\nu + n - \xi - P)| \leqslant |\Gamma(\nu + n - \xi - P + k)|$, for all $k \geqslant 0$, and by Stirling's formula

$$\begin{aligned} &\frac{(n + p - \xi - P)!}{(n + p - \xi - P + k)!} \ll \\ &\qquad \ll e^k (n + p - \xi - P + 1 + k)^{-k} \left(\frac{n + p - \xi - P + 1}{n + p - \xi - P + 1 + k} \right)^{n+p-\xi-P+k+\frac{1}{2}}. \end{aligned}$$

Since $n + p - \xi - P + 1 \leqslant n + p - \xi - P + 1 + k$ and $n + p - \xi - P + \frac{1}{2} \geqslant n + P + \frac{1}{2} > 0$, we have

$$\left(\frac{n + p - \xi - P + 1}{n + p - \xi - P + 1 + k} \right)^{n+p-\xi-P+k+\frac{1}{2}} < 1.$$

Furthermore, $n+p-\xi-P+1+k \geqslant n+P+1+k \geqslant 1+k$, so

$$(n+p-\xi-P+1+k)^{-k} \leqslant (1+k)^{-k}.$$

Collecting these gives $\frac{(n+p-\xi-P)!}{(n+p-\xi-P+k)!} \ll \left(\frac{e}{1+k}\right)^k$, which implies

$$\sum_{k=0}^{2P} \frac{(C_f^2/4)^k (n+p-\xi-P)!\,|\Gamma(\nu+n-\xi-P)|}{(n+p-\xi-P+k)!\,|\Gamma(\nu+n-\xi-P+k)|} \ll \sum_{k=0}^{2P} \left(\frac{C_f^2}{4}\right)^k \left(\frac{e}{1+k}\right)^k \cdot 1 \ll 1,$$

as $P \to \infty$. Hence,

$$\begin{aligned} S_2 &\ll_f \sum_{n=0}^{\infty} \frac{(C_f^2/4)^n (1+t)^{-B}}{n!|\Gamma(\nu+p+n+1)|} \frac{(C_f^2/4)^{n+p-\xi-P}}{(n+p-\xi-P)!\,|\Gamma(\nu+n-\xi-P)|} \\ &\ll_f \frac{(1+t)^{-B}}{|\Gamma(\nu+p+1)|} \frac{(C_f^2/4)^{p-\xi-P}}{(p-\xi-P)!\,|\Gamma(\nu-\xi-P)|}. \end{aligned} \tag{12.16}$$

From the definition of P we have $0 \leqslant P \leqslant p-\xi-P$, which implies $(p-\xi-P)! \geqslant P!$ and $|\Gamma(\nu-\xi-P)| \geqslant |\Gamma(\nu-p)|$. Thus

$$\frac{(C_f^2/4)^{p-\xi-P}}{(p-\xi-P)!|\Gamma(\nu-\xi-P)|} \ll \frac{(C_f^2/4)^P}{P!\,|\Gamma(\nu-p)|} \ll \left(\frac{C_f^2}{4eP}\right)^P \frac{|\nu-p|}{|\Gamma(\nu-p+1)|},$$

where

$$\left(\frac{C_f^2}{4eP}\right)^P \ll (1+P)^{-P} \ll \left\{ \begin{matrix} 1 & \text{if } P \leqslant 2 \\ \left(1+\frac{|p|}{2}\right)^{-|p|/2} & \text{if } P > 3 \end{matrix} \right\} \ll (1+|p|)^{-A}$$

for all $A \in \mathbb{N}$. This and (12.16) imply that

$$S_2 \ll_f \frac{(1+t)^{1-B}(1+|p|)^{-A}}{|\Gamma(\nu+p+1)||\Gamma(\nu-p+1)|}, \qquad \text{for each } A, B \in \mathbb{N}. \tag{12.17}$$

Because of (12.15) and (12.17), the total sum in (12.14) is estimated by

$$S_1 + S_2 \ll_f \frac{(1+|p|)^{-A}(1+t)^{1-B}}{|\Gamma(\nu+p+1)||\Gamma(\nu-p+1)|}. \tag{12.18}$$

The restriction $p \geqslant \xi$ is not essential, since for $p < \xi$ the estimate (12.15) still holds, and instead of (12.17), we have

$$\begin{aligned}
S_2 &\ll C_f^{2|\sigma|} \sum_{m=0}^{\infty} \frac{(C_f^2/4)^m (1+t)^{-B}}{m!|\Gamma(\nu-p+m+1)|} \sum_{n=m-p+\xi-P}^{m-p+\xi+P} \frac{(C_f^2/4)^n}{n!|\Gamma(\nu+p+n+1)|} \\
&\ll_f \sum_{m=0}^{\infty} \frac{(C_f^2/4)^m (1+t)^{-B}}{m!|\Gamma(\nu-p+m+1)|} \frac{(C_f^2/4)^{m-p+\xi-P}}{(m-p+\xi-P)!\,|\Gamma(\nu+m+\xi-P)|} \\
&\ll_f \frac{(1+t)^{-B}}{|\Gamma(\nu-p+1)|} \frac{(C_f^2/4)^{-p+\xi-P}}{(-p+\xi-P)!\,|\Gamma(\nu+\xi-P)|}.
\end{aligned}$$

Since in this case $0 \leqslant P \leqslant -p+\xi-P$, we have $(-p+\xi-P)! \geqslant P!$ and $|\Gamma(\nu+\xi-P)| \geqslant |\Gamma(\nu+p)|$, which implies

$$\ll \frac{(1+t)^{-B}}{|\Gamma(\nu-p+1)|} \frac{(C_f^2/4)^P}{P!\,|\Gamma(\nu+p)|} \ll \frac{(1+|p|)^{-A}(1+t)^{1-B}}{|\Gamma(\nu-p+1)||\Gamma(\nu+p+1)|},$$

for each $A \in \mathbb{N}$. Hence, (12.18) is still valid.

Let us now consider the product $|\Gamma(\nu+p+1)|^{-1}|\Gamma(\nu-p+1)|^{-1}$. First, we consider the case of small σ. On the strip $|\operatorname{Re}\nu| \leqslant \Sigma$ with some $\Sigma > 0$, for all $|p| \leqslant 2\Sigma+2$, we have the estimate

$$|\Gamma(\nu+p+1)|^{-1}|\Gamma(\nu-p+1)|^{-1} \ll (1+t)^{-2\sigma-1}e^{\pi t}. \tag{12.19}$$

For $|p| \geqslant 2\Sigma+2$, we use Stirling's formula in the following way:

$$\begin{aligned}
\left\{\Gamma(\nu+p+1)\Gamma(\nu-p+1)\right\}^{-1} &= \frac{-\sin\pi(\nu-|p|)\Gamma(|p|-\nu)}{\pi\,\Gamma(\nu+|p|+1)} \\
&\ll \frac{\big||p|-\nu\big|^{|p|-\sigma-\frac{1}{2}} e^{-t\arctan\frac{t}{|p|-\sigma}}}{\big||p|+\nu+1\big|^{|p|+\sigma+\frac{1}{2}} e^{-t\arctan\frac{t}{|p|+\sigma+1}}} e^{\pi t}.
\end{aligned}$$

Since $\left|\frac{|p|-\nu}{|p|+\nu+1}\right|^2 = \left|1 - \frac{(2\sigma+1)(2|p|+1)}{(|p|+\sigma+1)^2+t^2}\right| \leqslant 1 + \frac{C_\Sigma}{|p|}$ for some $C_\Sigma \geqslant 0$, we get

$$\left|\frac{|p|-\nu}{|p|+\nu+1}\right|^{|p|} \ll_\Sigma 1, \qquad \text{for each} \quad |p| \geqslant 2\Sigma+2.$$

By the mean value theorem, we know that

$$\begin{aligned}
\arctan\tfrac{t}{|p|+\sigma+1} - \arctan\tfrac{t}{|p|-\sigma} &= \frac{-t(2\sigma+1)}{(|p|-\sigma)(|p|+\sigma+1)} \cdot \frac{1}{1+(t/(|p|+\zeta))^2} \\
&\leqslant (2\Sigma+1)\frac{t}{(|p|+\zeta)^2+t^2} \cdot \frac{(|p|+\zeta)^2}{(|p|-\sigma)(|p|+\sigma+1)} \\
&\leqslant (2\Sigma+1)\frac{t}{(|p|-\Sigma)^2+t^2} \frac{(|p|+\Sigma+1)^2}{(|p|-\Sigma)(|p|-\Sigma+1)} < D_\Sigma \frac{t}{(|p|-\Sigma)^2+t^2},
\end{aligned}$$

for some $\zeta \in [-\Sigma, \Sigma+1]$ and some $D_\Sigma \geqslant 0$. Thus, in the expression

$$\left|\frac{|p|-\nu}{|p|+\nu+1}\right|^{|p|} |(|p|-\nu)(|p|+\nu+1)|^{-\sigma-\frac{1}{2}} e^{D_\Sigma \frac{t^2}{(|p|-\Sigma)^2+t^2}},$$

the exponential factor is $O(1)$, while

$$|(|p|-\nu)(|p|+\nu+1)|^{-\sigma-\frac{1}{2}} \ll (1+|p|)^{-\sigma-\frac{1}{2}}(1+t)^{-\sigma-\frac{1}{2}}.$$

Hence the product of the Γ-factors, for $|p| \geqslant 2\Sigma+2$ and $|\sigma| \leqslant \Sigma$, is estimated as follows:

$$|\Gamma(\nu-|p|+1)|^{-1}|\Gamma(\nu+|p|+1)|^{-1} \ll (1+|p|)^{-\sigma-\frac{1}{2}}(1+t)^{-\sigma-\frac{1}{2}}. \tag{12.20}$$

Next we consider large σ. For $\sigma \geqslant 2|p|+1$, Stirling's formula gives

$$\begin{aligned} &|\Gamma(\nu-p+1)|^{-1}|\Gamma(\nu+p+1)|^{-1} \ll \\ &\ll \frac{e^{2\sigma+2} e^{t\left\{\arctan \frac{t}{\sigma+|p|+1} + \arctan \frac{t}{\sigma-|p|+1}\right\}}}{|\nu+|p|+1|^{\sigma+|p|+\frac{1}{2}} |\nu-|p|+1|^{\sigma-|p|+\frac{1}{2}}} \\ &\ll e^{2\sigma}(\sigma-|p|+1)^{-2\sigma-1} e^{\pi t}, \end{aligned} \tag{12.21}$$

where we have used that $|\arctan x| \leqslant \frac{\pi}{2}$ for any $x \in \mathbb{R}$.

Collecting the results (12.18)–(12.21), and changing the A and B above, we obtain part of the following statement:

Lemma 12.1.1. *Let $\sigma > 0$, and let $f \in C^\infty_{c,ev}(\mathbb{C}^*)$. For $A, B \in \mathbb{N}$, $p \in \frac{1}{2}\mathbb{Z}$, ξ as in (4.27), and $\nu \in \mathbb{C}$, we have the estimates:*

$$\mathbf{J}f(\nu,p) \ll_{f,A,B,\sigma} \frac{C_f^{2|\operatorname{Re}\nu|}(1+|p|)^{-A}(1+|\operatorname{Im}\nu|)^{-B}}{(\operatorname{Re}\nu-|p|+1)^{2\operatorname{Re}\nu+1}} e^{\pi|\operatorname{Im}\nu|} \tag{12.22}$$

for $|\operatorname{Re}\nu| \geqslant 2|p|+1$,

$$\ll_{f,A,B,\sigma} (1+|p|)^{-A}(1+|\operatorname{Im}\nu)^{-B} e^{\pi|\operatorname{Im}\nu|} \tag{12.23}$$

for $|\operatorname{Re}\nu| \leqslant \sigma$,

$$\mathbf{K}f(\nu,p) \ll_{f,A,B,\sigma} (1+|p|)^{-A}(1+|\operatorname{Im}\nu|)^{-B} \tag{12.24}$$

for $|\operatorname{Re}\nu| \leqslant \sigma$.

Proof. The estimates (12.22) and (12.23) are proved by the discussion above in the case $|\operatorname{Im}\nu| \geqslant 1$. The statement (12.24) in this case follows immediately from the property (12.3) of $\mathbf{K}f(\nu,p)$.

If we take $B=0$ in the computations on page 125, the factor $(2(\nu+m+n))_B$ in the denominator is absent, and the reasoning goes through for $|\operatorname{Im}\nu| \leqslant 1$. The estimate of $\mathbf{K}f(\nu,p)$ follows, for ν staying away from the integers in the strip $|\operatorname{Re}\nu| \leqslant \sigma$. By representing $\mathbf{K}f(\nu,p)$ by an integral over a small circle on which the estimate holds already, we can extend the estimate of $\mathbf{K}f(\nu,p)$ to integers. ■

12.2 One-sided Bessel inversion

The main result in this section is the following

Theorem 12.2.1. *For any function f that is even, smooth and compactly supported on $\mathbb{C}^*$, the transformation*

$$\mathbf{K}f(\nu,p) := \frac{\pi}{2}\int_{\mathbb{C}^*} f(u)(u/|u|)^{4\xi}\mathcal{K}^*_{\nu,p}(u)d_*u$$

is a one-sided inverse of the Bessel transform $\mathbf{B}$ defined in (11.1), i.e.

$$\mathbf{BK}f = f. \tag{12.25}$$

Proof. The theorem is an immediate corollary of Proposition 12.2.2 below. Namely, for any $f, g \in C^\infty_{c,\mathrm{ev}}(\mathbb{C}^*)$, we have

$$\int_{\mathbb{C}^*} \mathbf{BK}f(u)\, g(u)(u/|u|)^{4\xi}d_*u = \frac{1}{\pi^2 i}\sum_{p\in\frac{1}{2}\mathbb{Z}}\int_{(0)} \mathbf{K}f(\nu,p)\mathbf{K}g(\nu,p)(p^2-\nu^2)d\nu.$$

The change in the order of integration is allowed because of the estimates (9.27) and (12.24). Proposition 12.2.2 implies that

$$\int_{\mathbb{C}^*}\Big(\mathbf{BK}f(u) - f(u)\Big)g(u)(u/|u|)^{4\xi}d_*u = 0, \qquad \forall g \in C^\infty_{c,\mathrm{ev}}(\mathbb{C}^*).$$

So, it must be $\mathbf{BK}f(u) = f(u)$ for all $u \in \mathbb{C}^*$, which proves the theorem. ■

Proposition 12.2.2. *For all $f, g \in C^\infty_{c,\mathrm{ev}}(\mathbb{C}^*)$, the functions $\mathbf{K}f$ and $\mathbf{K}g$ are square integrable for the Plancherel measure $(p^2-\nu^2)d\nu$, and the equality*

$$\sum_{p\in\frac{1}{2}\mathbb{Z}}\int_{(0)} \mathbf{K}f(\nu,p)\mathbf{K}g(\nu,p)(p^2-\nu^2)d\nu = \pi^2 i\int_{\mathbb{C}^*} f(u)g(u)\left(u/|u|\right)^{4\xi}d_*u$$

holds.

Proof. We fix $f, g \in C^\infty_{c,\mathrm{ev}}\mathbb{C}^*)$. The estimate (12.24) implies that the left side of the equality in Proposition 12.2.2 converges absolutely, and therefore we may move the line of integration to (σ), for a suitably chosen $\sigma \notin \frac{1}{2}\mathbb{Z}$. Then the left side of the equality in Proposition 12.2.2 is equal to

$$\begin{aligned}\sum_{p\in\frac{1}{2}\mathbb{Z}}\int_{(\sigma)} &\mathbf{K}f(\nu,p)\mathbf{K}g(\nu,p)(p^2-\nu^2)d\nu = \\ &= \frac{\pi^2}{4}\sum_{\pm}\sum_{p\in\frac{1}{2}\mathbb{Z}}\int_{(\pm\sigma)}\mathbf{J}f(\nu,p)\mathbf{J}g(\nu,p)\frac{p^2-\nu^2}{\sin^2\pi(\nu-p)}d\nu \\ &- \frac{\pi^2}{4}\sum_{\pm}\sum_{p\in\frac{1}{2}\mathbb{Z}}\int_{(\pm\sigma)}\mathbf{J}f(\nu,p)\mathbf{J}g(-\nu,-p)\frac{p^2-\nu^2}{\sin^2\pi(\nu-p)}d\nu.\end{aligned} \tag{12.26}$$

In the second term on the right side of (12.26), we replace $\mathbf{J}f(\nu,p)$ and $\mathbf{J}g(-\nu,-p)$ by the expansion (12.12) and get:

$$\begin{aligned}
&\frac{\pi^2}{4}\sum_{\pm}\sum_{p\in\frac{1}{2}\mathbb{Z}}\int_{(\pm\sigma)}\mathbf{J}f(\nu,p)\mathbf{J}g(-\nu,-p)\frac{p^2-\nu^2}{\sin^2\pi(\nu-p)}d\nu = \\
&\quad=\frac{\pi^2}{4}\sum_{\pm}\sum_{p\in\frac{1}{2}\mathbb{Z}}\int_{(\pm\sigma)}b_{0,0}(\nu,p)b_{0,0}(-\nu,-p)\frac{p^2-\nu^2}{\sin^2\pi(\nu-p)}\cdot \\
&\qquad\qquad\qquad\cdot\mathcal{M}_cf(\nu,p-\xi)\mathcal{M}_cg(-\nu,-p-\xi)d\nu \\
&\quad+\frac{\pi^2}{4}\sum_{\pm}\sum_{p\in\frac{1}{2}\mathbb{Z}}\sum_{m,n,k,l\geqslant 0}{}^{'}\int_{(\pm\sigma)}b_{m,n}(\nu,p)b_{k,l}(-\nu,-p)\frac{p^2-\nu^2}{\sin^2\pi(\nu-p)}\cdot \\
&\qquad\cdot\mathcal{M}_cf(\nu+m+n,p-\xi-m+n)\mathcal{M}_cg(-\nu+k+l,-p-\xi-k+l)d\nu,
\end{aligned}$$

where the prime in the $\sum'_{m,n,k,l\geqslant 0}$ means that the term with $m=n=k=l=0$ is to be omitted. The absolute convergence, implied from (12.9), allows us to exchange the order of integration and summation. Let us denote

$$\mathcal{B}_+^p(\sigma;f,g) := \int_{(\sigma)}\mathbf{J}f(\nu,p)\mathbf{J}g(\nu,p)\frac{p^2-\nu^2}{\sin^2\pi(\nu-p)}d\nu \tag{12.27}$$

and

$$\begin{aligned}
\mathcal{B}_-^p(\sigma;f,g) := \sum_{m,n,k,l\geqslant 0}{}^{'}\int_{(\sigma)}b_{m,n}(\nu,p)b_{k,l}(-\nu,-p)\frac{p^2-\nu^2}{\sin^2\pi(\nu-p)}\cdot \\
\cdot\mathcal{M}_cf(\nu+m+n,p-\xi-m+n)\mathcal{M}_cg(-\nu+k+l,-p-\xi-k+l)d\nu.
\end{aligned} \tag{12.28}$$

A simple calculation shows that $b_{0,0}(\nu,p)b_{0,0}(-\nu,-p) = -\frac{\sin^2\pi(\nu-p)}{\pi^2(p^2-\nu^2)}$. Therefore the right hand side of (12.26) further equals

$$\begin{aligned}
&\frac{\pi^2}{4}\sum_{\pm}\sum_{p\in\frac{1}{2}\mathbb{Z}}\mathcal{B}_+^p(\pm\sigma;f,g)-\frac{\pi^2}{4}\sum_{\pm}\sum_{p\in\frac{1}{2}\mathbb{Z}}\mathcal{B}_-^p(\pm\sigma;f,g)+ \\
&\qquad+\frac{1}{4}\sum_{\pm}\sum_{p\in\frac{1}{2}\mathbb{Z}}\int_{(\pm\sigma)}\mathcal{M}_cf_{(\xi)}(\nu,p+\xi)\mathcal{M}_cg(-\nu,-p-\xi)d\nu,
\end{aligned}$$

where we have used $\mathcal{M}_cf(\nu,p-\xi)=\mathcal{M}_cf_{(\xi)}(\nu,p+\xi)$. The Parseval-Plancherel formula (12.11) finally gives

$$\begin{aligned}
&\sum_{p\in\frac{1}{2}\mathbb{Z}}\int_{(0)}\mathbf{K}f(\nu,p)\mathbf{K}g(\nu,p)(p^2-\nu^2)d\nu = \pi^2 i\int_{\mathbb{C}^*}f(u)g(u)(u/|u|)^{4\xi}d_*u \\
&\qquad+\frac{\pi^2}{4}\sum_{\pm}\sum_{p\in\frac{1}{2}\mathbb{Z}}\mathcal{B}_+^p(\pm\sigma;f,g)-\frac{\pi^2}{4}\sum_{\pm}\sum_{p\in\frac{1}{2}\mathbb{Z}}\mathcal{B}_-^p(\pm\sigma;f,g).
\end{aligned}$$

From this we see that in order to prove Proposition 12.2.2, we need to show

$$\sum_{\pm}\sum_{p\in\frac{1}{2}\mathbb{Z}} \mathcal{B}_+^p(\pm\sigma; f,g) = \sum_{\pm}\sum_{p\in\frac{1}{2}\mathbb{Z}} \mathcal{B}_-^p(\pm\sigma; f,g), \tag{12.29}$$

for all $f,g \in C_{c,\mathrm{ev}}^\infty(\mathbb{C}^*)$.

Let us first consider $\mathcal{B}_+^p$. Estimate (12.22) shows that $\mathcal{B}_+^p(\sigma; f,g)$ tends to zero as $\sigma\to\infty$. Moving off the line of integration to the right leaves us with a sum of residues

$$\mathcal{B}_+^p(\sigma; f,g) = -2\pi i \sum_{\rho>\sigma} \mathrm{Res}_{\nu=\rho}\mathbf{J}f(\nu,p)\mathbf{J}g(\nu,p)\frac{p^2-\nu^2}{\sin^2\pi(\nu-p)},$$

where $\rho\in\frac{1}{2}\mathbb{Z}$ such that $\rho\equiv p \pmod 1$. Choosing $\sigma\in(0,\frac{1}{2})$, the only possible residue between $-\sigma$ and σ is at $\nu=0$. So, defining $a_0=1$ and $a_\rho=2$ for $\rho\neq 0$, we get

$$\begin{aligned}\sum_{\pm}\mathcal{B}_+^p(\pm\sigma; f,g) &= -2\pi i \sum_{0\leqslant\rho\equiv p\,(1)} a_\rho \mathrm{Res}_{\nu=\rho}\mathbf{J}f(\nu,p)\mathbf{J}g(\nu,p)\frac{p^2-\nu^2}{\sin^2\pi(\nu-p)}\\ &= \frac{2}{\pi i}\sum_{0\leqslant\rho\equiv p\,(1)} a_\rho\, \partial_\nu\Big((p^2-\nu^2)\mathbf{J}f(\nu,p)\mathbf{J}g(\nu,p)\Big)\Big|_{\nu=\rho}.\end{aligned}$$

From the definition of the transformation $\mathbf{J}$ we get that

$$\mathbf{J}f(\nu,p)\mathbf{J}g(\nu,p) = \int_{\mathbb{C}^*\times\mathbb{C}^*} f(u)g(w)(uw/|uw|)^{4\xi}\tilde{\mathcal{J}}_{\nu,p}(u)\tilde{\mathcal{J}}_{\nu,p}(w)d_*u\,d_*w,$$

where $\tilde{\mathcal{J}}_{\nu,p}(u) := |u/2|^{2\nu}(iu/|u|)^{-2p-2\xi}\mathcal{J}_{\nu,p}^*(u)$. Note that, for suitably chosen branches,

$$\tilde{\mathcal{J}}_{\nu,p}(u) = (-1)^{p+\xi}(u/|u|)^{-2\xi}J_{\nu-p}(u)J_{\nu+p}(\bar{u}). \tag{12.30}$$

Therefore we can easily derive the following properties for $p,\rho\in\frac{1}{2}\mathbb{Z}$ such that $\rho\equiv p \pmod 1$:

$$\tilde{\mathcal{J}}_{-\rho,-p} = \tilde{\mathcal{J}}_{\rho,p}, \qquad \text{and} \qquad \tilde{\mathcal{J}}_{\rho,p} = \tilde{\mathcal{J}}_{p,\rho}. \tag{12.31}$$

From the power series expansion (1.25) of the J-Bessel function, we easily get the following estimate

$$J_n(u) \ll \frac{|u/2|^{|n|}}{|n|!}e^{|u|^2/4}, \qquad \text{for any} \quad n\in\mathbb{Z}. \tag{12.32}$$

From the asymptotic expansion of the logarithmic derivative $\psi(z)$ of $\Gamma(z)$, for $z \to \infty$ and $|\arg z| < \pi$, given in [32] on page 18, we have for $z = n+1$:

$$\psi(n+1) = \log(1+n) - \frac{1}{2(n+1)} + O(n^{-2}), \qquad \text{as} \quad n \to \infty.$$

By termwise differentiation of (1.25), and using the estimate of ψ above, we obtain

$$\partial_\nu J_\nu(u)\big|_{\nu=n} \ll \begin{cases} \frac{|u/2|^n}{n!} e^{|u|^2/4} \big\{1 + \log(1+n)\big\}, & n \geqslant 0, \\ e^{|u|^2/4} \left\{ \frac{|u/2|^{|n|}}{|n|!} + (|n|-1)! \left|\frac{u}{2}\right|^{-|n|} \right\}, & n < 0. \end{cases} \tag{12.33}$$

The estimates (12.32) and (12.33) together with (12.30) imply for any $p, \rho \in \frac{1}{2}\mathbb{Z}$ such that $\rho \equiv p \pmod 1$:

$$\tilde{\mathfrak{J}}_{\rho,p}(u) \ll \left|\frac{u}{2}\right|^{|\rho-p|+|\rho+p|} \frac{e^{|u|^2/2}}{|\rho-p|!|\rho+p|!}, \tag{12.34}$$

and

$$\begin{aligned} \partial_\nu \tilde{\mathfrak{J}}_{\nu,p}(u)|_{\nu=\rho} &\ll \\ &\ll \left|\frac{u}{2}\right|^{|\rho-p|+|\rho+p|} e^{|u|^2/2} \frac{1 + \log(1+|\rho+p|) + \log(1+|\rho-p|)}{|\rho-p|!|\rho+p|!} \\ &\quad + \left|\frac{u}{2}\right|^{|\rho+p|-|\rho-p|} e^{|u|^2/2} \frac{(|\rho-p|-1)!}{|\rho+p|!} \qquad \text{only if } \rho - p \leqslant -1 \\ &\quad + \left|\frac{u}{2}\right|^{|\rho-p|-|\rho+p|} e^{|u|^2/2} \frac{(|\rho+p|-1)!}{|\rho-p|!} \qquad \text{only if } \rho + p \leqslant -1. \end{aligned} \tag{12.35}$$

We note that the last two terms in the estimate (12.35) are quite bigger than the first one, and they appear only in the indicated cases.

The estimates (12.34) and (12.35) further imply the following estimate, uniform in $(u,w) \in W$, for each compact subset $W \subset \mathbb{C}^* \times \mathbb{C}^*$:

$$\begin{aligned} \partial_\nu \Big((p^2 - \nu^2) \tilde{\mathfrak{J}}_{\nu,p}(u) \tilde{\mathfrak{J}}_{\nu,p}(w) \Big)\Big|_{\nu=\rho} &\ll \\ \ll_W \frac{C_W \left(p^2 + \rho^2 + \log(1+|\rho+p|) + \log(1+|\rho-p|)\right)}{\left(|\rho-p|!|\rho+p|!\right)^2}&. \end{aligned} \tag{12.36}$$

So, the sum

$$B_+^p(u,w) := \frac{2}{\pi i} \sum_{0 \leqslant \rho \equiv p\,(1)} a_\rho \partial_\nu \Big((p^2 - \nu^2) \tilde{\mathfrak{J}}_{\nu,p}(u) \tilde{\mathfrak{J}}_{\nu,p}(w) \Big)\Big|_{\nu=\rho} \tag{12.37}$$

converges absolutely on each W. The expression (12.37) defines a continuous function $B^p_+(u,w)$ on $\mathbb{C}^*\times\mathbb{C}^*$ such that

$$\sum_\pm \mathcal{B}^p_+(\pm\sigma;f,g) = \int_{\mathbb{C}^*\times\mathbb{C}^*} f(u)g(w)(uw/|uw|)^{4\xi} B^p_+(u,w) d_*u\, d_*w. \tag{12.38}$$

We define a function $B^p_-(u,w)$ by

$$B^p_-(u,w) := \frac{2}{\pi i} \sum_{0\leqslant\rho\equiv p\,(1)} a_\rho \partial_\nu \left((p^2-\nu^2)\tilde{\mathcal{J}}_{\nu,p}(u)\tilde{\mathcal{J}}_{-\nu,-p}(w)\right)\Big|_{\nu=\rho}. \tag{12.39}$$

The estimates (12.34) and (12.35) imply that an estimate as (12.36) holds also for $\partial_\nu \left((p^2-\nu^2)\tilde{\mathcal{J}}_{\nu,p}(u)\tilde{\mathcal{J}}_{-\nu,-p}(w)\right)\Big|_{\nu=\rho}$. This means that the sum in (12.39) converges absolutely on each compact subset $W\subset\mathbb{C}^*\times\mathbb{C}^*$, and defines a continuous function $B^p_-(u,w)$ on $\mathbb{C}^*\times\mathbb{C}^*$. We shall prove that this function satisfies

$$\sum_\pm \mathcal{B}^p_-(\pm\sigma;f,g) = \int_{\mathbb{C}^*\times\mathbb{C}^*} f(u)g(w)(uw/|uw|)^{4\xi} B^p_-(u,w) d_*u\, d_*w. \tag{12.40}$$

Going back to (12.28), we denote by $\tau_\pm(m,n,k,l)$ the term of order (m,n,k,l) in $\mathcal{B}^p_-(\pm\sigma;f,g)$. It is an integral over the line $\operatorname{Re}\nu=\pm\sigma$. Estimate (12.9) implies the absolute convergence of the integral. We may deform the path of integration to $C_\pm$, going up from $-i\infty$ to $-i\sigma$, moving via $\pm\sigma$ to $i\sigma$ along a half circle centered at 0, and finally going up to $i\infty$. Let $C_\pm(T)$ be the part of $C_\pm$ cut off between $-iT$ and iT, with $T\notin\frac12\mathbb{Z}$. We then have

$$\tau_\pm(m,n,k,l) = \int_{\mathbb{C}^*\times\mathbb{C}^*} f(u)g(w)(uw/|uw|)^{4\xi} K_\pm(u,w)\, d_*u\, d_*w, \tag{12.41}$$

where

$$\begin{aligned} K_\pm(u,w) &:= |u|^{2(m+n)}(u/|u|)^{-2(p+\xi-m+n)}|w|^{2(k+l)}(w/|w|)^{-2(-p+\xi-k+l)}\,\cdot \\ &\quad\cdot \lim_{T\to\infty}\int_{C_\pm(T)} b_{m,n}(\nu,p)b_{k,l}(-\nu,-p)\left|\frac{u}{w}\right|^{2\nu}\frac{p^2-\nu^2}{\sin^2\pi(\nu-p)}d\nu. \end{aligned} \tag{12.42}$$

Assume that $|u|\leqslant|w|$. We shift both integration lines $C_\pm(T)$ to the right into C_T, the right half of a circle centered at 0 with radius T. After picking up the residues, we have

$$\begin{aligned} &\sum_\pm \int_{C_\pm(T)} b_{m,n}(\nu,p)b_{k,l}(-\nu,-p)\left|\frac{u}{w}\right|^{2\nu}\frac{p^2-\nu^2}{\sin^2\pi(\nu-p)}d\nu \\ &\quad = 2\int_{C_T} b_{m,n}(\nu,p)b_{k,l}(-\nu,-p)\left|\frac{u}{w}\right|^{2\nu}\frac{p^2-\nu^2}{\sin^2\pi(\nu-p)}d\nu \\ &\quad - 2\pi i \sum_{\substack{\rho\equiv p\,(1)\\ 0\leqslant\rho<T}} a_\rho \operatorname{Res}_{\nu=\rho}\left\{ b_{m,n}(\nu,p)b_{k,l}(-\nu,-p)\left|\frac{u}{w}\right|^{2\nu}\frac{p^2-\nu^2}{\sin^2\pi(\nu-p)}\right\}, \end{aligned}$$

with $a_0 = 1$ and $a_\rho = 2$ for $\rho \in \frac{1}{2}\mathbb{Z} \setminus \{0\}$. We simplify the expression inside the integral over C_T:

$$b_{m,n}(\nu,p)b_{k,l}(-\nu,-p)\frac{p^2-\nu^2}{\sin^2 \pi(\nu-p)} = -\frac{1}{\pi^2}\frac{(-1)^{m+n+k+l}4^{-(m+n+k+l)}}{m!\,n!\,k!\,l!\,\lambda(\nu,p;m,n,k,l)}$$

where $\lambda(\nu,p;m,n,k,l) := (\nu-p+1)_m(\nu+p+1)_n(-\nu+p+1)_k(-\nu-p+1)_l$, and get

$$2\int_{C_T} b_{m,n}(\nu,p)b_{k,l}(-\nu,-p)\left|\frac{u}{w}\right|^{2\nu}\frac{p^2-\nu^2}{\sin^2\pi(\nu-p)}d\nu = \\ = -\frac{2}{\pi^2}\frac{(-1)^{m+n+k+l}4^{-(m+n+k+l)}}{m!\,n!\,k!\,l!}\int_{C_{P,T}}\frac{|u/w|^{2\nu}}{\lambda(\nu,p;m,n,k,l)}d\nu. \tag{12.43}$$

Since $(m,n,k,l) \neq (0,0,0,0)$, we have $\lambda(\nu,p;m,n,k,l) \gg T$ and the contribution of the integral in (12.43) is

$$\ll \int_0^{\pi/2}\frac{|u/w|^{2T\cos\varphi}}{T}T d\varphi \ll \min\left\{1, \frac{1}{T|\log|u/w||}\right\}.$$

Integrating this bound over any compact region in $\mathbb{C}^* \times \mathbb{C}^*$ where the condition $|u/w| \leqslant 1$ is satisfied leads to a bound $O(T^{-1}\log T)$, which is $o(1)$ as $T \to \infty$. Hence

$$\int_{|u|\leqslant|w|} f(u)g(w)(uw/|uw|)^{4\xi}\sum_{\pm}K_\pm(u,w)\,d_*u\,d_*w = \\ = \int_{|u|\leqslant|w|} f(u)g(w)(uw/|uw|)^{4\xi}\Bigg(-2\pi i\,|u|^{2(m+n)}(u/|u|)^{-2(p+\xi-m+n)}\,\cdot \\ \cdot\,|w|^{2(k+l)}(w/|w|)^{-2(-p+\xi-k+l)}\sum_{0\leqslant\rho\equiv p\,(1)} a_\rho \mathrm{Res}_{\nu=\rho}\Bigg\{b_{m,n}(\nu,p)\,\cdot \\ \cdot\, b_{k,l}(-\nu,-p)\left|\frac{u}{w}\right|^{2\nu}\frac{p^2-\nu^2}{\sin^2\pi(\nu-p)}\Bigg\}\Bigg)\,d_*u\,d_*w. \tag{12.44}$$

Summing in (12.44) over all $m,n,k,l \geqslant 0$ such that $(m,n,k,l) \neq (0,0,0,0)$ gives

$$\sideset{}{'}\sum_{m,n,k,l\geqslant 0}\int_{|u|\leqslant|w|} f(u)g(w)(uw/|uw|)^{4\xi}\sum_{\pm}K_\pm(u,w)\,d_*u\,d_*w = \\ = \int_{|u|\leqslant|w|} f(u)g(w)\,(uw/|uw|)^{4\xi}\Bigg(-2\pi i\sum_{0\leqslant\rho\equiv p\,(1)} a_\rho\mathrm{Res}_{\nu=\rho}\Bigg\{\sideset{}{'}\sum_{m,n,k,l\geqslant 0} \\ |u|^{2(m+n)}(u/|u|)^{-2(p+\xi-m+n)}|w|^{2(k+l)}(w/|w|)^{-2(-p+\xi-k+l)}\,\cdot \\ \cdot\, b_{m,n}(\nu,p)b_{k,l}(-\nu,-p)\left|\frac{u}{w}\right|^{2\nu}\frac{p^2-\nu^2}{\sin^2\pi(\nu-p)}\Bigg\}\Bigg)\,d_*u\,d_*w.$$

The formula for residues of a function at a double pole gives further

$$= \int_{|u|\leqslant|w|} f(u)g(w)(uw/|uw|)^{4\xi}\Big(\frac{2}{\pi i}\sum_{0\leqslant\rho\equiv p\,(1)} a_\rho \partial_\nu\Big((p^2-\nu^2)\,\cdot$$

$$\cdot \sum_{m,n,k,l\geqslant 0}{}' |u|^{2(m+n)}(u/|u|)^{-2(p+\xi-m+n)}|w|^{2(k+l)}(w/|w|)^{-2(-p+\xi-k+l)}\,\cdot$$

$$\cdot\, b_{m,n}(\nu,p)b_{k,l}(-\nu,-p)\,|u/w|^{2\nu}\Big)\Big|_{\nu=\rho}\Big)\, d_*u\, d_*w.$$

Inserting the defining expressions for $b_{m,n}(\nu,p)$ and $b_{k,l}(-\nu,-p)$, see (12.13), and some rearrangement gives

$$= \int_{|u|\leqslant|w|} f(u)g(w)(uw/|uw|)^{4\xi}\frac{2}{\pi i}\sum_{0\leqslant\rho\equiv p\,(1)} a_\rho\,\cdot$$

$$\cdot\,\partial_\nu\Big((p^2-\nu^2)\tilde{\mathcal{J}}_{\nu,p}(u)\tilde{\mathcal{J}}_{-\nu-p}(w)-(p^2-\nu^2)T_0\Big)\Big|_{\nu=\rho} d_*u\, d_*w,$$

where

$$T_0 = \frac{(-1)^{2\xi}|u/w|^{2\nu}(u/|u|)^{-2p-2\xi}(w/|w|)^{2p-2\xi}}{\Gamma(\nu-p+1)\Gamma(\nu+p+1)\Gamma(-\nu+p+1)\Gamma(-\nu-p+1)}.$$

Simplification in the expression for T_0 gives

$$-(p^2-\nu^2)T_0 = \pi^{-2}\sin^2\pi(\nu-p)|u/w|^{2\nu}(u/|u|)^{-2p-2\xi}(w/|w|)^{2p-2\xi},$$

and thus $\partial_\nu\big(-(p^2-\nu^2)T_0\big)\big|_{\nu=\rho}=0$. Therefore

$$\sum_{m,n,k,l\geqslant 0}{}' \int_{|u|\leqslant|w|} f(u)g(w)(uw/|uw|)^{4\xi}\sum_{\pm} K_\pm(u,w)\, d_*u\, d_*w =$$

$$= \int_{|u|\leqslant|w|} f(u)g(w)(uw/|uw|)^{4\xi}\,\cdot$$

$$\cdot\,\frac{2}{\pi i}\sum_{0\leqslant\rho\equiv p\,(1)} a_\rho\partial_\nu\Big((p^2-\nu^2)\tilde{\mathcal{J}}_{\nu,p}(u)\tilde{\mathcal{J}}_{-\nu,-p}(w)\Big)\Big|_{\nu=\rho} d_*u\, d_*w. \qquad (12.45)$$

If $|u|\geqslant|w|$ we go back to (12.42) and now shift the both integration lines $C_\pm(T)$ to the left into C'_T, the left half of a circle centered at 0 with radius T.

After picking up the residues, we have

$$\sum_{\pm}\int_{C_\pm(T)} b_{m,n}(\nu,p)b_{k,l}(-\nu,-p)\left|\frac{u}{w}\right|^{2\nu}\frac{p^2-\nu^2}{\sin^2\pi(\nu-p)}d\nu =$$
$$= 2\int_{C'_T} b_{m,n}(\nu,p)b_{k,l}(-\nu,-p)\left|\frac{u}{w}\right|^{2\nu}\frac{p^2-\nu^2}{\sin^2\pi(\nu-p)}d\nu$$
$$+ 2\pi i \sum_{\substack{\rho\in\frac{1}{2}\mathbb{Z}\\ 0\leqslant\rho<T}} a_\rho \mathrm{Res}_{\nu=\rho}\left\{b_{m,n}(-\nu,p)b_{k,l}(\nu,-p)\left|\frac{u}{w}\right|^{-2\nu}\frac{p^2-\nu^2}{\sin^2\pi(\nu-p)}\right\}.$$

In the same way as before we show that the integral over C'_T is negligible as $T\to\infty$, and that

$$\sum_{m,n,k,l\geqslant 0}{}' \int_{|u|\geqslant|w|} f(u)g(w)(uw/|uw|)^{4\xi}\sum_{\pm}K_\pm(u,w)\,d_*u\,d_*w =$$
$$= \int_{|u|\geqslant|w|} f(u)g(w)(uw/|uw|)^{4\xi}\cdot$$
$$\cdot\left(-\frac{2}{\pi i}\right)\sum_{0\leqslant\rho\equiv p\,(1)} a_\rho\partial_\nu\Big((p^2-\nu^2)\tilde{\mathfrak{J}}_{-\nu,p}(u)\tilde{\mathfrak{J}}_{\nu,-p}(w)\Big)\Big|_{\nu=\rho} d_*u\,d_*w$$
$$= \int_{|u|\geqslant|w|} f(u)g(w)(uw/|uw|)^{4\xi}\cdot$$
$$\cdot\frac{2}{\pi i}\sum_{0\leqslant\rho\equiv p\,(1)} a_\rho\partial_\nu\,\Big((p^2-\nu^2)\tilde{\mathfrak{J}}_{\nu,p}(u)\tilde{\mathfrak{J}}_{-\nu,-p}(w)\Big)\Big|_{\nu=\rho} d_*u\,d_*w. \qquad (12.46)$$

For the last equality, we have carried out the change of variables $\nu\mapsto-\nu$ and we then changed the summation over $-\rho$ instead of ρ. Summing the two identities (12.45) and (12.46), we obtain

$$\sum_{m,n,k,l\geqslant 0}{}'\sum_{\pm}\tau_\pm(m,n,k,l) = \int_{\mathbb{C}^*\times\mathbb{C}^*} f(u)g(w)(uw/|uw|)^{4\xi}\cdot$$
$$\cdot\frac{2}{\pi i}\sum_{0\leqslant\rho\equiv p\,(1)} a_\rho\partial_\nu\,\Big((p^2-\nu^2)\tilde{\mathfrak{J}}_{\nu,p}(u)\tilde{\mathfrak{J}}_{-\nu,-p}(w)\Big)\Big|_{\nu=\rho} d_*u\,d_*w, \qquad (12.47)$$

which is exactly (12.40).

Because of (12.38) and (12.40), proving equality (12.29) is equivalent to proving

$$\int_{\mathbb{C}^*\times\mathbb{C}^*} f(u)g(w)(uw/|uw|)^{4\xi}\cdot$$
$$\cdot\lim_{P\to\infty}\sum_{p=-P}^{P}\Big(B_+^p(u,w)-B_-^p(u,w)\Big)d_*u\,d_*w = 0, \qquad (12.48)$$

for all $f, g \in C^\infty_{c,\mathrm{ev}}(\mathbb{C}^*)$.

Let us consider the difference $B^p_+(u,w) - B^p_-(u,w)$. We have

$$\begin{aligned}
&B^p_+(u,w) - B^p_-(u,w) = \\
&\quad = \frac{2}{\pi i} \sum_{0\leqslant\rho\equiv p\,(1)} a_\rho\, \partial_\nu \Big((p^2-\nu^2)\tilde{\mathcal{J}}_{\nu,p}(u) \Big[\tilde{\mathcal{J}}_{\nu,p}(w) - \tilde{\mathcal{J}}_{-\nu,-p}(w) \Big] \Big) \Big|_{\nu=\rho} \\
&\quad = -\frac{2}{\pi i} \sum_{0\leqslant\rho\equiv p\,(1)} a_\rho (p^2-\rho^2) \tilde{\mathcal{J}}_{\rho,p}(u)\, \partial_\nu \Big(\tilde{\mathcal{J}}_{-\nu,-p}(w) - \tilde{\mathcal{J}}_{\nu,p}(w) \Big) \Big|_{\nu=\rho} \\
&\quad = 2i \sum_{0\leqslant\rho\equiv p\,(1)} a_\rho (-1)^{\rho-p} (p^2-\rho^2) \tilde{\mathcal{J}}_{\rho,p}(u) \mathcal{K}^*_{\rho,p}(w) =: C^p(u,w). \qquad (12.49)
\end{aligned}$$

Here we have used that for $\rho, p \in \frac{1}{2}\mathbb{Z}$ with $\rho \equiv p$ (mod 1),

$$\mathcal{K}^*_{\rho,p}(w) = \lim_{\nu\to\rho} \frac{\tilde{\mathcal{J}}_{-\nu,-p}(w) - \tilde{\mathcal{J}}_{\nu,p}(w)}{\sin \pi(\nu - p)} = \frac{(-1)^{\rho-p}}{\pi} \partial_\nu \Big(\tilde{\mathcal{J}}_{-\nu,-p}(w) - \tilde{\mathcal{J}}_{\nu,p}(w) \Big) \Big|_{\nu=\rho},$$

as well as the property $\tilde{\mathcal{J}}_{-\rho,-p} = \tilde{\mathcal{J}}_{\rho,p}$ in (12.31).

The expression of $\mathcal{K}^*_{\rho,p}(w)$ as a derivative of the difference $\tilde{\mathcal{J}}_{-\nu,-p}(w) - \tilde{\mathcal{J}}_{\nu,p}(w)$ at $\nu = \rho$ and the estimate (12.35) imply

$$\begin{aligned}
\mathcal{K}^*_{\rho,p}(w) \ll{}& \left|\frac{w}{2}\right|^{|\rho-p|+|\rho+p|} e^{|w|^2/2} \frac{1 + \log(1+|\rho+p|) + \log(1+|\rho-p|)}{|\rho-p|!|\rho+p|!} \\
&+ \left|\frac{u}{2}\right|^{|\rho+p|-|\rho-p|} e^{|u|^2/2} \frac{(|\rho-p|-1)!}{|\rho+p|!} \qquad \text{only if } \rho - p \neq 0 \\
&+ \left|\frac{u}{2}\right|^{|\rho-p|-|\rho+p|} e^{|u|^2/2} \frac{(|\rho+p|-1)!}{|\rho-p|!} \qquad \text{only if } \rho + p \neq 0. \qquad (12.50)
\end{aligned}$$

The estimates (12.35) and (12.50) suffice to obtain an estimate that is uniform on compact subsets in $\mathbb{C}^* \times \mathbb{C}^*$, and gives the convergence in ρ of the series defining $C^p(u,w)$ in (12.49).

Before we consider the sum of $C^p(u,w)$ over p, we shall prove a property of the Bessel kernel $\mathcal{K}^*_{\nu,p}$ which we shall use later.

Lemma 12.2.3. *For any $p, \rho \in \frac{1}{2}\mathbb{Z}$ with $\rho \equiv p$ (mod 1), we have*

$$\mathcal{K}^*_{\rho,p}(w) = \mathcal{K}^*_{|p|,\rho\,\mathrm{sign}(p)}(w) = \mathcal{K}^*_{p,\rho}(w).$$

Proof. The identity follows from the following expression:

$$\mathcal{K}^*_{\rho,p}(w) = -\frac{(-1)^{p+\xi}}{\pi} (u/|u|)^{-2\xi} \big\{ J_{p-\rho}(u) \mathbf{Y}_{p+\rho}(\bar{u}) + \mathbf{Y}_{p-\rho}(u) J_{p+\rho}(\bar{u}) \big\},$$

where

$$\mathbf{Y}_\nu(u) := 2\pi e^{i\pi\nu}\frac{J_\nu(u)\cos\pi\nu - J_{-\nu}(u)}{\sin 2\pi\nu}$$

is the Hankel function, which for $n \in \mathbb{Z}$ is

$$\mathbf{Y}_n(u) = \lim_{\nu\to n}\frac{J_\nu(u) - (-1)^n J_{-\nu}(u)}{\nu - n} = \partial_\nu J_\nu(u)|_{\nu=n} - (-1)^n \partial_\nu J_{-\nu}(u)|_{\nu=n},$$

and satisfies $\mathbf{Y}_{-n} = (-1)^n \mathbf{Y}_n$; see [43], p. 57–63. ■

We now have

$$\sum_{p=-P}^{P} C^p(u,w) = 2i \sum_{|p|\leqslant P}\sum_{\rho\equiv p\,(1)} (-1)^{\rho-p}(p^2-\rho^2)\tilde{\mathfrak{J}}_{\rho,p}(u)\mathcal{K}^*_{\rho,p}(w)$$

$$(\text{change} : \rho - p = a,\ \ \rho + p = b)$$

$$= -2i \sum_{\substack{a,b\in\mathbb{Z}\\ |b-a|\leqslant 2P}} (-1)^a ab\, \tilde{\mathfrak{J}}_{\frac{a+b}{2},\frac{b-a}{2}}(u)\mathcal{K}^*_{\frac{a+b}{2},\frac{b-a}{2}}(w).$$

Of course the term with $a = 0$ or $b = 0$ can be omitted. The contribution of the pairs (a,b) with $|a+b| \leqslant 2P$ and $|b-a| \leqslant 2P$ is zero because of (12.31) and Lemma 12.2.3. Moreover, the symmetry $(a,b) \mapsto (-a,-b)$ following from (12.31) and (K1) on p. 80, brings us to the following result:

$$\sum_{p=-P}^{P} C^p(u,w) = -4i \sum_{\substack{a,b\geqslant 1\\ |b-a|\leqslant 2P<a+b}} (-1)^a ab\, \tilde{\mathfrak{J}}_{\frac{a+b}{2},\frac{b-a}{2}}(u)\mathcal{K}^*_{\frac{a+b}{2},\frac{b-a}{2}}(w). \tag{12.51}$$

We use (12.34) and (12.50) to get an estimate for the latter sum in (12.51). For $(u,w) \in W$, with $W \subset \mathbb{C}^* \times \mathbb{C}^*$ arbitrary compact subset, the sum is bounded by

$$\sum_{p=-P}^{P} C^p(u,w) \ll \sum_{\substack{a,b\geqslant 1\\ |b-a|\leqslant 2P<a+b}} ab\,\frac{1+\log(1+a)+\log(1+b)}{(a!\,b!)^2}$$

$$= \sum_{n=2P+1}^{\infty}\sum_{\substack{a\geqslant 1\\ |n/2-a|\leqslant P}} a(n-a)\,\frac{1+\log(1+a)+\log(1+n-a)}{(a!\,(n-a)!)^2}.$$

Since $x! \geqslant x$ for all non-negative x, and $\log(1+a) < \log(1+n)$ as well as $\log(1+n-a) < \log(1+n)$, this is further

$$\ll \sum_{n=2P+1}^{\infty}\frac{1+\log(1+n)}{n!}\sum_{a=1}^{n}\binom{n}{a} = \sum_{n=2P+1}^{\infty}\frac{1+\log(1+n)}{n!}(2^n-1)$$

$$\ll \frac{\log(2P)}{\Gamma(2P+1)}\,2^{2P} \ll (2P)^{-1/2}\log(2P)\left(\frac{e}{P}\right)^{2P}.$$

Taking the limit as $P \to \infty$ yields

$$\lim_{P\to\infty} \sum_{p=-P}^{P} C^p(u,w) \ll \lim_{P\to\infty} \left((2P)^{-1/2}\log(2P)\right)\left(\frac{e}{P}\right)^{2P} = 0, \tag{12.52}$$

uniformly for (u,w) in any compact subset of $\mathbb{C}^* \times \mathbb{C}^*$. This implies (12.48), which finishes the proof of the proposition. ■

12.3 Kloosterman sum formula

We are now ready to give another form of the sum formula given in Theorem 11.3.3, which has an independent test function in the Kloosterman term. The following proposition shows that the delta term in the sum formula (11.34) vanishes for a test function of the form $\mathbf{K}f$, with $f \in C^\infty_{c,\mathrm{ev}}(\mathbb{C}^*)$.

Proposition 12.3.1. *If $f \in C^\infty_{c,\mathrm{ev}}(\mathbb{C}^*)$, then*

$$\sum_{p\in\frac{1}{2}\mathbb{Z}} \int_{(0)} \mathbf{K}f(\nu,p)(p^2-\nu^2)\,d\nu = 0.$$

Before we prove the proposition, it is worthwhile to mention that although $\int_Y \mathbf{K}f\, d\delta_{\omega,\omega'} = 0$, for an even, smooth and compactly supported f, there are test functions $\varphi \in \mathcal{H}^\sigma(a,b)$ for which the integral $\int_Y \varphi\, d\delta_{\omega,\omega'}$ is positive. This means that $\mathbf{B} : C^\infty_{c,\mathrm{ev}}(\mathbb{C}^*) \longrightarrow \mathcal{T}$, with $\mathcal{T}$ the class of test functions from Theorem 11.3.3, is not surjective. Therefore the transformation $\mathbf{K}$ is only a one-sided inversion of the Bessel transformation $\mathbf{B}$.

Proof. The estimates (12.22) and (12.24) allow us to shift the integration line $\mathrm{Re}\,\nu = 0$ to $\mathrm{Re}\,\nu = P + \frac{1}{4}$ for any $P \in \frac{1}{2}\mathbb{Z}$, $P \geqslant 1$, use the property (12.3) of the transform $\mathbf{K}f$, and picking up the residues to write:

$$\begin{aligned}\sum_{p\in\frac{1}{2}\mathbb{Z}} \int_{(0)} &\mathbf{K}f(\nu,p)(p^2-\nu^2)\,d\nu = \\ &= -2\pi(-1)^{2\xi} \lim_{P\to\infty} \sum_{|p|\leqslant P} \int_{(P+\frac{1}{4})} \mathbf{J}f(\nu,p)\frac{p^2-\nu^2}{\sin\pi(\nu-p)}\,d\nu \\ &\quad + 2\pi i(-1)^{2\xi} \sum_{\substack{p,q\in\frac{1}{2}\mathbb{Z}\\ \rho\equiv p\,(1)}} (-1)^{q-p}\mathbf{J}f(q,p)(q^2-p^2).\end{aligned} \tag{12.53}$$

The sum of residues vanishes because of the relation $\mathbf{J}f(q,p) = \mathbf{J}f(p,q)$, for all $p,q \in \frac{1}{2}\mathbb{Z}$ such that $p \equiv q \pmod 1$, which follows from (12.31) and the definition

(12.1) of the transform $\mathbf{J}f$. The estimate (12.22) implies that the latter integral in (12.53) is

$$\ll C_f^{2P}(1+|p|)^{2-A}\int_0^\infty (1+t)^{2-C}dt$$

for any fixed large $A, C > 0$. For $|p| \leqslant P$, we have $(1+|p|)^{-1} \geqslant (2P)^{-1}$, which implies that for any large enough $A > 0$ we have

$$\sum_{|p|\leqslant P}(1+|p|)^{2-A} \ll \sum_{|p|\leqslant P}(2P)^{-2P} = \frac{1+2P}{(2P)^{2P}} \ll (2P)^{-2P}.$$

Therefore

$$\sum_{p\in\frac{1}{2}\mathbb{Z}}\int_{(0)}\mathbf{K}f(\nu,p)(p^2-\nu^2)d\nu \ll \lim_{P\to\infty}\left(\frac{C_f}{2P}\right)^{2P} = 0. \qquad \blacksquare$$

Theorem 12.3.2. (Kloosterman sum formula) *Let $\omega, \omega' \in \mathcal{O}'\backslash\{0\}$ and ξ as in (4.27). Let f be an even, smooth, and compactly supported function on $\mathbb{C}^*$ such that $\mathbf{K}f(\nu,p)$ has at least double zeros at $\nu = \pm 1/2$ if $p \in \frac{1}{2}+\mathbb{Z}$. Then,*

$$\begin{aligned}\left(\frac{i\omega'}{|\omega'|}\right)^{2\xi}&\sum_{c\in I}{}' \left(\frac{c}{|c|}\right)^{-2\xi}\frac{S_\chi(\omega',\omega;c)}{|c|^2}f\left(\tfrac{4\pi}{c}\sqrt{\omega\omega'}\right) = \\ &= \frac{|d_F|}{2}\sum_V \overline{C_V(\omega;\nu_V,p_V)}C_V(\omega';\nu_V,p_V)\mathbf{K}f(\nu_V,p_V) \\ &\quad + \frac{1}{2\pi i}\sum_{\kappa\in\mathcal{C}_\chi}\frac{1}{[\Gamma_\kappa:\Gamma'_\kappa]|\Lambda_\kappa|}\sum_{p\in\frac{1}{2}\mathbb{Z}}^{\chi}\int_{(0)}\overline{B_{\kappa,\chi}(\omega;\nu,p)}\,\cdot \\ &\qquad\cdot B_{\kappa,\chi}(\omega';\nu,p)\mathbf{K}f(\nu,p)\,d\nu, \end{aligned} \tag{12.54}$$

where the transformation $\mathbf{K}$ is defined in (12.2), V runs over a maximal orthogonal system of irreducible cuspidal subspaces of $L^2(\Gamma\backslash G;\chi)$, and $|\Lambda_\kappa|$ is the Euclidean area of a fundamental domain for the lattice $\Lambda_\kappa \in \mathbb{C}$ corresponding to ${g_\kappa}^{-1}\Gamma'_\kappa g_\kappa$. Convergence of the expressions is absolute throughout.

Proof. For $f \in C^\infty_{c,\mathrm{ev}}(\mathbb{C}^*)$, the inverse Bessel transform $\mathbf{K}f$ satisfies the conditions in Theorem 11.3.3. Indeed, property (12.4) is condition (i), estimate (12.24) gives condition (iii), the holomorphy of the Bessel kernel $\mathcal{K}^*_{\nu,p}$ in ν (see Lemma 9.1.7) implies condition (ii), and condition (iv) is the requirement for f to have at least double zeros at $\nu = \pm\frac{1}{2}$ in the half-integer case. Therefore, for any non-zero $\omega, \omega' \in \mathcal{O}'$, we have

$$\int_Y \mathbf{K}f\,d\sigma_{\omega,\omega'} = \int_Y \mathbf{K}f\,d\delta_{\omega,\omega'} + \mathrm{Kl}(\omega,\omega';\mathbf{BK}f).$$

Proposition 12.3.1 implies that $\int_Y \mathbf{K}f\,d\delta_{\omega,\omega'} = 0$, and Theorem 12.2.1 then gives $\int_Y \mathbf{K}f\,d\sigma_{\omega,\omega'} = \mathrm{Kl}(\omega,\omega';f)$. This is exactly (12.54). $\blacksquare$

Remark 15. In the case of Gaussian number field $\mathbb{Q}(i)$, trivial character χ, and the discrete group $\mathrm{PSL}_2(\mathbb{Z}[i])$, Theorem 12.2.1 reduces to Theorem 11.1 in [9], and the result in Theorem 12.3.2 simplifies exactly to (13.1) in [9]. The class of test functions in [9], Theorem 13.1, is wider than our class of even, smooth and compactly supported functions on $\mathbb{C}^*$.

At the very end we note that a possible application of the formula given in Theorem 12.3.2 is deriving an estimate for the sum of Kloosterman sums

$$\sum_{\substack{c\in I\setminus\{0\}\\ |N(c)|\leqslant X}} \left(\frac{c}{|c|}\right)^{-2\xi} \frac{S_\chi(\omega',\omega;c)}{|N(c)|} \qquad \text{as} \quad X\to\infty.$$

The extra condition on the test function f concerning the zeros of $\mathbf{K}f$ at first sight seems insurmountable. However, we may choose the even, smooth, compactly supported function f arbitrarily on $|u| > 4\pi|\omega\omega'|^{1/2}$. The limited time for this thesis did not allow us to investigate whether this enables us to satisfy the condition $\mathbf{K}f(\frac{1}{2},p) = 0$ for all $p \equiv \frac{1}{2} \pmod 1$ without changing a given $f \in C^\infty_{c,\mathrm{ev}}(\mathbb{C}^*)$ on $|u| \leqslant 4\pi|\omega\omega'|^{1/2}$. We therefore stop here, and leave it as an open problem to return to at a future date.

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Selective Index of Terminology

Notation Index

Samenvatting

Het onderwerp van dit proefschrift is een generalisatie van de klassieke somformule van Bruggeman en Kuznetsov naar de bovenhalfruimte $\mathbb{H}^3$. Ten eerste zal ik de klassieke somformule en sommige toepassingen ervan kort uitleggen. Daarna volgt een overzicht van generalisaties van deze formule in verschillende richtingen. Ten slotte zal ik de somformule die het onderwerp van dit proefschrift is beschrijven en zowel op de overeenkomsten wijzen als de belangrijkste complicaties veroorzaakt door de generalisatie aangeven.

De somformule, zie Kuznetsov [25], en Bruggeman [2], die beide auteurs onafhankelijk van elkaar verkregen hebben, geeft een verband tussen de Fouriercoëfficiënten van de reëel-analytische spitsvormen op het bovenhalfvlak en de klassieke Kloostermansommen.

De verzameling $\mathbb{H}^2 = \{(x, y) \mid x \in \mathbb{R}, y > 0\}$ wordt het *bovenhalfvlak* genoemd. De groep $\mathrm{PSL}_2(\mathbb{R})$ werkt op deze ruimte via gebroken lineaire transformaties. Geef met $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ de modulaire groep aan. Op $\mathbb{H}^2$ is de standaard metriek met een constante kromming -1 gegeven. De differentiaal operator $-\Delta = -y^2(\partial_x^2 + \partial_y^2)$ is de Laplace operator op $\mathbb{H}^2$ die bij deze metriek hoort. Laat $\{\psi_j(z)\} \subset L^2(\Gamma\backslash\mathbb{H}^2)$ een compleet orthonormaal stelsel van Maass-spitsvormen met spectrale parameter $\nu_j \in i(0, \infty)$ zijn. Dit zijn eigenfuncties van de Laplace operator die tegelijkertijd eigenfuncties van de Heckeoperatoren zijn, geordend volgens de toenemende eigenwaarden $\lambda_j = \frac{1}{4} - \nu_j^2$. De Fourierontwikkeling van de functies $\psi_j(z)$ is van de vorm

$$\psi_j(x + iy) = \sum_{0 \neq n \in \mathbb{Z}} \rho_j(n) y^{1/2} K_{\nu_j}(2\pi|n|y) e^{2\pi i n x}, \tag{1}$$

waarbij K_ν de K-Besselfunctie is. Met $E(\nu, z)$ geef ik de Eisensteinreeks met parameter ν aan. De Fouriercoëfficiënten van de Eisensteinreeksen hebben een expliciete beschrijving in termen van de delersommen $\sigma_\nu(n) = \sum_{d|n} d^\nu$ en de zetafunctie van Riemann $\zeta(\nu)$.

Laat h een even holomorfe functie zijn die gedefinieerd is op de verzameling $\left\{\nu \in \mathbb{C} : |\operatorname{Re}\nu| < \frac{1}{2} + \varepsilon\right\}$, met $\varepsilon > 0$ zo dat $|h(\nu)| \ll (1 + |\nu|)^{-2-\delta} e^{-\pi|\operatorname{Im}\nu|}$ voor een $\delta > 0$ en alle ν met $|\operatorname{Re}\nu| \leqslant \frac{1}{2} + \varepsilon$. Voor dergelijke testfunctie h geldt de volgende gelijkheid voor alle gehele getallen $n, m \geqslant 1$ met absolute convergentie van alle sommen en integralen in de verschillende termen:

$$\sum_{j=1}^{\infty} \rho_j(n)\overline{\rho_j(m)} \frac{h(\nu_j)}{\cos(\pi\nu_j)} + \frac{1}{\pi i}\int_{\operatorname{Re}\nu=0} h(\nu)\left(\frac{n}{m}\right)^{-\nu} \frac{\sigma_{2\nu}(n)\sigma_{-2\nu}(m)}{|\zeta(1+2\nu)|^2} d\nu =$$

$$= \frac{\delta_{n,m}\, i}{\pi^2}\int_{\operatorname{Re}\nu=0} \nu\tan(\pi\nu)h(\nu)d\nu + \sum_{c=1}^{\infty}\frac{S(n,m;c)}{c}\varphi\left(\frac{4\pi\sqrt{nm}}{c}\right), \tag{2}$$

waarbij

$$\varphi(x) = \frac{2}{\pi i}\int_{\operatorname{Re}s=0} \frac{s\,J_{2s}(x)}{\cos(\pi s)} h(s)\,ds, \qquad \text{voor} \quad x > 0, \tag{3}$$

J_ν de klassieke J-Besselfunctie is, $S(n,m;c)$ de klassieke Kloostermansommen zijn en $\delta_{n,m}$ het Kronecker delta symbool is.

De linkerkant van de formule (2) komt uit de spectrale ontbinding van de Hilbertruimte $L^2(\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}^2)$ en wordt daarom de *spectrale kant* genoemd. De twee termen corresponderen met het discrete en het continu spectrum van de Laplace operator $-\Delta$. De rechterkant van (2) komt van de meetkunde van de ruimte $\{\left(\begin{smallmatrix}1 & *\\ 0 & 1\end{smallmatrix}\right)\}\backslash\mathrm{SL}_2(\mathbb{Z})$ die geïnduceerd wordt door de Bruhatdecompositie van de groep $\mathrm{SL}_2(\mathbb{R})$, en wordt daarom de *meetkundige kant* genoemd. De eerste, zogenaamde delta term, komt uit de representanten $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right)$ van de nevenklassen in $\{\left(\begin{smallmatrix}1 & *\\ 0 & 1\end{smallmatrix}\right)\}\backslash\mathrm{SL}_2(\mathbb{Z})$ met $c = 0$. Matrices met $c \neq 0$ die afkomstig zijn uit de grote cel in de Bruhatdecompositie van $\mathrm{SL}_2(\mathbb{Z}) \cap \mathrm{SL}_2(\mathbb{R})$ geven de tweede, zogenaamde Kloosterman term.

Somformules van dit type, zoals (2), kunnen op twee verschillende manieren gebruikt worden. Aan de ene kant, in de gegeven vorm (2) met de onafhankelijke testfunctie aan de spectrale kant, is het een middel om resultaten in verband met de spectrale data te krijgen. Bijvoorbeeld, Stelling 4.1 in [2] geeft het volgende dichtheidsresultaat:

$$\sum_{j=1}^{\infty} e^{-v\lambda_j}\frac{\rho_j(n)\overline{\rho_j(m)}}{\cos(\pi\nu_j)} = \frac{\delta_{n,m}}{\pi^2}\left|\frac{n}{m}\right|^{1/2} v^{-1} + O(v^{-1/2-\varepsilon}), \tag{4}$$

als $v \downarrow 0$ en $\varepsilon > 0$.

Aan de andere kant, als we weten hoe de Besseltransformatie (3) te inverteren, staat de onafhankelijke testfunctie aan de meetkundige kant. Dan kan de somfomule gebruikt worden om Kloostermansommen te schatten. In [25] geeft Kuznetsov een eenzijdig inverse van de Besseltransformatie en bewijst dat voor $m, n \geqslant 1$ en $X \to \infty$,

$$\sum_{c=1}^{X}\frac{S(n,m;c)}{c} \ll_{n,m} X^{1/6}(\log X)^{1/3}. \tag{5}$$

Weil's schatting van Kloostermansommen impliceert een schatting $O(X^{1/2+\varepsilon})$ voor de bovengenoemde som, zie [44], en de hypothese van Linnik voorspelt dat de schatting eigelijk $O(X^{\varepsilon})$ is, zie [29].

Generalisaties van de somformule (2) zijn op verschillende manieren verkregen. Zowel Bruggeman als Kuznetsov werken met de volle modulaire groep $\mathrm{SL}_2(\mathbb{Z})$, gewicht nul en het triviale multiplicator systeem. In [38] heeft Proskurin de aanpak van Kuznetsov gebruikt om de somformule te generaliseren naar alle coëindige discrete ondergroepen van $\mathrm{SL}_2(\mathbb{R})$ met niet-triviaal multiplicator systeem en algemeen gewicht. Hij werkt met Fouriercoëfficiënten die positieve orde hebben. De Fouriercoëfficiënten met algemene orde heeft Bruggeman in [3] behandeld. In tegenstelling tot Kuznetsov [25] en Proskurin [38], heeft de somformule in [3] een meer representationeel karakter.

In [35] geven Miatello en Wallach een formule van hetzelfde type voor reële samenhangende semisimpele Liegroepen met $\mathbb{R}$-rang 1. Daar is het bovenhalfvlak door een complete Riemannse niet-compacte symmetrische ruimte van rang één vervangen en de discrete ondergroep is een groep van isometries met eindig volume van het quotiënt. Er worden alleen maar de triviale K-types behandeld. In [42] generaliseren de auteurs de formule van [35] voor producten van groepen met rang één. De formule wordt nauwkeuriger bepaald als de groep die bekeken wordt een product is van groepen van de vorm $\mathrm{SL}_2(\mathbb{R})$ of $\mathrm{SL}_2(\mathbb{C})$. Bruggeman en Miatello, [4], gebruiken de formule in [35] om sommen van gegeneraliseerde Kloostermansommen voor deze klasse van groepen te bestuderen. Met een geschikte keuze van de testfunctie, krijgen ze een schatting van type (5) voor die sommen (zie [4], Hoofdstelling 1 in §4.3). In [5] geven dezelfde auteurs een somformule voor SL_2 over een willekeurig getallenlichaam met een beperking tot triviale K-types. Het geval van een totaal reëel getallenlichaam waarin alle K-types in aanmerking worden genomen, behandelen Bruggeman, Miatello en Pacharoni in [6].

In zijn boek [36] geeft Motohashi een expliciete formule voor het vierde moment van de zetafunctie van Riemann, waarvoor hij de somformule voor $\mathrm{SL}_2(\mathbb{R})$ gebruikt. Een overeenkomstige redenering leidt tot uitbreiding van deze resultaten tot sommige kwadratische getallenlichamen. Bruggeman en Motohashi laten in [8] zien hoe men hetzelfde voorbereidingswerk kan uitvoeren met de groep $\mathrm{SL}_2(\mathbb{C})$ in plaats van $\mathrm{SL}_2(\mathbb{R})$. Het uiteindelijke doel is een spectrale ontbinding van het vierde moment van de zetafunctie van Dedekind te geven. De somformule voor $\mathrm{SL}_2(\mathbb{Z}[i])\backslash\mathrm{SL}_2(\mathbb{C})$ voor alle K-types en tevens een expliciete formule voor het vierde moment van de zetafunctie van Dedekind leiden deze auteurs in [9] af, beperkt tot even functies.

In dit proefschrift generaliseer ik de somformule gegeven in [9] voor een algemeen imaginair kwadratisch getallenlichaam F en een willekeurige congruentieondergroep $\Gamma = \Gamma_0(I)$ met $I \subset \mathcal{O}$ een ideaal in de ring van gehelen van F ongelijk aan nul. Ik bekijk χ-automorfe functies met betrekking tot Γ, waarbij χ een karakter op Γ is dat triviaal op $\Gamma_1(I) \subset \Gamma$ is. Ik behandel ook het geval van oneven functies.

In het eerste hoofdstuk beschrijf ik enkele elementaire feiten over de meetkunde van de driedimensionale hyperbolische ruimte $\mathbb{H}^3$, transformatiegroepen op deze ruimte en klassieke Besselfuncties. In het tweede hoofdstuk is een klein stuk van de voorstellingstheorie van de Liegroepen $\mathrm{SL}_2(\mathbb{C})$ en $\mathrm{SU}(2)$ beschreven.

In het derde hoofdstuk voer ik de automorfe functies en automorfe voorstellingen in. Kies $l, q, p \in \frac{1}{2}\mathbb{Z}$ zodanig dat $|p|, |q| \leqslant l$ en $p \equiv q \equiv l \pmod 1$. Kies verder een karakter χ op $(\mathcal{O}/I)^*$ dat correspondeert met een karakter op $\Gamma_0(I)$ in de volgende manier: $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \chi(d)$ en $\nu \in \mathbb{C}$ zodanig dat (ν, p) spectrale parameter is. De gladde functie $f : \mathrm{SL}_2(\mathbb{C}) \to \mathbb{C}$ is een *χ-automorfe functie met betrekking tot $\Gamma_0(I)$ van type (l, q) met spectrale parameter* (ν, p) genoemd, als f aan de volgende voorwaarden voldoet:

(i) $f(\gamma g) = \chi(d) f(g)$, voor alle $\gamma = \begin{pmatrix} * & * \\ * & d \end{pmatrix} \in \Gamma_0(I)$, $g \in \mathrm{SL}_2(\mathbb{C})$,

(ii) $\Omega_{\mathfrak{k}} f = -\frac{l^2+l}{2} f$, $\quad \mathbf{H}_2 f = -iqf$,

(iii) $\Omega_{\pm} f = \frac{(\nu \mp p)^2 - 1}{8} f$.

Hier $\Omega_{\pm}$ zijn de Casimir elementen van de complexe Lie-algebra $\mathfrak{g}$ van $\mathrm{SL}_2(\mathbb{C})$ en $\Omega_{\mathfrak{k}}$ is het Casimir element van de complexe Lie-algebra $\mathfrak{k}$ van $\mathrm{SU}(2)$.

Met $H(\nu, p)$ duid ik de ruimte van K-eindige vectoren in de hoofdreeks voorstellingen aan. Elke twijnoperator voor de werking van $\mathfrak{g}$ vanuit $H(\nu, p)$ naar de ruimte van K-eindige vectoren in $C^\infty(\Gamma\backslash\mathrm{SL}_2(\mathbb{C}); \chi)$, gladde χ-automorfe functies op $\mathrm{SL}_2(\mathbb{C})$ met betrekking tot Γ, is een *automorfe voorstelling* voor $H(\nu, p)$.

In de beschrijving van de Fourierontwikkeling van de automorfe functies (voorstellingen) spelen de operatoren van Jacquet en Goodman-Wallach een belangrijke rol. Ze worden apart behandeld in het vierde hoofdstuk.

Een gedetailleerde beschrijving van Fouriercoëfficiënten van de Eisensteinreeksen en de spitsvormen geef ik in het vijfde hoofdstuk. De gelijkheid (5.6) laat zien dat de Fouriercoëfficiënten van een cuspidale automorfe voorstelling V met spectrale parameter (ν_V, p_V) tegelijkertijd de Fouriercoëfficiënten van de spitsvormen met dezelfde spectrale parameter en type (l, q), voor alle $l \geqslant |p_V|$, $|q| \leqslant l$ zijn.

Het zesde, het zevende en het achtste hoofdstuk bevatten bekende resultaten over Kloostermansommen, Poincaréreeksen en spectrale ontbinding van de Hilbertruimte $L^2(\Gamma\backslash\mathrm{SL}_2(\mathbb{C}))$. Ze worden gegeven in het meest geschikte vorm voor mijn doeleinden.

In het negende hoofdstuk definieer ik de Lebedevtransformatie, geef ik de eenzijdige inverse op een bepaalde klasse van testfuncties en beschrijf ik enige eigenschappen van beide. De inverse Lebedevtransformatie zal een bouwsteen zijn voor de constructie van de speciale Poincaréreeksen die gebruikt zullen worden om de somformule af te leiden. Mijn Lebedevtransformatie is een soort uitbreiding van de klassieke Lebedevtransformatie $f \mapsto \int_0^\infty f(r) K_\nu(r) \frac{dr}{r}$ die een belangrijke rol speelt in de theorie van somformules voor rationale Kloostermansommen. Een van de grootste problemen in dit proefschrift was de kwadratisch integreerbaarheid en begrensdheid van de gekozen Poincaréreeksen te bewijzen. Het bewijs steunt aanzienlijk zowel op de resultaten van Miatello en Wallach in [34] als op Lemma 5.2.1.

De afleiding van de voorlopige somformule behandel ik in het tiende hoofdstuk. Het bestaat uit de berekening van hetzelfde inproduct van twee speciale Poincaréreeksen op twee verschillende manieren: spectrale beschrijving waarin de Fouriercoëfficiënten $C_V(\omega; \nu_V, p_V)$ van de cuspidale automorfe voorstellingen verschijnen en meetkundige beschrijving waarin de sommen van Kloostermansommen verschijnen. Dit is ook de methode die Bruggeman [2], Kuznetsov [25] en Proskurin [38] gebruiken.

Het belangrijkste resultaat in dit proefschrift is de *spectrale somformule*, zie Hoofdstelling 11.3.3 in het elfde hoofdstuk. Deze wordt verkregen door de klasse van testfuncties in de voorlopige versie, Stelling 10.3.1, uit te breiden. De uitbreidingsmethode beschrijf ik in §11.2. Deze is analoog aan de methode van Miatello en Wallach in [35].

In §11.5 pas ik de somformule (11.34) toe om gewogen dichtheidsresultaten te krijgen voor cuspidale automorfe voorstellingen in $L^2(\mathrm{SL}_2(\mathcal{O})\backslash\mathrm{SL}_2(\mathbb{C}))$ met eigenwaarde $\lambda_V \leqslant X$ voor een voorgeschreven spectrale parameter p_V. Namelijk, voor willekeurige $\omega \in \mathcal{O}'\backslash\{0\}$, $p \in \frac{1}{2}\mathbb{Z}$ en $X \to \infty$ geldt

$$\sum_{\substack{V: p_V = \pm p \\ \lambda_V \leqslant X}} |C_V(\omega; \nu_V, p_V)|^2 \sim \frac{2\epsilon_p}{3\pi^3\sqrt{|d_F|}}\, X^{3/2}, \tag{6}$$

waarbij $\epsilon_0 = 1$ en $\epsilon_p = 2$ als $p \neq 0$.

In het laatste, twaalfde hoofdstuk, in Hoofdstelling 12.2.1, geef ik een eenzijdige inverse van de Besseltransformatie (11.1). Dit maakt het mogelijk om de somformule in de omgekeerde richting te schrijven, zie Hoofdstelling 12.3.2. De expliciete testfunctie is hier aan de meetkundige kant van de formule waar de Kloostermansommen verschijnen en daarom is deze vorm van de somformule ook bekend als *Kloosterman somformule*. Toepassingen van de formule (12.54) om schattingen voor sommen van de vorm

$$\sum_{c \in I}{}' (c/|c|)^{-2\xi}\, \frac{S_\chi(\omega, \omega'; c)}{|c|^2}, \tag{7}$$

waarbij $\xi = 0$ als p geheel is en $\xi = \frac{1}{2}$ als $p \in \frac{1}{2} + \mathbb{Z}$ te krijgen zijn allicht mogelijk.

Резиме

Целта на оваа дисертација е да установам аналог на класичната формула за сумирање на Бругеман и Кузнецов за горниот полупростор $\mathbb{H}^3$. Најпрво ќе дадам кусо објаснување на класичната формула за сумирање и некои нејзини примени. Потоа следи преглед на обопштувања на оваа формула во различни насоки. На крајот ќе ја опишам во кратки црти формулата за сумирање која е предмет на овој труд, нагласувајќи ги притоа аналогиите со класичната формула како и основните комликации кои произлегуваат од обопштувањето.

Формулата за сумирање, види Кузнецов [25] и Бругеман [2], која е независно изведена од двајцата автори, дава врска помеѓу Фуриевите коефициенти на каспидалните реално-аналитички модуларни форми на горната полурамнина и класичните суми на Клостерман.

Множеството $\mathbb{H}^2 = \{(x, y) \mid x \in \mathbb{R},\ y > 0\}$ се нарекува *горна полурамнина.* Групата $\mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm 1\}$ дејствува на овој простор преку дробно-рационалните линеарни трансформации. Нека $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$ е модуларната група. На просторот $\mathbb{H}^2$ е дадена стандардната метрика со константна кривина -1, така наречена *хиперболична метрика.* Лапласовиот оператор на $\mathbb{H}^2$ кој ѝ соодветствува на оваа метрика е означен со $-\Delta = -y^2(\partial_x^2 + \partial_y^2)$. Нека $\{\psi_j(z)\} \subset L^2(\Gamma\backslash\mathbb{H}^2)$ е комплетен ортонормален систем од каспидални Мас-форми со спектрален параметар $\nu_j \in i(0, \infty)$. Функциите ψ_j се сопствени функции на Лапласовиот оператор кои се истовремено и сопствени функции на Хекеовите оператори, индексирани според индексите на своите растечки сопствени вредности $\lambda_j = \frac{1}{4} - \nu_j^2$. Фуриевиот развој на $\psi_j(z)$ е даден со

$$\psi_j(x + iy) = \sum_{0 \neq n \in \mathbb{Z}} \rho_j(n) y^{1/2} K_{\nu_j}(2\pi|n|y) e^{2\pi i n x}, \qquad (1)$$

каде што K_ν е K-Беселовата функција. Нека со $E(\nu, z)$ е означен Ајзенштајновиот ред со параметар ν. Фуриевите коефициенти на Ајзенштајновите редови се експлицитно опишани со изрази во кои фигурираат сумите од делители $\sigma_\nu(n) = \sum_{d|n} d^\nu$ како и Римановата зета-функција $\zeta(\nu)$.

Нека h е парна холоморфна функција дефинирана на множеството $\left\{\nu \in \mathbb{C} : |\operatorname{Re}\nu| < \frac{1}{2} + \varepsilon\right\}$, со $\varepsilon > 0$ такво што $|h(\nu)| \ll (1+|\nu|)^{-2-\delta} e^{-\pi|\operatorname{Im}\nu|}$ за некое $\delta > 0$ и сите ν со $|\operatorname{Re}\nu| \leqslant \frac{1}{2} + \varepsilon$. За ваква тест-функција h важи следнава *формула за сумирање*, за сите цели броеви $n, m \geqslant 1$, при што конвергенцијата на сите суми и интеграли во поодделните членови е абсолутна:

$$\sum_{j=1}^{\infty} \rho_j(n)\overline{\rho_j(m)} \frac{h(\nu_j)}{\cos(\pi\nu_j)} + \frac{1}{\pi i}\int_{\operatorname{Re}\nu=0} h(\nu)\left(\frac{n}{m}\right)^{-\nu} \frac{\sigma_{2\nu}(n)\sigma_{-2\nu}(m)}{|\zeta(1+2\nu)|^2} d\nu =$$
$$= \frac{\delta_{n,m}\, i}{\pi^2}\int_{\operatorname{Re}\nu=0} \nu\tan(\pi\nu)h(\nu)d\nu + \sum_{c=1}^{\infty}\frac{S(n,m;c)}{c}\varphi\left(\frac{4\pi\sqrt{nm}}{c}\right). \qquad (2)$$

Тука

$$\varphi(x) = \frac{2}{\pi i}\int_{\operatorname{Re}s=0} \frac{s\, J_{2s}(x)}{\cos(\pi s)}\, h(s)\, ds, \qquad \text{за} \quad x > 0, \qquad (3)$$

J_ν е класичната Беселова функција, $S(n,m;c)$ е класична сума на Клостерман и $\delta_{n,m}$ е Кронекеровиот делта симбол.

Левата страна на формулата (2) произлегува од спектралната декомпозиција на Хилбертовиот простор $L^2(\Gamma\backslash\mathbb{H}^2)$ и затоа се нарекува *спектрална страна*. Нејзиниот прв член соодветствува на дискретниот, а вториот на непрекинатиот дел од спектарот на Лапласовиот оператор $-\Delta$. Десната страна на равенството (2) е поврзана со геометријата на просторот $\{\left(\begin{smallmatrix}1 & *\\ 0 & 1\end{smallmatrix}\right)\}\backslash\mathrm{PSL}_2(\mathbb{Z})$ индуциран од Брухатовото разложување на групата $\mathrm{PSL}_2(\mathbb{R})$ и затоа се нарекува *геометриска страна*. Првиот, така наречен делта член, произлегува од претставниците $\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ на комплексите во $\{\left(\begin{smallmatrix}1 & *\\ 0 & 1\end{smallmatrix}\right)\}\backslash\mathrm{PSL}_2(\mathbb{Z})$ со $c = 0$. Матриците со $c \neq 0$ кои доаѓаат од големата ќелија во Брухатовото разложување на $\mathrm{PSL}_2(\mathbb{Z}) \cap \mathrm{PSL}_2(\mathbb{R})$ го даваат вториот, така наречен Клостерманов член.

Формулите за сумирање од овој тип, како на пример формулата (2), може да бидат употребени на два различни начини. Од една страна во обликот (2), со независната тест функција на спектралната страна, таа е средство за добивање резултати во врска со спектралните податоци. На пример, Пропозицијата 4.1 од [2] го дава следниов резултат за распределба:

$$\sum_{j=1}^{\infty} e^{-v\lambda_j}\frac{\rho_j(n)\overline{\rho_j(m)}}{\cos(\pi\nu_j)} = \frac{\delta_{n,m}}{\pi^2}\left|\frac{n}{m}\right|^{1/2} v^{-1} + O(v^{-1/2-\varepsilon}), \qquad (4)$$

кога $v \downarrow 0$ и $\varepsilon > 0$.

Од друга страна, ако е познато како да се инвертира Беселовата трансформација (3), независната тест функција се добива на геометриската страна и тогаш формулата за сумирање може да се користи за добивање оценки за суми од Клостерманови суми. Во [25], Кузнецов дава еднострана инверзија на Беселовата трансформација и покажува дека за $m, n \geqslant 1$ и $X \to \infty$,

$$\sum_{c=1}^{X} \frac{S(n,m;c)}{c} \ll_{n,m} X^{1/6} (\log X)^{1/3}. \tag{5}$$

Границата на Веил за сумите на Клостерман имплицира оценка $O(X^{1/2+\varepsilon})$ за горе наведената сума (види [44]), додека хипотезата на Линик предвидува дека оценката е всушност $O(X^{\varepsilon})$, види [29].

Обопштувања на формулата за сумирање (2) се добиени на различни начини. И Бругеман и Кузнецов ја разгледуваат потполната модуларна група $\mathrm{SL}_2(\mathbb{Z})$, тежина нула и тривијален систем од мултипликатори. Во [38], Проскурин го користи пристапот на Кузнецов обопштувајќи ја формулата за сумирање на сите дискретни кофинитни подгрупи на $\mathrm{SL}_2(\mathbb{R})$, нетривијален систем од мултипликатори и општа тежина. Тој работи со фуриевите коефициенти чиј реден број е позитивен. Бругеман, во својата книга [3], ги третира сите фуриеви коефициенти, но за разлика од Кузнецов [25] и Проскурин [38], неговата формула е со повеќе репрезентациски карактер.

Миатело и Валах даваат во [35] формула од истиот тип за реални сврзани полуедноставни Ли-групи со $\mathbb{R}$-ранк 1. Тие ја заменуваат горната полурамнина со било кој комплетен Риманов некомпактен симетричен простор од ранк 1, дискретната подгрупа Γ е група од изометрии со конечен волумен на фактор просторот $\Gamma \backslash \mathbb{H}^2$ и ги разгледуваат само тривијалните K-типови. Во [42], авторите ја обопштуваат формулата од [35] на производи од групи со реален ранк 1. Формулата станува многу поексплицитна кога групата која се разгледува е производ од групи од обликот $\mathrm{SL}_2(\mathbb{R})$ или $\mathrm{SL}_2(\mathbb{C})$. Бругеман и Миатело во [4] ја употребуваат формулата за сумирање од [35] за проучување на суми од обопштени Клостерманови суми за оваа класа на групи. Избирајќи погодна тест-функција тие изведувааат оценка од типот (5) за овие суми (види [4], Теорема 1 во §4.3). Во [5], истите автори даваат формула за сумирање за SL_2 над произволно бројно поле, но со ограничување на тривијалните K-типови. Случајот над тотално реално бројно поле земајќи ги во предвид сите K-типови е третиран од Бругеман, Миатело и Пачарони во [6].

Мотохаши, во својата книга [36], дава експлицитна формула за моментот од четврти степен на Римановата зета-функција користејќи ја формулата за сумирање за $\mathrm{SL}_2(\mathbb{R})$. Аналогно размислување води кон обопштување на овие резултати за некои квадратични бројни полиња. Бругеман и Мотохаши покажуваат во [8] како еднаква подготвителна работа може да се изведе со групата $\mathrm{SL}_2(\mathbb{C})$ наместо $\mathrm{SL}_2(\mathbb{R})$, со крајна цел да се даде спектрално разложување на моментот од четврти степен на Дедекиндовата зета-функција. Формулата за сумирање за $\mathrm{SL}_2(\mathbb{Z}[i])\backslash\mathrm{SL}_2(\mathbb{C})$ која ги вклучува сите K-типови, како и експлицитна формула за моментот од четврти степен на Дедекиндовата зета-функција се изведени од овие автори во [9]. Таму е третиран само случајот на парни функции.

Во овој труд ја обопштувам формулата за сумирање дадена во [9] разгледувајќи произволно имагинарно квадратично бројно поле F, произволна конгруенциска подгрупа $\Gamma = \Gamma_0(I)$ со $I \subset \mathcal{O}$ ненулти идеал во прстенот од цели броеви на полето F и χ-автоморфни функции во однос на Γ, каде χ е карактер на Γ кој е тривијален на $\Gamma_1(I) \subset \Gamma$. Исто така го вклучувам и случајот на непарни функции воведувајќи така наречен централен карактер.

Во првите две глави се дадени некои основни факти за геометријата на тродимензионалниот хиперболичен простор $\mathbb{H}^3$, групата од трансформации на овој простор и Беселовите функции, како и мал дел од репрезентациската теорија на Ли-групите $\mathrm{SL}_2(\mathbb{C})$ и $\mathrm{SU}(2)$.

Во третата глава се воведени автоморфните функции и автоморфните репрезентации кои ќе бидат разгледувани. Нека $l, q, p \in \frac{1}{2}\mathbb{Z}$ се такви што $|p|, |q| \leqslant l$ и $l \equiv p \equiv q \pmod 1$. Со χ е означен карактер на $(\mathcal{O}/I)^*$ кој соодветствува на карактер на $\Gamma_0(I)$ на следниов начин: $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mapsto \chi(d)$. Нека $\nu \in \mathbb{C}$ е таков што (ν, p) е спектрален параметар. За глатката функција $f : \mathrm{SL}_2(\mathbb{C}) \to \mathbb{C}$ се вели дека е *χ-автоморфна функција во однос на $\Gamma_0(I)$ од тип (l, q) со спектрален параметар (ν, p)* ако ги задоволува следниве услови:

(i) $f(\gamma g) = \chi(d) f(g), \quad$ за сите $\gamma = \left(\begin{smallmatrix} * & * \\ * & d \end{smallmatrix}\right) \in \Gamma_0(I)$, $g \in \mathrm{SL}_2(\mathbb{C})$,

(ii) $\Omega_{\mathfrak{k}} f = -\frac{l^2+l}{2} f, \qquad \mathbf{H}_2 f = -iqf$,

(iii) $\Omega_{\pm} f = \frac{(\nu \mp p)^2 - 1}{8} f$.

Тука со $\Omega_{\pm}$ се означени Казимировите елементи на комплексната Ли-алгебра $\mathfrak{g}$ на $\mathrm{SL}_2(\mathbb{C})$, додека $\Omega_{\mathfrak{k}}$ е Казимировиот елемент на комплексната Ли-алгебра $\mathfrak{k}$ на групата $\mathrm{SU}(2)$.

Со $H(\nu, p)$ е означен просторот од K-конечни вектори во главното низовно претставување на $\mathrm{SL}_2(\mathbb{C})$. Секој линеарен оператор од просторот $H(\nu, p)$ во просторот од K-конечни елементи во $C^\infty(\Gamma\backslash\mathrm{SL}_2(\mathbb{C}); \chi)$, глатки χ-автоморфни функции на $\mathrm{SL}_2(\mathbb{C})$ во однос на Γ, кој комутира со дејството на Ли-алгебрата $\mathfrak{g}$, се нарекува *автоморфна репрезентација* за $H(\nu, p)$.

Во експлицитниот опис на фуриевиот развој на автоморфните функции (репрезентации) централно место завземаат операторите на Жаке и Гудман–Валах кои се одделно третирани во четвртата глава.

Подетален опис на фуриевите коефициенти на Ајзенштајновите редови и каспидалните автоморфни функции е даден во петтата глава. Равенството (5.6) укажува на фактот дека фуриевите коефициенти на каспидална автоморфна репрезентација V со спектрален параметар (ν_V, p_V) се истовремено и фуриеви коефициенти на автоморфните функции со истиот спектрален параметар и тип (l, q), за сите $l \geqslant |p_V|$, $|q| \leqslant l$.

Шестата, седмата и осмата глава содржат веќе познати резултати во врска со сумите на Клостерман, Поенкареовите редови и спектралната декомпозиција на просторот $L^2(\Gamma\backslash\mathrm{SL}_2(\mathbb{C}))$. Тие се дадени во облик кој е соодветен на потребите на овој труд.

Во деветтата глава е дефинирана Лебедевата трансформација, дадена е нејзина еднострана инверзија на одредена класа од тест функции и опишани се некои нивни својства. Инверзната Лебедева трансформација ќе игра главна улога во конструкцијата на специјалните Поенкареови редови кои пак подоцна ќе бидат употребени за изведување на формулата за сумирање. Мојата Лебедева трансформација е еден вид обопштување на класичната Лебедева трансформација $f \mapsto \int_0^\infty f(r) K_\nu(r) \frac{dr}{r}$ која е од огромно значење во теоријата на формули за сумирање за рационални суми на Клостерман. Еден од најголемите проблеми при пишувањето на тезава беше доказот на квадратно-интеграбилноста и ограниченоста на избраните Поенкареови редови. Тој цврсто се темели врз резултатите на Миатело и Валах во [34] како и врз Лема 5.2.1.

Изведувањето на прелиминарна формула за сумирање е опишано во десеттата глава. Тоа се состои од пресметување на истиот скаларен производ од два специјални Поенкареови редови на два различни начини: спектрален опис во кој се појавуваат фуриевите коефициенти $C_V(\omega; \nu_V, p_V)$ на каспидалните автоморфни репрезентации и геометриски опис во кој се појавуваат суми од Клостерманови суми. Ова е методот кој го користат и Бругеман [2], Кузнецов [25] и Проскурин [38].

Главен резултат во овој труд е *спектралната формула за сумирање* која е опишана во Теорема 11.3.3, во единаесеттата глава. Таа е добиена со проширување на класата од тест-функции во нејзината прелиминарна верзија, Пропозиција 10.3.1. Методот на екстензија е опишан во §11.2. Тој е потполно аналоген со методот на Миатело и Валах во [35].

Во §11.5 ја применувам формулата за сумирање (11.34) и добивам резултати во врска со распределбата на каспидалните автоморфни репрезентации во просторот $L^2(\mathrm{SL}_2(\mathcal{O})\backslash\mathrm{SL}_2(\mathbb{C}))$ чија сопствена вредност λ_V, при даден спектрален параметар p_V, не надминува X. Имено, за произволни $\omega \in \mathcal{O}'\backslash\{0\}$, $p \in \frac{1}{2}\mathbb{Z}$ и $X \to \infty$ важи

$$\sum_{\substack{V : p_V = \pm p \\ \lambda_V \leqslant X}} |C_V(\omega; \nu_V, p_V)|^2 \sim \frac{2\epsilon_p}{3\pi^3 \sqrt{|d_F|}} X^{3/2}, \tag{6}$$

каде што $\epsilon_0 = 1$ и $\epsilon_p = 2$ ако $p \neq 0$.

Во последната, дванаесетта глава, со Теорема 12.2.1 е дадена еднострана инверзија на Беселовата трансформација (11.1). Тоа овозможува да се докаже формулата за сумирање во обратна форма како во Теорема 12.3.2. Експлицитната тест-функција овој пат се наоѓа на геометриската страна во која фигурираат сумите на Клостерман, поради што овој облик на формулата за сумирање е уште познат под името *Клостерманова формула за сумирање*. Примени на оваа формула за добивање оценки на сумите од Клостерманови суми од обликот

$$\sum_{c \in I}{}' (c/|c|)^{-2\xi} \frac{S_\chi(\omega, \omega'; c)}{|c|^2}, \tag{7}$$

каде $\xi = 0$ ако p е цел број и $\xi = \frac{1}{2}$ ако $p \in \frac{1}{2} + \mathbb{Z}$, се можни, но ограниченоста на расположивото време за припрема на овој труд ни наложува да го одложиме нивното изведување за некоја друга прилика.

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* * * * *

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Curriculum Vitae

Hristina Lokvenec-Guleska was born on June 26th, 1974 in Skopje, Macedonia. She attended the primary school "Jochan Heinrich Pestalozzi", as well as the primary ballet school "Ilija Nikolovski–Luj", both in Skopje from September 1981 till June 1989, and the "Rade Jovčevski Korčagin" high school in Skopje from September 1989 till June 1993. In September 1993 Hristina started her study in theoretical mathematics at the University "St. Cyril and Methodius" in Skopje. On May 22, 1998 she graduated on the Faculty of Natural Sciences and Mathematics, Department of Mathematics, Theoretical Mathematics–Major and obtained the title Graduated mathematician. Her final exam was supervised by Professor Dončo Dimovski and is titled *Hyperbolic geometry*. During her studies Hristina was several times as an exchange student and member of the IAESTE student organization on practical training at universities around Europe. In September 1998 she started Master studies in Mathematics at the University "St. Cyril and Methodius"in Skopje. In the period April, 1999 to August, 1999 she worked as mathematics teacher in the "Rade Jovčevski Korčagin" high school. In September 1999 she got accepted for the Master Class in Arithmetic Algebraic Geometry at the Mathematical Institute of Utrecht University, The Netherlands. She completed it successfully in June 2000, and her finishing test-problem has title *Some examples of modular parameterizations*. She married Aleksandar Lokvenec on August 24th, 2000 in Skopje. A month later, in September 2000 Hristina started her Ph.D. studies at the Mathematical Institute of Utrecht University as an AiO (assistant in education) under the supervision of Dr. Roelof Bruggeman and Professor Hans Duistermaat. During her work, she was a member of the electing committee for three open post-doc positions at the Utrecht University, participated in the Summer School in *Automorphic forms and Shimura varieties* at the Fields Institute in Toronto, Canada (June 2003) and attended the International Conference on Topology and Applications in Skopje, Macedonia (September 2004).