Lamé Equations with Finite Monodromy

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Lamé Equations with Finite Monodromy

Lamé-vergelijkingen met eindige monodromie
(met een samenvatting in het Nederlands)

Proefschrift

ter verkrijging van de graad van doctor aan de Universiteit Utrecht op gezag van de Rector Magnificus, Prof. dr. W.H. Gispen, ingevolge het besluit van het College voor Promoties in het openbaar te verdedigen op woensdag 16 januari 2002 des middags te 12.45 uur

door

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geboren op 1 september 1970, te Nijmegen

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Dit proefschrift werd mede mogelijk gemaakt met financiële steun van de Nederlandse Organisatie voor Wetenschappelijk Onderzoek (NWO). Het onderzoek dat in dit proefschrift beschreven is, maakt deel uit van het project *Algoritmen in de algebra* van NWO, onder projectnummer 613.007.100.

Mathematics Subject Classification 2000: 12H20, 20F55, 34-04, 34L40, 34M15

ISBN 90-393-2927-3

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Introduction

We begin by discussing two differential equations to illustrate some of the ideas in this dissertation. The first is the second order differential equation

$$z^2y'' + \frac{1}{6}zy' + \frac{1}{6}y = 0, \tag{E1}$$

in which differentiation is with respect to z. Its coefficients are polynomials in z. Two solutions of the equation are $z^{1/2}$ and $z^{1/3}$. Moreover, $B_1 := [z^{1/2}, z^{1/3}]$ is an ordered basis of the solution space of the equation. We can continue $z^{1/2}$ and $z^{1/3}$ analytically in the complex plane $\mathbb{C}$ around 0. If we do this along any positively oriented single closed path, then B_1 becomes $[-z^{1/2}, \mathrm{e}^{2\pi \mathrm{i}/3}z^{1/3}]$. The effects of the path on B_1 can be represented by left multiplication with the 2×2 matrix

$$\gamma := \begin{pmatrix} -1 & 0 \\ 0 & \mathrm{e}^{2\pi \mathrm{i}/3} \end{pmatrix}.$$

Instead of focusing on one specific path, one might also consider all closed paths around 0 in $\mathbb{C}$. Each path gives rise to a matrix representing the change of basis from B_1. Such a matrix equals γ^n, where n is the winding number of the path around 0. All matrices that are obtained in this way form a group, which is generated by γ. This group is known as the *monodromy group* of Equation (E1). The solutions $z^{1/2}$ and $z^{1/3}$ are roots of the polynomials $T^2 - z$ and $T^3 - z$ in T, respectively, and so are typical examples of *algebraic functions*.

The second differential equation we consider is

$$z^2y'' - zy' + y = 0. \tag{E2}$$

It has a similar shape to the previous one. The two independent solutions z and $z\log(z)$ form the basis $B_2 := [z, z\log(z)]$ of the solution space of Equation (E2). The logarithmic solution $z\log(z)$ is not algebraic. Equation (E1) is an example of an algebraic differential equation, but the differential equation (E2) is not.

As before we can consider the analytical continuation of B_2 along any closed path in $\mathbb{C}^*$. Such a path changes $z\log(z)$ into $2\pi \mathrm{i}kz + z\log(z)$ for a certain integer k. It transforms the basis B_2 into $[z, 2\pi \mathrm{i} k z + z\log(z)]$. This transformation can again be viewed as a 2×2 matrix multiplication. The matrix obtained is an element of the group

$$\left\{ \begin{pmatrix} 1 & 2\pi \mathrm{i} k \\ 0 & 1 \end{pmatrix} : k \in \mathbb{Z} \right\},$$

which is the monodromy group of Equation (E2). Notice that the algebraic differential equation (E1) has a finite monodromy group of order 6. The monodromy group of the non-algebraic Equation (E2), however, is infinite. This is not a coincidence, but follows from the general theory on ordinary linear differential equations mentioned below.

In the first few chapters of this thesis we give an introduction to ordinary linear differential equations with coefficients in $\mathbb{C}(z)$. In particular, we study the Fuchsian equations, named after Lazarus Fuchs (1833-1902). A Fuchsian equation is characterised by the fact that each solution is bounded at every complex projective point $\alpha \in \mathbb{P}^1(\mathbb{C})$ by a polynomial in $1/|t|$ for a local parameter t at α. Classically one is interested in finding all Fuchsian equations that have a basis of algebraic solutions. Such Fuchsian equations are called *algebraic*. In the nineteenth century B. Riemann, H.A. Schwarz, L. Fuchs, P. Gordan, F. Klein, C. Jordan and others gave criteria for an n-th order Fuchsian equation to be algebraic, see for instance [Fuc75], [Fuc78], [BD79] and [Gra86]. A crucial idea was to relate a given Fuchsian equation to its monodromy group. In Theorem 2.1.8 we prove that the monodromy group of a Fuchsian equation is finite if and only if all solutions of the equation are algebraic.

In practice the problem of determining all algebraic Fuchsian equations explicitly is too complex. Specific conditions for them to be algebraic are only known for certain Fuchsian equations of low order. The first interesting Fuchsian equations are of order 2, of which (E1) and (E2) are examples. Our two equations belong to the family of *Euler equations* of order 2. An element of this family is of the form

$$z^2 y'' + azy' + by = 0, \tag{E}$$

with $a, b \in \mathbb{C}$. The coefficient of y'' has $z = 0$ as its only root. Together with the point at infinity ∞ of the complex projective plane $\mathbb{P}^1(\mathbb{C})$ they are known as the *singular points* of Equation (E). If the Euler equation (E) is algebraic, then its solution space is generated by solutions of the form z^{ρ_1} and z^{ρ_2}. The exponents ρ_1 and ρ_2 are distinct rational roots of the polynomial $X^2 + (a-1)X + b$. In particular, it follows that a and b are rational numbers in this case.

An important class of order 2 Fuchsian equations consists of the hypergeometric equations. The *hypergeometric equation* is defined as

$$z(z-1)y'' + [(a+b+1)z - c]y' + aby = 0, \tag{H}$$

with $a, b, c \in \mathbb{R}$. The famous hypergeometric function

$$F(a, b, c|z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n,$$

with $(x)_n = x(x+1)\cdots(x+n-1)$ for $n \in \mathbb{Z}_{\geq 0}$, is a solution of this equation when $c \notin \mathbb{Z}_{\leq 0}$. The roots $z = 0$ and $z = 1$ of $z(z-1)$ and the point at infinity are the 3 singular points of Equation (H). In [Sch73] H.A. Schwarz determined explicit criteria for the numbers a, b and c such that the corresponding hypergeometric equation has a basis of algebraic solutions. The resulting list is known as *Schwarz's list*, see Section 1.7. For example, it follows from Schwarz's results that $F(11/60, -1/60, 1/2|z)$ is an algebraic function. Schwarz's work also implies an enumeration of all Fuchsian equations of order 2 having 3 singular points. So it is known when this particular type of Fuchsian equations is algebraic.

For more general equations of order 2 there is a classical theorem of F. Klein. It states that any algebraic Fuchsian equation of order 2 has a solution of the form

$$Q(z)^q F(a, b, c|R(z))$$

for certain rational functions $Q(z), R(z) \in \mathbb{C}(z)$, $q \in \mathbb{Q}$ and rational numbers a, b and c from Schwarz's list of algebraic hypergeometric equations.

A more recent way of deciding whether or not a given second order Fuchsian equation is algebraic comes from J.J. Kovacic. In [Kov86] Kovacic gives an algorithm based on differential Galois theory, that computes a so-called Liouvillian solution of any given linear differential equation of order 2, if it exists. In particular, this algorithm can decide whether or not a given second order Fuchsian equation is algebraic. In the last decades various other contributions on the classification of the algebraic Fuchsian equations were made by the use of differential Galois theory and the invariant theory of finite groups. We refer the reader to the work of M. van der Put, M.F. Singer, F. Ulmer, M. van Hoeij and J.-A. Weil for more on this subject. They apply their general results specifically to equations of order 2 and 3. However, even with these results it is not obvious how to list all second order algebraic Fuchsian equations.

In this dissertation we treat one particular Fuchsian equation of order 2 that was introduced by the French mathematician and engineer Gabriel Lamé (1795-1870). The equation we consider in Chapters 4, 5 and 6 is the *Lamé (differential) equation*

$$p(z)y'' + \frac{1}{2}p'(z)y' - (n(n+1)z + B)y = 0.$$

It contains the square-free polynomial $p(z) = 4z^3 - g_2 z - g_3 \in \mathbb{C}[z]$ and the constants $n \in \mathbb{Q}$ and $B \in \mathbb{C}$. The Lamé equation has 4 singular points. They are the roots of $p(z)$ together with ∞. The general question we try to answer is for which n, g_2, g_3 and B the Lamé equation has an algebraic basis of solutions. In the work of F. Baldassarri ([Bal81] and [Bal87]) and B. Chiarellotto ([Chi95]) a systematic approach for solving this question was described for the first time. Two problems were not treated. The first is to give an explicit description of all finite monodromy groups that might occur. The second is to give a general algorithm that, given $n \in \mathbb{Q}$ and finite monodromy group, decides for which g_2, g_3 and B the Lamé equation is algebraic. These two questions are dealt with in this thesis.

In Chapters 4 and 5 we reprove some of the results of Baldassarri and Chiarellotto in a simplified way. Moreover, we give a complete list of the finite monodromy groups that may occur, together with the necessary values of n, see Table 5.1. An important ingredient we use is the classification of finite reflection groups in $\mathrm{GL}(2, \mathbb{C})$ of Shephard and Todd. In an unpublished manuscript B. Dwork [MR99, Prop.2.8] shows that up to scaling by the maps $z \mapsto \lambda z$, $\lambda \in \mathbb{C}^*$, there are finitely many algebraic Lamé equations with given monodromy group and index $n \notin 1/2 + \mathbb{Z}$. We reprove Dwork's statement in Theorems 5.4.4 and 6.7.9. In Chapter 6 we give an algorithm to carry out the procedure of determining all algebraic Lamé equations with a given monodromy group and index n. The underlying ideas in the algorithm are based on the invariant action of the monodromy group on polynomials in two variables and polynomial solutions of symmetric powers of L_n. This kind of technique has been used on severial previous occasions, including [vHW97], [SU97] and [vHRUW99] . To conclude this thesis, we present the beginning of the enumeration of the algebraic Lamé equations as tables in Appendix A.

Chapter 1

Basics

In this chapter we introduce some basic definitions, facts and preliminaries. Its purpose is to set the language for this dissertation.

1.1 Ordinary linear differential equations

The kind of differential equation that we consider is of the form

$$y^{(n)} + a_1(z)y^{(n-1)} + \cdots + a_{n-1}(z)y^{(1)} + a_n(z)y = 0 \tag{1.1}$$

with $a_i(z)$ rational functions in $\mathbb{C}(z)$ for $i = 1, 2, \ldots, n$ and derivative $y^{(j)} = \frac{d^j}{dz^j}(y)$ for $j = 1, 2, \ldots, n$. Such an equation is known as an *ordinary linear differential equation* of order n. We define the operator L by

$$L(y) := y^{(n)} + a_1(z)y^{(n-1)} + \cdots + a_{n-1}(z)y^{(1)} + a_n(z)y. \tag{1.2}$$

to distinguish the left hand side of (1.1) from the equation itself. The solution space of Equation (1.1) remains the same if we multiply the equation by a non-zero polynomial in $\mathbb{C}[z]$. This polynomial can be chosen in such a way that we obtain a linear differential equation of the form

$$p_0(z)y^{(n)} + p_1(z)y^{(n-1)} + \cdots + p_{n-1}(z)y^{(1)} + p_n(z)y = 0,$$

in which the $p_i(z)$, $i = 0, 1, \ldots, n$, are polynomials in $\mathbb{C}[z]$ and have greatest common divisor equal to 1. Such an equation is unique up to multiplication by an element in $\mathbb{C}^*$. The coefficient $a_i(z)$, $i \in \{1, 2, \ldots, n\}$, is then $p_i(z)/p_0(z)$.

An ordinary linear differential equation can alternatively be put into the form

$$D^n y + b_1(z)D^{n-1}y + \cdots + b_{n-1}(z)Dy + b_n(z)y = 0, \tag{1.3}$$

where D denotes $z\frac{d}{dz}$ and where $b_1(z), b_2(z), \ldots, b_{n-1}(z)$ are in $\mathbb{C}(z)$. It is obtained by multiplication of (1.1) by z^n and using the identity $z^r(d/dz)^r = D(D-1)\cdots(D-r+1)$.

The solution space of a linear differential equation is linear and at most n-dimensional over $\mathbb{C}$. One might wonder what solutions of such equations look like. For instance, solutions could be logarithmic, meromorphic, holomorphic or polynomial. Cauchy proved that solutions are locally holomorphic around any point $\alpha \in \mathbb{C}$ that is not a pole of any of the a_i's. Such points are known as regular points. We shall formulate Cauchy's theorem more precisely later.

Definition 1.1.1 The point $\alpha \in \mathbb{C}$ is called *regular* if $\lim_{z\to\alpha} a_i(z)$ exists and is finite for any $i \in \{1, 2, \ldots, n\}$. Analogously, ∞ is called *regular* if the limits $\lim_{z\to\infty} z^{2i}a_i(z)$ exist and are finite for $i = 1, 2, \ldots, n$.

Definition 1.1.2 A point in $\mathbb{P}^1(\mathbb{C})$ that is not regular is called *singular*.

The definition of a finite singular point implies that a singular point is a pole of at least one of the a_i's. In particular, it must be a zero of $p_0(z)$. We remark the following.

Remark 1.1.3 The set of singular points is finite.

It might seem strange at first to have the term z^{2i} in the definition of a regular point at $z = \infty$. Let us describe the reason for this.

Another definition that is common for the concept of regular and singular points is the following. Let $\alpha \in \mathbb{C}$ and choose a local parameter $t \in \mathbb{C}(z)$ at α (for instance, $t = z - \alpha$). The differential equation can be rewritten with respect to this new variable t. The point α is now regular or singular, exactly when this is the case for the newly obtained equation at $t = 0$. This coincides with our definition of regular and singular points. We mention that it does not matter which local parameter has been taken.
An analogous approach can be taken for $\alpha = \infty$. A local parameter is then $t = 1/z$. Differentiation with respect to z and t are related by $d/dz = -t^2 d/dt$. The substitution of $z = 1/t$ into Equation (1.1) results in a differential equation $L'_\infty(y) = 0$ that has $(-1)^n t^{2n} y^{(n)}$ as its n-th derivative term. The operator $L_\infty = (-1)^n t^{-2n} L'_\infty$ thus is as in (1.2). The coefficients of the derivatives $y^{(n-i)}$, $i > 0$, in L_∞ involve t^{-2i} and higher even order terms in t. The definition of a regular point in $t = 0$ exactly gives the condition for $\alpha = \infty$ to be regular as above. Again the chosen parameter t does not interfere with the definition.

For reasons that will become clear later one distinguishes a special kind of singular point, the regular singular point.

Definition 1.1.4 A complex number $\alpha \in \mathbb{C}$ is *regular singular* if the limits

$$\lim_{z \to \alpha} (z - \alpha)^i a_i(z)$$

exist and are finite for $i = 1, 2, \ldots, n$.
The point ∞ is a *regular singularity* if $\lim_{z \to \infty} z^i a_i(z)$ exists and is finite for all i.

We are now ready to formulate the above mentioned theorem of Cauchy. We refer to [Poo36, page 5] for a proof of this theorem. For convenience we write $\mathbb{P}^1$ instead of $\mathbb{P}^1(\mathbb{C})$ from now on.

Theorem 1.1.5 (Cauchy) *Let $\alpha \in \mathbb{P}^1$ be a regular point of* (1.1)*. Define t to be $1/z$ in the case of $\alpha = \infty$ and $z - \alpha$ otherwise. Then there exist n $\mathbb{C}$-linearly independent Taylor series solutions $f_1, f_2, \ldots, f_n$ of* (1.1) *in t with positive radius of convergence. Moreover, any Taylor series solutions around $z = \alpha$ is a linear combination of $f_1, f_2, \ldots, f_n$.*

The basis of local solutions around a regular point is holomorphic. They can be taken to be of a special form, as Cauchy pointed out.

Theorem 1.1.6 *Let the notation be as in Cauchy's Theorem* 1.1.5*. The Taylor series $f_1, f_2, \ldots, f_n$ can be chosen such that $\lim_{t \to 0} f_i/t^{i-1}$ is finite and non-zero for $i = 1, 2, \ldots, n$.*

Given a regular point α of $L(y) = 0$ there exists a basis of solutions of $L(y) = 0$ that is locally of the form

$$\begin{array}{rcl} f_1(t) & = & t^0 \cdot g_1(t) \\ f_2(t) & = & t^1 \cdot g_2(t) \\ \vdots & & \vdots \\ f_n(t) & = & t^{n-1} \cdot g_n(t) \end{array}$$

for certain power series $g_1(t), g_2(t), \ldots, g_n(t)$ in t with non-zero constant term and positive radius of convergence. In particular, there is a basis of n holomorphic local solutions at α. At a regular singular point there will also be series solutions. These solutions are holomorphic power series in $\mathbb{C}[[t]]$ that are multiplied by a certain (complex) power t^ρ. Their radii of convergence are positive. The occurring powers ρ are exactly known. They are roots of a specific polynomial, the so-called indicial polynomial.

Definition 1.1.7 Let $\alpha \in \mathbb{C}$ be a regular or regular singular point of (1.1). Let α_i be defined as $\lim_{z\to\alpha}(z-\alpha)^i a_i(z)$ for $i = 1, 2, \ldots, n$. Then the *indicial equation* at α is given by

$$\begin{aligned} X(X-1)\cdots(X-n+1) + \alpha_1 X(X-1)\cdots(X-n+2) & \\ + \cdots + \alpha_{n-1}X + \alpha_n &= 0. \end{aligned}$$

If ∞ is regular or regular singular, then the *indicial equation* at ∞ is

$$\begin{aligned} X(X+1)\cdots(X+n-1) - \alpha_1 X(X+1)\cdots(X+n-2) & \\ + \cdots + (-1)^{n-1}\alpha_{n-1}X + (-1)^n\alpha_n &= 0, \end{aligned}$$

where α_i is defined as $\lim_{z\to\infty} z^i a_i(z)$ for $i = 1, 2, \ldots, n$.

Definition 1.1.8 The left hand side of the indicial equation at $\alpha \in \mathbb{P}^1$ is called the *indicial polynomial* at α.

Definition 1.1.9 The roots of the indicial polynomial at $\alpha \in \mathbb{P}^1$ are called the *local exponents* at α.

Example 1.1.10 As a motivation for the indicial equation we consider a linear differential equation $L(y) = 0$ as in (1.1), and suppose that it has a solution $(z-\alpha)^\rho$ with $\rho \in \mathbb{C}$ for a certain regular singular point $\alpha \in \mathbb{C}$. Then $L((z-\alpha)^\rho) \equiv 0$ yields

$$\begin{aligned} (\rho(\rho-1)\cdots(\rho-n+1) + \rho(\rho-1)\cdots(\rho-n+2)(z-\alpha)a_1(z) & \\ + \cdots + \rho(z-\alpha)^{n-1}a_{n-1}(z) + (z-\alpha)^n a_n(z)\big)\,(z-\alpha)^{\rho-n} &\equiv 0. \end{aligned}$$

Hence, the coefficient of $z^{\rho-n}$ must be 0. In particular this is the case after taking the limit $\lim_{z\to\alpha}$. This means that ρ is a zero of the indicial equation at $z = \alpha$. Analogous reasoning works in the case where $(z-\alpha)^\rho g$ is a solution of $L(y) = 0$ for a certain power series g in $z-\alpha$ with $g(0) \neq 0$ and positive radius of convergence.

Remark 1.1.11 Let $\alpha \in \mathbb{P}^1$ be a regular singular point. Let t be $z-\alpha$ in case of $\alpha \in \mathbb{C}$ and $1/z$ otherwise. If $t^\rho g(t)$ is a solution $L(y) = 0$ for a certain power series g in t with $g(0) \neq 0$ and positive radius of convergence, then ρ is a local exponent of L at $z = \alpha$.

The definitions of the indicial equations are all based on an ordinary equation that is written in the form (1.1). There is an alternative definition that relates to differential equations of the kind

$$D^n y + b_1(z)D^{n-1}y + \cdots + b_{n-1}(z)Dy + b_n(z)y = 0$$

as in (1.3). If $z = 0$ is a regular singularity then the rational functions $b_1, b_2, \ldots, b_n$ in z are holomorphic at $z = 0$. The indicial equation at $z = 0$ is then given by

$$X^n + b_1(0)X^{n-1} + \cdots + b_{n-1}(0)X + b_n(0).$$

An advantage of this description can be found at $x = \infty$. In this case the local parameter $z = 1/t$ replaces $z\frac{d}{dz}$ by $-t\frac{d}{dt}$. The indicial polynomial at $z = \infty$ subsequently becomes

$$X^n - b_1(0)X^{n-1} + \cdots + (-1)^{n-1}b_{n-1}(0)X + (-1)^n b_n(0).$$

The differential operator D satisfies the relation $D^i(z^\mu y) = z^\mu(D+\mu)^i(y)$ for any positive integer i and power z^μ of z. The expression

$$D^n z^\mu y + b_1(z)D^{n-1}z^\mu y + \cdots + b_{n-1}(z)Dz^\mu y + b_n(z)z^\mu y$$

thus equals z^μ times

$$(D+\mu)^n y + b_1(z)(D+\mu)^{n-1}y + \cdots + b_{n-1}(z)(D+\mu)y + b_n(z)y.$$

Its indicial equation at $z = 0$ becomes

$$(X+\mu)^n + b_1(0)(X+\mu)^{n-1} + \cdots + b_{n-1}(0)(X+\mu) + b_n(0).$$

The conclusion we draw is that the local exponents at $z = 0$ of the original equation (1.3) have all decreased by μ. All local exponents at ∞ have been increased by μ. The exponents at any finite point other than 0 are left unchanged. The procedure of shifting exponents can be done at any $\alpha \in \mathbb{C}$. In this case the function $(z-\alpha)^\mu y(z)$ can be used. Changing exponents in this way is a basic operation for going from one equation to another.

Proposition 1.1.12 *Let α be in $\mathbb{C}$. Then the local exponents of α all decrease by $\mu \in \mathbb{C}$ if we replace y by $(z-\alpha)^\mu y$ in* (1.1)*. The exponents at ∞ increase by μ. All other exponents are unchanged. In particular, the difference of two exponents at one point remains the same.* □

A regular point has indicial equation $X(X-1)\cdots(X-n+1)$. The converse is not true. Look for instance at the differential equation $y^{(2)}-(1/z)y=0$. Here 0 is regular singular and has $X(X-1)$ as its indicial polynomial.
The indicial equation at a regular point α has $0,1,\ldots,n-1$ as its local exponents. These are exactly the lowest degrees of the holomorphic solutions at α as given in Theorem 1.1.6. More generally, at a regular singular point only some local exponents occur as the order of a local series solution. This is stated in the following theorem of Fuchs ([Poo36, V §16-17]).

Theorem 1.1.13 (Fuchs) *Let $\alpha\in\mathbb{P}^1$ be a regular singularity of* (1.1). *Define t to be $1/z$ in the case of $\alpha=\infty$ and $z-\alpha$ otherwise. Suppose that ρ is a local exponent at α such that none of the numbers $\rho+1,\rho+2,\ldots$ is also a local exponent. Then there exists a holomorphic power series $g(t)$ in t with non-zero constant term such that $t^\rho\cdot g(t)$ is a solution of* (1.1).

A regular singular point α of an ordinary linear differential equation has at least one series solution $t^{\rho_1}\cdot g_1(t)$ as in Fuchs' Theorem 1.1.13. The power series $g_1(t)$ is holomorphic in the neighbourhood of $t=0$. We can put an ordering on the set $\{\rho_1,\rho_2,\ldots,\rho_m\}$ of local exponents that differ by an integer from ρ_1. The one we take is $\rho_1\geq\rho_2\geq\cdots\geq\rho_m$. The differential equation then has m independent solutions

$$\begin{aligned}
f_1(t) &= t^{\rho_1}g_1(t)\\
f_2(t) &= t^{\rho_2}g_2(t)+p_{2,1}(t)f_1(t)\\
f_3(t) &= t^{\rho_3}g_3(t)+p_{3,2}(t)f_2(t)+p_{3,1}(t)f_1(t)\\
&\vdots\\
f_m(t) &= t^{\rho_m}g_m(t)+p_{m,m-1}(t)f_{m-1}(t)\cdots+p_{m,1}(t)f_1(t)
\end{aligned}$$

in which the g_i are holomorphic functions in t at with $g_i(0)\neq 0$ and the $p_{i,j}(t)$ are polynomials of degree at most $i-j$ in $\log(t)$, see [Inc44, §16.3]. For instance, if ρ_1 and ρ_2 are the same, then $p_{2,1}$ can be taken as $\log(z)$. In any case there are n $\mathbb{C}$-linearly independent local solutions around a regular singular point. Moreover, given a solution f of the equation, then f is locally at α of order $|t|^k$ for a certain $k\in\mathbb{Z}$.

Proposition 1.1.14 (Fuchs) *Let α be in $\mathbb{P}^1$. Let t be the local parameter $z-\alpha$ for $\alpha\in\mathbb{C}$ and $1/z$ otherwise. Then α is a regular singular point of the equation* (1.1) *if and only if each solution $f(z)$ of* (1.1) *at $t=0$ is $O(|t|^{k_f})$ for a certain integer k_f.*

Proof. For a proof of this proposition we refer the reader to [Inc44, §15.3] or [Poo36, IV §15]. □

In the case of non-regular singularities there is also a theory of normal forms of solutions. For an introduction on this topic we refer to Chapter XVII of [Inc44]. For more details on the theory of linear differential equations the reader is referred to [Inc44], [Poo36] or [Hil76].

1.2 Fuchsian differential equations

The ordinary linear differential equations we are interested in only have regular or regular singular points.

Definition 1.2.1 An ordinary linear differential equation is called *Fuchsian* if every point on $\mathbb{P}^1$ is regular or regular singular.

For every Fuchsian equation there is a relation between its degree and the sum of all local exponents. It is known as Fuchs' relation.

Theorem 1.2.2 (Fuchs' relation) *Let $\rho_1(\alpha), \rho_2(\alpha), \ldots, \rho_n(\alpha)$ denote the local exponents of a Fuchsian equation of order n at any $\alpha \in \mathbb{P}^1$. Then one has*

$$\sum_{\alpha\in\mathbb{P}^1}\left(\rho_1(\alpha)+\rho_2(\alpha)+\cdots+\rho_n(\alpha)-\binom{n}{2}\right)=-2\binom{n}{2}.$$

Proof. Notice that the sum in the theorem is in fact finite, since the sum of the local exponents at a regular point is $1+2+\cdots+n-1=\binom{n}{2}$. Now, let α be in $\mathbb{P}^1$. The sum $-\left(\rho_1(\alpha)+\rho_2(\alpha)+\cdots+\rho_n(\alpha)\right)$ equals the coefficient of X^{n-1} in the indicial equation. This gives

$$\begin{aligned}
&\rho_1(\alpha)+\rho_2(\alpha)+\cdots+\rho_n(\alpha)\\
&\quad= -1\left(-1-2-\cdots-(n-1)+\lim_{z\to\alpha}(z-\alpha)a_1(z)\right)\\
&\quad= \binom{n}{2}-\mathrm{res}_\alpha(a_1(z)dz)
\end{aligned}$$

for every finite α. Here $\mathrm{res}_\alpha(a_1(z)dz)$ denotes the residue of $a_1(z)$ at α. Analogously, the sum of exponents at ∞ is

$$\rho_1(\infty)+\rho_2(\infty)+\cdots+\rho_n(\infty)=-\binom{n}{2}-\mathrm{res}_\infty(a_1(z)dz).$$

The sum over all residues on $\mathbb{P}^1$ is 0 ([For81, 10.21]). The addition of all sums $\rho_1(\alpha) + \rho_2(\alpha) + \cdots + \rho_n(\alpha) - \binom{n}{2}$ then gives the theorem. □

Two examples of a Fuchsian equation are Euler's homogeneous equation and the hypergeometric differential equation.

1.3 Euler's homogeneous equation

A Fuchsian equation of the form

$$z^n y^{(n)} + c_1 z^{n-1} y^{(n-1)} + \cdots + c_{n-1} z y^{(1)} + c_n y = 0, \tag{1.4}$$

with $c_1, c_2, \ldots, c_n \in \mathbb{C}$, is known as *Euler's homogeneous equation* or the *Euler equation* of order n. The coefficients of the equation are just complex numbers. It has exactly two regular singular points: 0 and ∞. Substitution of $z = 1/t$ into Equation (1.4) yields another Euler equation in the variable t. A basis for the solution space of Euler's homogeneous equation can be given in the following way. If ρ is a local exponent at 0 of multiplicity r, then it gives rise to the r independent solutions $z^\rho, z^\rho \log(z), \ldots, z^\rho \log^{r-1}(z)$. Such solutions exist for every exponent ρ. Together they give an n-dimensional basis of the solution space.

1.4 The hypergeometric equation I

A well-known example of a Fuchsian equation is the hypergeometric (differential) equation (HGE). It is given by

$$z(z-1)F''(z) + [(a+b+1)z - c]F'(z) + abF(z) = 0, \tag{1.5}$$

where a, b and c are real numbers and where $'$ denotes the differentiation $\frac{d}{dz}$. The only non-regular points are 0, 1 and ∞; they are regular singular. Their local exponents are given in Table 1.1.

0	1	∞
0	0	a
$1-c$	$c-a-b$	b

Table 1.1: Local exponents of the hypergeometric equation.

The entries in the first row of Table 1.1 are all regular singular points of the hypergeometric equation (1.5). The column below such a regular singular point consists of its local exponents.

One solution of the hypergeometric equation around $z = 0$ is given by the *Gauss' hypergeometric function*

$$F(a, b, c|z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n n!} z^n, \quad c \notin \mathbb{Z}_{\leq 0}.$$

Here $(x)_n$ is the *Pochhammer symbol* that is defined as $(x)_0 = 1$ and $x(x+1)\cdots(x+n-1)$ for $n > 0$. The condition $c \notin \mathbb{Z}_{\leq 0}$ is necessary, since otherwise $(c)_n$ would be 0 for some positive integer n. The hypergeometric function converges for $|z| < 1$. The second solution of the hypergeometric equation around $z = 0$ is given by

$$z^{1-c} F(a - c + 1, b - c + 1, 2 - c|z), \quad c \notin \mathbb{Z}_{>0}.$$

The solution space of the hypergeometric equation around $z = 1$ is generated by

$$\begin{gathered} F(a, b, a + b - c + 1|1 - z), \\ (1 - z)^{c-a-b} F(c - a, c - b, c - a - b + 1|1 - z) \end{gathered}$$

whenever $c - a - b$ is not an integer. The solution space around $z = \infty$ is spanned by the functions

$$\begin{gathered} (1/z)^a F(a, a - c + 1, a - b + 1|1/z), \\ (1/z)^b F(b, b - c + 1, b - a + 1|1/z) \end{gathered}$$

for $b - a$ not in $\mathbb{Z}$.
In the case that one or more of the numbers $1 - c$, $c - a - b$ and $b - a$ are integers, there generally are some solutions that contain logarithmic terms locally around the corresponding point(s).

1.5 Algebraic solutions

The theorems of Cauchy and Fuchs give a clear picture of when to expect holomorphic or series solutions and what they look like. A much more difficult question is when to expect only algebraic solutions.

Definition 1.5.1 A function is called *algebraic* (over $\mathbb{C}(z)$) if it satisfies an irreducible polynomial equation $T^m + c_1(z)T^{m-1} + \cdots + c_m(z) = 0$ in T with coefficients in $\mathbb{C}(z)$.

Definition 1.5.2 We call an ordinary linear differential equation *algebraic* if its solution space has a basis of algebraic solutions.

In this thesis we are interested in finding equations with only algebraic solutions. The question we mostly concern ourselves with is the following.

Question 1.5.3 *When does an ordinary linear differential equation have n linearly independent algebraic solutions?*

A beautiful conjecture concerning this question is Grothendieck's Conjecture. He considered a linear differential operator

$$G(y) := y^{(n)} + a_1(z)y^{(n-1)} + \cdots + a_{n-1}(z)y^{(1)} + a_n(z)y$$

with all coefficients in $\mathbb{Q}(z)$. The rational functions a_i, $i = 1, 2, \ldots, n$, can be reduced modulo p for all but finitely many primes p. They belong to the field $\mathbb{F}_p(z)$ of the rational functions in z over the finite field of p elements $\mathbb{F}_p$. If we denote the reduction of G modulo p by G_p, then G_p is a linear operator over $\mathbb{F}_p(z)$. Grothendieck conjectured that there should be a connection between the global equation $G(y) = 0$ and the local equation $G_p(y) = 0$ for all but finitely many primes p.

Conjecture 1.5.4 (Grothendieck) *The following statements are equivalent:*

(1) The equation $G(y) = 0$ has n linearly independent solutions (over $(\overline{\mathbb{Q}})$) that are algebraic over $\mathbb{Q}(z)$.

(2) For almost all p the equation $G_p(y) = 0$ has n linearly independent solutions over the field $\mathbb{F}_p(z^p)$ in $\mathbb{F}_p(z)$.

The implication (1) ⇒ (2) in Grothendieck's Conjecture has been proved to be true. The validity of the converse is still an open problem. However, for Picard-Fuchs equations the converse is true, see [Kat72].

The conditions under which Euler's homogeneous equation has a basis of algebraic solutions are not difficult to find. First of all there should be no solutions with log-terms. This means that all n local exponents at 0 have to be distinct. If ρ is one of these exponents, then z^ρ is a solution. It must be a zero of an irreducible equation $T^m + c_1T^{m-1} + \cdots + c_m = 0$ with coefficients in $\mathbb{Q}(z)$. This only happens for $\rho \in \mathbb{Q}$. This is no coincidence. We shall shortly see that all local exponents of a Fuchsian equation with only algebraic solutions must be rational.

An algebraic function $f(z)$ is a root of a polynomial with coefficients in the field $\mathbb{C}\{\{z\}\}$ of meromorphic functions in z. If this polynomial is irreducible in $\mathbb{C}\{\{z\}\}$

and has degree m, then $f(z)$ has a series expansion in the variable $\sqrt[m]{z}$ defined by $(\sqrt[m]{z})^m = z$. More precisely, $f(z)$ can be written as the *Puiseux series*

$$f(z) = \sum_{\nu=k}^{\infty} c_\nu z^{\nu/m}$$

in which k is an integer and the c_ν are complex constants. This is an immediate consequence of the following theorem.

Theorem 1.5.5 (Puiseux) *Let $P(T) = T^m + c_1(z)T^{m-1} + \cdots + c_m(z)$ be an irreducible polynomial of degree m over $\mathbb{C}(z)$. Then $\mathbb{C}\{\{\sqrt[m]{z}\}\}$ is the splitting field of $P(T)$ over $\mathbb{C}\{\{z\}\}$.*

Proof. We refer to Theorem 8.14 of [For81] and the remarks after this theorem for a proof. □

We are going to prove that one can and may restrict to the Fuchsian equations in the search for ordinary linear differential equations that have bases of algebraic solutions.

Proposition 1.5.6 *Suppose that Equation* (1.1) *only has algebraic solutions. Then the equation is Fuchsian and all local exponents are rational.*

Proof. Let $\phi(z)$ be an algebraic solution of the differential equation (1.1). We can consider this function around any $\alpha \in \mathbb{P}^1$ in terms of the local parameter t at α. It follows from Theorem 1.5.5 that there exists an integer k such that ϕ is of order $|t|^k$. According to Proposition 1.1.14, α then is a regular singular point of the equation.
Let ρ be a local exponent of the equation at $z = \alpha$, such that there is no other local exponent λ at α satisfying $\lambda - \rho \in \mathbb{Z}_{>0}$. There is an algebraic solution ϕ that locally at α can be written as $\phi(t) = t^\rho g(t)$ for a certain holomorphic power series $g(t)$ in t, see Theorem 1.1.13. We then have the field extension $\mathbb{C}\{\{t\}\}(\phi(t))$ of $\mathbb{C}\{\{t\}\}$, that is $\mathbb{C}\{\{t\}\}(t^\rho)/\mathbb{C}\{\{t\}\}$. Now, if ρ is irrational, then this field extension would be infinite. This is however in contradiction to the assumption of ϕ being algebraic. We conclude that ρ is rational. Then all other local exponents of α that differ by an integer from ρ are also rational. An arbitrary local exponent at α satisfies either the conditions of ρ or differ by an integer from such an exponent. It follows that all local exponents of the equation at α are rational. □

1.6 The Wronskian determinant

The coefficients $a_1(z), a_2(z), \ldots, a_n(z)$ of Equation (1.1) can be entirely expressed in terms of a full basis of solutions. This description involves the Wronskian determinant.

Definition 1.6.1 Let $f_1, f_2, \ldots, f_n$ be meromorphic functions on some open non-empty subset V of $\mathbb{P}^1$. The *Wronskian (determinant)* of $f_1, f_2, \ldots, f_n$ is defined as

$$W(f_1, f_2, \ldots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

The Wronskian determinant $W(f_1, f_2, \ldots, f_n)$ expresses whether or not there exists a $\mathbb{C}$-linear relation between the functions $f_1, f_2, \ldots, f_n$. More precisely, if V is connected then there exists such a relation if and only if $W(f_1, f_2, \ldots, f_n) = 0$ holds, see for instance [Poo36, §I.4].

Suppose now that $f_1, f_2, \ldots, f_n$ is a basis of solutions of (1.1). The n-th order differential equation $W(y, f_1, f_2, \ldots, f_n) = 0$ has the same set of solutions as the original one. Up to a common non-zero factor they must coincide. Let $\Delta_i(f_1, f_2, \ldots, f_n)$, or Δ_i for short, be the determinant of the $n \times n$-matrix that is obtained by deleting the row of the i-th derivatives in

$$\begin{pmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & & \vdots \\ f_1^{(n)} & f_2^{(n)} & \cdots & f_n^{(n)} \end{pmatrix}.$$

Then $W(y, f_1, f_2, \ldots, f_n)$ equals $\sum_{i=0}^{n} (-1)^i \Delta_i y^{(i)}$. The coefficient of the n-th derivative $y^{(n)}$ is $(-1)^n \Delta_n$. It is non-zero, since Δ_n is exactly the Wronskian determinant $W(f_1, f_2, \ldots, f_n)$. This yields

$$a_{n-i}(z) = (-1)^{n-i} \Delta_i / \Delta_n$$

for $i = 1, 2, \ldots, n$. We obtain the following proposition.

Proposition 1.6.2 *The coefficients of the derivatives in Equation* (1.1) *satisfy* $a_{n-i}(z) = (-1)^{n-i} \Delta_i / \Delta_n$ *for all* i. □

The proposition in particular yields

$$a_1(z) = -\Delta_{n-1}/W(f_1, f_2, \ldots, f_n).$$

By some elementary calculations it can be shown that Δ_{n-1} is equal to the derivative $d/dz(W(f_1, f_2, \ldots, f_n))$ of the Wronskian determinant. Therefore one has

$$a_1(z) = -W'(f_1, f_2, \ldots, f_n)/W(f_1, f_2, \ldots, f_n).$$

Separating and integrating the variables then leads to the equation

$$W(f_1, f_2, \ldots, f_n)(z) = W(f_1, f_2, \ldots, f_n)(z_0) \cdot \mathrm{e}^{-\int_{z_0}^{z} a_1(t)dt}. \tag{1.6}$$

for a certain $z_0 \in \mathbb{C}$. This identity is known as the *Abel-Liouville formula.*

Theorem 1.6.3 (Heine) *Consider the ordinary linear differential equation*

$$y''(z) + p(z)y'(z) + q(z)y(z) = 0$$

with linearly independent solutions f_1 and f_2. Then f_1 and f_2 are algebraic if and only if $W(f_1, f_2)(z)$ and $(f_1/f_2)(z)$ are.

Proof. First of all, if $f_1(z)$ and $f_2(z)$ are algebraic, then so are f_1/f_2, f_1' and f_2'. From

$$W(f_1, f_2) = f_1 f_2' - f_2 f_1'$$

it follows that the Wronskian determinant is algebraic as well.
Conversely, suppose that $W(f_1, f_2)$ and f_1/f_2 are algebraic. Then the derivative $(f_1/f_2)'(z)$ is algebraic. From

$$(f_1/f_2)' = W(f_1, f_2)/f_2^2.$$

we derive that f_2^2 and thus also f_2 are algebraic. We conclude from $f_1 = f_2(f_1/f_2)$ that f_1 is an algebraic function as well. □

The algebraic ordinary linear equations of order 2 are treated in more detail in Chapter 3. We return to the hypergeometric equation in order to give an example.

1.7 The hypergeometric equation II

In his article [Sch73] H.A. Schwarz derived exact conditions for which the hypergeometric equation

$$z(z-1)F''(z) + [(a+b+1)z - c]F'(z) + abF(z) = 0,$$

$a, b, c \in \mathbb{R}$ is algebraic. We are going to give a very short outline of this work. It is not our intention to prove or state specific details.

The results in [Sch73] are obtained by considering the circumstances in which the quotient $(y_1/y_2)(z)$ of two independent solutions y_1 and y_2 of the HGE is algebraic. As Schwarz points out the statement that y_1/y_2 is algebraic is closely related to y_1 and y_2 being algebraic functions.

Theorem 1.7.1 *The hypergeometric equation has a basis (y_1, y_2) of algebraic solutions if and only if y_1/y_2 is algebraic and a, b and c are rational.*

Proof. The local exponents of the hypergeometric equation at $z = 0$ are 0 and $1-c$. At $z = \infty$ they are a and b. Due to Proposition 1.5.6 the numbers a b and c must be rational if y_1 and y_2 are algebraic. In that case y_1/y_2 is obviously also an algebraic function.
Conversely, suppose that a, b and c are rational numbers and that in addition y_1/y_2 is algebraic. The Wronskian determinant of y_1 and y_2 is

$$\begin{aligned} W(y_1, y_2) &= C \cdot \exp\left(\int_{z_0}^{z} \frac{(a+b+1)t - c}{t(t-1)} dt\right) \\ &= C \cdot \exp\left(\int_{z_0}^{z} \frac{-c}{t} + \frac{c-a-b-1}{t-1} dt\right) \\ &= D z^{-c} (z-1)^{c-a-b-1} \end{aligned}$$

for a certain $C, D \in \mathbb{C}$ and $z_0 \in \mathbb{C}$. It is an algebraic function over $\mathbb{C}(z)$, since $-c$ and $c - a - b - 1$ are rational. Finally, Theorem 1.6.3 completes the proof. □

The most general case occurs when the exponent differences

$$\begin{aligned} \lambda &:= |1-c| \\ \mu &:= |c-a-b| \\ \nu &:= |a-b| \end{aligned}$$

at $z = 0$, 1 and ∞, respectively, are not integers. Then there are no solutions that locally contain a log-term, as mentioned in Section 1.4. H.A. Schwarz showed that only the exponent differences are of importance for a hypergeometric equation to have an algebraic ratio of solutions $(y_1/y_2)(z)$. Moreover, he gave all triples (λ, μ, ν) belonging to such equations. These differences λ, μ, and ν correspond to a triple $(\lambda'', \mu'', \nu'')$ in the so-called *Schwarz's list* Table 1.2 as is outlined below.

Schwarz's list should be read as follows.

- In the first column the *Schwarz number* is given. It denotes the case of λ'', μ'' and ν'' we are in.

Sch. no.	λ'', μ'', ν''	polyhedron
I.	$\frac{1}{2}, \frac{1}{2}, \nu$	dihedron (regular double-pyramid)
II.	$\frac{1}{2}, \frac{1}{3}, \frac{1}{3}$	tetrahedron
III.	$\frac{2}{3}, \frac{1}{3}, \frac{1}{3}$	
IV.	$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$	cube and octahedron
V.	$\frac{2}{3}, \frac{1}{4}, \frac{1}{4}$	
VI.	$\frac{1}{2}, \frac{1}{3}, \frac{1}{5}$	dodecahedron and icosahedron
VII.	$\frac{2}{5}, \frac{1}{3}, \frac{1}{3}$	
VIII.	$\frac{2}{3}, \frac{1}{5}, \frac{1}{5}$	
IX.	$\frac{1}{2}, \frac{2}{5}, \frac{1}{5}$	
X.	$\frac{3}{5}, \frac{1}{3}, \frac{1}{5}$	
XI.	$\frac{2}{5}, \frac{2}{5}, \frac{2}{5}$	
XII.	$\frac{2}{3}, \frac{1}{3}, \frac{1}{5}$	
XIII.	$\frac{4}{5}, \frac{1}{5}, \frac{1}{5}$	
XIV.	$\frac{1}{2}, \frac{2}{5}, \frac{1}{3}$	
XV.	$\frac{3}{5}, \frac{2}{5}, \frac{1}{3}$	

Table 1.2: Schwarz's list.

- The second column concerns the rational numbers λ'', μ'' and ν''. They are associated with λ, μ and ν as follows. Define λ' as the minimum of $\lambda \bmod 2$ and $2 - \lambda \bmod 2$ in the closed interval $[0, 1]$. The numbers μ' and ν' are defined analogously. Then choose from the four rows

$$\begin{array}{rcc} \lambda' & \mu' & \nu' \\ \lambda' & 1-\mu' & 1-\nu' \\ 1-\lambda' & \mu' & 1-\nu' \\ 1-\lambda' & 1-\mu' & \nu' \end{array}$$

the one in which the sum is smallest. The entries of this row now denote λ'', μ'' and ν''. We remark that the numbers in a row of the second column of Schwarz's list can be given in any order.

- The map y_1/y_2 maps $\mathbb{P}^1$ to $\mathbb{P}^1$. If it is algebraic then, y_1/y_2 induces a finite curvilinear triangulation on the sphere. By definition such triangles have open segments of circles or lines as edges. One of the triangles has $\lambda''\pi$,

$\mu''\pi$ and $\nu''\pi$ as its angles. The angles are determined by some planes of symmetry of a regular concentric polyhedron. The third column describes these polyhedrons. The reader is referred to [Sch73] for more details.

Theorem 1.7.2 *The hypergeometric equation is algebraic if and only if λ, μ and ν correspond to a triple $(\lambda'', \mu'', \nu'')$ in Schwarz's list.*

Proof Let y_1 and y_2 be two independent solutions of the hypergeometric equation. We have seen that y_1/y_2 is algebraic if and only if $(\lambda'', \mu'', \nu'')$ appears in the Schwarz list. Any such triple has rational entries. Then the numbers λ, μ and ν and thus a, b and c are rational as well. Finally, Theorem 1.7.1 proves the theorem. $\square$

Example 1.7.3 Consider the hypergeometric equation with parameters

$$\begin{aligned} a &= 1/3 \\ b &= -1/6 \\ c &= 1/2. \end{aligned}$$

One then has $\lambda = 1/2$, $\nu = 1/2$ and $\nu = 1/3$. We are in case I. of Schwarz's list. The equation thus is algebraic. Its solution space is generated by

$$F(1/3, -1/6, 1/2|z) = \frac{1}{2}\left((1+\sqrt{z})^{1/3} + (1-\sqrt{z})^{1/3}\right)$$

and

$$z^{1/2}F(1/3, 5/6, 3/2|z) = \frac{3}{2}\left((1+\sqrt{z})^{1/3} - (1-\sqrt{z})^{1/3}\right).$$

In Chapter 3 we shall see that the platonic solids of $\mathbb{P}^1$ and the hypergeometric equation are important for general algebraic Fuchsian equations of order 2.

Chapter 2

Monodromy groups and equivalence

2.1 Monodromy groups

In this section we associate a group to a linear differential equation $L(y) = 0$ or order n as in Equation (1.1). This group will be called the monodromy group of the equation.

Let $S \subset \mathbb{P}^1$ be the set of singularities of Equation (1.1). We have seen that S is finite and only contains all roots of $p_0(z)$ and possibly ∞. There exists a basis $f_1, f_2, \ldots, f_n$ of holomorphic solutions at a fixed regular point z_0. These functions can be continued analytically along any closed path u in $\mathbb{P}^1 \setminus S$ that begins and ends in z_0. The functions f_i, $i = 1, 2, \ldots, n$ have then changed into new functions $\tilde{f}_i$, $i = 1, 2, \ldots, n$. The newly obtained functions are still independent solutions of the differential equation. Therefore, they must be linear combinations of the original basis $\mathbf{f} := (f_1, f_2, \ldots, f_n)$. This implies that there exists an invertible matrix $M_{\mathbf{f}}(u) \in \mathrm{GL}(n, \mathbb{C})$ with the property

$$\begin{pmatrix} \tilde{f}_1 \\ \tilde{f}_2 \\ \vdots \\ \tilde{f}_n \end{pmatrix} = M_{\mathbf{f}}(u) \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix}.$$

Notice that $M_{\mathbf{f}}(u)$ depends on the given differential equation (1.1), since $\mathbf{f}$ does. For convenience we write $M(u)$ instead of $M_{\mathbf{f}}(u)$ if it is clear that the involving matrix is taken with respect to the ordered basis $\mathbf{f}$.

If u is a closed path with none of the singular points enclosed, then $M(u)$ is the identity-matrix. The matrix $M(u)$ is in fact determined by the class $[u]$ of u in the fundamental group $\pi_1(\mathbb{P}^1 \setminus S, z_0)$ of $\mathbb{P}^1 \setminus S$ with base point z_0. In this way one obtains the so-called *monodromy representation*

$$\begin{array}{rcl} M_{\mathbf{f}} : \pi_1(\mathbb{P}^1 \setminus S, z_0) & \rightarrow & \mathrm{GL}(n, \mathbb{C}) \\ {[u]} & \mapsto & M_{\mathbf{f}}(u). \end{array}$$

Any solution f of $L(y) = 0$ is a $\mathbb{C}$-linear combination $\sum_{i=1}^n a_i f_i$ of the basis vectors of $\mathbf{f}$. The analytic continuation of f after completing a closed path u as before becomes $\sum_{i=1}^n a_i \tilde{f}_i$. This is exactly $\sum_{i=1}^n a_i (M_{\mathbf{f}}(u)\mathbf{f}^T)_i$, in which the summation is taken over the n entries of $M_{\mathbf{f}}(u)\mathbf{f}^T$. This description of the analytic continuation of f gives rise to the natural action of $M_{\mathbf{f}}$ on the solution space of $L(y) = 0$.

Notation 2.1.1 Let f be in the solution space of $L(y) = 0$. Then the natural action of a monodromy matrix $\gamma \in M_{\mathbf{f}}$ on f is denoted by ${}^{\gamma}f$.

By construction the function ${}^{\gamma}f$ is the analytic continuation of $f = \sum_{i=1}^n a_i f_i$ along the path corresponding to γ. It satisfies

$${}^{\gamma}f = \sum_{i=1}^{n} a_i (\gamma \mathbf{f}^T)_i.$$

The monodromy representation depends on the chosen basis $\mathbf{f}$ and on the base point z_0. Let us first describe what happens if we take another ordered basis of solutions at z_0. Suppose that $\mathbf{g}$ is such a basis. Then there exists a matrix $C \in \mathrm{GL}(n, \mathbb{C})$ such that $M_{\mathbf{f}}(u) = C^{-1} M_{\mathbf{g}}(u) C$ holds for all paths u as before. This matrix C represents the change of basis of $\mathbf{f}$ into $\mathbf{g}$. Conversely, any invertible matrix $C \in \mathrm{GL}(n, \mathbb{C})$ defines a new ordered basis $\mathbf{g}$ at z_0. The monodromy matrices $M_{\mathbf{f}}(u)$ and $M_{\mathbf{g}}(u)$ are then related by conjugation as above.

Lemma 2.1.2 *Let $c\mathbf{f}$ be the ordered basis $(cf_1, cf_2, \ldots, cf_n)$ for a constant $c \in \mathbb{C}^*$. Then one has $M_{c\mathbf{f}}(\pi_1(\mathbb{P}^1 \setminus S, z_0)) = M_{\mathbf{f}}(\pi_1(\mathbb{P}^1 \setminus S, z_0))$.*

Proof. Let I_n be the $n \times n$ identity matrix. We then have

$$\begin{array}{rcl} M_{c\mathbf{f}}(\pi_1(\mathbb{P}^1 \setminus S, z_0)) & = & c^{-1} I_n \cdot M_{\mathbf{f}}(\pi_1(\mathbb{P}^1 \setminus S, z_0)) \cdot c I_n \\ & = & M_{\mathbf{f}}(\pi_1(\mathbb{P}^1 \setminus S, z_0)). \end{array}$$

□

For another base point $z_1 \in \mathbb{P}^1 \setminus S$, let $\mathbf{g}$ be a basis of solutions around z_1. We then have the monodromy map

$$\begin{array}{rcl} M_{\mathbf{g}} : \pi_1(\mathbb{P}^1 \setminus S, z_1) & \to & \mathrm{GL}(n, \mathbb{C}) \\ [v] & \mapsto & M_{\mathbf{g}}(v). \end{array}$$

Next we take a path w from z_0 to z_1. This path gives rise to the isomorphism

$$\begin{array}{rcl} \phi : \pi_1(\mathbb{P}^1 \setminus S, z_0) & \to & \pi_1(\mathbb{P}^1 \setminus S, z_1) \\ [u] & \mapsto & [w^{-1}uw]. \end{array}$$

A basis element of $\mathbf{f}$ is a linear combination of the elements of $\mathbf{g}$ after continuation along w. This defines the transition matrix $D \in \mathrm{GL}(n, \mathbb{C})$ of $\mathbf{f}$ into $\mathbf{g}$. We obtain the commutative diagram

$$\begin{array}{ccc} \pi_1(\mathbb{P}^1 \setminus S, z_0) & \xrightarrow{M_{\mathbf{f}}} & \mathrm{GL}(n, \mathbb{C}) \\ \downarrow \phi & & \downarrow c_D \\ \pi_1(\mathbb{P}^1 \setminus S, z_1) & \xrightarrow{M_{\mathbf{g}}} & \mathrm{GL}(n, \mathbb{C}) \end{array}$$

in which the conjugation map c_D is defined by $c_D(A) := DAD^{-1}$. Notice that D depends on the chosen path w. Another path $\tilde{w}$ has $DM_{\mathbf{f}}(w\tilde{w}^{-1})$ as its conjugation matrix.

Definition 2.1.3 The conjugacy class of $M_{\mathbf{f}}(\pi_1(\mathbb{P}^1 \setminus S, z_0))$ in $\mathrm{GL}(n, \mathbb{C})$ is called the *monodromy group* $M_{\mathbf{f}}$ of Equation (1.1).

Remark 2.1.4 We abuse the terminology and refer to any representative of the conjugacy class of $M_{\mathbf{f}}$ as the monodromy group as well.

Every conjugate group of a given monodromy group of an equation can also be seen as the monodromy group of that equation, just by making a simple change of basis of solutions. That is why we shall choose a convenient basis whenever appropriate. The remark that the monodromy group is given up to conjugacy will sometimes be left out.

Proposition 2.1.5 *The monodromy group of a linear differential equation is generated by* $|S|$ *matrices* $\gamma_1, \gamma_2, \ldots, \gamma_{|S|}$ *that satisfy* $\gamma_1\gamma_2 \cdots \gamma_{|S|} = I_n$.

Proof. The fundamental group $\pi_1(\mathbb{P}^1 \setminus S, z_0)$ is generated by positively oriented single loops that have only one singular point inside. Moreover, let $[1] \in \pi_1(\mathbb{P}^1 \setminus S, z_0)$ be the identity element. The loops can be chosen as $u_1, u_2, \ldots, u_{|S|}$ having the property $[u_1u_2\cdots u_{|S|}]=[1]$. The monodromy group of a linear differential equation is then generated by the matrices $M_{\mathbf{f}}(u_1), M_{\mathbf{f}}(u_2), \ldots, M_{\mathbf{f}}(u_{|S|})$ for any basis $\mathbf{f}$ of the equation. These matrices satisfy $M_{\mathbf{f}}(u_1)M_{\mathbf{f}}(u_2)\cdots M_{\mathbf{f}}(u_{|S|})=I_n$ by construction. □

Let $\alpha\in\mathbb{P}^1$ be a regular or a regular singular point of Equation (1.1). There are n local exponents at α, say $\rho_1,\rho_2,\ldots,\rho_n$. If they do not differ by an integer in the case of a singular point, then there are n independent local solutions given by

$$\begin{array}{rcl} f_1(t) & = & t^{\rho_1}\cdot g_1(t) \\ f_2(t) & = & t^{\rho_2}\cdot g_2(t) \\ \vdots & & \vdots \\ f_n(t) & = & t^{\rho_n}\cdot g_n(t) \end{array}$$

for certain power series $g_1(t), g_2(t), \ldots, g_n(t)$ in t by Fuchs' Theorem 1.1.13. A simple positive circuit u on $\mathbb{P}^1\setminus S$ that contains no irregularities except possibly α changes $f_1, f_2, \ldots, f_n$ into $\mathrm{e}^{2\pi\mathrm{i}\rho_1}f_1, \mathrm{e}^{2\pi\mathrm{i}\rho_2}f_2, \ldots, \mathrm{e}^{2\pi\mathrm{i}\rho_n}f_n$, respectively. Then $M(u)$ with respect to $f_1, f_2, \ldots, f_n$ is the diagonal matrix

$$\begin{pmatrix} \mathrm{e}^{2\pi\mathrm{i}\rho_1} & 0 & \cdots & 0 \\ 0 & \mathrm{e}^{2\pi\mathrm{i}\rho_2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \mathrm{e}^{2\pi\mathrm{i}\rho_n} \end{pmatrix}.$$

If $M(u)$ is the monodromy matrix of a general basis at α, then it is diagonalisable and has $\mathrm{e}^{2\pi\mathrm{i}\rho_1}, \mathrm{e}^{2\pi\mathrm{i}\rho_2}, \ldots, \mathrm{e}^{2\pi\mathrm{i}\rho_n}$ as its n eigenvalues. In particular we see that the monodromy at a regular point is the identity matrix. In general there is no diagonalisable monodromy matrix $M(u)$ at a regular (singular) point $\alpha\in\mathbb{P}^1$ when two or more of its local exponents differ by an integer.

Our object in the remainder of this section is to show that the finiteness of a monodromy group of a Fuchsian equation is equivalent to the existence of a basis of algebraic solutions.

Lemma 2.1.6 *Let $g(z)$ be an algebraic function on $\mathbb{P}^1$. Let S be a finite set containing the branch points and the poles of $g(z)$. Choose a branch of $g(z)$ on an open set $U\subset\mathbb{P}^1\setminus S$. Take $z_0\in U$ and suppose that the analytic continuation of $g(z)$ along any closed path in $\pi_1(\mathbb{P}^1\setminus S, z_0)$ yields the same $g(z)$. Then one has $g(z)\in\mathbb{C}(z)$.*

Proof. The function $g(z) \in U$ can be continued analytically to a one-valued holomorphic function on $\mathbb{P}^1 \setminus S$, since its monodromy is trivial. So the points of S are isolated singularities of the univalent function $g(z)$. Moreover, $g(z)$ being algebraic, the growth order of $g(z)$ around $\alpha \in S$ is bounded by some power of $|t|^{-1}$, where t is a local parameter at α. It follows that the points of S are regular points or poles of $g(z)$. We conclude that $g(z)$ is a meromorphic funtion on $\mathbb{P}^1$. Therefore, one has $g(z) \in \mathbb{C}(z)$. □

Lemma 2.1.7 *Let f be an algebraic solution of $L(y) = 0$. Then every algebraic conjugate of f over $\mathbb{C}(z)$ also is a solution. In particular, ${}^\gamma f$ is an algebraic solution of $L(y) = 0$ for every monodromy matrix γ.*

Proof. Let f be an algebraic solution of $L(y) = 0$. By definition there exists a polynomial

$$P(T) := T^m + c_1(z)T^{m-1} + \cdots + c_m(z)$$

in $\mathbb{C}(z)[T]$ such that $P(f) \equiv 0$ holds. Let σ be a Galois element of $\overline{\mathbb{C}(z)}/\mathbb{C}(z)$. The conjugate of ${}^\sigma f$ of f induced by σ is also a root of $P(T)$. Hence, it is algebraic. Moreover, one has ${}^\sigma(L(f)) = L({}^\sigma f)$. This gives $L({}^\sigma f) = 0$.
Now let γ be a monodromy matrix of the equation. The analytic continuation of $P(f)$ along γ on one hand is $P({}^\gamma f)$ and on the other 0. This yields $P({}^\gamma f) = 0$. The function ${}^\gamma f$ thus is an algebraic conjugate of f. □

Theorem 2.1.8 *A Fuchsian equation is algebraic if and only if its monodromy group is finite.*

Proof. Let M be the monodromy group of the Fuchsian equation $L(y) = 0$. Suppose that $\{f_1, f_2, \ldots, f_n\}$ is a basis of algebraic solutions of the equation. Let S be the finite set of singular points of L. Under analytic continuation along any closed loop on $\mathbb{P}^1 \setminus S$ any algebraic solution f is changed into one of its conjugates. There is a finite number of conjugates. Hence, the number of images of the n-tuple $\{f_1, f_2, \ldots, f_n\}$ under monodromy is finite. It follows that the monodromy group of L is finite.
Conversely, suppose that M is a finite. Let f be a solution of the Fuchsian equation. We can construct the polynomial $P(T) := \prod_{\gamma \in M}(T - {}^\gamma f)$ in T, since M is finite. Due to Lemma 2.1.7 it only has algebraic functions as roots. Every coefficient of $P(T)$ is a symmetric polynomial in the roots of $P(T)$. It thus is invariant under the action of $M \subset \pi_1(\mathbb{P}^1 \setminus S, z_0)$. A point $z \in \mathbb{C}$ having the property ${}^\gamma f(z) = \infty$ for a $\gamma \in M$, is contained in S. Lemma 2.1.6 now implies that each coefficient is a rational function. One zero of $P(T) = 0$ is f itself. This means that f is algebraic by definition. □

Theorem 2.1.9 *The monodromy group of the hypergeometric equation is finite if and only if λ, μ and ν correspond to a triple $(\lambda'', \mu'', \nu'')$ in the Schwarz list (Table* 1.2*) in the way that is outlined in Section* 1.7*.*

Proof This is an immediate consequence of Theorems 1.7.2 and 2.1.8. □

Theorem 2.1.8 is an important tool for the research on Fuchsian equations that have a basis of algebraic solutions.

2.2 Projective monodromy groups

Not only the matrices in $\mathrm{GL}(n, \mathbb{C})$ play an import role in the theory of differential equations. Their projections in $\mathrm{PGL}(n, \mathbb{C})$ are also of importance. The group $\mathrm{PGL}(n, \mathbb{C})$ is the quotient group of $\mathrm{GL}(n, \mathbb{C})$ by the group of scalar elements $\Lambda I := \{\lambda I_n : \lambda \in \mathbb{C}^*\}$. The natural projective map is given by

$$\begin{array}{rcl} P : \mathrm{GL}(n, \mathbb{C}) & \rightarrow & \mathrm{PGL}(n, \mathbb{C}) \\ \gamma & \mapsto & [\gamma] := \gamma \cdot \Lambda I. \end{array}$$

The homomorphism P maps a group $G \subset \mathrm{GL}(n, \mathbb{C})$ to $PG := G \cdot \Lambda I / \Lambda I$ The projective group PG is group isomorphic to $G/G \cap \Lambda I$. Usually we consider PG to be of the latter form.

Definition 2.2.1 Let n be an element of $\mathbb{Z}_{>0}$. The *kernel of projection* Z_G of a group $G \subset \mathrm{GL}(n, \mathbb{C})$ is the subgroup $G \cap \Lambda I$ of scalar elements of G.

In the search for Fuchsian equations having a basis of algebraic solutions we may restrict ourselves to those with finite monodromy groups. Finite subgroups of $\mathrm{GL}(n, \mathbb{C})$ have cyclic kernels of projection.

Lemma 2.2.2 *Suppose that G is a finite subgroup of* $\mathrm{GL}(n, \mathbb{C})$*. Then there exists a $|Z_G|$-th primitive root of unity $\zeta \in \mathbb{C}$ such that Z_G is generated by ζI_n.*

Proof. Suppose that G is a finite subgroup of $\mathrm{GL}(n, \mathbb{C})$. Its projective group PG is one of the finite subgroups of $\mathrm{PGL}(n, \mathbb{C})$. The kernel Z_G consists of scalar matrices of the form λI_n, with $\lambda \in \mathbb{C}$. Such a matrix satisfies $\lambda^{|Z_G|} = 1$. Hence λ is a $|Z_G|$-th root of unity. The kernel Z_G is thus a cyclic group generated by ζI_n for a certain primitive $|Z_G|$-th root of unity $\zeta \in \mathbb{C}$.

Definition 2.2.3 Let M be the monodromy group of a Fuchsian equation of order n. Then its natural image $PM \in \mathrm{PGL}(n, \mathbb{C})$ is called the *projective monodromy group* of the equation.

The projective monodromy group is actually determined up to conjugacy in $\mathrm{PGL}(n, \mathbb{C})$. This is due to the fact that the monodromy group itself is defined up to conjugation by any invertible matrix. Notice that in general any two conjugate groups in $\mathrm{GL}(n, \mathbb{C})$ have the same kernels of projection. Their projective images are necessarily conjugate in $\mathrm{PGL}(n, \mathbb{C})$.

Definition 2.2.4 We call two subgroups of $\mathrm{GL}(n, \mathbb{C})$ *projectively conjugate* if their projective groups are conjugate by an element of $\mathrm{PGL}(n, \mathbb{C})$.

2.3 Equivalent equations

An equivalence relation can be put on the set of linear differential equations of a specific order n.

Definition 2.3.1 A function $\theta(z)$ on $\mathbb{P}^1$ is called a *radical function* if one has $\theta(z) = \prod_{i=1}^{r}(z - \alpha_i)^{\lambda_i}$ for certain $\lambda_i, \alpha_i \in \mathbb{C}$ and $r \in \mathbb{Z}_{\geq 0}$.

Definition 2.3.2 Let L_1 and L_2 be two ordinary linear differential operators, both with differentiation with respect to z. Then L_1 and L_2 are called *(projectively) equivalent* by the radical function $\theta(z)$ on $\mathbb{P}^1$ if the solution space of $L_2(Y) = 0$ is equal to $\{\theta y : L_1(y) = 0\}$.

Notice that the definition of projective equivalence is indeed an equivalence relation on ordinary linear differential operators. The following proposition shows that the notion of projective equivalence is also an equivalence relation on Fuchsian equations.

Proposition 2.3.3 *Suppose that the two ordinary linear differential equations $L_1(y(z)) = 0$ and $L_2(y(z)) = 0$ are projectively equivalent. If L_1 is Fuchsian, then so is L_2.*

Proof. Let $\theta(z)$ be the function such that the solution space of L_2 is $\{\theta y : L_1(y) = 0\}$. So we have $\theta(z) = \prod_{i=1}^{r}(z-\alpha_i)^{\lambda_i}$ with $\lambda_i, \alpha_i \in \mathbb{C}$ for all i. Let α be a point in $\mathbb{P}^1$. A solution f of L_2 is θy_f for a certain solution y_f of L_1. If t denotes the local parameter at $z = \alpha$ as usual, then by Proposition 1.1.14 the function y_f is of order $|t|^k$ at α for a certain integer k. There also exits an integer m such that θ is of order $|t|^m$. It follows that $f = \theta y_f$ is of order $O(|t|^{k+m})$. The function f is an arbitrary solution of L_2. We deduce from Proposition 1.1.14 that α is a regular singular point of L_2. $\square$

There is a straightforward relation between two equivalent operators. It is described by the following proposition.

Proposition 2.3.4 *Let L_1 and L_2 be equivalent operators by the radical function θ. Then the following holds.*

(i) One has $L_1 = \theta^{-1} L_2 \theta$.

(ii) The Wronskian determinant of L_2 with respect to $\theta f_1, \theta f_2, \ldots, \theta f_n$ satisfies

$$W(\theta f_1, \theta f_2, \ldots, \theta f_n) = \theta^n W(f_1, f_2, \ldots, f_n).$$

Proof. Let L_1 and L_2 be equivalent operators as defined above. If y is a solution of the equation $L_1(Y) = 0$ then so is θy for L_2. This means that L_1 is equal to $L_2\theta$ up to multiplication by a factor. The coefficient in front of the n-th derivative of the variable Y of $L_1(Y) = 0$ is 1. The one that belongs to the n-th derivative in $L_2\theta$ is θ itself. This implies $L_1 = \theta^{-1} L_2 \theta$, as had to be proved. Item *(ii)* of the proposition is basic linear algebra. □

The local exponents of two equivalent operators L_1 and L_2 can easily be expressed in terms of one another.

Lemma 2.3.5 *Let L_1 and L_2 be two equivalent operators by $\theta(z) = \prod_{i=1}^{r} (z - \alpha_i)^{\lambda_i}$, with distinct $\alpha_1, \alpha_2, \ldots, \alpha_r$. Let $\rho_1, \rho_2, \ldots, \rho_n$ denote the local exponents of L_1 at $\alpha_i \in \mathbb{P}^1$ for a given $i \in \{1, 2, \ldots, r\}$. Then:*

- *$\rho_1 + \lambda_i, \rho_2 + \lambda_i, \ldots, \rho_n + \lambda_i$ are the exponents of L_2 at α_i.*
- *The local exponents of L_1 and L_2 at a finite point other than the points $\alpha_1, \alpha_2, \ldots, \alpha_r$ coincide.*
- *If $\nu_1, \nu_2, \ldots, \nu_n$ are the local exponents of L_1 at ∞, then the set $\{\nu_i - \lambda_1 - \lambda_2 - \cdots - \lambda_r : i = 1, 2, \ldots, n\}$ consists of the local exponents of L_2 at ∞.*

Proof. Let L_1 and L_2 be projectively equivalent operators by $\theta(z)$ as stated in the lemma. Repeated use of Proposition 1.1.12 shows that any exponent τ at α_i of $L_2(Y) = 0$ changes into the exponent $\tau - \lambda_i$ at α_i of the equation $\theta^{-1} L_2 \theta(Y) = 0$. The latter equation is just $L_1(Y) = 0$. Therefore, the numbers $\rho_1 + \lambda_i, \rho_2 + \lambda_i, \ldots, \rho_n + \lambda_i$ are the local exponents of L_2 at α_i, if $\rho_1, \rho_2, \ldots, \rho_n$ are the local exponents of L_1 at α_i. Analogously, all local exponents of $L_1(Y) = 0$ at ∞ decrease by $\lambda_1 + \lambda_2 + \cdots + \lambda_r$. The same reasoning as for the α_i's shows that the local exponents at all other points remain the same. □

Corollary 2.3.6 *Suppose that L_1 and L_2 are projectively equivalent operators of order at least 2. Let $\rho_1, \rho_2, \cdots, \rho_n$ denote the local exponents of L_1 at $\alpha \in \mathbb{P}^1$. Then the set $\{\rho_2 - \rho_1, \rho_3 - \rho_1, \ldots, \rho_n - \rho_1\}$ is the set of local exponent differences of L_1 as well as of L_2 at $\alpha \in \mathbb{P}^1$.* □

We have seen that there is a straightforward relation between θ and the local exponents of the involved equivalent equations L_1 and L_2. On one hand a given θ completely describes all local exponents. On the other hand, the function θ is completely known if all local exponents of L_1 and L_2 are given. The following proposition describes the special case in which the local exponents of a given $\alpha \in \mathbb{P}^1$ of L_1 and L_2 are the same.

Proposition 2.3.7 *Suppose that the local exponents of two equivalent equations L_1 and L_2 coincide at every point in $\mathbb{P}^1$. Then one has $L_1 = L_2$.*

Proof. Let $L_1(Y) = 0$ and $L_2(Y) = 0$ be two equivalent equations. By definition there exists a function $\theta(z) = \prod_{i=1}^{r}(z-\alpha_i)^{\lambda_i}$ such that y is a solution of $L_1(Y) = 0$ if and only if θy is one of $L_2(Y) = 0$. Suppose that local exponents of L_1 and L_2 at an arbitrary point of $\mathbb{P}^1$ are the same. By Lemma 2.3.5 we then have $\theta(z) = 1$. Proposition 2.3.4 finally gives $L_1 = \theta^{-1}L_2\theta = L_2$. □

Apart from the relation concerning the local exponents of two equivalent equations there also is a relation between their monodromy groups.

Proposition 2.3.8 *Let L_1 and L_2 be equivalent operators by the function θ. Let $\mathbf{f} = (f_1, f_2, \ldots, f_n)$ be a vector of n independent solutions of $L_1(Y) = 0$. Define S to be the set consisting of all singularities of L_1, L_2 and the roots of θ. Then for every closed path u in $\mathbb{P} \setminus S$ there exists a $\lambda_u \in \mathbb{C}^*$ satisfying $M_{\theta\mathbf{f}}(u) = \lambda_u M_{\mathbf{f}}(u)$.*

Proof. For any closed path u in $\mathbb{P} \setminus S$ there is a constant $\lambda_u \in \mathbb{C}^*$ such that θ changes into $\lambda_u\theta$ after analytic continuation along u. Now, let $\tilde{f}$ be the analytical continuation of a solution f of $L_1(Y) = 0$ along u. Then $\lambda_u\theta\tilde{f}$ is the continuation that belongs to θf. One therefore has

$$M_{\theta\mathbf{f}}(u)\begin{pmatrix}\theta f_1\\ \theta f_2\\ \vdots\\ \theta f_n\end{pmatrix} = \lambda_u\theta \cdot M_{\mathbf{f}}(u)\begin{pmatrix}f_1\\ f_2\\ \vdots\\ f_n\end{pmatrix}.$$

This yields $M_{\theta\mathbf{f}}(u) = \lambda_u M_{\mathbf{f}}(u)$, since the first vector is

$$M_{\theta\mathbf{f}}(u)\begin{pmatrix}\theta f_1\\ \theta f_2\\ \vdots\\ \theta f_n\end{pmatrix} = \theta \cdot M_{\theta\mathbf{f}}(u)\begin{pmatrix}f_1\\ f_2\\ \vdots\\ f_n\end{pmatrix}.$$

This proves the proposition. □

Corollary 2.3.9 *The monodromy groups of two equivalent equations are projectively conjugate.*

Proof. Let L_1 and L_2 be two equivalent equations by the radical function θ. We take the basis of solutions $\mathbf{f}$ of $L_1(Y) = 0$ as in Proposition 2.3.8. This proposition shows that any monodromy matrix of L_2 with respect to its basis $\theta\mathbf{f}$ is a scalar multiple of the corresponding monodromy matrix of L_1 with respect to $\mathbf{f}$. The projective groups of these monodromy groups are therefore equal. Another basis of solutions of e.g. L_1 yields a conjugate (projective) group of the original (projective) monodromy group of L_1. The analogue is true for L_2. The projective monodromy groups of L_1 and L_2 are therefore conjugate in general. □

If $f_1, f_2, \ldots, f_n$ is a basis of solutions for the linear operator L_1, then the function $\theta = W(f_1, f_2, \ldots, f_n)^{-1/n}$ yields an equivalent operator $L_2 := \theta L_1 \theta^{-1}$ of L_1. The operator L_2 contains no $(n-1)$-th derivative. This observation yields a nice consequence in the case of a Fuchsian operator L_1. The sum of the local exponents of $L_2(Z) = 0$ at any point in $\mathbb{P}^1$ then is $\sum_{i=0}^{n-1} i = n(n-1)/2$. In particular it is an integer. The determinant of the monodromy matrix at an arbitrairy point has become $\mathrm{e}^{2\pi\mathrm{i}\cdot n(n-1)/2} = 1$. The monodromy group of $L_2(Z) = 0$ not only lies in $\mathrm{GL}(n, \mathbb{C})$ but is in fact a subgroup of $\mathrm{SL}(n, \mathbb{C})$. We have proved the following theorem.

Theorem 2.3.10 *Let L be a Fuchsian operator of order n. Then there exists a projectively equivalent operator of L such that its monodromy group is contained in* $\mathrm{SL}(n, \mathbb{C})$. □

2.4 Reducibility

In this section we collect some standard facts on representation theory of finite groups. More details about this topic can for instance be found in [Isa94].

The monodromy group M acts faithfully on the space of solutions of a Fuchsian equation. This space is $\mathbb{C}$-linear and so is the action. Sometimes the monodromy group leaves a proper non-trivial subspace invariant. In this case we call the action of M reducible.

Definition 2.4.1 Let G be a group that acts on a linear vector space $V \neq \{0\}$. If V is the only non-trivial subspace that is invariant under G, then V is called *irreducible* (under G). Otherwise we call V *reducible*. We shall call a reducible space V *completely reducible* if for every invariant subspace $V_1 \subset V$, there exists an invariant subspace V_2 satisfying $V = V_1 \oplus V_2$.

Notice that we assume a completely reducible group to be reducible. Some other authors prefer not to have this restriction.

Definition 2.4.2 Let G be a subgroup of $\mathrm{GL}(n, \mathbb{C})$ that acts on a $\mathbb{C}$-linear space V of dimension n. Then G is called *totally reducible* if there exists n G-invariant subspaces $V_1, V_2, \ldots, V_n$ of dimension 1 satisfying $V = V_1 \oplus V_2 \oplus \cdots \oplus V_n$.

A well-known theorem concerning the irreducibility of spaces under the action of finite groups is Maschke's theorem.

Theorem 2.4.3 (Maschke) *Let G be a finite group that acts on a finite dimensional vector space over $\mathbb{C}$. Then the action is either irreducible or completely reducible.*

Corollary 2.4.4 *Let $L(y) = 0$ be a Fuchsian equation with finite monodromy group. Then the monodromy group acts either irreducibly or completely reducibly on the solution space of the equation.*

Theorem 2.4.5 *Let $G \subset \mathrm{GL}(n, \mathbb{C})$ be a finite subgroup that acts naturally on a n-dimensional $\mathbb{C}$-linear space V. Then G acts totally reducibly on V if and only if G is abelian.*

From Chapter 3 on we shall be interested in Fuchsian equations of order 2. So let us specify what kind of finite monodromy group exists for such equations. Any finite group $G \subset \mathrm{GL}(2, \mathbb{C})$ that acts naturally on a 2-dimensional $\mathbb{C}$-linear space V is either irreducible or completely reducible. If it is completely reducible then there exist subspaces V_1 and V_2 of V that are invariant under the group action and both are of dimension 1. This means that G is totally reducible, or equivalently, abelian, see Theorem 2.4.5. This yields the following theorem.

Theorem 2.4.6 *Let M be a finite monodromy group of a Fuchsian equation of order 2. Then M acts reducibly on the solution space of the equation if and only if this action is totally reducible. This is equivalent to M being abelian.* $\square$

A completely reducible group is a specific case of the more general notion of an imprimitive group.

Definition 2.4.7 Let G be a group that acts on a vector space V. Then G is called *imprimitive* if V is a direct sum $V_1 \oplus V_2 \oplus \cdots \oplus V_r$, $r \geq 2$, of non-trivial proper subspaces such that $\{gV_1, gV_2, \ldots, gV_r\} = \{V_1, V_2, \ldots V_r\}$ holds for every $g \in G$. If such a splitting into subspaces does not exist, then G is called *primitive*.

A direct observation from the definition of imprimitive groups is that a completely reducible group is also imprimitive. The converse is in general not true.

2.5 The ring of invariants

Any subgroup G of $\mathrm{GL}(n, \mathbb{C})$ acts naturally on the multivariate polynomial ring $\mathbb{C}[X_1, X_2, \ldots, X_n]$ by left multiplication of an element $g^{-1} \in G$ on the column vector $(X_1, X_2, \ldots, X_n)^T$. So by definition $gI(X_1, X_2, \ldots, X_n)$ will be $I(g^{-1}(X_1, X_2, \ldots, X_n)^T)$ for every polynomial $I(X_1, X_2, \ldots, X_n)$ having coefficients in $\mathbb{C}$.

Definition 2.5.1 Let $G \subset \mathrm{GL}(n, \mathbb{C})$ act on $\mathbb{C}[X_1, X_2, \ldots, X_n]$. Then a polynomial $I \in \mathbb{C}[X_1, X_2, \ldots, X_n]$ is called an *invariant of* G if $gI = I$ holds for every $g \in G$. It is called a *semi-invariant* if :

(i) for every $g \in G$ there exists a $c_g \in \mathbb{C}^*$, depending on g, satisfying $gI = c_g I$;

(ii) at least one c_g is not 1.

Notice that if $I(X_1, X_2, \ldots, X_n)$ is an invariant for G then $\tau G \tau^{-1}$ with $\tau \in \mathrm{GL}(n, \mathbb{C})$, has $\tau I(X_1, X_2, \ldots, X_n)^t$ as an invariant. The degrees of I and τI coincide. This follows from $\tau^{-1}\tau I = I$ and the fact that the degree of τI never exceeds the degree of I itself.

Definition 2.5.2 Let G be a subgroup of $\mathrm{GL}(n, \mathbb{C})$. The ring of all invariant polynomials of G in $\mathbb{C}[X_1, X_2, \ldots, X_n]$ is called the *ring of invariants* of G. It will be denoted by $\mathbb{C}[X_1, X_2, \ldots, X_n]^G$.

2.6 Equations as rational pull-backs

We have seen in Proposition 2.3.3 that a Fuchsian equation yields other Fuchsian equations by considering projective equivalence. In this section we introduce another operation that carry Fuchsian differential equations with finite monodromy into other such equations.

We let L be the operator

$$L := \frac{d^n}{dx^n} + b_1(x)\frac{d^{n-1}}{dx^{n-1}} + \cdots + b_{n-1}(x)\frac{d}{dx} + b_n(x),$$

corresponding to a Fuchsian equation with finite monodromy. If x is replaced by a non-constant rational function $\xi(z) \in \mathbb{C}(z)$, then L changes into the differential operator

$$L_\xi := \left(\frac{d}{\xi'(z)dz}\right)^n + \cdots + b_{n-1}(\xi)\frac{d}{\xi'(z)dz} + b_n(\xi) \qquad (2.1)$$

of order n with differentiation with respect to z.

Definition 2.6.1 Let the notation be as above. Then L_ξ is called a *proper (rational) pull-back of L by $x = \xi(z)$.*

Definition 2.6.2 If L_ξ is a proper rational pull-back of L by ξ, then ξ is called the *(rational) pull-back function of L to L_ξ.*

Notice that $f \circ \xi(z)$ is a solution of $L_\xi(y(z)) = 0$ if and only if $f(x)$ is a solution of $L(y(x)) = 0$.

Proposition 2.6.3 *Let L_ξ be a proper pull-back of the Fuchsian operator L by ξ. Then L_ξ is Fuchsian.*

Proof. Let $\tilde{\alpha}$ be a point in $\mathbb{P}^1$ with $\alpha = \xi(\tilde{\alpha})$. Let $\tilde{f}(z)$ be a solution of $L_\xi(y(z)) = 0$. Then there exists a solution $f(x)$ of $L(y(x)) = 0$ such that $\tilde{f}(z) = f(\xi(z))$ holds. Now let t and s denote local parameters at $\tilde{\alpha}$ and α, respectively. The function $f(x)$ is of order $|s|^k$ around $x = \alpha$ for a certain integer k, because of Proposition 1.1.14. It follows from $\xi(z) \in \mathbb{C}(z)$ that there also exists an integer m such that ξ is of order $|t|^m$ at $z = \tilde{\alpha}$. Therefore, $\tilde{f}(z)$ can be written as a Laurent series of order $|t|^{-|km|}$ in an open neighbourhood around $z = \tilde{\alpha}$. The lemma now follows from Proposition 1.1.14, since $\tilde{\alpha}$ and $\tilde{f}$ are arbitrarily. □

We extend the notion of a proper rational pull-back of L by ξ to a rational pull-back which is defined on projectively equivalent operators.

Definition 2.6.4 Let L' be a Fuchsian operator of degree n with differentiation with respect to z. Then L' is called a *(rational) pull-back of* L *by* $x = \xi(z)$ if L' is projectively equivalent to the proper rational pull-back L_ξ of L by $x = \xi(z)$.

Notice that in this definition the assumption of L' being Fuchsian may be omitted in virtue of Proposition 2.3.3. For $n = 2$ the definition of a rational pull-back immediately yields the following proposition.

Proposition 2.6.5 *Let* $L'(y(z)) = 0$ *and* $L(y(x)) = 0$ *be two Fuchsian equations of order* 2*. Then* L' *is a rational pull-back of* L *by* $\xi(z) \in \mathbb{C}(z)$ *if there are ratios* $\tilde{\tau}(z)$ *and* $\tau(x)$ *of two independent solutions of* L' *and* L*, respectively, with the property* $\tilde{\tau}(z) = \tau \circ \xi(z)$*.* □

The rational function ξ ramifies in a finite number of points. The so-called Riemann-Hurwitz formula relates their ramification indices and the degree of ξ in z. It is given in the following theorem.

Theorem 2.6.6 (Riemann-Hurwitz) *Let* $\xi : \mathbb{P}^1 \to \mathbb{P}^1$ *be a rational function in* $\mathbb{C}(z)$ *of degree* d*. Let* e_P *be the ramification index of* ξ *in* $P \in \mathbb{P}^1$*. Then one has*

$$\sum_{P \in \mathbb{P}^1} (e_P - 1) = 2d - 2. \tag{2.2}$$

Proof. For a proof in an algebraic geometric setting we refer to [Sil86, II.5.9] and [Sil86, II.5.6]. For a more analytical formulation one could consult [For81, §17.14]. □

We are now going to describe what the rational map ξ does on solutions of L. Let $\phi_1(x)$ and $\phi_2(x)$ be two such solutions that have Puiseux series solutions at a certain $x = \alpha \in \mathbb{P}^1$ with respect to the local exponents $\rho_1(\alpha)$ and $\rho_2(\alpha)$, respectively. We let $s = x - \alpha$ and $s = 1/x$ be the local parameters for successively a finite or infinite point $\alpha \in \mathbb{P}^1$. In particular, we can think of s as a function $s(x)$ in x. The solutions ϕ_1 and ϕ_2 can locally be given as

$$\begin{aligned} \phi_1(x) &= (s)^{\rho_1(\alpha)} g_1(s) \\ \phi_2(x) &= (s)^{\rho_2(\alpha)} g_2(s) \end{aligned}$$

at $x = \alpha$ for certain power series $g_1(s)$ and $g_2(s)$ in s with non-zero constant terms and positive radii of convergence. Then

$$\begin{aligned} \phi_1(\xi(z)) &= (s(\xi(z))^{\rho_1(\alpha)} g_1(s(\xi(z)) \\ \phi_2(\xi(z)) &= (s(\xi(z))^{\rho_2(\alpha)} g_2(s(\xi(z)) \end{aligned}$$

are solutions of $L_\xi(y(z)) = 0$. These series can locally be expanded at any $z = \tilde{\alpha} \in \mathbb{P}^1$ with $\xi(\tilde{\alpha}) = \alpha$ as follows. If ξ ramifies at a given $\tilde{\alpha}$ with ramification index e, then one has

$$\begin{aligned} \phi_1(\xi(z)) &= (t)^{e\rho_1(\alpha)}\tilde{g}_1(t) \\ \phi_2(\xi(z)) &= (t)^{e\rho_2(\alpha)}\tilde{g}_2(t), \end{aligned}$$

for the appropriate local parameter $t = z - \tilde{\alpha}$ or $t = 1/z$. The Taylor series $\tilde{g}_1$ and $\tilde{g}_2$ have positive radii of convergence around $t = 0$ and have non-zero constant terms. The series $\phi_1(\xi(z))$ and $\phi_2(\xi(z))$ thus are series solutions of L_ξ at $z = \tilde{\alpha}$ that belong to the local exponents $e\rho_1(\alpha)$ and $e\rho_2(\alpha)$, respectively. It can be proven analogously that the same result holds in the case that ϕ_1 contains some logarithmic terms.

Theorem 2.6.7 *Let L_ξ be a proper rational pull-back of L by $x = \xi(z)$. Let $\rho_1, \rho_2, \ldots, \rho_n$ denote the local exponents of L at $\alpha = \xi(\tilde{\alpha}) \in \mathbb{P}^1$. Let e denote the ramification index of ξ at $\tilde{\alpha}$. Then $e\rho_1, e\rho_2, \ldots, e\rho_n$ are the local exponents of L_ξ at $\tilde{\alpha}$.*

Proof. We choose to give a proof without using the local solution expansions of L. Let t and s be the usual local parameters at $x = \alpha$ and $z = \tilde{\alpha}$, respectively. The differential operator L in the variable t can be written as

$$D^n + b_1(t)D^{n-1} + \cdots + b_{n-1}(t)D + b_n(t),$$

with $D = t\frac{d}{dt}$ and certain holomorphic functions $b_1, b_2, \ldots, b_n \in \mathbb{C}[[t]]$. The indicial equation at $t = 0$ is then given by

$$X^n + b_1(0)X^{n-1} + \cdots + b_{n-1}(0)X + b_n(0).$$

If we replace t by s^e then $t\frac{d}{dt}$ becomes $\left(\frac{s}{e}\right)\frac{d}{ds}$. Therefore, the indicial equation at $\tilde{\alpha}$ is

$$(X/e)^n + b_1(0)\,(X/e)^{n-1} + \cdots + b_{n-1}(0)\,(X/e) + b_n(0).$$

It follows that ρ is a local exponent of L at α precisely when $e\rho$ is a local exponent of L_ξ at $\tilde{\alpha}$. □

Proposition 2.6.8 *Let L' be a rational pull-back of the Fuchsian operator L by $x = \xi(z) \in \mathbb{C}(z)$. Let $\rho_1(\alpha)$ and $\rho_2(\alpha)$ be any two local exponents of L at $\alpha = \xi(\tilde{\alpha}) \in \mathbb{P}^1$. Let e denote the ramification index of ξ at $\tilde{\alpha}$. Then there are local exponents $\rho_1(\tilde{\alpha})$ and $\rho_2(\tilde{\alpha})$ of L' at $\tilde{\alpha}$ that satisfy*

$$\rho_1(\tilde{\alpha}) - \rho_2(\tilde{\alpha}) = e\left(\rho_1(\alpha) - \rho_2(\alpha)\right).$$

Proof. Let L_ξ be a proper pull-back of L by ξ. It follows from Theorem 2.6.7 that the statement is true for L_ξ and L instead of L' and L, respectively. According to Corollary 2.3.6, the difference $e\,(\rho_1(\alpha) - \rho_2(\alpha))$ is not only a difference of exponents of L_ξ at $z = \tilde{\alpha}$, but also one of L' at $\tilde{\alpha}$. This concludes our proof. □

Proposition 2.6.8 is in particular applicable on those Fuchsian differential equations $L(y(x)) = 0$ and $L_\xi(y(z)) = 0$ that have finite monodromy. The monodromy groups of L_ξ and L are directly related to each other.

Theorem 2.6.9 *Let L_ξ be a proper pull-back of the Fuchsian operator L by $x = \xi(z)$. Let $\mathbf{f}$ and $\mathbf{f} \circ \xi$ be the bases of solutions of $L(y) = 0$ and $L_\xi(y) = 0$, respectively. Suppose that M and M_ξ denote the successive monodromy groups of L and L_ξ with respect to these bases. Then M_ξ is a subgroup of M.*

Proof. Let S be the set of singular points of L. Analogously, let the set S_ξ consist of all singularities of L_ξ. We then define the finite set $\xi^{-1}(S) \subset \mathbb{P}^1$ to be

$$\xi^{-1}(S) := \left\{ z \in \mathbb{P}^1 : \xi(z) \in S \right\}.$$

The group M_ξ describes the analytical continuations of $\mathbf{f} \circ \xi$ along a closed path $u \subset \mathbb{P}^1 \setminus S_\xi$ with a fixed initial point z_0. This point may assumed not to lie in $\xi^{-1}(S) \cup S_\xi$. We can move u a bit such that the obtained path in $\mathbb{P}^1 \setminus S_\xi$ is equivalent to u and does not contain any element of $\xi^{-1}(S)$.
If f_i is basis element of $\mathbf{f}$, then $f_i \circ \xi$ is single-valued in any point of $\xi^{-1}(S) \setminus (\xi^{-1}(S) \cap S_\xi)$. So an analytical continuation of $f_i \circ \xi$ along a closed path in $\mathbb{P}^1 \setminus S_\xi$ may also considered to be in $\mathbb{P}^1 \setminus (\xi^{-1}(S) \cup S_\xi)$. Now the analytical continuation of $f_i \circ \xi$ along $u \subset \mathbb{P}^1 \setminus (\xi^{-1}(S) \cup S_\xi)$ is the same as the one of f_i along $\xi(u)$. We need to prove that the continuation is independent on the class $[u] \in \mathbb{P}^1 \setminus (\xi^{-1}(S) \cup S_\xi)$ of u.
The rational function $\xi(z)$ maps z_0 to $x_0 \in \mathbb{P}^1 \setminus (S \cup \xi(S_\xi))$. This leads to the map

$$\begin{aligned} \phi : \pi_1\left(\mathbb{P}^1 \setminus (\xi^{-1}(S) \cup S_\xi), z_0\right) &\to \pi_1\left(\mathbb{P}^1 \setminus (S \cup \xi(S_\xi)), x_0\right) \\ [u] &\mapsto [\xi(u)]. \end{aligned}$$

It needs to be proved that ϕ is well-defined. Let u be a closed path in $\mathbb{P}^1 \setminus (\xi^{-1}(S) \cup S_\xi)$ that is contractible to z_0. Then u lies in an open subset $U \subset \mathbb{P}^1$ with $z_0 \in U$ and $U \cap (\xi^{-1}(S) \cup S_\xi) = \emptyset$. The path $\xi(u)$ is closed and is contained in $\xi(U)$. It has an empty intersection with $S \cup \xi(S_\xi)$. Hence $[\xi(u)]$ is trivial.
The map

$$\begin{aligned} j : \pi_1\left(\mathbb{P}^1 \setminus (S \cup \xi(S_\xi)), x_0\right) &\to \pi_1\left(\mathbb{P}^1 \setminus S, x_0\right) \\ [v] &\mapsto [v] \end{aligned}$$

now induces the well-defined function $\psi := j \circ \phi$ that is given as

$$\begin{array}{rcl}\psi : \pi_1\left(\mathbb{P}^1 \setminus (\xi^{-1}(S) \cup S_\xi), z_0\right) & \to & \pi_1\left(\mathbb{P}^1 \setminus S, x_0\right) \\ [u] & \mapsto & [\xi(u)].\end{array}$$

Consequently, the analytical continuation of $f_i \circ \xi$ along $[u]$ is the same as the one of f_i along $[\xi(u)]$. We conclude that $M_{\mathbf{f}\circ\xi}[u]$ equals $M_{\mathbf{f}}[\xi(u)] \in M$. In particular, M_ξ is a subgroup of M. □

Corollary 2.6.10 *Suppose that L' is a pull-back of the Fuchsian operator L by $x = \xi(z) \in \mathbb{C}(z)$. Let M' and M denote the monodromy groups of L' and L, respectively. Then PM' is conjugate to a subgroup of PM.*

Proof. There exist a Fuchsian operator L_ξ as in Equation (2.1) such that it is projectively equivalent to L'. The projective monodromy group PM_ξ of L_ξ is projectively conjugate to PM', see Corollary 2.3.9. Therefore, it is sufficient to prove the statement for L_ξ and L instead of L' and L. The fact that PM_ξ is conjugate to a subgroup of PM however is an immediate consequence of Theorem 2.6.9. □

The theory of monodromy groups and rational pull-backs will be illustrated in more detail in the next chapter.

Chapter 3

Equations of order 2

This chapter is mostly devoted to Fuchsian equations $L(y) = 0$ of order 2. For these equations we shall describe all finite monodromy groups that are possible. In particular we discuss the projective monodromy groups of dimension 2. Most of the theory we give is based on the work of Felix Klein and of G.C. Shephard and J.A. Todd. Unless stated otherwise we let $L(y) = 0$ be a Fuchsian differential equation order 2, with differentiation with respect to the variable z.

3.1 Finite subgroups of $\mathrm{PGL}(2, \mathbb{C})$

Any Fuchsian differential equation with a basis of algebraic solutions has a finite monodromy group, (and vice versa, see Theorem 2.1.8). In particular this is the case for second order Fuchsian equations, to which we restrict ourselves for now. The monodromy group for such an equation is contained in $\mathrm{GL}(2, \mathbb{C})$. Any subgroup $G \subset \mathrm{GL}(2, \mathbb{C})$ acts on the Riemann sphere $\mathbb{P}^1$. More specifically, a matrix $\gamma \in \mathrm{GL}(2, \mathbb{C})$ with

$$\gamma := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

has an action on $\mathbb{C}$ defined as

$$\gamma * t := \frac{at + b}{ct + d}. \tag{3.1}$$

for all appropriate $t \in \mathbb{C}$. The action is extended to $\mathbb{P}^1$ by $\gamma * \infty = a/c$ in the case of $ac \neq 0$ and $\gamma * (-d/c) = \infty$ when c is non-zero. The action of G on $\mathbb{P}^1$ factors through to the projective group $PG \subset \mathrm{PGL}(2, \mathbb{C})$. In other words we have the well-defined action

$$\begin{array}{rcl} PG : \mathbb{P}^1 & \to & \mathbb{P}^1 \\ \gamma \cdot \Lambda I_2 : t & \mapsto & \gamma * t. \end{array}$$

PG	name	$\|PG\|$
C_m	Cyclic group	m
D_m	Dihedral group	$2m$
A_4	Tetrahedral group	12
	(Alternating group on 4 elements)	
S_4	Octahedral group	24
	(Symmetric group on 4 elements)	
A_5	Icosahedral group	60
	(Alternating group on 5 elements)	

Table 3.1: The finite subgroups of $\mathrm{PGL}(2,\mathbb{C})$.

The projective group PG thus can be seen as a subgroup of the automorphism group of $\mathbb{P}^1$. It turns out that the finite subgroups of $\mathrm{PGL}(2,\mathbb{C})$ can be obtained by choosing the symmetry subgroup of one of the platonic solids with vertices on the sphere. We refer to [Kle84] (or its English translation [Kle56]) and in particular to its Chapters I.1 and I.2 and Section I.5.2 for a proof of this assertion. The complete list of these groups is given in Table 3.1.

One can consider t as a variable rather than a point in Definition (3.1). The action of PG on $\mathbb{C}$ as in (3.2) then induces one on the field $\mathbb{C}(t)$ as

$$\begin{aligned} PG : \mathbb{C}(t) &\to \mathbb{C}(t) \\ \gamma : h(t) &\mapsto \gamma * h(t) := h\left(\gamma^{-1} * t\right). \end{aligned}$$

Again the action is also valid for G instead of PG. F. Klein proved that the subfield $\mathbb{C}(t)^{PG} = \mathbb{C}(t)^G$ of invariants in $\mathbb{C}(t)$ for finite PG is generated by a single rational function $j_G(t)$. In particular we then have

$$j_G(\gamma * t) = j_G(t)$$

for every $\gamma \in G$. Notice that $j_G(t)$ is not unique.

Theorem 3.1.1 (Klein) *Suppose that PG is finite. Then there exists a function $j_G(t) \in \mathbb{C}(t)$ such that $\mathbb{C}(t)^{PG} = \mathbb{C}(j_G(t))$ holds. Moreover, its degree is* $\deg_t(j_G(t)) = |PG|$.

Proof. A first proof that $j_G(t)$ exists is given in [Kle84, I,Ch.2]. The same sort of proof, but in a more modern setting, can be found in [BD79]. Their given descriptions of $j_G(t)$ shows that $j_G(t)$ is of degree $|PG|$. □

Theorem 3.1.2 (Klein) *Suppose that PG is a finite subgroup of* $\mathrm{PGL}(2,\mathbb{C})$ *acting on $\mathbb{P}^1$. Then a coordinate t of $\mathbb{P}^1$ can be chosen such that PG is the projective image of one of the subgroups $\widehat{C}_m$, $\widehat{D}_{2m}$, T, O and I of* $\mathrm{SL}(2,\mathbb{Z})$*, as in the following examples.*

Example 3.1.3 In this example we consider the cyclic group

$$\widehat{C}_m := \left\{ \begin{pmatrix} \zeta_{2m}^k & 0 \\ 0 & \zeta_{2m}^{-k} \end{pmatrix} : k = 0, 1, \ldots, 2m-1 \right\}$$

with $m \in \mathbb{Z}_{>0}$ and $\zeta_{2m} := \mathrm{e}^{2\pi\mathrm{i}/2m}$. The projective group $P\widehat{C}_m$ is cyclic of order m. It reflects the rotations around the origin by angles that are multiples of $2\pi/m$. For this particular representation of $\widehat{C}_m$ one has

$$j_{\widehat{C}_m}(t) = t^m.$$

The points 0 and ∞ are the only points fixed by C.

Example 3.1.4 The next group we encounter is the dihedral group

$$\widehat{D}_{2m} := \left\langle \begin{pmatrix} \zeta_{2m} & 0 \\ 0 & \zeta_{2m}^{-1} \end{pmatrix}, \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \right\rangle$$

of order $4m$ with $m \in \mathbb{Z}_{\geq 2}$. The center of $\widehat{D}_{2m}$ is $\pm I_2$ and hence $P\widehat{D}_{2m}$ is dihedral with group order $2m$. The group $P\widehat{D}_{2m}$ can be seen as the symmetry group of the dihedron, or regular double-pyramid, with vertices on the unit-sphere as follows. The point $(0,0,1)$ corresponds to $z = \infty$. We then have a central projection from this point of $\mathbb{C}$ on the surface of the unit sphere. On the ground circle, that consists of all points on the sphere with last coordinate equal to 0, we can put m equidistant vertices. The summits of the pyramids then correspond to $z = 0$ and $z = \infty$. The diagonal matrices in $P\widehat{D}_{2m}$ represent rotations of the circle by the angles $2\pi k/m$, $k \in \mathbb{Z}$. They leave the two summits unchanged. The other matrices represent rotations of the dihedron around the diameters of the ground circle, containing a vertex or the middle of two consecutive vertices. For these matrices two points on the circle are fixed and the summits are interchanged. The invariant polynomial $J_{\widehat{D}_{2m}}$ has been found to be

$$j_{\widehat{D}_{2m}}(t) = -\frac{(t^m-1)^2}{4t^m}.$$

Notice that its degree is $2m$ in t, as it should by Klein's Theorem 3.1.1.

Examples 3.1.5 As an example of a group such that its projective group is the symmetry group of a tetrahedron we take

$$T := \left\langle \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \frac{1}{2}\begin{pmatrix} 1+\mathrm{i} & -1+\mathrm{i} \\ 1+\mathrm{i} & 1-\mathrm{i} \end{pmatrix} \right\rangle.$$

The order of this matrix group is 24. One can show that PT is indeed isomorphic to A_4. The invariant polynomial $j_T(t)$ is

$$j_T(t) = \frac{(t^4 - 2\sqrt{3}\mathrm{i}t^2 + 1)^3}{(t^4 + 2\sqrt{3}\mathrm{i}t^2 + 1)^3}.$$

If one suitably embeds the octahedron in a tetrahedron, then PT becomes a subgroup of the symmetry group of the octahedron. Such a group is for instance

$$O := \left\langle T, \frac{1}{\sqrt{2}}\begin{pmatrix} 1+\mathrm{i} & 0 \\ 0 & 1-\mathrm{i} \end{pmatrix} \right\rangle.$$

We then have $|O| = 48$, $PO \cong S_4$ and

$$j_O(t) = \frac{(t^8 + 14t^4 + 1)^3}{108t^4(t^4 - 1)^4}.$$

Example 3.1.6 Finally, the group

$$I := \left\langle \begin{pmatrix} \zeta_5^3 & 0 \\ 0 & \zeta_5^2 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \frac{1}{\sqrt{5}}\begin{pmatrix} -\zeta_5 + \zeta_5^4 & \zeta_5^2 - \zeta_5^3 \\ \zeta_5^2 - \zeta_5^3 & \zeta_5 - \zeta_5^4 \end{pmatrix} \right\rangle$$

of order 120 has an icosahedral projective group. Its invariant polynomial is

$$j_I(t) = \frac{(-t^{20} + 228t^{15} - 494t^{10} - 228t^5 - 1)^3}{1728t^5(t^{10} + 11t^5 - 1)^5}.$$

The invariant $j_G(t)$ is a Galois covering from $\mathbb{P}^1$ to $\mathbb{P}^1$ with group PG. It ramifies above either 2 or 3 points, see [BD79, p.47-48]. This proof, which is based on the Hurwitz formula, separates these two cases. If $j_G(t)$ ramifies above 2 points, then PG must be cyclic. If not, PG is dihedral, tetrahedral, octahedral or icosahedral. After applying a projective linear map one may assume that all ramification points of $j_G(t)$ are above 0, 1 and ∞. In fact, if $\alpha_1, \alpha_2 \in \mathbb{P}^1$ are branch points of $j_G(t)$ and satisfy $j_G(\alpha_1) = j_G(\alpha_2)$, then their multiplicities, or ramification indices, are the same. If e_0, e_1 and e_∞ denote the ramification indices of the ramification points above 0, 1 and ∞, respectively, then Table 3.2 gives all ramifications that occur.

In our examples above the rational function $j_G(t)$ is chosen in such a way that its branch points are mapped to at least two of 0, 1 and ∞. Its ramification indices satisfy $e_1 \leq e_0 \leq e_\infty$.

PM	$\{e_0, e_1, e_\infty\}$
C_m	$\{1, m, m\}$
D_m	$\{2, 2, m\}$
A_4	$\{2, 3, 3\}$
S_4	$\{2, 3, 4\}$
A_5	$\{2, 3, 5\}$

Table 3.2: The ramification indices of $j_M(t)$ above 0, 1 and ∞.

Example 3.1.7 For the octahedral group O as given in Examples 3.1.5 we have

$$\begin{aligned} j_O(t) &= \frac{(t^8 + 14t^4 + 1)^3}{108t^4(t^4 - 1)^4} \\ &= 1 + \frac{(z^{12} - 33z^8 - 33z^4 + 1)^2}{108t^4(t^4 - 1)^4}. \end{aligned}$$

It has ramification indices 3, 2 and 4 above 0, 1 and ∞, respectively. For all the other invariant functions in the Examples 3.1.3 through 3.1.6 there are similar results.

3.2 A theorem of Felix Klein

The monodromy group M of the equation $L(y(z)) = 0$ may be assumed to be taken on a specific ordered basis of solutions $(y_1(z), y_2(z))$. We also may assume that $j_M(t)$ only ramifies above 0, 1 and ∞, for finite M. We first prove that

$$R(z) := j_M(y_1/y_2(z))$$

is actually a rational function in z.

Proposition 3.2.1 *Suppose that M is finite. Then one has $R(z) \in \mathbb{C}(z)$. It is independent of the chosen basis (y_1, y_2) of the solution space of $L(y(z)) = 0$.*

Proof. Let $\gamma \in M$ be as before. We derive

$$\begin{aligned} {}^{\gamma}j_M\left(\frac{y_1}{y_2}\right) &= j_M\left(\frac{ay_1 + by_2}{cy_1 + dy_2}\right) \\ &= j_M\left(\frac{a(y_1/y_2) + b}{c(y_1/y_2) + d}\right) \\ &= \gamma^{-1} * j_M\left(\frac{y_1}{y_2}\right) \\ &= j_M\left(\frac{y_1}{y_2}\right). \end{aligned}$$

Hence, the algebraic function $j_M(y_1/y_2)$ in z is invariant under the monodromy action of M. It follows from Lemma 2.1.6 that $R(z) = j_M(y_1/y_2)$ is a rational function in z.
For the second assertion we now assume that $B' := (y_1', y_2')$ is an other basis of solutions of $L(y) = 0$. There exists a matrix $\tau \in \mathrm{GL}(2, \mathbb{C})$ such that $(y_1', y_2')^T = \tau(y_1, y_2)^T$ holds. The monodromy group M' of L with respect to B' then is $\tau M \tau^{-1}$. It follows that an invariant for M' is $\tau * j_M(t)$. It has the same degree in t as $j_M(t)$. Therefore $j_{M'}(t) := \tau * j_M(t)$ is a generator of $\mathbb{C}(t)^{M'}$. Altogether we obtain

$$\begin{aligned} j_{M'}(y_1'/y_2') &= \tau * j_M\left(\tau^{-1} * (y_1/y_2)\right) \\ &= j_M(y_1/y_2) \\ &= R(z) \end{aligned}$$

which concludes our proof. □

The explicit ramifications of $j_M(t)$ above 0, 1 and ∞ induce specific ramification properties on $R(z)$. The following proposition shows how.

Proposition 3.2.2 *Suppose that M is finite. Let S be the set of singular points of $L(y) = 0$ with $\infty \in S$. Suppose that the local exponents $\rho_1(s)$ and $\rho_2(s)$ for $s \in S$ satisfy*

$$\rho_1(s) - \rho_2(s) > 0 \quad \textit{and} \quad \rho_1(s) - \rho_2(s) \notin \mathbb{Z}.$$

Then:

(i) *$R(z)$ only ramifies above 0, 1 and ∞.*

(ii) *One has $R(S) \subset \{0, 1, \infty\}$.*

(iii) *For $s \in S$ with $R(s) = \alpha$ we have $(\rho_1(s) - \rho_2(s))e_\alpha \in \mathbb{Z}$.*

(iv) *$R(z)$ has ramification order e_α for $z_0 \notin S$ with $R(z_0) = \alpha \in \{0, 1, \infty\}$.*

Proof. Consider $z_0 \notin S$. Then the local exponents of z_0 are 0 and 1. We can choose two solutions of $L(y) = 0$ as $f_1(z) = (z - z_0)g_1(z)$ and $f_2(z) = g_2(z)$ with $g_1(z)$ and $g_2(z)$ holomorphic around z_0 and $g_1(z_0) = g_2(z_0) = 1$. Therefore, the quotient $f_1/f_2(z)$ is locally biholomorphic at z_0. It follows that the ramification index of $R(z) = j_M(f_1/f_2)$ at z_0 equals the ramification index of $j_M(t)$ at $t = 0$. In particular this shows that the ramification index of $R(z)$ at z_0 is 1 in case of $j_M(0) \notin \{0, 1, \infty\}$. For $j_M(0) = \alpha \in \{0, 1, \infty\}$ the ramification index is e_α. This proves the parts (i) and (iv) of the proposition.
Suppose now $z_0 \in S$. Then the equations has solutions of the form $f_1(z) = (z - z_0)^{\rho_1(z_0)}g_1(z)$ and $f_2(z) = (z - z_0)^{\rho_2(z_0)}g_2(z)$ with g_1 and g_2 as before. The function $f_1/f_2(z)$ can be written as $(z - z_0)^{\rho_1(z_0) - \rho_2(z_0)}g_1/g_2(z)$. The ramification

index of $R(z)$ at z_0 therefore is equal to $\rho_1(z_0) - \rho_2(z_0)$ times the ramification index e of $j_M(t)$ at $t = 0$. From $\rho_1(z_0) - \rho_2(z_0) \notin \mathbb{Z}$ and $(\rho_1(z_0) - \rho_2(z_0))\, e \in \mathbb{Z}$ we conclude $e > 1$. This gives $j_M(0) = \{0, 1, \infty\}$ and thus $R(z_0) = \alpha \in \{0, 1, \infty\}$. We also see that then $e = e_\alpha$ and $(\rho_1(z_0) - \rho_2(z_0))\, e_\alpha \in \mathbb{Z}$ hold.
The proposition has been proven for $z \neq \infty$. The consideration for the singularity at infinity is similar, when we use the local parameter $1/z$ instead of $z - z_0$. □

In his book on the icosahedron Felix Klein has proved that a second order Fuchsian equation with the finite monodromy group M as above, is in fact the pull-back of a certain hypergeometric equation by $R(z)$. We only state this theorem and omit its proof.

Theorem 3.2.3 (Klein) *Let $L(y(z)) = 0$ be a second order Fuchsian equation on $\mathbb{P}^1$ with finite monodromy group. Then L is the rational pull-back of a hypergeometric operator H by $R(z)$ such that H has local exponent differences $1/e_0$, $1/e_1$ and $1/e_\infty$ at successively 0, 1 and ∞, for a certain $\{e_0, e_1, e_\infty\}$ as in Table 3.2.*

Proof. For a proof one could consult [Kle84, I,Ch.3] and [BD79]. □

3.3 Complex reflection groups

We return to our general differential equation $L(y) = 0$ of order 2. Its monodromy group M is generated by 2×2-matrices in $\mathrm{GL}(2, \mathbb{C})$. The generators come, for instance, from the positively oriented single closed paths around the finite singular points. Let us fix for now a singular point $\alpha \in \mathbb{C}$ that has local exponents ρ_1 and ρ_2. If we replace $y(z)$ by $(z - \alpha)^{-\rho_1}\tilde{y}(z)$, then the newly obtained Fuchsian equation for $\tilde{y}$ is equivalent to $L(y) = 0$ and has local exponents 0 and $\rho_2 - \rho_1$ at α. The local exponents of all other complex numbers remain the same. The exponents at ∞ do change; the quantity ρ_1 is added to each of them.

This construction can be done at all finite singular points of L. It yields an equivalent equation $L'(\tilde{y}) = 0$ to $L(y) = 0$ for which all complex points have at least one exponent equal to 0. The monodromy group M' of L' is generated by the monodromy matrices around the singular points of L'. Each of these matrices has one eigenvalue equal to $e^0 = 1$. The remaining eigenvalue may differ from 1. The group M' is an example of a complex reflection group in dimension 2.

Definition 3.3.1 An element $g \in \mathrm{GL}(n, \mathbb{C})$ is called a *(complex) reflection* if the rank of $g - I_n$ has rank 1. A *(complex) reflection group* is a group that is generated by complex reflections.

Notice that a group conjugate to a reflection group is again a reflection group. We may therefore speak of the monodromy group as a reflection group.

The fundamental paper [ST54] of Shephard and Todd gives a description of all finite irreducible unitary reflection groups. The restriction that the finite groups be unitary is not necessary however, since a finite subgroup of $\mathrm{GL}(2,\mathbb{C})$ is conjugate to a unitary group ([Bur55, §196]). The paper can also be applied to a reducible group, since it is a direct sum of its irreducible components, see Maschke's Theorem 2.4.3. Besides an explicit description of the finite irreducible reflection groups as matrices, a characterization of these groups in terms of invariant polynomials is given.

Theorem 3.3.2 (Shephard-Todd) *Let G be a finite subgroup of* $\mathrm{GL}(n,\mathbb{C})$. *Then G is a complex reflection group if and only if $\mathbb{C}[X_1, X_2, \ldots, X_n]^G$ is generated by n algebraically independent homogeneous polynomials. If their degrees are $d_1, d_2, \ldots, d_n$, then one has $\prod_{i=1}^n d_i = |G|$.*

Theorem 3.3.2 is proved for any unitary group in Part II of [ST54] of Shephard and Todd. As remarked there is no restriction in assuming unitarity. For other properties of unitary reflection groups we refer the reader to [ST54] and [Coh76]. They contain for instance an extensive description of all finite unitary complex reflection groups of dimension 2.

3.4 Finite reflection groups in $\mathrm{GL}(2,\mathbb{C})$

In general Shephard and Todd [ST54] considered the action of finite reflection groups on the Euclidean space $\mathbb{C}^n$ of dimension n. The primitive groups are cyclic in the case $n = 1$. If n is greater than 8, then the reflection group is the symmetric group on n symbols S_n. In addition to S_n more reflection groups occur in the case $2 \leq n \leq 8$.

We shall be concerned with the case $n = 2$. The action of every reflection group on $\mathbb{C}^2$ we use is the natural action. We begin by considering the finite primitive reflection groups for dimension 2. These reflection groups are irreducible and modulo scalars are the tetrahedral, octahedral and icosahedral groups. We put some of their properties in tables to give an orderly overview. For more details we refer to [ST54] or [Coh76].

The reflection groups that reduce to a tetrahedral group are listed in Table 3.3. Each reflection group G is uniquely associated to a number. The enumeration is done in accordance with [ST54]. The first column of Table 3.3 gives these numbers. The second column gives the order $|G|$ of G. By definition the cyclic

ST no.	$\|G\|$	$\|Z\|$	determinants	deg.'s inv
4	24	2	$\langle\zeta_3\rangle$	$4, 6$
5	72	6	$\langle\zeta_3\rangle$	$6, 12$
6	48	4	$\langle\zeta_6\rangle$	$4, 12$
7	144	12	$\langle\zeta_6\rangle$	$12, 12$

Table 3.3: The finite reflection groups with tetrahedral projection.

subgroup $Z = Z_G$ of G consists of all diagonal matrices in G. Any determinant that occurs in G must be a root of unity. The smallest group in which all occurring determinants in G can be put, is given in column nr. 4. Finally, the last column gives the two degrees of the generating invariant polynomials under G. The Tables 3.4 and 3.5 are the analogous tables for the octahedral and icosahedral groups.

ST no.	$\|G\|$	$\|Z\|$	determinants	deg.'s inv
8	96	4	$\langle \mathrm{i}\rangle$	$8, 12$
9	192	8	$\langle \mathrm{i}\rangle$	$8, 24$
10	288	12	$\langle\zeta_{12}\rangle$	$12, 24$
11	576	24	$\langle\zeta_{12}\rangle$	$24, 24$
12	48	2	$\langle -1\rangle$	$6, 8$
13	96	4	$\langle -1\rangle$	$8, 12$
14	144	6	$\langle\zeta_6\rangle$	$6, 24$
15	288	12	$\langle\zeta_6\rangle$	$12, 24$

Table 3.4: The finite reflection groups with octahedral projection.

ST no.	$\|G\|$	$\|Z\|$	determinants	deg.'s inv
16	600	10	$\langle\zeta_5\rangle$	$20, 30$
17	1200	20	$\langle\zeta_{10}\rangle$	$20, 60$
18	1800	30	$\langle\zeta_{15}\rangle$	$30, 60$
19	3600	60	$\langle\zeta_{30}\rangle$	$60, 60$
20	360	6	$\langle\zeta_3\rangle$	$12, 30$
21	720	12	$\langle\zeta_6\rangle$	$12, 60$
22	240	4	$\langle -1\rangle$	$12, 20$

Table 3.5: The finite reflection groups with icosahedral projection.

Some examples of complex reflection groups with octahedral or icosahedral projective groups are considered in Section 3.6. There the groups with ST-numbers

12, 13 and 22 are given in terms of generating matrices.

We move to the irreducible complex reflection groups that are imprimitive. Following [ST54, §I.2] a finite irreducible imprimitive group G has a dihedral group as its projective group. There exist a basis and a primitive m-th rooth of unity ζ_m such that G contains the dihedral subgroup

$$G^* = \left\{ \begin{pmatrix} \zeta_m^k & 0 \\ 0 & \zeta_m^{-k} \end{pmatrix}, \begin{pmatrix} 0 & \zeta_m^k \\ \zeta_m^{-k} & 0 \end{pmatrix} : k = 0, 1 \ldots m-1 \right\}$$

of order $2m$. The relation

$$\begin{pmatrix} 0 & \zeta_m \\ \zeta_m^{-1} & 0 \end{pmatrix} = \begin{pmatrix} \zeta_m & 0 \\ 0 & \zeta_m^{-1} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

shows that

$$G^* = \left\langle \begin{pmatrix} \zeta_m & 0 \\ 0 & \zeta_m^{-1} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle.$$

is a better description; it reveals the structure of a dihedral group. The generators of G are the generators of G^* together with the matrices of the form

$$\begin{pmatrix} \zeta_m^k & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and/or} \quad \begin{pmatrix} 1 & 0 \\ 0 & \zeta_m^l \end{pmatrix}$$

with $k, l | m$. It follows from

$$\begin{pmatrix} \zeta_m^k & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \zeta_m^k \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

and $\langle \zeta_m^k, \zeta_m^l \rangle = \langle \zeta_m^{\gcd(k,l)} \rangle$ that G is the finite group $G(m, p, 2)$ defined below, for the divisor $p = \gcd(k, l)$ of m.

Definition 3.4.1 Let m be a positive integer with positive divisor p. Then the group $G(m, p, 2) \subset \mathrm{GL}(2, \mathbb{C})$ is defined as

$$G(m, p, 2) := \left\langle \begin{pmatrix} \zeta_m & 0 \\ 0 & \zeta_m^{-1} \end{pmatrix}, \begin{pmatrix} \zeta_m^p & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle.$$

Theorem 3.4.2 *Suppose that $G \subset \mathrm{GL}(2, \mathbb{C})$ is an irreducible imprimitive reflection group. Then G is conjugate to $G(m, p, 2)$ for some $m, p \in \mathbb{Z}_{>0}$ with $p|m$. Moreover, one has $m > 1$ and $(m, p, 2) \neq (2, 2, 2)$.*

Proof. The first assertion is sketched above. The possibility $m = 1$ implies that G is abelian and hence is totally reducible. The same is true for $m = p = 2$, but not for the case $m = 2$ with $p = 1$. For another proof we refer to Theorem 2.4 of [Coh76]. □

We are going to determine the number of scalar matrices in G even though [ST54] provides us with the answer. A matrix $\lambda I_2 \in G$ is of the form

$$\begin{pmatrix} \zeta_m & 0 \\ 0 & \zeta_m^{-1} \end{pmatrix}^a \begin{pmatrix} \zeta_m^p & 0 \\ 0 & 1 \end{pmatrix}^b$$

for certain $a, b \in \mathbb{Z}$. This is equivalent to $2a \equiv -pb \bmod m$. There are m/p possibilities for b. In the case of odd m each $b \bmod m/p$ leads to exactly one $a \bmod m$, since 2 is invertible in $\mathbb{Z}/m\mathbb{Z}$. It follows that Z is cyclic of order m/p. For m even and p odd the integer b should be even. There are $m/2p$ of those. Each b gives rise to 2 possibilities for a, namely $a = -pb/2 \bmod m$ and $a = -pb/2 + m/2 \bmod m$. Altogether we have m/p diagonal matrices in Z.
Finally suppose that m and p are even. Then a is either $-pb/2 \bmod m$ or $-pb/2 + m/2 \bmod m$ for any given b. We derive $|Z| = 2m/p$. The properties of G are listed in Table 3.6, just as before. An addition to the table is the order of PG.

ST no.	$\lvert G = G(m,p,2)\rvert$	$\lvert Z\rvert$	$\lvert PG\rvert$	det.'s	deg.'s inv	remark
2	$2m(m/p)^*$	m/p	$2m$	$\langle\zeta_{2m/p}\rangle$	$m, 2m/p$	m odd
		m/p	$2m$	$\langle\zeta_{m/p}\rangle$	$m, 2m/p$	m even p odd
		$2m/p$	m	$\langle\zeta_{2m/p}\rangle$	$m, 2m/p$	m, p even m/p odd
		$2m/p$	m	$\langle\zeta_{m/p}\rangle$	$m, 2m/p$	m, p even m/p even

$^*m \in \mathbb{Z}_{>1}$, $p|m$ and $p \geq 1$

Table 3.6: The finite reflection groups with dihedral projection.

The group $G(m,m,2)$, $m > 1$, is a dihedral group of order m. Conversely, it follows from Theorem 3.4.2 that every finite dihedral reflection group in $\mathrm{GL}(2,\mathbb{C})$ is conjugate to $G(m,m,2)$ for a certain $m \in \mathbb{Z}_{>1}$. The projective group $PG(m,m,2)$ is also dihedral but might be of a smaller order.

Definition 3.4.3 Let K be in $\mathbb{Z}_{>1}$. Then the dihedral group of order $2K$ is denoted by D_K.

We are left with the description of the finite reducible subgroups of $G \subset \mathrm{GL}(2,\mathbb{C})$. Any finite reducible $G \subset \mathrm{GL}(2,\mathbb{C})$ is abelian and completely reducible, see Theorem 2.4.6. In particular it is isomorphic to the direct product of two finite cyclic groups. More specifically, there exist a basis and two roots of unity ζ_k and ζ_l of order k and l, respectively, with

$$G = \left\langle \begin{pmatrix} \zeta_k & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \zeta_l \end{pmatrix} \right\rangle,$$

since G is a reflection group. Its projective group is abelian and could a priori be either cyclic or dihedral of order 4 (Klein's Four group). By Proposition 3.4.5 we know that PG is in fact cyclic. It is generated by the two matrices

$$\begin{pmatrix} \zeta_k & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \zeta_l^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

This yields

$$PG = \left\langle \begin{pmatrix} \zeta_m & 0 \\ 0 & 1 \end{pmatrix} \right\rangle$$

for a certain primitive m-th root of unity with $m = \mathrm{lcm}(k,l)$. Notice that G is also imprimitive. The two generating polynomial invariants of G are X_1^k and X_2^l. The results obtained are given in Table 3.7.

G	$\lvert Z\rvert$	$\lvert PG\rvert$	determinants	deg.'s inv
$\mathbb{Z}/k\mathbb{Z} \times \mathbb{Z}/l\mathbb{Z}$	$\gcd(k,l)$	$\mathrm{lcm}(k,l)$	$\langle \zeta_{\mathrm{lcm}(k,l)} \rangle$	k, l

$^*k, l \in \mathbb{Z}_{>0}$

Table 3.7: The finite reflection groups with cyclic projection.

Proposition 3.4.4 *Suppose that $G \subset \mathrm{GL}(2,\mathbb{C})$ is a finite reducible reflection group. Then one has $G \cong \mathbb{Z}/k\mathbb{Z} \times \mathbb{Z}/l\mathbb{Z}$ for certain $k, l \in \mathbb{Z}_{>0}$.* □

We have seen that a finite reducible subgroup of $\mathrm{GL}(2,\mathbb{C})$ has a cyclic projective group. The following proposition is a generalisation of this observation.

Proposition 3.4.5 *Suppose that the subgroup $H \subset \mathrm{GL}(2,\mathbb{C})$ has a finite projective group. Then H is completely reducible if and only if its projective group is cyclic.*

Proof. Let $H \subset \mathrm{GL}(2,\mathbb{C})$ be a completely reducible group that has a finite projective group. Up to a change of basis one has

$$H \subset \left\langle \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \middle| \lambda, \mu \in \mathbb{C}^* \right\rangle,$$

Consider the homomorphism $\phi : H \to \mathbb{C}^*$ defined as

$$\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \mapsto \lambda\mu^{-1}.$$

The kernel $Z(H)$ of ϕ consists of the multiples of the identity matrix in H. So we have $H/Z(H) = PH$. Hence, by the hypothesis $\phi(H)$ is a finite subgroup of $\mathbb{C}^*$. Therefore, the projective group PH is cyclic.
Conversely, suppose that PH is cyclic. Then there exist a basis and $\zeta \in \mathbb{C}^*$ such that PH is generated by the matrix

$$\begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & \zeta \\ 0 & 1 \end{pmatrix},$$

respectively. However, the latter matrix has infinite order in $\mathrm{PGL}(2,\mathbb{C})$ and is therefore impossible. The assumption $|PH| < \infty$ also implies that ζ is a root of unity. We deduce that H is of the form

$$H = \left\langle \lambda \begin{pmatrix} \zeta & 0 \\ 0 & 1 \end{pmatrix}, \mu \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \middle| \lambda, \mu \in \mathbb{C}^* \right\rangle.$$

In particular we conclude that H is totally reducible. □

In the following two sections we give some examples of finite complex reflection groups in $\mathrm{GL}(2,\mathbb{C})$. These reflection groups are not chosen arbitrarily, but are the most interesting ones that may occur as the monodromy group of the Lamé differential equation, see Section 5.2. For each of the groups we also give the invariant homogeneous polynomials, that generate the ring of invariants of the group. We refer the reader to [Kle84, I,Ch.2] for a detailed description of the invariant theory of the finite binary linear groups.

3.5 Invariants for $G(N,N,2)$

The explicit description of $G(m,p,2)$ immediately gives the ring of invariants $\mathbb{C}[X,Y]^{G(m,p,2)}$. It is generated by the polynomials

$$X_1^m + X_2^m \quad \text{and} \quad (X_1X_2)^{m/p}$$

by Theorem 3.3.2 or Remark 2.5.*ii* of [Coh76]. For the finite dihedral group

$$G(N,N,2) = \left\langle \begin{pmatrix} \zeta_N & 0 \\ 0 & \zeta_N^{-1} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle,$$

with $N \geq 2$ and $\zeta_N = e^{2\pi \mathrm{i}/N}$, this implies the following theorem.

Proposition 3.5.1 *The ring of invariants of $D_N \subset \mathrm{GL}(2,\mathbb{C})$ is generated by homogeneous polynomials of degree 2 and N. In particular they are XY and $X^N + Y^N$ for $D_N = G(N,N,2)$.*

Proof. The proposition follows immediately from the remarks made above and Theorem 3.4.2. □

$$m_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Proposition 3.5.2 *Let D_N be a finite subgroup of $\mathrm{GL}(2,\mathbb{C})$. Then there exists a semi-invariant polynomial of D_N of degree N with multiplication factor -1. For $D_N = G(N,N,2)$ such a semi-invariant is $X^N - Y^N$.*

Proof. It is sufficient to prove the corollary for $D_N = G(N,N,2)$. The element

$$m_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

of $G(N,N,2)$ satisfies $m_2(X^N - Y^N) = -(X^N - Y^N)$. The diagonal matrices contained in $G(N,N,2)$ act as invariants on $X^N - Y^N$. □

3.6 Invariants for G_{12}, G_{13} and G_{22}

Consider the reflection group G of order 48 generated by

$$g_1 := \begin{pmatrix} 0 & \zeta_8 \\ \zeta_8^{-1} & 0 \end{pmatrix}, \quad g_2 := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{and} \quad g_3 := \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & \mathrm{i} \\ -\mathrm{i} & -1 \end{pmatrix}.$$

The occurring diagonal matrices in G are $\pm I_2$. The factor group $G/\langle -I_2\rangle$ is isomorphic to S_4. Table 3.4 implies that G is conjugated to G_{12}, as given by Shephard and Todd. In fact G and G_{12} are identical. The determinants of g_1, g_2 and g_3 are -1. Each of them has eigenvalues -1 and 1 has order are 2. The invariants in the variables X and Y that generate the ring of invariants of G_{12} are

$$J_6(X,Y) := XY(X^4 - Y^4)$$

and

$$J_8(X,Y) := X^8 + 14X^4Y^4 + Y^8,$$

see [Beu98]. They are square-free polynomials on $\mathbb{P}^1$. The polynomials J_6 and J_8 are also obtained by Felix Klein as invariants of the octahedral group in [Kle84, §I.2.10].

The group G_{13} is $\{G_{12}, \mathrm{i}G_{12}\}$. Simultaneous multiplication of X and Y by a 4-th root of unity does not alter the invariant J_8 of G_{12}. Therefore, the invariant J_8 remains an invariant for G_{13}. The monomial XY, however, obtains a factor -1. The invariant J_6 of degree 6 of G_{12} becomes a semi-invariant of order 2 for $G_{13} \setminus G_{12}$. In any case J_6^2 is an invariant for G_{13}. The group of invariants of G_{13} is generated by homogeneous polynomials that have degree 8 and 12, see Table 3.4. We have obtained the following proposition.

Proposition 3.6.1 *Let $G \subset \mathrm{GL}(2, \mathbb{C})$ be G_{12} or G_{13}. Then*

(i) the ring of invariants of G is generated by the square-free polynomials J_6 and J_8 on $\mathbb{P}^1$ for $G = G_{12}$.

(ii) The ring of invariants of G is generated by J_8 and J_6^2 in the case $G = G_{13}$. All matrices in $G_{13} \setminus G_{12}$ act on J_6 by multiplication with -1.

□

Notice that the inhomogeneous forms of J_6 and J_8, in which (X, Y) is replaced by $(t, 1)$ appear in j_O of Example 3.1.5. It can be shown that the group O mentioned there is generated by $\mathrm{i}g_1$, $\mathrm{i}g_2$ and $\mathrm{i}g_3$. The polynomial J_6 turns into a semi-invariant for O with multiplication factor -1. However the homogenised function

$$j_O(X, Y) := \frac{(X^8 + 14X^4Y^4 + Y^8)^3}{108X^4Y^4(X^4 - Y^4)^4}.$$

of $j_O(t)$ is unchanged by the action of O. Then so does the action of PO on $j_O(t)$, as we have stated before.

The example of a group G_{22} we consider here comes from [Beu98]. Let G be the group that is generated by

$$g_1 := \begin{pmatrix} \mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix}, \quad g_2 := \frac{1}{\sqrt{5}} \begin{pmatrix} \zeta_5 - \zeta_5^4 & \zeta_5^2 - \zeta_5^3 \\ \zeta_5^2 - \zeta_5^3 & \zeta_5^4 - \zeta_5 \end{pmatrix}$$

$$\text{and} \quad g_3 := \frac{1}{\sqrt{5}} \begin{pmatrix} \zeta_5^3 - \zeta_5 & 1 - \zeta_5 \\ \zeta_5^4 - 1 & \zeta_5^2 - \zeta_5^4 \end{pmatrix}.$$

Here ζ_5 denotes a primitive 5-th root of unity. The successive determinants of g_1, g_2 and g_3 are -1, 1 and 1. The eigenvalues of g_2 are i and $-$i, The ones for g_3 are the two primitive 3-rd roots of unity ζ_3 and ζ_3^2. Therefore, the matrices g_1, g_2 and g_3 are not reflections. They do however generate a reflection group G_{22} of order 240. The generating invariants of G are

$$XY(X^{10} - 11X^5Y^5 - Y^{10})$$

and

$$X^{20} + 288X^{15}Y^5 + 494X^{10}Y^{10} - 288X^5Y^{15} + Y^{20}.$$

This is in accordance with Table 3.5. Notice that both invariants have no multiple zeros in $\mathbb{P}^1$. For later convenience we state the following proposition.

Proposition 3.6.2 *The ring of invariants of G_{22} is generated by two square-free homogeneous polynomials whose degrees are* 12 *and* 20*, respectively.* □

Chapter 4

The Lamé equation

In this chapter we introduce the Lamé (differential) equation $L_n(y) = 0$ and consider its monodromy group in general. It turns out that a finite monodromy group of the Lamé equation does not act reducibly on the solution space of the equation. The Lamé equations with reducible monodromy groups will have an integer index n. That is why we consider Lamé equations with $n \in \mathbb{Z}$ separately.

4.1 The Lamé equation

The Lamé equation $L_n(y) = 0$ is a Fuchsian equation with four singular points. It involves a rational number n and the *Lamé polynomial*

$$\begin{aligned} p(z) &= 4z^3 - g_2 z - g_3 \\ &= 4\prod_{i=1}^{3}(z - z_i), \end{aligned}$$

with $g_2, g_3 \in \mathbb{C}$. The Lamé polynomial $p(z)$ is assumed to have three distinct roots z_1, z_2 and z_3 in $\mathbb{C}$.

Definition 4.1.1 The *Lamé differential operator* L_n (of index n) is defined as

$$L_n := p(z)\frac{d^2}{dz^2} + \frac{1}{2}p'(z)\frac{d}{dz} - (n(n+1)z + B).$$

The constant $B \in \mathbb{C}$ is called the *accessory parameter* of L_n. The equation $L_n(y) = 0$ is the *Lamé equation* (of index n).

Notice that $L_{-n-1} = L_n$ holds. Therefore we restrict ourselves to the case $n + 1/2 \geq 0$ to study the Lamé equation.

Assumption 4.1.2 *We assume* $n \geq -1/2$.

The Lamé equation has exactly four singular points; the only singular points of $L_n(y) = 0$ are z_1, z_2, z_3 and ∞. The singular points of the Lamé equation have the following scheme of local exponents.

z_1	z_2	z_3	∞
0	0	0	$-n/2$
1/2	1/2	1/2	$(n+1)/2$

Conversely, every Fuchsian equation with such a scheme of singular points and their local exponents is a Lamé equation.

In the next section and in a large part of the rest of this dissertation we shall take a look at the monodromy group of the Lamé equation. This is why we introduce the following notation.

Notation 4.1.3 For the remaining part of this dissertation the monodromy group of the Lamé equation $L_n(y) = 0$ is denoted by M. The natural image of M in $\mathrm{PGL}(2,\mathbb{C})$ is denoted by PM.

4.2 About the monodromy group

The local exponents at each of the three finite singular points are 0 and 1/2. Therefore there are three elements γ_1, γ_2 and γ_3 in M, each of order 2, and a $\gamma_\infty \in M$ such that the product $\gamma_1\gamma_2\gamma_3\gamma_\infty$ yields the unity matrix I_2. More precisely, γ_1, γ_2 and γ_3 are conjugate to

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

since their eigenvalues are $\mathrm{e}^{0(2\pi\mathrm{i})} = 1$ and $\mathrm{e}^{1/2(2\pi\mathrm{i})} = -1$. They correspond to positively oriented single closed paths around the finite singular points.

Proposition 4.2.1 *The monodromy group M of the Lamé equation is generated by reflections of order* 2*. All elements of M have determinant* ± 1*. In particular, the scalar elements of M form a subgroup of* $\langle \mathrm{i}I_2 \rangle$.

Proof. The monodromy group M is generated by the matrices γ_1, γ_2 and γ_3. These matrices are complex reflections of order 2. Therefore, an element of M has determinant -1 or 1. It follows that the subgroup of M consisting of scalar matrices is contained in $\langle \mathrm{i}I_2 \rangle$. □

The matrix γ_∞ corresponds to a single closed path around ∞. It is conjugate to the matrix

$$\begin{pmatrix} e^{-n\pi i} & 0 \\ 0 & e^{(n+1)(\pi i)} \end{pmatrix}$$

unless $n + 1/2$ is an integer. In that case the two local exponents at ∞ differ by an integer. Hence there are two independent solutions of the Lamé equation at infinity of the form $t^{\rho_1} g_1$ and $t^{\rho_2} g_2 + c \log(t) t^{\rho_1} g_1$, for the local exponents ρ_1 and ρ_2, convergent power series $g_1(t)$ and $g_2(t)$ and constant c. There is only one eigenvalue of γ_∞; it is $e^{-n\pi i}$. The Jordan form of γ_∞ is

$$\begin{pmatrix} e^{-n\pi i} & \beta \\ 0 & e^{-n\pi i} \end{pmatrix}$$

for a certain $\beta \in \mathbb{C}$. From the restriction $n + 1/2 \in \mathbb{Z}$ we deduce $e^{-n\pi i} \in \{i, -i\}$.

The projective images of γ_1, γ_2, γ_3 and γ_∞ in $\mathrm{PGL}(2, \mathbb{C})$ will successively be written as σ_1, σ_2, σ_3 and σ_∞. The matrices γ_1, γ_2 and γ_3 each have distinct eigenvalues. Therefore, the matrices σ_1, σ_2 and σ_3 thus should also be of order 2 and are still conjugate to

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

but now in $\mathrm{PGL}(2, \mathbb{C})$. They generate PM, since γ_1, γ_2 and γ_3 do so for M. The matrix σ_∞ is a conjugate in $\mathrm{PGL}(2, \mathbb{C})$ of

$$\begin{pmatrix} 1 & 0 \\ 0 & e^{(n+1/2)(2\pi i)} \end{pmatrix}$$

when $n - 1/2$ is not an integer. Otherwise, there exists a $\beta \in \mathbb{C}$ such that σ_∞ is conjugate to

$$\begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix}$$

in $\mathrm{PGL}(2, \mathbb{C})$.

4.3 Solutions at infinity

The local exponents of the Lamé equation at a finite point α do not differ by an integer. Therefore, there exists a basis (f_1, f_2) of the solution space that for $t = z - \alpha$ looks like $f_1 = t^{\rho_1} g_1(t)$ and $f_2 = t^{\rho_2} g_2(t)$ for certain power series $g_1(t)$ and $g_2(t)$ in t. The exponents ρ_1 and ρ_2 are the local exponents at α. However, in the remaining part of this dissertation the local solutions at $z = \infty$ will be of most importance. That is why we introduce them specifically.

For any Lamé equation with $n - 1/2 \notin \mathbb{Z}$ there are two independent solutions at infinity, one belonging to the exponent $-n/2$, the other one to $(n+1)/2$. They can locally be given as a Laurent series in the local parameter $t = 1/z$. The solutions then are $t^{-n/2}s_1(t)$ and $t^{(n+1)/2}s_2(t)$ for certain holomorphic power series $s_1(t)$ and $s_2(t)$ in t with non-trivial constant term. These series are determined up to multiplication by a non-zero constant. The constant terms of $s_1(t)$ and $s_2(t)$ may therefore be assumed to be 1.

Definition 4.3.1 For a Lamé equation with $n \notin \{\frac{1}{2}\} + \mathbb{Z}$ the two independent solutions y_1 and y_2 at $z = \infty$ are defined as

$$\begin{aligned} y_1(t) &= t^{-n/2}s_1(t) \\ y_2(t) &= t^{(n+1)/2}s_2(t). \end{aligned}$$

Here, $s_1(t)$ and $s_2(t)$ are certain holomorphic power series in t with $s_1(0) = s_2(0) = 1$.

We would like to mention that in general the solution space at $z = \infty$ for a Lamé equation with integral $n + 1/2$ is generated by

$$\begin{aligned} f_1(t) &= t^{(n+1)/2}g_1(t) \\ f_2(t) &= t^{-n/2}g_2(t) + c\log(t)t^{(n+1)/2}g_1(t) \end{aligned}$$

for certain power series $g_1(t)$ and $g_2(t)$ with non-zero constant term and constant c. If M is finite, then all solutions are algebraic. In this situation the log-term of f_2 vanishes, since γ_∞ must be of finite order. We then have two independent solutions of the differential equation that at ∞ resemble y_1 and y_2.

4.4 Reducibility and Lamé solutions

In this section we show that a finite monodromy group of the Lamé equation is not reducible. This diminishes the number of possible monodromy groups of the Lamé equations having only algebraic solutions. For the definition of reducibility and other useful statements we refer to Section 2.4.

Theorem 4.4.1 *The monodromy group of a Lamé equation is not completely reducible.*

Proof. Suppose that M is completely reducible. The Lamé equation with respect to M then has two linearly independent solutions of the form

$$f_1(z) = (z - z_1)^{\epsilon_1}(z - z_2)^{\epsilon_2}(z - z_3)^{\epsilon_3}p_1(z)$$

and

$$f_2(z) = (z - z_1)^{(1/2-\epsilon_1)}(z - z_2)^{(1/2-\epsilon_2)}(z - z_3)^{(1/2-\epsilon_3)}p_2(z).$$

Here one has $\epsilon_i \in \{0, 1/2\}$ for $i = 1, 2, 3$, in accordance with the local exponents at z_1, z_2 and z_3, respectively. The functions $p_1(z)$ and $p_2(z)$ are fixed under M. So they are contained in $\mathbb{C}(z)$. Moreover p_1 and p_2 have no finite poles. Therefore p_1 and p_2 are polynomials. We denote their degrees in z by m_1 and m_2, respectively. Then we have

$$\begin{aligned} -\epsilon_1 - \epsilon_2 - \epsilon_3 - m_1 &= -n/2 \\ \epsilon_1 + \epsilon_2 + \epsilon_3 - 3/2 - m_2 &= (n+1)/2, \end{aligned}$$

since the left hand sides of these equalities each belong to one of the local exponents at ∞. Addition of the equations however yields the contradiction $-3/2 > 1/2$. We conclude that M is not completely reducible. □

Corollary 4.4.2 *A finite monodromy group of the Lamé equation is neither abelian nor reducible.*

Proof. This follows directly from the Theorems 2.4.6 and 4.4.1. □

Any reducible monodromy group M of the Lamé operator is necessarily infinite. In this case there is a one dimensional subspace of the space of solutions that is invariant under M. This subspace is generated by a solution of the shape

$$f(z) := (z - z_1)^{\epsilon_1}(z - z_2)^{\epsilon_2}(z - z_3)^{\epsilon_3}q(z) \tag{4.1}$$

for certain $\epsilon_i \in \{0, 1/2\}$ and a polynomial q of degree m, say.

Proposition 4.4.3 *Let M be a reducible monodromy group of a Lamé operator L_n is infinite. Then M is infinite and n is an integer. Moreover, the proper M-invariant subspace is generated by $f(z)$.* □

Definition 4.4.4 The function $f(z)$ as in (4.1) is known as *the Lamé solution* or *Lamé function* of degree n (of L_n) of the $2(\epsilon_1 + \epsilon_2 + \epsilon_3)$-th kind, see [Poo36] or [WW50].

Proposition 4.4.5 *Let $f(z) = q(z)\prod_{i=1}^{3}(z-z_i)^{\epsilon_i}$ be a Lamé solution of degree n as in* (4.1)*. Let m be the degree of q in z. Then one has*

$$\epsilon_1 + \epsilon_2 + \epsilon_3 + m = n/2. \tag{4.2}$$

In particular, n is an integer.

Proof. The local exponent of f at infinity is either $-n/2$ or $(n+1)/2$. One of these should be negative and belong to f. Therefore, we have either $n \geq 0$ or $n \leq -1$. By the assumption $n \geq -1/2$ this gives $n \geq 0$. The function f then satisfies Equation (4.2). Hence, the index $n = 2(\epsilon_1 + \epsilon_2 + \epsilon_3 + m)$ is a non-negative integer. □

Notice that exactly 0 or 2 of the ϵ_i's of the Lamé function f are 1/2 if n is even. In case of n odd, the number of the ϵ_i's that are non-zero is 1 or 3.

The Lamé function f is an element of the $\mathbb{C}$-vector space of functions

$$W := \left\{ Q(z)\prod_{i=1}^{3}(z-z_i)^{\epsilon_i} \ : \ Q(z) \in \mathbb{C}[z],\ \deg(Q) \leq m \right\}.$$

We are going to prove that the operator

$$O_n := p(z)\left(\frac{d}{dz}\right)^2 + \frac{1}{2}p'(z)\left(\frac{d}{dz}\right) - n(n+1)z.$$

acts on W. Notice that O_n is just L_n without the term involving B.

Lemma 4.4.6 *Let the notation be as above. Then O_n acts on W. The characteristic polynomial $Q_{(\epsilon_1,\epsilon_2,\epsilon_3)}(T)$ of O_n on W has degree $m+1$ in T. It is contained in $\mathbb{Q}[z_1,z_2,z_3][T]$. Moreover, one has $Q_{(\epsilon_1,\epsilon_2,\epsilon_3)}(T) \in \mathbb{Q}[g_2,g_3][T]$ in the case $\epsilon_1 = \epsilon_2 = \epsilon_3$.*

Proof. The dimension of W is $m+1$; a basis is given by $\Phi = (\phi_0, \phi_1, \ldots, \phi_m)$ with

$$\phi_i := z^i \prod_{j=1}^{3}(z-z_j)^{\epsilon_j}$$

for every appropriate i. A long and tedious calculation shows that for $i = 0, 1, \ldots, m$ one has

$$\begin{aligned} O_n(\phi_i) \;=\; & -i(i-1)g_3\phi_{i-2} \\ & +[i(-i+1/2)g_2 + 8i(\epsilon_1 z_2 z_3 + \epsilon_2 z_1 z_3 + \epsilon_3 z_1 z_2)]\phi_{i-1} \\ & -[(8i+2)(\epsilon_1(z_2+z_3) + \epsilon_2(z_1+z_3) + \epsilon_3(z_1+z_2)) \\ & \quad +8(\epsilon_1\epsilon_2 z_3 + \epsilon_1\epsilon_3 z_2 + \epsilon_2\epsilon_3 z_1)]\phi_i \\ & +[-n(n+1) + 2i(2i+1) + (8i+4)(\epsilon_1+\epsilon_2+\epsilon_3) \\ & \quad +8(\epsilon_1\epsilon_2 + \epsilon_1\epsilon_3 + \epsilon_2\epsilon_3)]\phi_{i+1}. \end{aligned}$$

The last term vanishes for $i = m$. This is also due to the relation between m and the local exponent $-n/2$ at infinity. It follows that O_n maps W into itself. The specific description of the action of O_n on each ϕ_i shows that the characteristic polynomial $Q_{(\epsilon_1,\epsilon_2,\epsilon_3)}(T)$ of O_n on W in T has coefficients in $\mathbb{Q}[z_1, z_2, z_3]$. It is of degree $m+1$. In the case $\epsilon_1 = \epsilon_2 = \epsilon_3$, each coefficient in the expansion of $O_n(\phi_i)$ is symmetric in z_1, z_2 and z_3. Therefore, each of these coefficients is contained in $\mathbb{Q}[g_2, g_3]$. Then so are the coefficients of $Q_{(\epsilon_1,\epsilon_2,\epsilon_3)}(T)$. □

Along the way we have obtained most ingredients for the following theorem.

Theorem 4.4.7 *Let M be the monodromy group of L_n. Then the following statements are equivalent.*

(i) M is reducible.

(ii) $L_n(y) = 0$ has a Lamé solution $f(z) = q(z)\prod_{i=1}^{3}(z - z_i)^{\epsilon_i}$ of degree n for a certain square-free polynomial $q(z)$ of degree m in z and $\epsilon_1, \epsilon_2, \epsilon_3 \in \{0, 1/2\}$.

(iii) There exist $\epsilon_1, \epsilon_2, \epsilon_3 \in \{0, 1/2\}$ such that B is a root of $Q_{(\epsilon_1,\epsilon_2,\epsilon_3)}(T)$ of degree $m + 1$ in T.

Moreover, in each case one has $m = n/2 - \epsilon_1 - \epsilon_2 - \epsilon_3$. In the equivalence between (ii) and (iii) the tuples $(\epsilon_1, \epsilon_2, \epsilon_2)$ coincide.

Proof. First we remark that a Lamé solution $f(z)$ as in item (*ii*) satisfies (4.2) because of Proposition 4.4.5.
The implication from (*i*) to (*ii*) is provided for by Proposition 4.4.3, except for one detail. We still have to prove that $q(z)$ has no double roots. If $q(z)$ were to have a multiple root at $z = \alpha$, then α would have a local exponent at least 2. This is never the case.
Suppose that $f(z)$ is a Lamé solution as in (*ii*). Then for an arbitrary $\gamma \in M$ one has ${}^{\gamma}g(z) = \pm g(z)$. Hence $f(z)$ generates a proper invariant subspace of M. In other words, M is reducible by definition.
We have shown that (*i*) and (*ii*) are equivalent. It remains to prove their equivalence with statement (*iii*). Let ϵ_1, ϵ_2 and ϵ_3 be elements of $\{0, 1/2\}$ such that $m = n/2 - \epsilon_1 - \epsilon_2 - \epsilon_3$ is a non-negative integer. We have

$$\begin{aligned} Q_{(\epsilon_1,\epsilon_2,\epsilon_3)}(B) = 0 \quad &\iff \quad B \text{ is an eigenvalue of } O_n \text{ on } W \\ &\iff \quad \exists g(z) \in W^* \text{ with } O_n(g) = B(g) \\ &\iff \quad \exists g(z) \in W^* \text{ with } L_n(g) = 0. \end{aligned}$$

We see that B is an eigenvalue of $Q_{(\epsilon_1,\epsilon_2,\epsilon_3)}(T)$ exactly if there is a non-trivial solution Lamé solution $q(z)\prod_{i=1}^{3}(z - z_i)^{\epsilon_i} \in W^*$ of $L_n(y) = 0$. The polynomial $q(z)$ has degree $\deg_z(q) \leq m$. Finally, if we apply Proposition 4.4.5 on $\deg_z(q)$ instead of m, then equality follows. □

Corollary 4.4.8 *Suppose that L_n has monodromy group M. Let ϵ be $1/2$. Define $Q_n(T)$ as the polynomial*

- $Q_{(0,0,0)}(T)$ *for* $n = 0$,
- $Q_{(\epsilon,0,0)}Q_{(0,\epsilon,0)}Q_{(0,0,\epsilon)}(T)$ *for* $n = 1$,
- $Q_{(0,0,0)}Q_{(\epsilon,\epsilon,0)}Q_{(\epsilon,0,\epsilon)}Q_{(0,\epsilon,\epsilon)}(T)$ *for* $n \geq 2$ *even, and*
- $Q_{(\epsilon,0,0)}Q_{(0,\epsilon,0)}Q_{(0,0,\epsilon)}Q_{(\epsilon,\epsilon,\epsilon)}(T)$ *for* $n \geq 3$ *odd.*

Then one has

$$M \text{ is reducible} \iff Q_n(B) = 0.$$

In each case $Q_n(T) \in \mathbb{Q}[g_2, g_3][T]$ is a polynomial of degree $2n + 1$ in T.

Proof. In this proof we use Lemma 4.4.6 and Theorem 4.4.7. We first prove the corollary in the case where n is odd and at least 3. The number $m := n/2 - \epsilon_1 - \epsilon_2 - \epsilon_3$ is an integer exactly if an odd number of the ϵ_i's is $1/2$. It must be non-negative. If M is reducible, then B is a root of one of the polynomials $Q_{(\epsilon,0,0)}$, $Q_{(0,\epsilon,0)}$, $Q_{(0,0,\epsilon)}$ and $Q_{(\epsilon,\epsilon,\epsilon)}$ and thus of their product $Q_n(T)$. The converse is also true. A priori the coefficients of $Q_n(T)$ are in $\mathbb{Q}[z_1, z_2, z_3]$. Any automorphism of $\mathbb{P}^1$ that permutes z_1, z_2 and z_3 induces a permutation of $Q_{(\epsilon,0,0)}$, $Q_{(0,\epsilon,0)}$ and $Q_{(0,0,\epsilon)}$. The polynomial $Q_n(T)$ is therefore fixed. It follows that its coefficients are in $\mathbb{Q}[g_2, g_3]$. We derive from the degrees of the components of Q_n that $\deg_T(Q_n)$ is $3(n/2 - 1/2 + 1) + (n/2 - 3/2 + 1) = 2n + 1$.
The case for $n = 1$ is analogous to the the previous situation. The only difference is that $Q_{(\epsilon,\epsilon,\epsilon)}$ does not occur, since m is not allowed to be negative. The degree of $Q_n(T)$ in T is 3. This also is $2n + 1$.
The case of n even is similar to the odd case. The general degree of $Q_n(T)$ in T then is $(n/2 + 1) + 3(n/2 - 1 + 1) = 2n + 1$. For $n = 0$ it is 1. We leave further details to the reader. □

Example 4.4.9 Consider $L_0(y) = 0$. Then M is reducible if and only if the space of solutions of L_n contains all constant functions. But then we must have $B = 0$. This is in accordance with the fact that there is only one eigenvalue for O_n ($\deg_T(Q_0) = 1$). This eigenvalue is 0 and belongs to the space of constant functions W. To conclude, we have $Q_0(T) = -T$.

Example 4.4.10 Let n be 1. First assume $(\epsilon_1, \epsilon_2, \epsilon_3) = (1/2, 0, 0)$. Then m is 0. A direct calculation for this specific situation (or using the proof of Lemma 4.4.6) leads to $O_n((z - z_1)^{1/2}) = -(z_2 + z_3)$. It follows from $z_1 + z_2 + z_3 = 0$ that

$Q_{(1/2,0,0)}(T)$ is $z_1 - T$. The analogous results for the other possibilities of the ϵ_i's yield

$$\begin{aligned} Q_1(T) &= (-T+z_1)(-T+z_2)(-T+z_3) \\ &= -\frac{1}{4}p(T) \end{aligned}$$

We conclude that L_1 has a reducible monodromy group if and only if B satisfies $p(B) = 0$.

The solution space of the Lamé equation is generated by two functions. If one of these is a Lamé solution, then the other must be non-algebraic. The following proposition describes such a solution.

Proposition 4.4.11 *Let $f(z)$ be a Lamé solution of $L_n(y) = 0$. Then*

$$y = f \cdot \int_z^\infty \frac{1}{f^2(x)\sqrt{p(x)}}\, dx$$

is another solution of the Lamé equation.

Proof. Let f be a Lamé solution of L_n. The substitution of $y := fu$ in the Lamé equation yields

$$p\,(2f'u' + fu'') + \frac{1}{2}p'fu' = 0.$$

Hence, $u'f^2\sqrt{p}$ must be a constant, c. Therefore,

$$u = \int_z^\infty \frac{c}{f^2(x)\sqrt{p(x)}}\, dx.$$

The proposition immediately follows if we substitute this identity together with $c = 1$ into $y = fu$. □

For a more classical treatment of the Lamé solutions we refer to [Poo36, §36] and to [WW50].

4.5 The monodromy groups for integral n

In the previous section we have seen that a reducible monodromy group M of the Lamé equation is necessarily infinite and not completely reducible. Moreover, the index n of the Lamé equation is an integer. In this section we shall give a complete description of the monodromy groups for the Lamé equations with integral index n.

Theorem 4.5.1 *Suppose that n is an integer. Then the subgroup $\langle \gamma_1\gamma_2, \gamma_2\gamma_3 \rangle$ of M is abelian and consists of all matrices in M having determinant* 1. *In particular, it has index* 2 *in M.*

Proof. Let n be an integer. The matrix γ_∞ is of order 2 and has eigenvalues -1 and 1. The monodromy group M is generated by the 4 involutions γ_1, γ_2, γ_3 and γ_∞ in such a way that their product $\gamma_1\gamma_2\gamma_3\gamma_\infty$ yields the identity element $1 \in M$. Its subgroup $H := \langle \gamma_1\gamma_2, \gamma_2\gamma_3 \rangle$ consists of matrices of determinant 1. Moreover, it is abelian. This can be seen from

$$\gamma_1\gamma_2\gamma_3 \cdot \gamma_1\gamma_2\gamma_3 = \gamma_\infty^2 = 1$$

which yields

$$\gamma_2\gamma_3 \cdot \gamma_1\gamma_2 = \gamma_1\gamma_3 = \gamma_1\gamma_2 \cdot \gamma_2\gamma_3.$$

The group H contains $\gamma_2\gamma_1 = (\gamma_1\gamma_2)^{-1}$ and $\gamma_1\gamma_3$. The matrix $\gamma_3\gamma_1 = (\gamma_1\gamma_3)^{-1}$ is also an element of H. Therefore, every monodromy matrix of determinant 1 is contained in H, as such a matrix is an even product of the matrices γ_1, γ_2 and γ_3. It follows that H is the kernel of the determinant map from M to $\langle -1 \rangle$. □

Corollary 4.5.2 *Suppose that n is an integer. Then $\langle \gamma_1\gamma_2, \gamma_2\gamma_3 \rangle$ acts reducibly on $\mathbb{C}^2$.*

Proof. The corollary follows directly from Theorem 4.5.1 and the fact that an abelian group is reducible. □

We can distinguish two cases for

$$H := \langle \gamma_1\gamma_2, \gamma_2\gamma_3 \rangle.$$

Either H acts completely reducibly or it does not. The following proposition deals with the second case.

Proposition 4.5.3 *Let n be an integer and suppose that H does not act completely reducibly. Then M is reducible. In particular, M is infinite.*

Proof. The group $H \subset M$ is reducible. Suppose that it is not completely reducible. There exists a certain basis (f_1, f_2) on which H satisfies

$$H \subset \left\{ \pm \begin{pmatrix} 1 & \lambda \\ 0 & 1 \end{pmatrix} : \lambda \in \mathbb{C}^* \right\}.$$

In particular H is an infinite subgroup of M. The set of functions

$$\langle f_1 \rangle := \{ c f_1 : c \in \mathbb{C} \}$$

is the only H-invariant $\mathbb{C}$-linear subspace of the solutions space of $L_n(y) = 0$. We derive from $\gamma_\infty H = H\gamma_\infty$ that $\gamma_\infty f_1$ generates an invariant subspace of H as well. One then has $\gamma_\infty f_1 = \mu f_1$ for a certain $\mu \in \mathbb{C}^*$. Therefore, the subspace $\langle f_1 \rangle$ is M-invariant, since M equals $\langle H, \gamma_\infty \rangle$. □

We move to the situation in which H is completely reducible. It follows that there are solutions g_1 and g_2 that each form a 1-dimensional invariant subspace of H. For now we consider M with respect to the ordered basis (g_1, g_2). The matrices of H are then diagonal. This yields

$$H \subset \left\{ \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} : \lambda \in \mathbb{C}^* \right\}.$$

In particular, for finite M one has

$$H = \langle \begin{pmatrix} \zeta_N & 0 \\ 0 & \zeta_N^{-1} \end{pmatrix} \rangle \tag{4.3}$$

for a certain primitive N-th root of unity ζ_N. It follows that H is cyclic. We have proved the following lemma.

Lemma 4.5.4 *Suppose that n is an integer and that M is finite. Then H is a cyclic subgroup of M.* □

The functions $\gamma_\infty g_1$ and $\gamma_\infty g_2$ are stable under the action of H, since we have $M = \langle H, \gamma_\infty \rangle$. This implies either $\gamma_\infty g_1 = \lambda g_1$ or $\gamma_\infty g_1 = \lambda g_2$ for a certain $\lambda \in \mathbb{C}^*$. The case $\gamma_\infty g_1 = \lambda g_1$ yields the complete reducibility of the Lamé equation. This however is impossible by Theorem 4.4.1. Therefore one has $\gamma_\infty g_1 = \lambda g_2$, which yields $\gamma_\infty^2 g_1 = \lambda \gamma_\infty g_2$ and $\gamma_\infty g_2 = (1/\lambda) g_1$. If we define f_1 and f_2 to be g_1 and λg_2, respectively, then the base change of (g_1, g_2) in to (f_1, f_2) does not alter H. We do however get $\gamma_\infty f_1 = f_2$ and $\gamma_\infty f_2 = f_1$. With respect to the ordered basis (f_1, f_2) we have

$$M \subset \left\langle \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : \lambda \in \mathbb{C}^* \right\rangle.$$

Moreover, the matrix γ_∞ is then the anti-diagonal matrix with non-zero entries 1.

Theorem 4.5.5 *Let n be an integer. Suppose that H acts completely reducibly on $\mathbb{C}^2$. Then M is irreducible and satisfies*

$$M \subset \left\langle \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : \lambda \in \mathbb{C}^* \right\rangle \quad \textit{and} \quad \gamma_\infty = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{4.4}$$

with respect to a certain basis (f_1, f_2). If in particular M is finite, then one has $M = G(N, N, 2)$ for a certain $N \in \mathbb{Z}_{\geq 3}$.

Proof. The shape of M and γ_∞ on a certain basis (f_1, f_2) has been proven just above this theorem. There exist diagonal matrices in M other than $\pm I_2$, since otherwise M would be abelian. Henceforth, M is irreducible. The assertions $M = G(N, N, 2)$ and $N \geq 3$ are an immediate consequence of Equality (4.3) and $M = \langle H, \gamma_\infty \rangle$. □

Corollary 4.5.6 *Let M be as in* (4.4)*. Then γ_1, γ_2, γ_3 and γ_∞ are anti-diagonal. In particular this holds for $M = G(N, N, 2)$.*

Proof. If M is as in (4.4), then a matrix having determinant -1 is a product of an odd number of the anti-diagonal matrix γ_∞ and some diagonal matrices. The matrix then is anti-diagonal. In particular this is the case for γ_1, γ_2 and γ_3. □

Remark 4.5.7 Notice that Proposition 4.5.3 and Theorem 4.5.5 imply a division of M and H into the following two cases.

1. The monodromy group M is irreducible and H is totally reducible. Moreover, if M is finite, then it is dihedral.

2. Both M and H are reducible, but not completely reducible.

We have described all monodromy groups of the Lamé equation when n is an integer. The monodromy group in particular is dihedral if it is of finite order. The chosen basis (f_1, f_2) puts M in a simple and explicit form. For future use we describe the functions f_1 and f_2.

4.6 Solutions at infinity for integral n

In this section we give a specific basis of solutions for the monodromy group of the Lamé equation with $n \in \mathbb{Z}_{\geq 0}$. This will be done with respect to the standard solutions y_1 and y_2 at $z = \infty$ as in Definition 4.3.1. We consider two cases, corresponding to M being irreducible or not.

We begin by considering the first case in which M is irreducible. Due to Remark 4.5.7 we may suppose the basis of solutions (f_1, f_2) and M to be as in (4.4). So we have

$$M \subset \left\langle \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} : \lambda \in \mathbb{C}^* \right\rangle \quad \text{and} \quad \gamma_\infty = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

on the ordered basis (f_1, f_2). The product $f_1 f_2$ is a solution of the so-called second symmetric power of L_n, see [WW50, §23.7] or [Poo36, §39].

Definition 4.6.1 The *second symmetric power* $L_n^{(2)}(y) = 0$ *of* $L_n(y) = 0$ is the third order linear differential equation

$$py''' + \frac{3}{2}p'y'' + \frac{1}{2}p''y' - 4\left(n(n+1)z + B\right)y' - 2n(n+1)y = 0, \tag{4.5}$$

where $p(z)$, n and B are as in the given Lamé operator L_n.

The equation $L_n^{(2)}(y) = 0$ has the same finite singular points z_1, z_2 and z_3 as the Lamé equation. They each have local exponents 0, 1/2 and 1. All other complex points are regular. Also ∞ is singular; its local exponents are $n+1$, $1/2$ and $-n$. If $(\tilde{f}_1, \tilde{f}_2)$ is a basis of the Lamé equation, then $(\tilde{f}_1^2, \tilde{f}_1\tilde{f}_2, \tilde{f}_2^2)$ is a basis of $L_n^{(2)}(y) = 0$. The specific product $f_1 f_2$ turns out to be a polynomial solution.

Theorem 4.6.2 *Let n be an integer. Suppose that M acts irreducibly and is given on the basis (f_1, f_2) as in Theorem* 4.5.5. *Then $f_1 f_2$ is a square-free polynomial of degree n in z.*

Proof. Let n be an integer and suppose that M and f_1, f_2 are as in Theorem 4.5.5. Any matrix of M leaves $f_1 f_2$ invariant. It follows that $f_1 f_2$ is a rational function. Since the local exponents at the finite points are all non-negative, we conclude that $f_1 f_2$ is a polynomial in z. The local exponents of L_n at infinity imply that there is only one polynomial solution of $L_n^{(2)}(y) = 0$, up to multiplication by constants. Its degree in z is n.
It remains to prove that $f_1 f_2$ is square-free in z. Suppose that $f_1 f_2$ contains a double root $z = \alpha$. Then either α is a common root of f_1 and f_2, or it is a double root of one of the f_i's. If it is a common root, then the local exponents at α are both greater than 0. But then there is no solution of the Lamé equation that corresponds to the local exponent 0 at α, since f_1 and f_2 form a basis of the solution space. This leaves us with the possibility of α being a double root. However, if α were regular, then it would have a local exponent that is at least 2. The exponent is at least 1 in case of α being singular. We conclude that our assumption is invalid. The polynomial $f_1 f_2(z)$ is thus square-free in z. □

Proposition 4.6.3 *Suppose that n is an integer and that M and (f_1, f_2) are as in Theorem* 4.5.5. *Let $t = 1/z$ be the local parameter at $z = \infty$. Then we may take*

$$f_1(t) = y_1(t) + \gamma \cdot y_2(t) \tag{4.6}$$
$$f_2(t) = (-1)^n \left(y_1(t) - \gamma \cdot y_2(t)\right) \tag{4.7}$$

for a unique $\gamma \in \mathbb{C}^*$ depending on B, g_2, g_3 and n.

Proof. We choose $t = 1/z$ as local parameter at $z = \infty$. The basis elements f_1 and f_2 are of the form

$$\begin{aligned} f_1(t) &= c_1 t^{-n/2} s_1(t) + d_1 t^{(n+1)/2} s_2(t) \\ f_2(t) &= c_2 t^{-n/2} s_1(t) + d_2 t^{(n+1)/2} s_2(t) \end{aligned}$$

for certain constants $c_1, d_1, c_2, d_2 \in \mathbb{C}$. Notice that $-n/2$ is an integer if and only if the integer n is even. By assumption we have

$$\gamma_\infty = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

On one hand this gives

$$\gamma_\infty(f_1) = (-1)^n \left(c_1 t^{-n/2} s_1(t) - d_1 t^{(n+1)/2} s_2(t) \right)$$

due to the action of γ_i on the $t^{-n/2}$ and $t^{(n+1)/2}$. On the other hand $\gamma_\infty(f_1)$ is the second vector f_2, because of the explicit description of the matrix. The action of γ_∞ on f_1 thus also gives

$$\gamma_\infty(f_1) = f_2.$$

This yields $c_2 = (-1)^n c_1$ and $d_2 = -(-1)^n d_1$. Notice that c_1 and d_1 are both non-trivial, since otherwise f_1 and f_2 would be dependent solutions of the Lamé equation. We deduce

$$f_2(t) = (-1)^n \left(c_1 t^{-n/2} s_1(t) - d_1 t^{(n+1)/2} s_2(t) \right).$$

The basis functions are determined up to simultaneous multiplication by a non-zero constant in $\mathbb{C}$. Hence, we may assume $c_1 = 1$. The first part of the proposition now follows if we take $\gamma = d_1$. The constant γ depends on the parameters B, g_2, g_3 and n, since the solutions f_1 and f_2 of the Lamé equation do so. □

Corollary 4.6.4 *Suppose that M is irreducible. Let M, (f_1, f_2) and $\gamma \in \mathbb{C}^*$ be as in Proposition* 4.6.3 *and its identities* (4.6) *and* (4.7)*. Then*

$$(-1)^n f_1 f_2(t) = y_1^2(t) - \gamma^2 y_2^2(t)$$

is a monic square-free polynomial of degree n in $z = 1/t$ that generates the polynomial solution space of $L_n^{(2)}(y) = 0$. In particular, it is independent of any representation of M and is invariant under monodromy.

Proof. It follows from Proposition 4.6.3 and Theorem 4.6.2 that

$$\begin{aligned} (-1)^n f_1 f_2(t) &= y_1^2(t) - \gamma^2 y_2^2(t) \\ &= t^{-n} s_1^2(t) - \gamma^2 t^{n+1} s_2^2(t) \end{aligned}$$

is a square-free polynomial whose degree in $z = 1/t$ is n. It is monic, since $(-1)^n f_1 f_2(t)$ has lowest order term $t^{-n} s_1^2(0) = t^{-n}$. The function $f_1 f_2$ is a solution of the second symmetric power (4.5). The local exponents for this differential equation are $-n$, $1/2$ and $n+1$ at $z = \infty$. Therefore, the polynomial solution space of $L_n^{(2)} y = 0$ is 1-dimensional and has the generating function $f_1 f_2$. The polynomial $f_1 f_2$ is independent of the representation of the monodromy group. It is fixed by monodromy, since it is a polynomial in z. □

In the case of irreducible M we have given an explicit basis of solutions of $L_n(y) = 0$ in terms of y_1 and y_2. This has led to an explicit description of a polynomial in z such that it is a solution of $L_n^{(2)} y = 0$. We are going to derive similar statements for a reducible monodromy group.

According to Remark 4.5.7 and Proposition 4.5.3 we may assume the reducible monodromy group M satisfies

$$M \subset \left\{ \begin{pmatrix} \pm 1 & \lambda \\ 0 & \pm 1 \end{pmatrix} : \lambda \in \mathbb{C}^* \right\}$$

on a certain ordered basis $(f_1(z), f_2(z))$. The only invariant subspace of the solution space of $L_n(y) = 0$ is 1-dimensional and is generated by a Lamé solution $f_1(z)$. We can expand $f_1(z)$ at $z = \infty$ by using $z = 1/t$ as usual. It follows from Theorem 4.4.7 that the first power of the Puiseux series of f_1 in t then is $t^{-n/2}$.

Proposition 4.6.5 *Let M be reducible. Let M, f_1 and f_2 be as above. We may take $f_1(t)$ as*

$$f_1(t) = y_1(t).$$

There exist $\alpha, \beta \in \mathbb{C}$ such that one has

$$f_2(t) = \alpha y_1(t) + \beta y_2(t).$$

Proof. In general f_1 and f_2 can be written as

$$\begin{aligned} f_1(t) &= c_1 y_1(t) + d_1 y_2(t) \\ f_2(t) &= c_2 y_1(t) + d_2 y_2(t) \end{aligned}$$

for certain $c_1, d_1, c_2, d_2 \in \mathbb{C}$. The function

$$f_1^2(t) = c_1^2 y_1^2(t) + c_1 d_1 y_1 y_2(t) + d_1^2 y_2^2(t)$$

is fixed by M. It is thus a non-trivial rational function in $\mathbb{C}(t)$. The local exponents of L_n at all finite points are non-negative. Therefore f_1^2 is a polynomial in $z = 1/t$. Hence f_1^2 has a Laurent series expansion in t that contains no powers t^i for $i > 0$. It follows from

$$\begin{aligned} y_1^2(t) &= t^{-n} s_1^2 \\ y_1 y_2(t) &= t^{1/2} s_1 s_2(t) \\ y_2^2(t) &= t^{(n+1)} s_2^2 \end{aligned}$$

that $c_1 \neq 0$ and $d_1 = 0$ holds. Replacing (f_1, f_2) by $(c_1^{-1} f_1, c_1^{-1} f_2)$ does not alter M and yields the proposition. □

Corollary 4.6.6 *If M is reducible, then $y_1^2(z)$ is a monic polynomial of degree n in $z = 1/t$ that is invariant under M. It generates the polynomial solution space of $L_n^{(2)}(y) = 0$.*

Proof. Let M and f_1 be as in Proposition 4.6.5. The function $f_1^2(t) = y_1^2(t)$ is fixed by M and has lowest term t^{-n}. It is not only rational but also polynomial, since there are no non-negative local exponents of L_n at the finite points. This proves the first assertion. The remaining statement can be proven in a way that is similar to the proof of Corollary 4.6.4. □

Whether or not M is reducible, we have shown that there always exists a unique monic polynomial of degree n that is fixed by monodromy and is a solution of $L_n^{(2)}(y) = 0$. We emphasise the relation between the shape of the polynomial and the (ir)reducibility of M in the following theorem.

Theorem 4.6.7 *Let $L_n(y) = 0$ be a Lamé equation with given parameters $n \in \mathbb{Z}$, g_2, g_3 and B. Let $y_1(t)$ and $y_2(t)$ be solutions of the equation as in Definition* 4.3.1. *Then there exist a unique $\gamma \in \mathbb{C}$ such that*

$$y_1^2(t) - \gamma^2 y_2^2(t)$$

is a polynomial of degree n in $z = 1/t$ that is an invariant of any representation of M. Moreover, one has

$$M \text{ is reducible} \iff \gamma = 0.$$

Proof. This follows directly from the Corollaries 4.6.4 and 4.6.6. □

Definition 4.6.8 Let L_n with parameters $n \in \mathbb{Z}$, g_2, g_3 and B be given. Let M be its monodromy group. The *invariant polynomial P_n of degree n in* $1/t$ of M is defined as

$$P_n(1/t) := y_1^2(t) - \gamma^2 y_2^2(t)$$

as in the previous theorem.

In Section 6.6 we calculate $P_n(1/t)$ and γ^2 for various integers n.

Chapter 5

The finite monodromy groups of L_n

In this chapter we specify the finite monodromy groups that may occur for the Lamé equation. Specific relations between the groups and the rational index n of their Lamé equations will be established. We shall also give some explicit families of algebraic Lamé equations with accessory parameter $B = 0$.

5.1 The finite projective monodromy groups

A priori a finite projective monodromy group of the Lamé equation is cyclic, dihedral, tetrahedral, octahedral or icosahedral. F. Baldassarri proved in his article [Bal81] that certain of these groups cannot occur. We state some of his results and give alternative proofs.

Proposition 5.1.1 (Baldassarri) *The finite cyclic group does not occur as the projective monodromy group of a Lamé equation.*

Proof. Suppose that PM is finite and cyclic. Then M is completely reducible by Proposition 3.4.5. This is however in contradiction with Theorem 4.4.1. □

Lemma 5.1.2 *The group A_4 is not generated by its order 2 elements.*

Proof. All elements of order 2 in A_4 lie in Klein's Four group. Then so does the group that they generate. Klein's Four group is however a proper subgroup of A_4. □

Proposition 5.1.3 *Let $Ly = 0$ be a Fuchsian equation of order 2, such that its projective monodromy group PM is generated by elements of order 2. Then PM is not tetrahedral.*

Proof. Suppose that PM is the tetrahedral group. Then it is isomorphic to the alternating group on 4 elements A_4. However, A_4 is never generated by its elements of order 2. This gives a contradiction. □

Corollary 5.1.4 (Baldassarri) *The tetrahedral group does not occur as the projective monodromy group of a Lamé equation.* □

The remaining finite projective monodromy groups of the Lamé equation are the dihedral, octahedral and icosahedral groups. As we are interested in the finite monodromy groups we shall mainly focus on these groups instead of on their projective groups.

5.2 A correspondence between M and n

In this section we consider the determination of all finite monodromy groups of the Lamé equation. At the same time we relate the rational indices n of the algebraic Lamé operators to those finite monodromy groups. In particular we prove that $60n$ is an integer, whenever M is finite. This result has previously been proved by F. Baldassarri in [Bal81]. First we show that $2n \in \mathbb{Z}$ yields a dihedral projective monodromy group.

Theorem 5.2.1 *Suppose that n is contained in $\{\frac{1}{2}\}+\mathbb{Z}$. Then γ_∞ is semi-simple if and only if M is finite. In that case PM is Klein's Four group.*

Proof. Let n be contained in $\{1/2\} + \mathbb{Z}$. In general σ_∞ is a conjugate of

$$\tau := \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \tag{5.1}$$

for a certain complex number β. Suppose that γ_∞ is semi-simple. We then have $\beta = 0$ and trivial matrices τ and σ_∞. It follows that the three generators σ_1, σ_2 and σ_3 of PM satisfy $\sigma_1\sigma_2\sigma_3 = 1$. Each one is of order 2. The only group that meets these conditions is Klein's Four group. The group M is generated by the reflections γ_1, γ_2, γ_3 of determinant -1. The kernel of projection of M is thus contained in the finite group

$$\left\langle \begin{pmatrix} \mathrm{i} & 0 \\ 0 & \mathrm{i} \end{pmatrix} \right\rangle$$

Then in particular M is finite.
Conversely, suppose that M is finite. Then σ_∞ is finite and conjugate to a matrix τ of the form (5.1) with $\beta = 0$. It follows that σ_∞ is semi-simple. □

Recall that logarithmic solutions can only occur in the solution space of when there are local exponents that differ by an integer. This only happens for $n \in \{1/2\}+\mathbb{Z}$. In Section 1.1 we describe what such solutions look like. It follows that there appear logarithmic terms in the solution space of a specific Lamé equation if its γ_∞ is not semi-simple.

Lemma 5.2.2 *The Lamé equation is not algebraic for $n = -1/2$.*

Proof. For $n = -1/2$ there is always a solution with a logarithmic term, because $-1/4$ is then a double local exponent at $z = \infty$. □

It is due to the following classical theorem of Brioschi and Halphen that every $n \in \{1/2\} + \mathbb{Z}_{\geq 0}$ leads to infinitely many algebraic Lamé equations.

Theorem 5.2.3 (Brioschi-Halphen) *Let n be contained in $\{\frac{1}{2}\} + \mathbb{Z}_{\geq 0}$. Then there exists a polynomial R_n of degree $n + 1/2$ with coefficients in $\mathbb{Z}[g_2/4, g_3/4]$, such that $L_n(y) = 0$ is algebraic if and only if $R_n(B) = 0$.*

Proof. We refer to [Poo36, §37] or [Bal81, Th. 2.6] for the proof of this theorem. A slightly different proof is given in Section 6.3. □

Corollary 5.2.4 *Let $n \in \{\frac{1}{2}\} + \mathbb{Z}_{\geq 0}$ and $g_2, g_3 \in \mathbb{C}$ be given. Then there exists a $B \in \mathbb{C}$ such that Klein's Four group occurs as the projective monodromy group of the Lamé equation $L_n(y) = 0$ with the given parameters.*

Proof. Take n in $\{1/2\} + \mathbb{Z}_{\geq 0}$ and $g_2, g_3 \in \mathbb{C}$. According to Theorem 5.2.3 there exists a Lamé equation that only has algebraic solutions. It follows that M is finite. Theorem 5.2.1 then yields the statement. □

For finite M we have proved that the projective monodromy group is V_4 if n differs by 1/2 from an integer. It turns out that the projective group PM is dihedral if more generally $2n$ is an integer.

Theorem 5.2.5 *Suppose that n is an integer and that M is finite. Then PM is dihedral.*

Proof. Suppose that n is integral and that the monodromy group is finite. Due to Theorem 4.5.5 we know that M is dihedral. The projective group PM is then either dihedral or cyclic of order 2. The latter however is impossible because of Proposition 5.1.1. □

Our object for now is to prove that $n \in \{1/2\} + \mathbb{Z}$ or $n \in \mathbb{Z}$ are the only cases when PM is dihedral for finite M. Since this concerns finite monodromy groups, we assume in the remainder of this section that M is of finite order.

Shephard and Todd [ST54, §I.2] gave a classification of the finite reflection groups in $\mathrm{GL}(2,\mathbb{C})$ that reduce to a dihedral group. These groups are imprimitive reflection groups. They are conjugate to the groups called $G(m,p,2)$ as in Definition 3.4.1. A priori every one of the groups $G(m,p,2)$ is a candidate for being the monodromy group of a Lamé equation. The restriction that the matrices in M have determinant ± 1 diminishes the number of possibilities for p. It follows from Table 3.6 that the index p is either m or $m/2$. If p is m, then M is dihedral. It is conjugate to

$$G(m,m,2) = \left\langle \begin{pmatrix} \mathrm{e}^{2\pi\mathrm{i}/m} & 0 \\ 0 & \mathrm{e}^{-2\pi\mathrm{i}/m} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle .$$

Moreover m is at least 3, since otherwise M would be abelian.
For $p = m/2$ the monodromy group is

$$G(m,m/2,2) = \left\langle \begin{pmatrix} \mathrm{e}^{2\pi\mathrm{i}/m} & 0 \\ 0 & \mathrm{e}^{-2\pi\mathrm{i}/m} \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle ,$$

with $m \in 2\mathbb{Z}_{>0}$ up to conjugation with a matrix in $\mathrm{GL}(2,\mathbb{C})$.

Theorem 5.2.6 *Suppose that the Lamé equation has a finite monodromy group with a dihedral projective monodromy group. Then we are in one of the following two cases.*

(i) *M is conjugate to*

$$G(m,m,2) = \left\langle \begin{pmatrix} \mathrm{e}^{2\pi\mathrm{i}/m} & 0 \\ 0 & \mathrm{e}^{-2\pi\mathrm{i}/m} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle ,$$

with $m \in \mathbb{Z}_{\geq 3}$. The index n is an integer.

(ii) *M is conjugate to*

$$G(4,2,2) = \left\langle \begin{pmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle .$$

One then has $PM = V_4$ and $n \in \{\frac{1}{2}\} + \mathbb{Z}$.

Moreover, γ_∞ is anti-diagonal in case (i). It is $\pm\mathrm{i}I_2$ for case (ii).

Proof. As we have pointed out, the monodromy group M is either conjugate to $G(m,m,2)$ with $m \geq 3$, or to $G(m,m/2,2)$ with m positive and even. We may assume that M is exactly given as one of these groups, as a monodromy group is defined up to conjugacy. An arbitrary matrix in M then either is diagonal or anti-diagonal. In particular this is true for γ_∞. We shall prove that the distinction between γ_∞ being diagonal or not corresponds uniquely to the separation between

M being dihedral or $G(m, m/2, 2)$.
First we consider the case of γ_∞ being anti-diagonal. An anti-diagonal matrix in M is of the form

$$\begin{pmatrix} 0 & \pm\zeta_m^k \\ \zeta_m^{-k} & 0 \end{pmatrix}.$$

for a certain m-th primitive root of unity ζ_m and $k \in \mathbb{Z}$. The determinant of γ_∞ is -1. This implies that the non-zero entries of γ_∞ are ζ_m^k and ζ_m^{-k} for an integer k. The eigenvalues of $\mathrm{e}^{-n\pi\mathrm{i}}$ and $\mathrm{e}^{(n+1)\pi\mathrm{i}}$ of γ_∞ therefore are 1 and -1. This yields $n \in \mathbb{Z}$. It follows from Theorem 4.5.5 that the monodromy group is dihedral.
Suppose that γ_∞ is diagonal. Every diagonal matrix of $G(m, m, 2)$ has determinant 1. Therefore, the monodromy group must be of the form $G(m, m/2, 2)$. In particular, $\det(\gamma_\infty) = -1$ implies

$$\gamma_\infty = \begin{pmatrix} -\zeta_m^k & 0 \\ 0 & \zeta_m^{-k} \end{pmatrix}$$

for certain integers $m \in 2\mathbb{Z}$ and $k \in \mathbb{Z}$. Any product of one diagonal and one anti-diagonal matrix is again anti-diagonal. Secondly, the product $\gamma_1\gamma_2\gamma_3\gamma_\infty$ yields the identity matrix. Hence, the number of diagonal matrices in $\{\gamma_1, \gamma_2, \gamma_3, \gamma_\infty\}$ is even. In fact it is 2 since otherwise M would only consist of diagonal matrices. We conclude that exactly one matrix in $\{\gamma_1, \gamma_2, \gamma_3\}$ is diagonal. It has diagonal entries 1 and -1, because the γ_i's are reflections. From

$$\begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & \alpha \\ \beta & 0 \end{pmatrix},$$

$\alpha, \beta \in \mathbb{C}$, we see that we may assume γ_1 to be diagonal, i.e.

$$\gamma_1 = \pm \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

There exist integers s and t such that

$$\gamma_2 = \begin{pmatrix} 0 & \zeta_m^s \\ \zeta_m^{-s} & 0 \end{pmatrix} \quad \text{and} \quad \gamma_3 = \begin{pmatrix} 0 & \zeta_m^t \\ \zeta_m^{-t} & 0 \end{pmatrix}$$

hold. The explicit description of γ_∞ shows that there are two solutions f_1 and f_2 of the Lamé equation that satisfy $\gamma_\infty f_1 = -\zeta_m^k f_1$ and $\gamma_\infty f_2 = \zeta_m^{-k} f_2$. The set $\{f_1, f_2\}$ is just a basis of solutions on which the monodromy matrices are given. The action of γ_∞ of f_1 and f_2 implies $\gamma_\infty(f_1 f_2) = -f_1 f_2$. By the same reasoning we have $\gamma_1(f_1 f_2) = -f_1 f_2$. The two matrices γ_2 and γ_3 leave $f_1 f_2$ invariant. Altogether this shows that $g(z) := f_1 f_2/(z - z_1)^{1/2}$ is invariant under the action of M and thus is rational. It contains no finite complex pole other than z_1, since all local exponents of the Lamé equation are non-negative. The local exponents

at z_1 are 0 and 1/2. The action of γ_1 on $f_1 f_2$ shows that its series expansion in $z - z_1$ is $(z-z_1)^{1/2}$ times a power series. This implies that $g(z)$ has no complex poles. Therefore $g(z)$ is not only rational but also a polynomial.
The local exponents of the Lamé equation at ∞ are $-n/2$ and $(n+1)/2$. The substitution $z = 1/t$ shows that $f_1 f_2$ has lowest degree $-n$, 1/2 or $n+1$ in t. On the other hand, the degree is $-(1/2 + \deg_z(g(z))$. This implies $\deg(g(z)) = n - 1/2$. It follows that n is an element of $\{1/2\} + \mathbb{Z}$. According to Theorem 5.2.1, PM must be Klein's Four group. Table 3.6 yields only one possibility for the monodromy group. It is $G(4,2,2)$. The eigenvalues of γ_∞ are both i or both $-$i. We derive from γ_∞ being diagonal that γ_∞ is $\pm \mathrm{i} I_2$. Notice that these matrices are indeed elements of $G(4,2,2)$. □

It turns out that the dihedral group D_4 does not occur as the monodromy group of a Lamé equation.

Theorem 5.2.7 *The monodromy group of a Lamé equation is not D_4.*

Proof. Suppose that M is D_4. According to Theorem 5.2.6 it is $G(4,4,2)$ up to an isomorphism. Its projective group is V_4. Due to Theorem 5.2.6 the index n is an integer. It follows from Proposition 4.5.3 and Theorem 4.5.5 that γ_∞ may be assumed to be

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We are entirely in the situation of Theorem 4.5.5. It follows from Proposition 4.6.3 that the basis (f_1, f_2) of M can be written as

$$\begin{aligned} f_1(t) &= y_1(t) + \gamma \cdot y_2(t) \\ f_2(t) &= (-1)^n \left(y_1(t) - \gamma \cdot y_2(t)\right) \end{aligned}$$

for certain $\gamma \in \mathbb{C}^*$ and non-zero power series $y_1(t) = t^{-n/2} s_1(t)$ and $y_2(t) = t^{(n+1)/2} s_2(t)$ in t. The product $f_1 f_2$ is a polynomial in $z = 1/t$, see theorem 4.6.2. The expression $f_1^4 + f_2^4$ is invariant under the action of M. It is a polynomial in $\mathbb{C}[1/t]$, since all of its local exponents at the finite points are non-negative. The subtraction of

$$\begin{aligned} 2(f_1 f_2)^2 &= (-1)^{2n} 2(y_1^2 - \gamma^2 y_2^2)^2 \\ &= 2(y_1^4 + \gamma^4 y_2^4) - 4\gamma^2 y_1^2 y_2^2 \end{aligned}$$

from

$$\begin{aligned} f_1^4 + f_2^4 &= (y_1 + \gamma y_2)^4 + (-1)^{4n} (y_1 - \gamma y_2)^4 \\ &= 2(y_1^4 + \gamma^4 y_2^4) + 12\gamma^2 y_1^2 y_2^2 \end{aligned}$$

yields $y_1^2 y_2^2 \in \mathbb{C}[1/t]$. The lowest order of t in $y_1^2 y_2^2$ is $-n + n + 1 = 1$. We derive that $y_1^2 y_2^2$ is identical to 0. Then so is y_1 or y_2. This gives a contradiction. □

Corollary 5.2.8 *Suppose that the projective monodromy group of the Lamé equation is Klein's Four group and that M is finite. Then we have $M = G(4, 2, 2)$ and $n \in \{\frac{1}{2}\} + \mathbb{Z}$.*

Proof. Let M be finite. Suppose that PM is V_4. By theorem 5.2.6 we have either $M \cong G(4, 2, 2)$ or $M \cong G(4, 4, 2)$. The latter, however, is impossible due to Theorem 5.2.7. Finally we conclude $n - 1/2 \in \mathbb{Z}$ from Theorem 5.2.6. □

Corollary 5.2.9 *Let L_n be a Lamé operator with given g_2 and g_3. Then $G(4, 2, 2)$ occurs as the monodromy group of L_n for every $n \in \{\frac{1}{2}\} + \mathbb{Z}_{\geq 0}$.*

Proof. Corollary 5.2.4 states that V_4 is the projective monodromy group for any given $n \in \{1/2\} + \mathbb{Z}$ and $g_2, g_3 \in \mathbb{C}$. According to Corollary 5.2.8 the monodromy group then is $G(4, 2, 2)$. □

We shall see that that only a few finite monodromy groups M can occur if the projective monodromy group PM is a predescribed non-abelian dihedral group.

Theorem 5.2.10 *Suppose that the finite monodromy group of a Lamé equation has a dihedral projective group of order $2K > 4$. Then M is dihedral and n is an integer. For K odd M is D_K or D_{2K} with $K \geq 3$. For K even one has $M = D_{2K}$ and $K \geq 4$.*

Proof. If PM is dihedral of order at least 6, then M is dihedral by Theorem 5.2.6. Also n is an integer. More precisely, $M = D_m$ has order at least 6. From Theorem 5.2.7 it follows that m is not 4. Therefore, m is at least 6, whenever it is even.
If m is odd then the only multiple of I_2 that M contains is I_2 itself. Hence M and PM are isomorphic. The assumption that PM is dihedral of order $2K$, now implies $m = K$. The multiples of I_2 in M are $\pm I_2$ in the case of m even. The projective monodromy group is still dihedral, but of order $2K = m$. So in general we must have $m = K$ or $m = 2K$. The case $m = K$, however, only occurs for K odd. □

The dihedral projective groups have now been dealt with. This leaves the octahedral and icosahedral projective monodromy groups. The following theorem relates n to the octahedral and the icosahedral groups.

Theorem 5.2.11 (Baldassarri) *The following holds.*

(a) *Suppose that the projective monodromy group PM is octahedral. Then one has $n \in \{\frac{1}{6}, \frac{5}{6}\} + \mathbb{Z}$ for $|\sigma_\infty| = 3$. Otherwise we have $|\sigma_\infty| = 4$ and $n \in \{\frac{1}{4}, \frac{3}{4}\} + \mathbb{Z}$.*

(b) *Suppose that the projective monodromy group PM is icosahedral. Then one has $n \in \{\frac{1}{6}, \frac{5}{6}\} + \mathbb{Z}$ for $|\sigma_\infty| = 3$. Otherwise we have $|\sigma_\infty| = 5$ and $n \in \{\frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}\} + \mathbb{Z}$.*

Proof. The matrix σ_∞ is conjugate to

$$\begin{pmatrix} 1 & 0 \\ 0 & \mathrm{e}^{(n+1/2)(2\pi \mathrm{i})} \end{pmatrix}$$

unless $n + 1/2$ is an integer. For the octahedral and icosahedral groups however, $n + 1/2 \in \mathbb{Z}$ never happens, see theorem 5.2.1. The order $|\sigma_\infty|$ is the smallest positive integer such that $|\sigma_\infty|(n + 1/2)$ is an integer. An element of S_4 is of order 1, 2, 3 or 4. The orders that occur in A_5 are 1, 2, 3 or 5. If σ_∞ would be the identity then n should be contained in $\{1/2\} + \mathbb{Z}$. For σ_∞ of order 2 it would give $n \in \mathbb{Z}$. Both cases cannot occur for an octahedral and icosahedral projective monodromy group PM. This leaves the necessary condition $|\sigma_\infty| \in \{3, 4, 5\}$. Suppose that one has $|\sigma_\infty| = 3$. Then this leads to

$$\begin{aligned} & 3(n + 1/2) \in \mathbb{Z} \text{ and } n \notin \frac{1}{2}\mathbb{Z} \\ \iff \quad & (n + 1/2) \in \frac{1}{3}\mathbb{Z} \text{ and } n \notin \frac{1}{2}\mathbb{Z} \\ \iff \quad & n \in \{1/6, 5/6\} + \mathbb{Z}. \end{aligned}$$

For $|\sigma_\infty| = 4$ one has

$$\begin{aligned} & 4(n + 1/2) \in \mathbb{Z} \text{ and } n \notin \frac{1}{2}\mathbb{Z} \\ \iff \quad & (n + 1/2) \in \frac{1}{4}\mathbb{Z} \text{ and } n \notin \frac{1}{2}\mathbb{Z} \\ \iff \quad & n \in \{1/4, 3/4\} + \mathbb{Z}. \end{aligned}$$

The possibility $|\sigma_\infty| = 5$ yields

$$\begin{aligned} & 5(n + 1/2) \in \mathbb{Z} \text{ and } n \notin \frac{1}{2}\mathbb{Z} \\ \iff \quad & (n + 1/2) \in \frac{1}{5}\mathbb{Z} \text{ and } n \notin \frac{1}{2}\mathbb{Z} \\ \iff \quad & n \in \{\pm 1/10, \pm 3/10\} + \mathbb{Z}. \end{aligned}$$

□

In Item (3.b) of [Bal81] it is stated that $|\sigma_\infty| = 3$ does not occur if PM is an octahedral group. However, this is not true as is shown by the following example.

Example 5.2.12 Consider the hypergeometric operator

$$L_{(1/4,1/2,1/3)} := x(x-1)\frac{d^2}{dx^2} + (5/4x - 3/4)\frac{d}{dx} - 7/(24)^2$$

that has local exponent difference 1/4, 1/2 and 1/3 at successively 0, 1 and ∞. It is known that the monodromy group G of this operator is finite, see Theorem 2.1.9. In particular PG is isomorphic to S_4. Now replace x by z^2. Then we obtain the Lamé operator

$$L_{1/6} = (4z^3 - 4z)\frac{d^2}{dz^2} + (6z^2 - 2)\frac{d}{dz} - 7/36z$$

whose accessory parameter B is 0. It has the local exponent differences 1/2 and 2/3 at $z = 0, \pm 1$ and $z = \infty$, respectively. The group M is a subgroup of G by Theorem 2.6.9. It follows that PM is a subgroup of $PG \cong S_4$. It is the icosahedral group itself, since the index $n = 1/6$ only applies to the octahedral or icosahedral group.

Analogously to the case of the finite monodromy groups with dihedral projective groups we want to establish a relation between the monodromy groups with octahedral or icosahedral projections and n. First we determine which monodromy groups may occur.

Theorem 5.2.13 *Suppose that M is finite and that PM is octahedral or icosahedral. Then*

(i) M is G_{12} or G_{13} if PM is octahedral

(ii) M is G_{22} if PM is icosahedral.

Proof. The monodromy group of the Lamé equation is a complex reflection group. An octahedral projective monodromy group PM a priori has 8 possible finite monodromy groups, see Table 3.4. However, the fact that M consists of matrices of determinant ± 1 yields only two possibilities. They are G_{12} and G_{13}; the indices 12 and 13 refer to the group number in the before-mentioned table. Analogously, Table 3.5 implies that G_{22} is the only possible finite monodromy group of L_n with an icosahedral projective group.

In order to obtain a relation between $M \in \{G_{12}, G_{13}, G_{22}\}$ and n we begin by considering the groups G_{12} and G_{13}. The case $M = G_{22}$ is treated later. We are going to prove that G_{12} corresponds to σ_∞ having order 4 and that G_{13} only occurs for $|\sigma_\infty| = 3$. According to [ST54, Table *II*] there exists a basis of $\mathbb{C}^2$, such that G_{12} is generated by the two matrices

$$S_{12} := \frac{\mathrm{i}}{\sqrt{2}}\begin{pmatrix} \mathrm{i} & 1 \\ -1 & -\mathrm{i} \end{pmatrix} \quad \text{and} \quad T_{12} := \frac{1}{\sqrt{2}}\begin{pmatrix} \zeta_8 & \zeta_8 \\ \zeta_8^3 & \zeta_8^7 \end{pmatrix}$$

Their orders are successively 2 and 6. The same table shows that G_{13} is generated by the matrices

$$S_{13} := S_{12} \quad \text{and} \quad T_{13} := \mathrm{i}T_{12}.$$

The matrix T_{13} is of order 12. The group G_{12} is a subgroup of G_{13}.

Lemma 5.2.14 *One has $G_{12} \subset G_{13}$ of index 2 and $G_{13} = \{G_{12}, \mathrm{i}G_{12}\}$.*

Proof. For the inclusion it is sufficient to prove that T_{12} is contained in G_{13}. The diagonal matrices in G_{13} generate $Z = \langle \mathrm{i}I_2 \rangle$, see Table 3.4. Therefore $-\mathrm{i}T_{13} = T_{12}$ is an element of G_{13}. It follows directly from $|G_{13}| = 96$ and $|G_{12}| = 48$ that the index $[G_{13} : G_{12}]$ is 2. The diagonal matrix $\mathrm{i}I_2$ is not an element of G_{12}. This yields $G_{13} = \{G_{12}, \mathrm{i}G_{12}\}$. □

The factor group G_{13}/Z, with $Z = \langle \mathrm{i}I_2 \rangle$ is isomorphic to S_4. We take such an isomorphism and identify herewith G_{13}/Z and S_4. Notice that the canonical image $G_{12}Z/Z$ of G_{12} in G_{13}/Z is G_{13}/Z itself, as G_{13} is $\{G_{12}, \mathrm{i}G_{12}\}$. We define the two homomorphisms ϕ_1 and ϕ_2 on G_{13} as

$$\begin{array}{rccl} \phi_1, \phi_2 : & G_{13} & \rightarrow & \{\pm 1\} \\ \phi_1 : & g & \mapsto & \det(g) \\ \phi_2 : & g & \mapsto & \mathrm{sgn}(gZ) \end{array}$$

where sgn denotes the sign of a permutation of S_4. By restriction the maps are also homomorphisms on G_{12}.

Lemma 5.2.15 *One has $\phi_1 = \phi_2$ on G_{12}. Moreover, G_{12} is the kernel of the map $\phi_1/\phi_2 : G_{13} \rightarrow \{\pm 1\}$.*

Proof. For the first statement it is sufficient to prove the equality of ϕ_1 and ϕ_2 on the generators S_{12} and T_{12} of G_{12}. The matrix T_{12} is of order 6 and satisfies $T_{12}^3 = -I_2$. This implies that the order of $T_{12}Z$ in S_4 is 3. Hence, it is an even permutation. The determinant of T_{12} is 1. We conclude that $\phi_1(T_{12})$ and $\phi_2(T_{12})$ are both 1. The sign of $S_{12}Z$ is -1, since otherwise $G_{12}Z/Z$ would be a subgroup of A_4. The determinant of S_{12} is -1. We see that ϕ_1 and ϕ_2 also have the same value on S_{12}. The group G_{12} is thus contained in the kernel of ϕ_1/ϕ_2 on G_{13}. The kernel is then either G_{12} or G_{13}, as the index of G_{12} in G_{13} is 2. The sign of $T_{13}Z = T_{12}Z$ is 1. The determinant of T_{13} is -1. We conclude that $\phi_1/\phi_2(T_{13})$ is -1. Therefore, the kernel of ϕ_1/ϕ_2 is G_{12}. □

The observation that G_{12} is the kernel of the homomorphism ϕ_1/ϕ_2 will be responsible for the distinction between G_{12} and G_{13}, in terms of the order of σ_∞. We first prove a lemma before giving the theorem on G_{12}, G_{13} and σ_∞.

Lemma 5.2.16 *Let g_1, g_2 and g_3 be elements of S_4. Suppose that g_1, g_2 and g_3 are of order 2 and that exactly one of them is a 2-cycle. Then $H := \langle g_1, g_2, g_3 \rangle$ is either $C_2 \times C_2$ or D_4.*

Proof. Let g_1, g_2, g_3 and H be as in the lemma. We may assume that g_1 and g_2 are contained in Klein's Four group and that the remaining g_3 is a 2-cycle. If g_1 and g_2 are different, then H contains V_4. This gives $H = \langle V_4, g_3 \rangle$. Klein's Four group is a normal subgroup of S_4. Hence, for any permutation $v \in V_4$ there exists a $w \in V_4$ that satisfies $g_3 v = w g_3$. This implies $H = \{V_4, g_3 V_4\}$, since g_3 has order 2. In particular we see that H consists of 8 elements. It follows that H is a Sylow-subgroup of S_4. All Sylow-subgroups of the same order are conjugate. Therefore H is a dihedral group of order 8.
We still have to prove the statement for equal g_1 and g_2. If g_1 and g_2 are the same then $H = \langle g_1, g_3 \rangle$ holds. One can distinguish two cases. Either H is of the form $\langle (12)(34), (12) \rangle$ or it can be assumed to be $\langle (12)(34), (14) \rangle$. In the first case H is isomorphic to the abelian group $C_2 \times C_2$. Conjugation of $(12)(34)$ with (14) in the second case yields $(13)(24) \in H$. We deduce that H is $\langle V_4, (14) \rangle$. We have already shown that H is then D_4. □

Theorem 5.2.17 *Suppose that the finite monodromy group of the Lamé equation has an octahedral projective group. Then we are in one of the following two cases.*

(i) The matrix σ_∞ is of order 3, M is G_{13} and γ_∞ is contained in $G_{13} \setminus G_{12}$.

(ii) The matrix σ_∞ is of order 4 and M is G_{12}.

Proof. Let M be contained in G_{13}. We are going to treat the cases of $|\sigma_\infty| = 3$ and $|\sigma_\infty| = 4$ separately. Due to Theorem 5.2.11 there are no other possibilities for $|\sigma_\infty|$. We first consider the case in which σ_∞ has order 3. Suppose one has $\gamma_\infty \in G_{12}$. The element $\sigma_\infty \in G_{12}/\langle -1 \rangle \cong S_4$ is of order 3. Then so is the image of γ_∞ in $G_{12}/\langle -I_2 \rangle \cong G_{13}/Z$. Therefore one has $\phi_2(\gamma_\infty) = 1$, whereas the determinant of γ_∞ always is -1. This gives a contradiction with the assumption that γ_∞ is an element of G_{12}. It follows that γ_∞ is contained in $G_{13} \setminus G_{12}$. In particular, the monodromy group is G_{13}.
Suppose that σ_∞ of order 4. It corresponds to σ_∞ being a 4-cycle. So $\phi_2(\gamma_\infty) = -1$ holds. From $\phi_2(\gamma_1\gamma_2\gamma_3\gamma_4) = 1$ we deduce that an even number of these matrices has sign 1. If their signs all are -1 then M is G_{12}, because of Lemma 5.2.15. Then we are done. The other possibility is the situation in which exactly two out of the four matrices γ_1, γ_2, γ_3 and γ_∞ correspond to an even permutation. If M is G_{13}, then exactly two of the matrices σ_1, σ_2 and σ_3 of G_{13}/Z have sign 1. They are elements of Klein's Four group, since they have order 2. The remaining matrix corresponds to a 2-cycle. We are exactly in the situation of Lemma 5.2.16. According to this lemma the projective monodromy group $PM = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$ is $C_2 \times C_2$ or D_4 rather then S_4 itself. In fact it follows from $|\sigma_\infty| \neq 2$ that PM is

D_4. This is a contradiction to the assumption $M = G_{13}$. We conclude that M is G_{12} whenever the order $|\sigma_\infty|$ is 4. □

Corollary 5.2.18 *Suppose that the Lamé equation has a finite monodromy group with octahedral projection. Then we have*

(i) the index n is $1/6$ or $5/6$ modulo $\mathbb{Z}$ and M is G_{13}

or

(ii) the index n is $1/4$ or $3/4$ modulo $\mathbb{Z}$ and M is G_{12}.

Proof. This follows directly from Theorems 5.2.11 and 5.2.17. □

Similar statements can be made for a finite monodromy group M that has an icosahedral projective group.

Theorem 5.2.19 *Suppose that the Lamé equation has a finite monodromy group with icosahedral projection. Then the monodromy group is G_{22} such that σ_∞ has order 3 or 5.*

Proof. We have shown that the finite monodromy group that has A_5 as its projective group is G_{22}, see Theorem 5.2.13. The two orders for σ_∞ follow from Theorem 5.2.11. □

In this chapter we have obtained a lot of information about the finite monodromy groups of L_n. We summarise these results in Table 5.1, which gives all finite groups that are likely to be the monodromy group of an algebraic Lamé equation. It is not implied that every group G and given corresponding n in this table indeed belongs to an algebraic Lamé operator L_n with monodromy group G. We know by Theorem 5.2.3 that $G(4, 2, 2)$ is a monodromy group of infinitely many Lamé equations with given $n \in \{1/2\} + \mathbb{Z}_{\geq 0}$. In Section 5.5 we prove that D_3, G_{12}, G_{13} and G_{22} are also monodromy groups for certain algebraic Lamé equations.

Remark 5.2.20 Notice that the indices $n \geq -1/2$ in Table 5.1 are all positive. We proved this in Theorem 6.8.9 of Chapter 6. To be safe one may always take $\mathbb{Z}$ instead of $\mathbb{Z}_{\geq 0}$ and $\mathbb{Z}_{>0}$ in the last column of the table. Then the indices $n < 1/2$ are also taken into account.

M	PM	$\|M\|$	$n \in$
$G(4,2,2)$	V_4	16	$\{\frac{1}{2}\} + \mathbb{Z}_{\geq 0}$
$D_N,\ \ N \in \{3\} \cup \mathbb{Z}_{\geq 5}$	D_N for N odd $D_{N/2}$ for N even	$2N$	$\mathbb{Z}_{>0}$
G_{12}	S_4	48	$\{\frac{1}{4}, \frac{3}{4}\} + \mathbb{Z}_{\geq 0}$
G_{13}	S_4	96	$\{\frac{1}{6}, \frac{5}{6}\} + \mathbb{Z}_{\geq 0}$
G_{22}	A_5	120	$\{\frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}\} + \mathbb{Z}_{\geq 0}$ or $\{\frac{1}{6}, \frac{5}{6}\} + \mathbb{Z}_{\geq 0}$

Table 5.1: The possible finite monodromy groups M of L_n with $n \geq -1/2$.

5.3 Scaling of the Lamé equation

Given a Lamé equation $L_n(y) = 0$, the substitution $z \mapsto \lambda z$, $\lambda \in \mathbb{C}^*$, in L_n gives another Lamé equation up to a scalar multiplication. The newly obtained Lamé operator is given by

$$\tilde{L}_n(y) := \tilde{p}(z)y'' + \frac{1}{2}\tilde{p}'(z)y' - (n(n+1)z + \tilde{B})y$$

with $\tilde{p}(z) = 4z^3 - \tilde{g}_2 z - \tilde{g}_3$. It has parameters

$$\begin{aligned} \tilde{B} &= B/\lambda \\ \tilde{g}_2 &= g_2/\lambda^2 \\ \tilde{g}_3 &= g_3/\lambda^3. \end{aligned}$$

The map $z \mapsto \lambda z$, $\lambda \in \mathbb{C}^*$, is a specific example of a projective linear map on $\mathbb{P}^1$. Such a map ϕ is given by

$$\begin{aligned} \phi : \mathbb{P}^1 &\rightarrow \mathbb{P}^1 \\ z &\mapsto \frac{az+b}{cz+d}, \end{aligned}$$

in which $ad - bc$ is non-zero. As before we can consider the equation $\tilde{L}(y) = 0$ that is obtained after the substitution of $\phi(z)$ in z in the Lamé equation. In this way $\tilde{L}$ is a rational pull-back of L_n by ϕ.

Proposition 5.3.1 *Let the notation be as above. Suppose* $n \neq 0$. *Suppose that* $\tilde{L}$ *is again a Lamé operator* L_m. *Then we have* $\phi : z \mapsto \lambda z$, *for a certain* $\lambda \in \mathbb{C}^*$, *and* $n = m$. *Moreover, the monodromy groups* M *and* $\tilde{M}$ *of* L_n *and* $\tilde{L}$, *respectively, are conjugate in* $\mathrm{GL}(2, \mathbb{C})$.

Proof. Let ω be in $\mathbb{P}^1$. Then the local exponents of L_n at ω and of L_m at $\phi(\omega)$ coincide. So do their local exponent differences. The local exponent difference of L_n at $z = \infty$ is unequal to $1/2$ if n is not equal to 0. Therefore, one has $\phi(\infty) = \infty$ and thus $c = 0$. The three finite singular points z_1, z_2 and z_3 are sent to the three finite singular points of $\tilde{L}$. Their sum

$$\begin{aligned}\phi(z_1) + \phi(z_2) + \phi(z_3) &= a/d(z_1 + z_2 + z_3) + b/d \\ &= b/d\end{aligned}$$

must be 0. This yields $b = 0$. We see that ϕ maps z to $(a/d)z$. It is a scalar multiplication as in the beginning of this section. There we have already mentioned that L_m has index $m = n$. The last statement follows from Proposition 2.6.9, as L_n and $\tilde{L}$ are proper pull-backs of each other by ϕ and ϕ^{-1}. □

We prove in Corollary 6.7.5 that $L_0 = L_{-1}$ never has a basis of algebraic solutions. The only linear pull-backs of an algebraic Lamé equation to another is thus a scalar multiplication.

Definition 5.3.2 Let L_m and L_n be two Lamé operators. Suppose that L_m is a pull-back of L_n by a non-zero linear map. Then L_n and L_m are called *scalar equivalent*.

Another possible way to create one Lamé equation from another is to consider projectively equivalent equations (see Section 2.3). However, due to Proposition 2.3.7 no two different Lamé equations are projectively equivalent.

5.4 Lamé equations as rational pull-backs

In this section we again assume the Lamé operator L_n has a finite monodromy group. According to Klein's Theorem 3.2.3 there exists a rational pull-back function $R(z)$ of a certain hypergeometric operator

$$H := x(x-1)\frac{d^2}{dx^2} + [(a+b+1)x - c]\frac{d}{dx} + ab, \tag{5.2}$$

$a, b, c \in \mathbb{R}$, to L_n. This function was obtained by taking M with invariant function j_M, that only ramifies above 0, 1 and ∞. If (y_1, y_2) is a basis of solutions on

PG	$\{e_0, e_1, e_\infty\}$
C_m	$\{1, m, m\}$
D_m	$\{2, 2, m\}$
A_4	$\{2, 3, 3\}$
S_4	$\{2, 3, 4\}$
A_5	$\{2, 3, 5\}$

Table 5.2: The ramification indices of the HGE with monodromy group G.

which M is defined, then the rational function $R(z)$ is $j_M(y_1/y_2)$. We are going to describe $R : \mathbb{P}^1 \to \mathbb{P}^1$ for the case $n + 1/2 \notin \mathbb{Z}$.

The local exponent differences of H at its singular points 0, 1 are ∞ are of the form $1/e_0$, $1/e_1$ and $1/e_\infty$, respectively, with $\{e_0, e_1, e_\infty\}$ as in Table 5.2. It follows from Theorem 2.1.9 that the monodromy group G of H is finite. Its projective group PG depends on e_0, e_1 and e_∞ and is also given in the Table 5.2.

Lemma 5.4.1 *Suppose $n + 1/2 \notin \mathbb{Z}$. Let $R(z)$ be a pull-back function from the HGE with exponent differences $1/e_0$, $1/e_1$ and $1/e_\infty$ to $L_n(y) = 0$ as above. Then we have*

(i) $R(z)$ only ramifies above 0, 1 and ∞.

(ii) The set $\{z_1, z_2, z_3, \infty\}$ is mapped into $\{0, 1, \infty\}$ by R.

(iii) The ramification index of $R(z)$ at $z_0 \notin \{z_1, z_2, z_3, \infty\}$ with $R(z_0) = \alpha \in \{0, 1, \infty\}$ is e_α.

(iv) The ramification index at $z_i \in \{z_1, z_2, z_3\}$ with $R(z_i) = \alpha$ is $e_\alpha/2$. In particular, e_α is even.

(v) The ramification index at $z = \infty$ with $R(\infty) = \beta$ is $e_\beta(n + 1/2)$.

In particular, the map $R(z) : \mathbb{P}^1 \to \mathbb{P}^1$ is a Belyi-map.

Proof. We use Proposition 3.2.2. The fact that $R(z)$ only ramifies above 0, 1 and ∞ is exactly item (i) of this proposition. The covering $R(z)$ of $\mathbb{P}^1$ is thus a Belyi map by definition. The singular points z_1, z_2, z_3 and ∞ of L_n are mapped into $\{0, 1, \infty\}$ because of Proposition 3.2.2(ii). Part (iii) of the lemma coincides with Proposition 3.2.2(iv).
A finite singularity z_i has exponent difference 1/2. For $\alpha = R(z_i)$ Proposition 2.6.8 yields $1/2 = e/e_\alpha$ in which e denotes the ramification index of R at z_i. Therefore, we have $e = e_\alpha/2$. In particular $2|e_\alpha$ holds, since e must be an integer.

The ramification index of R in $z = \infty$ follows analogously by using the local exponent difference $n + 1/2$ at ∞ instead of $1/2$. □

Proposition 5.4.2 *Suppose $n + 1/2 \notin \mathbb{Z}$. Let $R(z)$ be a pull-back function of degree d in z from the HGE with exponent differences $1/e_0$, $1/e_1$ and $1/e_\infty$ as above to $L_n(y) = 0$. Then one has*

$$\left(\frac{1}{e_0} + \frac{1}{e_1} + \frac{1}{e_\infty} - 1\right) d = n. \tag{5.3}$$

Proof. This proof is based on the Riemann-Hurwitz Formula (2.2) applied to $R(z)$. It follows from Lemma 5.4.1 that any regular point z_0 of the Lamé equation with $R(z_0) = \alpha \in \{0, 1, \infty\}$ has ramification index e_α. Let δ_α denote the number of singularities $z_i \in \{z_1, z_2, z_3\}$ with $R(z_i) = \alpha$. The number of points above α counted with multiplicities is d. One of them could be ∞. If k_α denotes the number of regular points that are mapped to α, then the previous lemma yields

$$d = k_\alpha e_\alpha + \frac{\delta_\alpha e_\alpha}{2} + \epsilon_\alpha e_\alpha \left(n + \frac{1}{2}\right)$$

with $\epsilon_\alpha = 1$ in case of $R(\infty) = \alpha$ and $\epsilon_\alpha = 0$ otherwise. We thus have

$$k_\alpha = \frac{d}{e_\alpha} - \frac{\delta_\alpha}{2} - \epsilon_\alpha \left(n + \frac{1}{2}\right).$$

Again by the lemma, there is no ramification above the points other than 0, 1 and ∞. Let β be defined by $R(\infty) = \beta$. The Riemann-Hurwitz Formula (2.2) together with the obvious result $\sum_{\alpha=0,1,\infty} \delta_\alpha = 3$ yields

$$\begin{aligned} 2d - 2 &= \sum_{\alpha=0,1,\infty} k_\alpha (e_\alpha - 1) + \sum_{\alpha=0,1,\infty} \delta_\alpha \left(\frac{e_\alpha}{2} - 1\right) + e_\beta \left(n + \frac{1}{2}\right) - 1 \\ &= \sum_{\alpha=0,1,\infty} \frac{d}{e_\alpha} (e_\alpha - 1) - \sum_{\alpha=0,1,\infty} \frac{\delta_\alpha}{2} + \left(n + \frac{1}{2}\right) - 1 \\ &= 3d - \sum_{\alpha=0,1,\infty} \frac{d}{e_\alpha} + n - 2. \end{aligned}$$

The proposition follows after the separation of the terms concerning the degree d from the others. □

Example 5.4.3 In the case $PM \cong PG = D_m$ Equation (5.3) becomes

$$\frac{d}{m} = n,$$

which has previously been determined by B. Chiarellotto in [Chi95]. In this article Chiarellotto relates finding the number of Lamé operators with finite dihedral projective group to counting the number of rational pull-backs $R(z)$.

The following theorem proves that the number of scalar equivalent algebraic Lamé equations with given finite monodromy group and index n, $2n \notin \mathbb{Z}$, is finite. This result had previously been obtained by B. Dwork [MR99, Prop.2.8], but is unpublished.

Theorem 5.4.4 *Let $M \in \{G_{12}, G_{13}, G_{22}\}$ and n be given. Then up to scalar equivalence the number of algebraic Lamé operators L_n with monodromy group M is finite.*

Proof. We may assume that M is an explicitly given monodromy group such that j_M only ramifies above 0, 1 and ∞. The Lamé operator L_n with monodromy group M and given index n is then projectively equivalent to the rational pull-back $L_{R(z)}$ of a certain algebraic hypergeometric operator H by $x = R(z) = j_M(y_1/y_2)$ as in Proposition 5.4.2. The projective group PM is thus isomorphic to a subgroup of PG. The projective groups PM and PG are octahedral or icosahedral. Then PM and PG are isomorphic, as an octahedral group is not a subgroup of A_5. Therefore, there is exactly one possibility for $\{e_0, e_1, e_\infty\}$ for given n and finite monodromy group. The Riemann-Hurwitz Formula (5.3) implies that the the degree d of $R(z)$ in z is fixed. According to Lemma 5.4.1 the map $R(z) : \mathbb{P}^1 \to \mathbb{P}^1$ is a Belyi-map of degree d. If $R(z)$ and $R(\frac{az+b}{cz+d})$ are assumed to be equivalent for every $a, b, c, d, \in \mathbb{C}$ with $ad - bc \neq 0$, then the number of Belyi-maps is finite.
The rational function $R(z)$ maps ∞ to 0, 1 or ∞. After applying an appropriate linear transformation we may assume $R(\infty) = \infty$. This does not affect the finiteness of the set of functions $R(z)$. It follows that the number of maps $R(z)$ up to linear transformations $z \mapsto az + b$, $a \neq 0$ is finite.
Consider the Lamé equations $L_n(y) = 0$ and $\tilde{L}_n(y) = 0$ that are rational pull-backs by $R(z)$ and $R(az + b)$, respectively. If the set of finite singular points of L_n is $\{z_1, z_2, z_3\}$ then the set of singular points of $\tilde{L}_n$ is $\{az_i + b : i = 1, 2, 3\}$. In both cases the sum of the singular points should be 0. This implies $b = 0$.
It remains to prove that there are finitely many Lamé operators that lead to the same $R(z)$. Suppose that $L_n(y) = 0$ and $\tilde{L}_n(y) = 0$ give rise to the same quotient $y_1/y_2(z) = \tilde{y}_1/\tilde{y}_2(z)$ of solutions. Differentiation with respect to z leads to $W(y_1, y_2)/y_1^2 = W(\tilde{y}_1, \tilde{y}_2)/\tilde{y}_2^2$, where $W(y_1, y_2)$ and $W(\tilde{y}_1, \tilde{y}_2)$ are the Wronskians of L_n and $\tilde{L}_n$, respectively. It follows from the Abel-Liouville formula (1.6) that $W(y_1, y_2) = p(z)^{-1/2}$. Similarly, if $\tilde{p}(z)$ is the Lamé polynomial of $\tilde{L}_n$ then we have $W(\tilde{y}_1, \tilde{y}_2) = \tilde{p}(z)^{-1/2}$. This yields $\tilde{y}_2 = \pm\tilde{p}(z)^{-1/4}\tilde{p}(z)^{1/4}y_2$. We conclude that L_n and $\tilde{L}_n$ are projectively equivalent. It follows from Proposition 2.3.7 that they coincide. □

Remark 5.4.5 A more specific version of Klein's Theorem 3.2.3 states that PM and PG may always be assumed to be isomorphic, see [Bal80, §1]. With this given, Theorem 5.4.4 also holds for $2n \in \mathbb{Z}$. The proof is similar to the one presented above. In Theorem 6.7.9 we explicitly see why there are finitely many algebraic Lamé equations (up to scalar equivalence) that have a given finite dihedral group as their monodromy groups.

5.5 Algebraic Lamé equations with $B = 0$

In this section we consider the two specific rational pull-back functions

$$\begin{array}{rcl} f : z & \mapsto & z^2 \\ g : z & \mapsto & z^3 \end{array}$$

on $\mathbb{P}^1$ of the hypergeometric equation and derive when they result in an algebraic Lamé equation. In particular, it turns out that such Lamé equations have accessory parameter $B = 0$.

Recall that the hypergeometric operator H (5.2) has real parameters a, b and c. It has singular points $x = 0$, 1 and ∞ and corresponding local exponents as in Table 5.3. The non-negative exponent differences at $x = 0$, 1 and ∞ will be denoted by Δ_0, Δ_1 and Δ_∞, respectively. If the Lamé operator is a rational pull-back of H, then H is determined on projective equivalence. That is why we may and shall assume the following.

0	1	∞
0	0	a
$1-c$	$c-a-b$	b

Table 5.3: Local exponents of the hypergeometric equation.

Assumption 5.5.1 *We assume $1 - c \geq 0$ and $c - a - b \geq 0$ for the parameters a, b and c of H.*

The functions f and g ramify in $z = 0$ and $z = \infty$. There are no other points of ramification. The Tables 5.4 and 5.5 depict the local exponents of the singular points of the proper rational pull-backs L_f and L_g, of H by $x = f(z)$ and $x = g(z)$, respectively.

Theorem 5.5.2 *The Lamé operator L_n is a rational pull-back of the hypergeometric operator H by $\xi(z) = f(z)$ or $\xi(z) = g(z)$ if and only if we have*

0	1	-1	∞
0	0	0	$2a$
$2(1-c)$	$c-a-b$	$c-a-b$	$2b$

Table 5.4: Local exponents of L_f.

0	1	ζ_3	ζ_3^2	∞
0	0	0	0	$3a$
$3(1-c)$	$c-a-b$	$c-a-b$	$c-a-b$	$3b$

Table 5.5: Local exponents of L_g.

(i) $L_n = 4(z^3 - z)\frac{d^2}{dz^2} + (6z^2 - 2)\frac{d}{dz} - n(n+1)z$ *with* $\xi = f$, $c = 3/4$ *and* $\{a, b\} = \{-n/4, (n+1)/4\}$,

or

(ii) $L_n = 4(z^3 - 1)\frac{d^2}{dz^2} + 6z^2\frac{d}{dz} - n(n+1)z$ *with* $\xi = g$, $c = 2/3$ *and* $\{a, b\} = \{-n/6, (n+1)/6\}$.

Moreover, in both cases L_n *is a proper pull-back of* H *by* ξ *and has accessory parameter* $B = 0$.

Proof. We first consider $\xi(z)$ to be $f(z)$. By definition L_n is projectively equivalent to L_f if it is a rational pull-back of H by f. So suppose that L_n and L_f are projectively equivalent. Then the local exponent difference of L_n and L_f at an arbitrary projective point coincide. The singular points of L_n and L_f thus are 0, ± 1 and ∞. From Assumption 5.5.1 we derive $2(1-c) = 1/2$ and $c - a - b = 1/2$. Hence one has $c = 3/4$. Not only the exponent differences but also the local exponents themselves of L_n and L_f at a arbitrary finite point coincide. Then so do the exponents at ∞, since L_f and L_n are supposed to be equivalent, see Lemma 2.3.5. This leaves no other possibility than $L_n = L_f$ and $\{2a, 2b\} = \{-n/2, (n+1)/2\}$. The operator L_f equals

$$4(z^3 - z)\frac{d^2}{dz^2} + (6z^2 - 2)\frac{d}{dz} - n(n+1)z, \tag{5.4}$$

as can be calculated after the direct substitution of $x = z^2$ in H having parameters $c = 3/4$ and $\{a, b\} = \{-n/4, (n+1)/4\}$. Notice that the accessory parameter B of L_n is 0 and that the values of a and b satisfy $c - a - b = 1/2$.
Conversely, if H has parameters $c = 3/4$ and $\{a, b\} = \{-n/4, (n+1)/4\}$ and if L_n is as in equation (5.4), then we already mentioned that L_n is the proper pull-back of H by $f(z) = z^2$. This finishes the proof of the theorem for $\xi = f$.
The proof of the theorem for $\xi(z) = g(z)$ is almost similar. There is one different

detail. If L_g is projectively equivalent to a Lamé equation, than precisely one of its finite singular points 0, 1, ζ_3 and ζ_3^2 has ± 1 as its local exponent difference. The only possibility is $z = 0$. Assumption 5.5.1 then implies $3(1 - c) = 1$ and $c - a - b = 1/2$. In particular this gives $c = 2/3$. □

Lemma 5.5.3 *Let $\xi(z)$ denote $f(z)$ or $g(z)$. Then there exists a Lamé operator that is a (proper) pull-back of an algebraic hypergeometric operator H by ξ if and only if the the exponent differences $(\Delta_0, \Delta_1, \Delta_\infty)$ of H equals*

(i) $(1/4, 1/2, k/2)$, *with* $\xi = f$ *and* $k \in \mathbb{Z} \setminus 2\mathbb{Z}$, *or*

(ii) $(1/4, 1/2, k/3)$, *with* $\xi = f$ *and* $k \in \mathbb{Z} \setminus 3\mathbb{Z}$, *or*

(iii) $(1/3, 1/2, k/2)$, *with* $\xi = g$ *and* $k \in \mathbb{Z} \setminus 2\mathbb{Z}$, *or*

(iv) $(1/3, 1/2, k/4)$, *with* $\xi = g$ *and* $k \in \mathbb{Z} \setminus 2\mathbb{Z}$, *or*

(v) $(1/3, 1/2, k/5)$, *with* $\xi = g$ *and* $k \in \mathbb{Z} \setminus 5\mathbb{Z}$.

Proof. Consider Theorem 5.5.2. The exponent differences $\Delta_0 = 1/4$ and $\Delta_1 = 1/2$ for $\xi = f$ coincide with the identities $c = 3/4$ and $\{a, b\} = \{-n/4, (n+1)/4\}$ of Theorem 5.5.2(i) for any n. The analogue holds for $\xi = g$, as $\Delta_0 = 1/3$ and $\Delta_1 = 1/2$ comes down to $c = 2/3$ and $\{a, b\} = \{-n/6, (n + 1)/6\}$. If we now prove that H being finite corresponds to $(\Delta_0, \Delta_1, \Delta_\infty)$ and ξ being as in one of the five cases of the lemma, then we are done. The operator H is algebraic precisely when $\{\Delta_0, \Delta_1, \Delta_\infty\}$ belongs to the Schwarz's list 1.2. In Section 1.7 we described how to interpret the list. We adopt the notation that is used there.
Suppose that H has exponent differences $\Delta_0 = 1/4$ and $\Delta_1 = 1/2$. In the Schwarz's list we have to look for those Schwarz numbers such that 2 and 4 occur in two out the three denominators of the corresponding λ'', μ'' and ν''. Therefore, we are in case I with $\nu = 1/4$, or in case IV. Schwarz number I implies that the denominator of Δ_∞ is 2. Hence, we have $\Delta_\infty = k/2$ for a certain $k \in \mathbb{Z} \setminus 2\mathbb{Z}$. We see that $(\Delta_0, \Delta_1, \Delta_\infty)$ is of the form as in (i) of the lemma. We still have to prove the converse. So let $(\Delta_0, \Delta_1, \Delta_\infty)$ be $(1/4, 1/2, k/2)$, with odd $k \in \mathbb{Z}$. Then we have $\nu' = \nu'' = 1/2$. We conclude that each choice of k belongs to a hypergeometric equation with a finite monodromy group.
The denominator of Δ_∞ must be 3, if H has Schwarz number IV. Therefore we have $\Delta_\infty = k/3$ for an integer k with $3 \not| k$. Conversely, suppose that $(\Delta_0, \Delta_1, \Delta_\infty)$ equals $(1/4, 1/2, k/3)$, with $k \in \mathbb{Z} \setminus 3\mathbb{Z}$. For k satisfying $k \equiv \pm 1 \bmod 6$ one has $\nu' = 1/3$. If we have $k \equiv \pm 2 \bmod 6$, then $\nu' = 2/3$ holds. This leads to the parameters $\mu'' = 1 - 1/2 = 1/2$ and $\nu'' = 1 - 2/3 = 1/3$. We see that H belongs to Schwarz number IV. The monodromy group G of H is thus finite.
This leaves us with the investigation of H with $\Delta_0 = 1/3$ and $\Delta_1 = 1/2$. A priori the Schwarz's list shows that H has Schwarz number II, IV, VI or XIV.

However, PG is tetrahedral if H belongs to II. But then PM would be a subgroup of S_3, which is impossible. Exploration of the Schwarz's list at the numbers IV, VI and XIV of H as before yields k as wanted. For instance, consider Δ_∞ to be $k/5$ with $k \equiv \pm 1, \pm 2 \bmod 5$. After a short calculation it turns out that the integers k satisfying $k \equiv \pm 1, \pm 4 \bmod 10$ belongs to Schwarz number VI. If k is $\pm 2, 3 \bmod 10$, then H has Schwarz number XIV. □

Theorem 5.5.4 *Let L_n be the Lamé operator*

$$L_n = 4(z^3 - z)\frac{d^2}{dz^2} + (6z^2 - 2)\frac{d}{dz} - n(n+1)z.$$

Then L_n is algebraic in the following two cases.

(i) $n \in \{\frac{1}{2}\} + 2\mathbb{Z}$.

(ii) $n \in \{\frac{1}{6}, \frac{5}{6}\} + 2\mathbb{Z}$.

With these n, the operators L_n are the (proper) pull-backs of an algebraic hypergeometric operator by $f(z) = z^2$. Moreover, we have $M = G(4, 2, 2)$ and G_{13} for n as in (i) and (ii), respectively.

Proof. We are going to classify all Lamé equations that are rational pull-backs of algebraic hypergeometric equations by $f(z) = z^2$. It follows from Theorem 5.5.2, that such an operator L_n is in fact a proper pull-back of H of the desired form. According to Lemma 5.5.3 exactly two possibilities for $(\Delta_0, \Delta_1, \Delta_\infty)$ specify when L_n is a pull-back of an algebraic hypergeometric operator H by f. The first one is $(1/4, 1/2, k/2)$ with odd k. In this case the local exponent difference $a - b$ of H at $x = \infty$ is $\pm k/2$. On the other hand we have $\pm(2a - 2b) = n + 1/2$. This yields $n = -1/2 \pm k$ and thus $M = G(4, 2, 2)$. Both signs of k lead to the same Lamé operator L_n as $-n-1$ is just $-1/2 \mp k$. The set $\{1/2\} + 2\mathbb{Z}$ is precisely covered in this way as the odd k can be chosen arbitrarily. Item (i) of the theorem follows. This leaves us with the situation in which $(\Delta_0, \Delta_1, \Delta_\infty)$ is $(1/4, 1/2, k/3)$ for an arbitrary $k \in \mathbb{Z}$ with $k \equiv \pm 1 \bmod 3$. Looking at Δ_∞ we may assume $a - b = k/3$ as H is symmetric in a and b. Together with the result $\{a, b\} = \{-n/4, (n+1)/4\}$ of Theorem 5.5.2(i) this leads to

$$\begin{aligned} n &= -\frac{1}{2} \pm \frac{2k}{3} \\ &= \frac{1}{6} + \frac{2(\pm k - 1)}{3}, \end{aligned}$$

in which the two signs coincide with each choice for a. As before, both signs belong to the same Lamé equation. In particular we have $n \in \{1/6, 5/6\} + 2\mathbb{Z}$, as 3 does not divide k. The fact that every n will occur follows directly from our construction. This proves item (*ii*). According to Theorem 5.2.11, PM then is octahedral or icosahedral. It is also a subgroup of S_4 because of Theorem 2.6.9. Hence PM is S_4. Finally M is G_{13} by Corollary 5.2.18. □

Corollary 5.5.5 *Let the notation be as in Theorem* 5.5.4. *Then the Lamé operator*

$$L_n = \left(4z^3 - g_2 z\right)\frac{d^2}{dz^2} + \left(6z^2 - \frac{1}{2}g_2\right)\frac{d}{dz} - n(n+1)z,$$

with n and M as in the theorem, is algebraic for every $g_2 \in \mathbb{C}^*$.

Proof. The Lamé operators as in Theorem 5.5.4 have parameters $g_2 = 4$ and $g_3 = B = 0$. We can scale the equations by replacing 4 by $g_2 \in \mathbb{C}^*$. □

Examples 5.5.6 For $n = 1/6$ ($k = 1$), the Lamé equation

$$L_{1/6} = \left(4z^3 - g_2 z\right)\frac{d^2}{dz^2} + \left(6z^2 - \frac{1}{2}g_2\right)\frac{d}{dz} - \frac{7}{36}z$$

has monodromy group G_{13} for every $g_2 \in \mathbb{C}^*$. The same is true for

$$L_{5/6} = \left(4z^3 - g_2 z\right)\frac{d^2}{dz^2} + \left(6z^2 - \frac{1}{2}g_2\right)\frac{d}{dz} - \frac{55}{36}z$$

for which we have used $n = 5/6$ ($k = 2$).

We have explicitly determined all algebraic operators L_n which are pull-backs of hypergeometric operators by f. We state the similar result for g instead of f.

Lemma 5.5.7 *Let H be a hypergeometric operator with exponent differences* $1/3$, $1/2$ *and* $k/2$ *at $x = 0$, 1 and ∞, respectively, for a $k \in \mathbb{Z} \setminus 3\mathbb{Z}$. Then G is isomorphic to* $G(3, 1, 2)$.

Proof. The monodromy group G is generated by the monodromy matrices g_0 and g_1 around $x = 0$ and $x = 1$, respectively. These matrices are complex reflections. It follows that G is a complex reflection group. Consider Table 3.6 that describes all finite reflection groups in $\mathrm{GL}(2, \mathbb{C})$ having dihedral projective groups. We deduce from $PG = D_6$ that there are four candidates for G. They are D_3, $G(3, 1, 2)$, $G(6, 6, 2)$ and

$$G(6, 2, 2) = \left\langle \begin{pmatrix} \zeta_6 & 0 \\ 0 & \zeta_6^{-1} \end{pmatrix}, \begin{pmatrix} \zeta_3 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle,$$

with $\zeta_6 := \mathrm{e}^{2\pi \mathrm{i}/6}$ and $\zeta_3 := \zeta_6^2$. Notice that $G(6, 6, 2)$ contains each of the other groups. Therefore, it suffices to show that g_0 and g_1 generate a group that is isomorphic to $G(3,1,2)$ if they are assumed to be in $G(6,2,2)$. So let g_0 and g_1 be in $G(6,2,2)$. The successive (projective) orders of g_0 and g_1 are 3 and 2. A short calculation shows that g_0 is contained in

$$S := \left\{ \begin{pmatrix} \zeta_3 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & \zeta_3 \end{pmatrix} : i = 1, 2 \right\},$$

where ζ_3 is defined to be ζ_6^2. Analogously, the matrix g_1 is an element of

$$T := \left\{ \pm \begin{pmatrix} 0 & \zeta_3^j \\ \zeta_3^{-j} & 0 \end{pmatrix} : j = 0, 1, 2 \right\}.$$

The local exponents at $x = \infty$ are $\pm k/4 + 1/12$. Therefore, the eigenvalues of the monodromy matrix g_∞ are ζ_6^{-1} and ζ_3. It follows that $g_\infty^{-1} = g_0 g_1$ has eigenvalues ζ_6 and ζ_3^{-1}, or equivalently, determinant $\zeta_6^{-1} = -\zeta_3$ and trace 0. It is not hard to see that each choice of $g_0 \in S$ and $g_1 \in T$ satisfy this property. We may even assume

$$g_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

as we prove next. Suppose that G is given on the basis (f_1, f_2). Then it can also be taken with respect to $(f_1, \pm\zeta_3^j f_2)$, $j = 0, 1, 2$. This base change leaves the elements of S unaltered and permutes the ones in T. The simplification of g_1 finally yields $G = G(3,1,2)$. □

Theorem 5.5.8 *Let L_n be the Lamé operator*

$$L_n = 4(z^3 - 1)\frac{d^2}{dz^2} + 6z^2\frac{d}{dz} - n(n+1)z.$$

Then L_n is algebraic in the following three cases.

(i) $n \in \{1\} + 3\mathbb{Z}$.

(ii) $n \in \{\frac{1}{4}, \frac{7}{4}\} + 3\mathbb{Z}$.

(iii) $n \in \{\frac{1}{10}, \frac{7}{10}, \frac{13}{10}, \frac{19}{10}\} + 3\mathbb{Z}$.

With these n, the operators L_n are the (proper) pull-backs of algebraic hypergeometric operators by $g(z) = z^3$. Moreover, we have $M = D_3$, G_{12} and G_{22} for n as in (i), (ii) and (iii), respectively.

Proof. The proof is analogous to the one of Theorem 5.5.4. The operator L_n is a (proper) rational pull-back of the algebraic hypergeometric operator H by $g(z) = z^3$ if and only if $(\Delta_0, \Delta_1, \Delta_\infty)$ is as in case *(iii)*, *(iv)* or *(v)* of Lemma 5.5.3. These cases correspond to exactly three possibilities for Δ_∞. In general

Δ_∞ can be written as k/s for $s = 2, 4, 5$. We may suppose $a - b = k/s$ to be true. According to Theorem 5.5.2.(*ii*) we have $\{a, b\} = \{-n/2, (n+1)/6\}$. and hence

$$n = -\frac{1}{2} \pm \frac{3k}{s}.$$

Both signs lead to the same Lamé equation. First consider the case $s = 2$. Then k is congruent to 1 mod 2. The index n can be rewritten as $n = 1 + 3(\pm k - 1)/2$. This yields $n \equiv 1 \bmod 3$ as in item (*i*) of the theorem. Moreover, the set $\{1\} + 3\mathbb{Z}$ is fully covered in this way, as k can be any odd integer. The monodromy group M then is finite, dihedral and non-abelian. Theorem 2.6.9 and Lemma 5.5.7 now imply $M = D_3$.
For $s = 4$ we have $n = 1/4 + 3(\pm k - 1)/4$ with $k \in \mathbb{Z} \setminus 2\mathbb{Z}$. Again both signs of the $\pm$-sign yield the same Lamé equation. One has $n \in \{1/4\} + 3\mathbb{Z}$ if $(\pm k)$ is equivalent to 1 mod 4. For $(\pm k) \equiv 3 \bmod 4$ we deduce $n \in \{7/4\} + 3\mathbb{Z}$. Altogether this gives $n \in \{1/4, 7/4\} + 3\mathbb{Z}$. As before our construction confirms that each of the n's as above indeed occurs.
Finally, consider $s = 5$. Then one has $n = 1/10 + 3(\pm k - 1)/5$ for $k \in \mathbb{Z}$ with $5 \not| k$. In other words we have $n \in \{\frac{1}{10}, \frac{3}{10}, \frac{7}{10}, \frac{9}{10}\} + 3\mathbb{Z}$ as wanted. Conversely, by the same remarks made above for the other values of s, each choice of n in this set is valid.
The monodromy groups G_{12} and G_{22} for n as in the items (*ii*) and (*iii*), respectively, follow immediately from Table 5.1 and Remark 5.2.20. □

The Lamé operators L_n that appear in Theorem 5.5.8 are scaled by $g_3 = 4$ and $g_2 = B = 0$. If we use the non-scaled version of the operator we immediately obtain the following corollary.

Corollary 5.5.9 *Let the notation be as in Theorem* 5.5.8*. Then the Lamé operator*

$$L_n = \left(4z^3 - g_3\right) \frac{d^2}{dz^2} + 6z^2 \frac{d}{dz} - n(n+1)z,$$

with n and M as in the theorem, is algebraic for every constant $g_3 \in \mathbb{C}^*$. □

Example 5.5.10 For $n = 1$ the Lamé equation

$$(4z^3 - g_3)y'' + 6z^2 y' - 2zy = 0,$$

with $g_3 \in \mathbb{C}^*$ has a (projective) monodromy group D_3. This result was obtained by F. Baldassarri in Theorem 2.16 of [Bal87]. He proved that this is in fact the only Lamé equation with $n = 1$ and $PM = D_3$ with an algebraic solution space.

In this section we have found certain algebraic Lamé equations with $B = 0$. The next chapter contains explicit algorithms that determine the Lamé equations with a given finite monodromy group M, n and B. The Lamé equations with $B = 0$ should appear as output of the algorithms.

Chapter 6

Algorithms

In this chapter we construct algorithms that determine all algebraic Lamé equations with given monodromy group M and index n. We also give examples of algebraic Lamé equations that we obtained from the algorithms. Most of the algorithms use group invariants, but all involve explicit series expansions of solutions of the Lamé equation at $z = \infty$.

6.1 The Lamé equation at infinity

The Lamé operator L_n can be developed around $z = \infty$, by substituting $z = 1/t$ into L_n and constructing a new operator around $t = 0$. Differentiation with respect to z and t are related by $d/dz = -t^2(d/dt)$. The new differential operator in t is

$$p(1/t)\left(-t^2 \cdot \frac{d}{dt}\right)^2 + \frac{1}{2}p'(1/t)\left(-t^2 \cdot \frac{d}{dt}\right) - (n(n+1)\frac{1}{t} + B).$$

Working out the expression $(-t^2 \cdot d/dt)^2$ yields $2t^3(d/dt) + t^4(d/dt)^2$. The coefficient in front of $(d/dt)^2$ then becomes $p(1/t)t^4 = t(4 - g_2t^2 - g_3t^3)$. The coefficient of d/dt is

$$\begin{aligned} p(1/t)(2t^3) - \frac{1}{2}t^2p'(1/t) &= 8 - 2g_2t^2 - 2g_3t^3 - 6 + \frac{1}{2}g_2t^2 \\ &= 2 - \frac{3}{2}g_2t^2 - 2g_3t^3. \end{aligned}$$

Multiplication of all the coefficients by t yields the Lamé operator

$$\begin{aligned} L_{n,\infty} \;:=\; & t^2(4 - g_2t^2 - g_3t^3)\frac{d^2}{dt^2} + t(2 - 3/2g_2t^2 - 2g_3t^3)\frac{d}{dt} \\ & -(n(n+1) + Bt) \end{aligned}$$

at ∞. The Lamé equation at ∞ is then defined as

$$L_{n,\infty}y = 0.$$

6.2 A recursive relation at infinity

In Chapter 4 we introduced the series expansion of the local solutions that belong to the local exponents $-n/2$ and $(n+1)/2$ at $z = \infty$, respectively, for $n \notin \{1/2\} + \mathbb{Z}$. They are

$$\begin{aligned} y_1(t) &= t^{-n/2} s_1(t) \\ y_2(t) &= t^{(n+1)/2} s_2(t), \end{aligned}$$

where $s_1(t)$ and $s_2(t)$ are power series in $t = 1/z$ with constant term 1, see Definition 4.3.1. The functions $y_1(t)$ and $y_2(t)$ are independent solutions of $L_{n,\infty} y = 0$. They have a series expansion around $t = 0$. Applying $L_{n,\infty}$ to $y_1(t)$ and $y_2(t)$ will result in two recursive relations for the coefficients of the power series involved.

Proposition 6.2.1 *Let e be an element of $\{-n/2, (n+1)/2\}$. Let y be the series $t^e \sum_{i=0}^{\infty} u_i t^i$ with $u_i \in \mathbb{C}$ for $i \in \mathbb{Z}_{\geq 0}$. Define u_{-2} and u_{-1} to be 0. Then one has $L_{n,\infty} y = 0$ if and only if the recursive relation*

$$\begin{aligned} 4k(k - 1/2 + 2e)u_k &= Bu_{k-1} + g_2(k - 3/2 + e)(k - 2 + e)u_{k-2} \\ &\quad + g_3(k - 2 + e)(k - 3 + e)u_{k-3}. \end{aligned} \tag{6.1}$$

holds for every $k \in \mathbb{Z}_{>0}$. The coefficient u_0 can be chosen arbitrarily. Moreover, the coefficient $4k(k - 1/2 + 2e)$ is 0 only for $k = \pm(n + 1/2)$.

Proof. The proof is rather straightforward. We use the notation as in the proposition and apply the operator $L_{n,\infty}$ to $y(t) = t^e \sum_{i=0}^{\infty} u_i t^i$. We have

$$\begin{aligned} y'(t) &= \sum_{i=0}^{\infty} u_i (t^{i+e})' \\ &= \sum_{i=0}^{\infty} u_i (i + e) t^{i-1+e} \end{aligned}$$

for the first derivative of y. Similarly the second derivative is

$$y''(t) = \sum_{i=0}^{\infty} u_i (i + e)(i - 1 + e) t^{i-2+e}.$$

The term $t^2(4 - g_2 t^2 - g_3 t^3) y''$ that appears in $L_{n,\infty}$ then gives rise to

$$\begin{aligned} t^2(4 - g_2 t^2 - g_3 t^3) y'' &= 4 \sum_{i=0}^{\infty} u_i (i + e)(i - 1 + e) t^{i+e} \\ &\quad - g_2 \sum_{i=0}^{\infty} u_i (i + e)(i - 1 + e) t^{i+2+e} \\ &\quad - g_3 \sum_{i=0}^{\infty} u_i (i + e)(i - 1 + e) t^{i+3+e}. \end{aligned}$$

After some shifts involving the summation indices this is

$$4\sum_{i=0}^{\infty} u_i(i+e)(i-1+e)t^{i+e} - g_2\sum_{j=2}^{\infty} u_{j-2}(j-2+e)(j-3+e)t^{j+e}$$
$$-g_3\sum_{j=3}^{\infty} u_{j-3}(j-3+e)(j-4+e)t^{j+e}.$$

The terms involving y' and y also need to be written down. The one that belongs to y' becomes

$$\begin{aligned} t(2-3/2g_2t^2-2g_3t^3)y' &= 2\sum_{i=0}^{\infty} u_i(i+e)t^{i+e} - 3/2g_2\sum_{i=0}^{\infty} u_i(i+e)t^{i+2+e} \\ &\quad -2g_3\sum_{i=0}^{\infty} u_i(i+e)t^{i+3+e} \\ &= 2\sum_{i=0}^{\infty} u_i(i+e)t^{i+e} \\ &\quad -3/2g_2\sum_{j=2}^{\infty} u_{j-2}(j-2+e)t^{j+e} \\ &\quad -2g_3\sum_{j=3}^{\infty} u_{j-3}(j-3+e)t^{j+e}. \end{aligned}$$

For the term involving y one has

$$\begin{aligned} -(n(n+1)+Bt)y &= -n(n+1)\sum_{i=0}^{\infty} u_i t^{i+e} - B\sum_{i=0}^{\infty} u_i t^{i+1+e} \\ &= -n(n+1)\sum_{i=0}^{\infty} u_i t^{i+e} - B\sum_{j=1}^{\infty} u_{j-1} t^{j+e}. \end{aligned}$$

All coefficients that occur in front of a fixed exponent t^{i+e} have to be added together. Each of the resulting coefficients is 0 if and only $L_{n,\infty}y = 0$ holds. The coefficient in front of t^e is $4u_0e(e-1) + 2u_0e - n(n+1)u_0$. Up to multiplication by u_0 this is $2e(2e-1) - n(n+1)$ that in both cases of e is always 0. In general the coefficient of t^{e+k} with $k \geq 3$ is

$$\begin{aligned} &4u_k(k+e)(k-1+e) + 2u_k(k+e) - n(n+1)u_k \\ &-\quad Bu_{k-1} \\ &-\quad g_2u_{k-2}(k-2+e)(k-3+e) - 3/2g_2u_{k-2}(k-2+e) \\ &-\quad g_3u_{k-3}(k-3+e)(k-4+e) - 2g_3u_{k-3}(k-3+e). \end{aligned}$$

A short calculation shows that the coefficient in front of u_k is exactly $4k(k-1/2+2e)$. We then obtain

$$\begin{aligned}4k(k-1/2+2e)u_k &= Bu_{k-1}+g_2(k-3/2+e)(k-2+e)u_{k-2}\\ &\quad +g_3(k-2+e)(k-3+e)u_{k-3}.\end{aligned}$$

This recursive relation is also valid for $k=1$ and $k=2$ if u_{-2} and u_{-1} are taken to be 0. There is no condition on u_0. To conclude the proof we remark

$$\begin{aligned}4k(k-1/2+2e)=0 &\iff 2e=1/2-k\\ &\iff n+1=1/2-k \text{ or } -n=1/2-k\\ &\iff k=\pm(n+1/2).\end{aligned}$$

□

Corollary 6.2.2 *Let the notation be as in Proposition* 6.2.1*. Then the coefficients u_1 and u_2 satisfy*

$$(2+8e)u_1=Bu_0 \quad \textit{and} \quad (12+16e)u_2=Bu_1+g_2(1/2+e)eu_0.$$

For $n\notin\{\frac{1}{2}\}+\mathbb{Z}$ and $k\in\mathbb{Z}_{>0}$ one has $u_k/u_0\in\mathbb{Q}[g_2,g_3,B]$ and

$$\begin{aligned}\deg_{g_3}(u_k/u_0) &\le \left\lfloor\frac{k}{3}\right\rfloor\\ \deg_{g_2}(u_k/u_0) &\le \left\lfloor\frac{k}{2}\right\rfloor\\ \deg_B(u_k/u_0) &= k.\end{aligned}$$

Moreover, the coefficient of B^k in u_k/u_0 is then contained in $\mathbb{Q}^*$.

Proof. The first two identities follow immediately from the recursive relation (6.1). Let k be in $\mathbb{Z}_{>0}$. It follows from Proposition 6.2.1 that the coefficient of u_k in Equation (6.1) is non-zero. We are going to use the recursive relation throughout the proof. The statements are true for $k=1$ and $k=2$. Next take $k>2$ and suppose that we have proven the assertion for all indices smaller than k. First of all we see that by induction u_k/u_0 is in $\mathbb{Q}[g_2,g_3,B]$. Following the recursive relation we get

$$\begin{aligned}\deg_{g_3}(u_k/u_0) &\le \max\left(\left\lfloor\frac{k-1}{3}\right\rfloor,\left\lfloor\frac{k-2}{3}\right\rfloor,\left\lfloor\frac{k-3}{3}\right\rfloor+1\right)\\ &\le \left\lfloor\frac{k}{3}\right\rfloor.\end{aligned}$$

The same sort of reasoning gives the desired bound for g_2. The highest power of B in $4k(k-1/2+2e)u_k/u_0$ is B times the highest term of B in u_{k-1}. We obtain $\deg_B(u_k/u_0) = k$ by induction. Moreover, the coefficient of B^k in u_k/u_0 is a non-zero rational number. □

The recursive relation (6.1) shows that all coefficients u_k are in fact a constant times a polynomial expression in $\mathbb{Q}[B, g_2, g_3]$. This corresponds to the fact that the solution space of a linear differential equation is linear over $\mathbb{C}$; any multiple of a solution still is a solution.

6.3 An algorithm for $M = G(4,2,2)$

In this section we give an algorithm that determines the Lamé equations with monodromy group $G(4,2,2)$ and given n. Such Lamé equations have index $n \in \{1/2\} + \mathbb{Z}$. That is why we assume n to be of this form in this section. Given n, g_2 and g_3, Brioschi and Halphen showed that there exists a polynomial $R_n(X) \in \mathbb{Z}[g_2/4, g_3/4][X]$ of degree $n + 1/2$ with the property

$$L_n \text{ is algebraic} \iff R_n(B) = 0,$$

see Theorem 5.2.3. In particular this shows that there are finitely many Lamé equations with monodromy group $G(4,2,2)$ if g_2, g_3 and n are fixed. We give a proof of the theorem of Brioschi and Halphen that, in addition, proves the validity of the algorithm.

According to Theorem 5.2.1 the Lamé operator L_n is algebraic iff it has a power series solution $y(t)$ in $t = 1/z$ that belongs to the local exponent $-n/2$ at ∞. This power series must satisfy the recursive relation (6.1). The coefficient $4k(k - 1/2 - n)$ of u_k vanishes for $k := n + 1/2$. It follows from the recursive relation that we then have the system

$$\begin{array}{rcl} Bu_0 - (2+8e)u_1 & = & 0 \\ g_2(1/2+e)eu_0 + Bu_1 - (12+16e)u_2 & = & 0 \\ \vdots & \vdots & \vdots \\ g_3(k-2+e)(k-3+e)u_{k-3} & & \\ +g_2(k-3/2+e)(k-2+e)u_{k-2} + Bu_{k-1} & = & 0 \end{array}$$

of k linear equations in the k unknowns $u_0, u_1, \ldots, u_{k-1}$. Its coefficient matrix C has entries in $\mathbb{Z}[g_2/4, g_3/4, B]$. The existence of the series solution $y(t)$ of L_n thus amounts to the disappearance of the determinant of the $k \times k$-matrix C. The determinant of C is a polynomial R_n of degree $k = n + 1/2$ in B with coefficients in $\mathbb{Z}[g_2/4, g_3/4]$. It follows that L_n has a basis of algebraic solutions exactly if $R_n(B) = 0$. This is precisely the content of Theorem 5.2.3 .

The algorithm lameG422, that calculates all Lamé equations for $M = G(4,2,2)$ and given n, is contained in the above mentioned proof of the theorem of Brioschi and Halphen. Its outline is as follows.

Algorithm 1: lameG422(n)
Input: $n \in \{\frac{1}{2}\} + \mathbb{Z}_{\geq 0}$
Output: The array $[n, R_n(X)]$ corresponding to the algebraic Lamé operators L_n with $M = G(4,2,2)$, arbitrary g_2 and g_3 and polynomial $R_n(X)$.

1. Check the input on $n \in 1/2 + \mathbb{Z}_{\geq 0}$.
2. Construct C with B replaced by X.
3. Compute $R_n := \det(C)$.
4. **Return** $[n, R_n]$.

Examples 6.3.1 For $n = 1/2$ we obtain

$$R_{1/2}(X) = X.$$

So the assumption $B = 0$ describes all Lamé equations with an algebraic solution space for $n = 1/2$. The next two polynomials are

$$R_{3/2}(X) = X^2 - \frac{3}{4}g_2$$

and

$$R_{5/2}(X) = X^3 - 7g_2X + 20g_3.$$

We have implemented the algorithm in Maple 5.3. In Table A.1 of Appendix A its computational results can be found for $n = 1/2, 3/2, \ldots, 19/2$.

6.4 The general strategy

In the previous section we gave an easy algorithm that determines the algebraic Lamé equations in the case $M = G(4,2,2)$. For other monodromy groups we follow a different approach. The new algorithms determine the algebraic Lamé equations up to scalar equivalence. In practice this means that we restrict ourselves to the cases $B = 0$ and $B = 1$.

Strategy 6.4.1 *In order to find all algebraic Lamé equations with a given finite monodromy group M, we use the following general strategy.*

- *Find all such Lamé equations with $B = 0$.*
- *Find all such Lamé equations $L_n(y) = 0$ for a fixed $B \in \mathbb{C}^*$. Then an algebraic Lamé operator with monodromy group M and a given accessory parameter $\tilde{B}$ is obtained after the substitution of $z \mapsto (B/\tilde{B})z$ in one of the L_n's.*

Another essential ingredient in most of the algorithms to come is the use of group invariants.

6.5 Group invariants of M

We give some general theory on the invariant theory of groups and apply the results to the finite groups we are interested in.

There exist two solutions $f_1(z)$ and $f_2(z)$ of the Lamé equation $L_n(y) = 0$ in the universal covering of $\mathbb{C} \setminus \{z_1, z_2, z_3\}$, such that $(f_1(z), f_2(z))$ is a basis of the solutions space of L_n. We let $M \subset \mathrm{GL}(2, \mathbb{C})$ be explicitly given with respect to this basis. Let $I \in \mathbb{C}[X, Y]^M$ be an invariant polynomial. So we have $I(X, Y) = I(\gamma(X, Y)^T)$ for every matrix $\gamma \in M$. Then $I(f_1, f_2)$ is fixed under the monodromy action. It follows that $I(f_1, f_2)$ is a rational function of $\mathbb{C}(z)$ by Lemma 2.1.6. Moreover, $I(f_1, f_2)$ contains no finite poles since all local exponents of the Lamé equation at the finite points are non-negative. The expression $I(f_1, f_2)$ is therefore a polynomial in z.

Proposition 6.5.1 *Let M be given on the basis $(f_1(z), f_2(z))$. Let $J(X, Y)$ be a homogeneous polynomial in $\mathbb{C}[X, Y]$. Then one has:*

$$J(X, Y) = J(\gamma(X, Y)^T) \text{ for all } \gamma \in M \iff J(f_1, f_2) \in \mathbb{C}[z].$$

Proof. The 'only if' side has just been proved. Therefore let $J(f_1, f_2)$ be a polynomial in $\mathbb{C}[z]$. Suppose that there exists a matrix $\gamma \in M$ with $\gamma J(X, Y) \neq J(X, Y)$. Then $R(X, Y) := \gamma J(X, Y) - J(X, Y)$ is a non-zero homogeneous polynomial that satisfies $R(f_1, f_2) \equiv 0$. At least one of $R(X, 1)$ and $R(1, X)$ in X is non-constant. By symmetry we may assume that it is $R(X, 1)$. This yields $R(f_1/f_2, 1) \equiv 0$. Consequently f_1/f_2 must be constant on its domain, since f_1/f_2 is analytic in all but finitely many points. This is a contradiction to the assumption of (f_1, f_2) being a basis. We conclude that $\gamma J(X, Y)$ and $J(X, Y)$ are equal for every $\gamma \in M$. □

Definition 6.5.2 Let $J(X, Y) \in \mathbb{C}[X, Y]$ be a homogeneous polynomial in X and Y. Then *the automorphism group of* $J(X, Y)$ is the largest group $G \subset \mathrm{GL}(2, \mathbb{C})$ of which $J(X, Y)$ is an invariant.

Lemma 6.5.3 *Let $J(X,Y)$ be a homogeneous polynomial of degree ≥ 3 with automorphism group G. Suppose that $J(X,Y)$ has at least 3 distinct zeros in $\mathbb{P}^1$. Then G is finite.*

Proof. The projective group PG acts on the zeros of $J(X,Y)$ in $\mathbb{P}^1$. It is a group of automorphisms of $\mathbb{P}^1$. Then PG is finite, since there are only finitely many projective linear maps that permute at least three but finitely many distinct projective points.
Now let $g, h \in G$ have the same image in PG. Then we have $h^{-1}g = \lambda I_2$ for a certain $\lambda \in \mathbb{C}^*$. On one hand this gives $gJ = hJ$ and on the other $gJ = \lambda^d hJ$. Hence, λ is a d-th root of unity. So given a $g \in G$, there exists finitely many h that differ by a scalar matrix from g. There are finitely many matrices in G such that their projective equivalents generate PG. Therefore G itself is finite. □

Proposition 6.5.4 *Let f_1 and f_2 be two independent solutions of $L_n(y) = 0$ with monodromy group M. Let $J(X,Y)$ be a homogeneous polynomial of degree ≥ 3 with three distinct zeros in $\mathbb{P}^1$. Suppose that $J(f_1, f_2) \in \mathbb{C}[z]$ holds. Then M is a finite reflection group which is a subgroup of the automorphism group of $J(X,Y)$.*

Proof. According to Proposition 6.5.1 the polynomial $J(X,Y)$ is invariant under the action of M. The previous lemma implies that M is finite. In addition it is a complex reflection group, as the monodromy group of any Lamé equation is. □

We restrict Proposition 6.5.4 to some of the monodromy groups of L_n in which we are interested.

Theorem 6.5.5 *Let n be an integer. Let f_1 and f_2 be two independent solutions of $L_n(y) = 0$. Suppose that $P(X,Y) := XY$ and $J(X,Y) := X^N + Y^N$, $N \geq 3$, satisfy $P(f_1, f_2), J(f_1, f_2) \in \mathbb{C}[z]$. Then M is a subgroup of $G(N,N,2)$.*

Proof. Let the group G be the intersection of the automorphism groups of $P(X,Y)$ and $J(X,Y)$ in $\mathrm{GL}(2,\mathbb{C})$. Then M is a subgroup of G because of Proposition 6.5.4. The matrices that fix XY are the diagonal matrices of determinant 1 and the anti-diagonal matrices of determinant -1. The group G thus consists of these matrices that also act invariant on $X^N + Y^N$. This implies

$$G = \left\{ \begin{pmatrix} \mathrm{e}^{2\pi \mathrm{i}k/N} & 0 \\ 0 & \mathrm{e}^{-2\pi \mathrm{i}k/N} \end{pmatrix}, \begin{pmatrix} 0 & \mathrm{e}^{2\pi \mathrm{i}k/N} \\ \mathrm{e}^{-2\pi \mathrm{i}k/N} & 0 \end{pmatrix} : k = 0, 1, \ldots, N-1 \right\},$$

which is exactly $G(N,N,2)$. Therefore M is a subgroup of $G(N,N,2)$. □

Theorem 6.5.6 *Let the notation and assumptions be as in Proposition 6.5.4. Then M is G_{13} in the case $n \in \{1/6, 5/6\} + \mathbb{Z}$ and $\deg(J(X,Y)) = 8$.*

Proof. Table 5.1 and Remark 5.2.20 show that $n \in \{1/6, 5/6\} + \mathbb{Z}$ corresponds to $M = G_{13}$ or $M = G_{22}$. However, according to Propositions 3.6.1 and 3.6.2 there is only an invariant of degree 8 for G_{13}. □

Theorem 6.5.7 *Let notation and assumptions be as in Proposition* 6.5.4. *In addition, suppose that we have* $n \in \{1/6, 5/6\} + \mathbb{Z}$ *and* $\deg(J(X,Y)) = 12$. *Then one has:*

(i) $M = G_{13}$ *if* $J(X,Y)$ *only has double zeros in* $\mathbb{P}^1$, *and*

(ii) $M = G_{22}$ *if* $J(X,Y)$ *is square-free.*

Proof. There are two possibilities if the index n is an element of $\{1/6, 5/6\} + \mathbb{Z}$, namely $M = G_{13}$ and $M = G_{22}$. According to Proposition 3.6.1 an invariant of degree 12 of G_{13} is a square of a homogeneous square-free polynomial of degree 6 in $\mathbb{C}[X,Y]$. However, it follows from Proposition 3.6.2 that every invariant of G_{22} having degree 12 is square-free. This concludes our proof. □

The smallest invariant polynomials $J(X,Y)$ that can be taken for $M = G_{12}$, G_{13} and G_{22} are of degree $m = 6$, 8, and 12, respectively. The function $J(f_1, f_2)$ is always a solution of the m-th symmetric power $L_n^{(m)}(y) = 0$ of the Lamé equation. This is a linear differential equation with solution space generated by the products $f_1^i f_2^{m-i}$ for $i = 0, 1, \ldots, m$. The symmetric power $L_n^{(m)}(y) = 0$ is a linear differential equation of order $m + 1$.

Proposition 6.5.8 *Let* m *be* 6, 8 *or* 12. *Then the* m*-th symmetric power* $L_n^{(m)}(y) = 0$ *of* $L_n(y) = 0$ *is a Fuchsian equation of order* $m+1$. *If it is given as an equation with polynomial coefficients having* gcd $= 1$, *then the coefficient of* $\frac{d^{(m+1)}}{dz^{(m+1)}}$ *is* $p^{m/2}(z)$ *up to scalar multiplication.*

Proof. Every solution $f_1^i f_2^{m-i}$ of $L_n^{(m)}(y) = 0$ is locally of the order $O(|t|^k)$ for the local parameter t and a certain integer k. Therefore $L_n^{(m)}$ is Fuchsian. The polynomial coefficient of $\frac{d^{(m+1)}}{dz^{(m+1)}}$ in $L_n^{(m)}$ a priori is $p^m(z)$. However, after using the package DEtools of Maple 6.0 and in particular its routine symmetric_power on the Lamé equation we conclude that the exponent of $p(z)$ may be assumed to be $m/2$ for $m = 6$, 8 or 12. □

Theorem 6.5.9 *Let* $f_1(1/t)$ *and* $f_2(1/t)$ *be two independent Puiseux series solutions of* $L_n(y) = 0$ *at* $z = \infty$. *Let* $J(X,Y)$ *be a homogeneous polynomial of degree* $m \in \{6, 8, 12\}$, *such that* $J(f_1, f_2)(1/t)$ *is a Laurent series in* t. *If* $J(f_1, f_2)(1/t)$ *is a polynomial in* $1/t$ *up to degree* $\frac{3m}{2}+2$ *in* t *then it is a polynomial in* $z = 1/t$ *itself.*

Proof. By Proposition 6.5.8 the m-th symmetric power is a Fuchsian equation with leading polynomial coefficient $p^{m/2}(z)$. Any Puiseux series solution at $z=\infty$ thus satisfies a recursive relation of order $1+3m/2$. Therefore, if $1+3m/2$ consecutive coefficients of $J(f_1,f_2)(1/t)$ in t are 0, then so are the ones for the higher powers of t. □

6.6 The polynomial P_n of degree n

This section concerns the monodromy groups M of the Lamé equation with $n\in\mathbb{Z}_{\geq 0}$. We consider the Lamé equation at infinity. As before $t=1/z$ is the local parameter at $z=\infty$. Given $n\in\mathbb{Z}$, L_n and M, we determined the polynomial $P_n(1/t)$ in $1/t$ of degree n that is fixed by monodromy in Section 4.6. This polynomial is given by

$$\begin{aligned} P_n(1/t) &= y_1^2(t)-\gamma^2 y_2^2(t) \\ &= t^{-n}s_1^2(t)-\gamma^2 t^{n+1}s_2^2(t) \end{aligned} \tag{6.2}$$

for a unique γ, see Definition 4.6.8. If we substitute $t=1/z$ into $P_n(1/t)$ then $P_n(z)$ is a polynomial of degree n in z.

Remark 6.6.1 The polynomial $P_n(1/t)$ will also be denoted by $P_n(z)$.

In this section we give two algorithms that determine $P_n(z)$ and γ^2 for a given index $n\in\mathbb{Z}_{\geq 0}$.

Lemma 6.6.2 *One has* $P_n(1/t)=t^{-n}s_1^2(t)+O(t)$.

Proof. We assumed n to be non-negative. The lemma then follows from Equation (6.2) and the fact $t^{n+1}s_2^2(t)=O(t)$. □

Lemma 6.6.2 implies that the power series s_1^2 in t up to degree n leads to an explicit description of $P_n(z)$. Hence, we only need the first $n+1$ terms of $s_1=\sum_{i=0}^{\infty}u_it^i$ for the determination of P_n. The coefficients $u_0,u_1,\ldots,u_n$ can be obtained by using the recursive relation (6.1) and the choice $u_0=s_1(0)=1$. This leads to the polynomial $q(t)=\sum_{i=0}^{n}u_it^i$ of degree n in t with coefficients in $\mathbb{Q}[g_2,g_3,B]$. The polynomial $P_n(1/t)$ then is $t^{-n}q^2(t)$ without its positive powers of t. Substitution of $t=1/z$ in $P_n(1/t)$ finally puts P_n into the form $P_n(z)$. We see that Lemma 6.6.2 proves the validity of the following algorithm.

Algorithm 2: lamePn(n)
Input: $n\in\mathbb{Z}_{\geq 0}$.
Output: The invariant polynomial $P_n(z)\in\mathbb{Q}[g_2,g_3,B][z]$.

1. Check the input on $n \in \mathbb{Z}_{\geq 0}$.
2. Calculate u_0, u_1 up to u_n by using the recursive relation (6.1) with $\epsilon = -n/2$ and $u_0 = 1$.
3. Define $q := \sum_{i=0}^{n} u_i t^i$.
4. Determine q^2 as polynomial in t up to degree n.
5. Define $I(t) := t^{-n} q^2$.
6. Set $P_n(z) = I(1/z)$ as polynomial in $z = 1/t$.
7. **Return** $P_n(z)$.

Examples 6.6.3 Three invariant polynomials that the algorithm gives are

$$\begin{aligned} P_0(z) &= 1, \\ P_1(z) &= z - B \end{aligned}$$

and

$$P_2(z) = z^2 - \frac{1}{3}Bz + \frac{1}{9}B^2 - \frac{1}{4}g_2.$$

These computational results coincide with the invariant polynomials as given in (2.2) of [Bal87].

More of these invariant polynomials are listed in Table $A.2$ of Appendix A. In Table $A.3$ we give γ^2 for some indices n. The latter table shows what complex number γ^2 would be if a Lamé equation has an explicit Lamé polynomial and accessory parameter B. On the other hand, it also indicates that γ^2 can be seen as a polynomial in the variables g_2, g_3 and B. We abuse the notation and write γ^2 for both situations.

Proposition 6.6.4 *Let n be in $\mathbb{Z}_{\geq 0}$. Then the coefficient γ^2 is contained in $\mathbb{Q}[g_2, g_3, B]$ and is not identically 0. It is the coefficient of t^{2n+1} of the Taylor series $s_1^2(t)$. Moreover, one has*

$$\begin{aligned} \deg_{g_3}(\gamma^2) &\leq \left\lfloor \frac{2n+1}{3} \right\rfloor \\ \deg_{g_2}(\gamma^2) &\leq n \\ \deg_B(\gamma^2) &\leq 2n+1. \end{aligned}$$

Proof. Suppose that we have $\gamma \equiv 0$. Equation (6.2) then yields $t^{-n/2}s_1(t) = \sqrt{P_n(1/t)}$ for all choices of g_2, g_3 and B. In other words, the operator L_n is reducible. This is a contradiction to Corollary 4.4.8. The series $\gamma^2 t^{n+1} s_2^2$ thus gives a non-trivial contribution to the expansion of $P_n(1/t)$. On the other hand we know that $P_n(1/t)$ is a polynomial in $1/t$. Hence γ^2 is the coefficient of t^{n+1} in $t^{-n}s_1^2$ or, equivalently, the coefficient of t^{2n+1} in $s_1^2(t)$. Therefore,

$$\gamma^2 = u_0 u_{2n+1} + u_1 u_{2n} + \cdots + u_{2n+1} u_0.$$

It now follows from Corollary 6.2.2 and $u_0 = 1$ that γ^2 is a polynomial in $\mathbb{Q}[g_2, g_3, B]$. The upper bounds for the degrees in the proposition are valid, if they are for each term $u_i u_{2n+1-i}$ with $i = 0, 1, \ldots, 2n+1$. We deduce from Corollary 6.2.2 that each $u_i u_{2n+1-i}$ satisfies

$$\begin{aligned} \deg_{g_3}(u_i u_{2n+1-i}) &\leq \left\lfloor \frac{i}{3} \right\rfloor + \left\lfloor \frac{2n+1-i}{3} \right\rfloor \\ &\leq \left\lfloor \frac{2n+1}{3} \right\rfloor. \end{aligned}$$

In the same way we deduce

$$\deg_{g_2}(u_i u_{2n+1-i}) \leq \left\lfloor \frac{i}{2} \right\rfloor + \left\lfloor \frac{2n+1-i}{2} \right\rfloor = n$$

and $\deg_B(u_i u_{2n+1-i}) = 2n+1$. □

In addition to Proposition 6.6.4 two more conclusions can be drawn. The first one is that the series $t^{-n}s_1^2$ does not contain the powers $t, t^2, \ldots, t^n$, whatever g_2, g_3 and B are. Secondly, the series $\gamma^2 t^{n+1} s_2^2$ should be totally annihilated by $t^{-n}s_1^2 - P_n(1/t)$.

Instead of developing $s_1(t)$ up to degree n as for P_n by using the recursive relation (6.1), we can develop $s_1(t)$ up to degree $2n+1$ in the same way. Then the square of s_1 is correct up to the same degree. Its coefficient of t^{2n+1} exactly is γ^2 by Proposition 6.6.4. This idea is the content of the following algorithm.

Algorithm 3: lamesquaregamma(n)
Input: $n \in \mathbb{Z}_{\geq 0}$.
Output: The coefficient $\gamma^2 \in \mathbb{Q}[g_2, g_3, B]$ of P_n.

1. Check the input on $n \in \mathbb{Z}_{\geq 0}$.
2. Calculate u_0, u_1 up to u_{2n+1} by using the recursive relation (6.1) with $\epsilon = -n/2$ and $u_0 = 1$.
3. Define $q := \sum_{i=0}^{2n+1} u_i t^i$.
4. Determine q^2 as polynomial in t up to degree $2n+1$.

5. Compute γ^2 as the coefficient of t^{2n+1} of q^2.
6. **Return** γ^2.

Examples 6.6.5 We ran the algorithm in Maple 5.3 for several n. This gave

$$\gamma^2 = \frac{1}{36}(4B^3 - g_2 B - g_3)$$

for $n = 1$. In case of $n = 2$ we obtained

$$\gamma^2 = \frac{1}{8100}(B^2 - 3g_2)(4B^3 - 9g_2 B + 27g_3).$$

For $n = 3$ one has

$$\begin{aligned}\gamma^2 &= \frac{1}{39690000}B(16B^6 - 36450Bg_3g_2 - 504B^4g_2 + 2376g_3B^3 \\ &\quad +4185B^2g_2^2 - 3375g_2^3 + 91125g_3^2).\end{aligned}$$

Notice that γ^2 is a scalar multiple of $Q_0(B)$ and $Q_1(B)$ as in Examples 4.4.9 and 4.4.10 for $n = 0$ and $n = 1$, respectively. For more examples of γ^2 we refer the reader to Table A.3 of Appendix A.

Theorem 6.6.6 *Let n be in $\mathbb{Z}_{\geq 0}$. Let γ^2 be evaluated in the parameters g_2, g_3 and B of an explicit Lamé operator L_n. Then we have*

$$\begin{aligned}\gamma^2(g_2, g_3, B) = 0 &\iff M \text{ is reducible} \\ &\iff Q_n(B) = 0.\end{aligned}$$

Proof. The theorem follows from Theorem 4.6.7 and Corollary 4.4.8. □

Notice that in this section we made no explicit assumptions about the cardinality or (ir)reducibility of M.

6.7 Algebraic solutions for $M = D_N$

In this section we assume $N \in \mathbb{Z}_{\geq 3}$. We shall give an algorithm that for fixed $n \in \mathbb{Z}_{\geq 0}$, N and $B \in \mathbb{C}$ determines all Lamé operators L_n with accessory parameter B and monodromy group contained in D_N. Unless stated otherwise we let M be dihedral of order $2N$. According to Section 4.6 we may assume

$$M = \left\langle \begin{pmatrix} e^{2\pi i/N} & 0 \\ 0 & e^{-2\pi i/N} \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right\rangle \quad \text{and} \quad \gamma_\infty = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with respect to the basis (f_1, f_2) defined by

$$\begin{aligned} f_1(t) &= y_1(t) + \gamma \cdot y_2(t) \\ f_2(t) &= (-1)^n \left(y_1(t) - \gamma \cdot y_2(t)\right) \end{aligned}$$

and $\gamma \in \mathbb{Q}[g_2, g_3, B]$. Notice that by definition M is equal to $G(N, N, 2)$. We shall produce a polynomial in $z = 1/t$, other than $P_n(z)$, that is invariant under the action of the monodromy group M.

Proposition 6.7.1 *Consider $L_n(y) = 0$ with $n \in \mathbb{Z}_{\geq 0}$. Let M and (f_1, f_2) be as above. Then $f_1^N + f_2^N$ is a polynomial in z. It has highest order term $2z^{nN/2}$ in the case of nN even. The highest order term is $2N\gamma z^{(nN-2n-1)/2}$ when nN is odd.*

Proof. One fixed polynomial by $M = G(N, N, 2)$ in $\mathbb{C}[X, Y]$ is $X^N + Y^N$. Therefore $f_1^N + f_2^N$ should be fixed by the action of M as well. Thus it is a rational function in t. In fact, the function $f_1^N + f_2^N$ is a polynomial in $z = 1/t$, since the local exponents at all finite points are non-negative.
One has

$$\begin{aligned} (f_1^N + f_2^N)(t) &= (y_1 + \gamma y_2)^N + (-1)^{nN} (y_1 - \gamma y_2)^N \\ &= \left(t^{\frac{-n}{2}} s_1 + \gamma t^{\frac{n+1}{2}} s_2\right)^N + (-1)^{nN} \left(t^{\frac{-n}{2}} s_1 - \gamma t^{\frac{n+1}{2}} s_2\right)^N \\ &= \sum_{i=0}^{N} \binom{N}{i} \left(1 + (-1)^{nN+i}\right) t^{\frac{-n(N-i)+i(n+1)}{2}} s_1^{N-i} \gamma^i s_2^i \\ &= \sum_{i=0}^{N} \binom{N}{i} \left(1 + (-1)^{nN+i}\right) \gamma^i t^{\frac{-nN+2ni+i}{2}} s_1^{N-i} s_2^i. \end{aligned} \tag{6.3}$$

Suppose that nN is even. We then have $1 + (-1)^{nN+i} \neq 0$ if and only if i is even. This corresponds with $t^{(-nN+2ni+i)/2}$ being an integral power of t, as it should be. The lowest term in t belongs to $i = 0$ and is $2t^{-nN/2}$.
For odd nN the coefficient $1 + (-1)^{nN+i}$ vanishes if and only if i even. All non-integral powers in the expansion of $f_1^N + f_2^N$ are thus annihilated. We see that $f_1^N + f_2^N$ is contained in $\mathbb{C}((t))$. The lowest non-trivial power of t occurs for $i = 1$. It is $2N\gamma t^{(-nN+2n+1)/2}$. □

Corollary 6.7.2 *Let the polynomial $(f_1^N + f_2^N)(z)$ be given as $\sum_{j=0}^{d} c_j z^j$, such that each c_j does not contain z. Then for $j = 0, 1, \ldots, d$ one has*

$$c_j \in \begin{cases} \mathbb{Q}[g_2, g_3, B] & \text{for } nN \text{ even} \\ \gamma\mathbb{Q}[g_2, g_2, B] & \text{for } nN \text{ odd} \end{cases}.$$

Proof. The power series $s_1(t)$ and $s_2(t)$ have coefficients in $\mathbb{Q}[g_2, g_3, B]$. Therefore, each factor $s_1^{N-i}s_2^i$ in Equation (6.3) has coefficients in $\mathbb{Q}[g_2, g_3, B]$. Equation (6.3) and the fact $\gamma^2 \in \mathbb{Q}[g_2, g_3, B]$ then yield the corollary. $\square$

We have proved that $f_1 f_2$ and $f_1^N + f_2^N$ are fixed by the action of M. The expression $f_1^N - f_2^N$, however, is a semi-invariant with multiplication factors 1 and -1. This can be shown for instance by direct verification of the action of the matrices of M on $f_1^N - f_2^N$, or by using Corollary 3.5.2.

Lemma 6.7.3 *Let M, f_1 and f_2 be as before. Then the actions of the matrices γ_1, γ_2, γ_3 and γ_∞ each multiply $f_1^N - f_2^N$ by a factor -1.*

Proof. The matrices γ_1, γ_2, γ_3 and γ_∞ are anti-diagonal for $M = G(N, N, 2)$, since its elements of determinant -1 are exactly the anti-diagonal ones. Such a matrix γ satisfies $\gamma(X^N - Y^N) = Y^N - X^N$. $\square$

Theorem 6.7.4 *Let M be $G(N, N, 2)$ with respect to the basis (f_1, f_2). Then there exist polynomials $A(z)$ and $B(z)$ in z that satisfy*

$$\begin{aligned} f_1^N(z) &= A(z) + \sqrt{p}B(z) \\ f_2^N(z) &= A(z) - \sqrt{p}B(z). \end{aligned}$$

Their degrees are

$$\deg_z(A) = nN/2 \quad \text{and} \quad \deg_z(B) = nN/2 - n - 2$$

if nN is even. The highest order term of $A(z)$ then is $z^{nN/2}$. In the case of nN odd one has

$$\deg_z(A) = (nN - 2n - 1)/2 \quad \text{and} \quad \deg_z(B) = (nN - 3)/2$$

with highest order term $\pm\frac{1}{2}z^{(nN-3)/2}$ of $B(z)$.

Proof. Proposition 6.7.1 shows that $f_1^N + f_2^N$ is a polynomial in z. We define this polynomial to be $2A(z)$. Let γ_i be one of the matrices γ_1, γ_2 or γ_3. It follows from Lemma 6.7.3 that γ_i acts trivially on $(f_1^N - f_2^N)/\sqrt{p(z)}$. But then $(f_1^N - f_2^N)/\sqrt{p(z)}$ is an invariant for M. The local exponents at the finite singular points are integral and at least $-1/2$. It follows that they are non-negative. The expression $(f_1^N - f_2^N)/\sqrt{p(z)}$ therefore is a polynomial $2B(z)$ in z. This yields $(f_1^N - f_2^N) = 2\sqrt{p}B(z)$. We then have

$$\begin{aligned} 2f_1^N(z) &= (f_1^N + f_2^N) + (f_1^N - f_2^N) \\ &= 2A(z) + 2\sqrt{p}B(z) \end{aligned}$$

and

$$\begin{aligned} 2f_2^N(z) &= (f_1^N + f_2^N) - (f_1^N - f_2^N) \\ &= 2A(z) - 2\sqrt{p}B(z) \end{aligned}$$

as wanted.
The degrees and the highest order term for $A(z)$ follow directly from Proposition 6.7.1. We need to obtain are the degrees and the highest order term of B in z. By definition we have

$$\begin{aligned} (f_1^N - f_2^N)(t) &= (y_1 + \gamma y_2)^N - (-1)^{nN}(y_1 - \gamma y_2)^N \\ &= \left(t^{\frac{-n}{2}} s_1 + \gamma t^{\frac{n+1}{2}} s_2\right)^N + (-1)^{nN+1}\left(t^{\frac{-n}{2}} s_1 - \gamma t^{\frac{n+1}{2}} s_2\right)^N \\ &= \sum_{i=0}^{N} \binom{N}{i} \left(1 + (-1)^{nN+i+1}\right) t^{\frac{-n(N-i)+i(n+1)}{2}} s_1^{N-i} \gamma^i s_2^i \\ &= \sum_{i=0}^{N} \binom{N}{i} \left(1 + (-1)^{nN+i+1}\right) \gamma^i t^{\frac{-nN+2ni+i}{2}} s_1^{N-i} s_2^i. \end{aligned}$$

The lowest order term in t of $f_1^N - f_2^N$ comes from $i = 1$ if nN is even. It has order $(-nN + 2n + 1)/2$ and non-zero coefficient $2N\gamma$. For odd nN the lowest order term comes from $i = 0$ and is $2t^{-nN/2}$. Notice that all occurring exponents of t in $f_1^N - f_2^N$ are contained in $\{1/2\} + \mathbb{Z}$. This should be case, since $f_1^N - f_2^N$ is a semi-invariant for γ_∞ with multiplication factor $-1 = \mathrm{e}^{\pi \mathrm{i}}$. Both $(f_1^N - f_2^N)^2(z)$ and $p(z)$ are fixed by M. We deduce

$$2\deg_z(B) = \deg_z(f_1^N - f_2^N)^2 - \deg_z(p)$$

and in particular

$$\begin{aligned} 2\deg_z(B) &= -2(-nN + 2n + 1)/2 - 3 \\ &= nN - 2n - 4 \end{aligned}$$

for nN even. In the case of odd nN we derive

$$\begin{aligned} 2\deg_z(B) &= -2(-nN) - 3 \\ &= nN - 3. \end{aligned}$$

We have already seen that the lowest order term of $(f_1^N - f_2^N)(t)$ is $2t^{-nN/2}$. It entirely comes from $2\sqrt{p}B(1/t)$. The lowest term in t of $pB^2(1/t)$ thus is t^{-nN}. We derive from $p(1/t) = 4t^{-3} - g_2 t^{-1} - g_3$ that $B^2(1/t)$ has $1/4$ as the coefficient of t^{-nN+3}. Hence $B(z)$ has $\pm 1/2 z^{(nN-3)/2}$ as leading term. This finishes the proof of the theorem. $\square$

Corollary 6.7.5 *One has* $n \notin \{-1, 0\}$ *if* L_n *has monodromy group* D_N.

Proof. Consider the degrees of the polynomials $A(z)$ and $B(z)$ of Theorem 6.7.4. They are $nN/2$ and $nN/2-n-2 = n(N/2-1)-2$ in the case of even nN. These degrees should also be non-negative. It follows from the condition $N \geq 3$ that n is at least 1 when nN is even. For nN odd the expressions $nN-2n-1 = n(N-2)-1$ and $nN-3$ should at least be 0. Again this yields $n \geq 1$. □

Definition 6.7.6 The polynomials $A(z)$ and $B(z)$ in z are defined as

$$\begin{aligned} A(z) &:= \left(f_1^N(z) + f_2^N(z)\right)/2 \\ B(z) &:= \left(f_1^N(z) - f_2^N(z)\right)/2\sqrt{p(z)}, \end{aligned}$$

in accordance with Theorem 6.7.4.

Proposition 6.7.7 *The polynomials* $A(z)$, $pB(z)$ *and* $P_n(z)$ *are pairwise coprime. Moreover, they satisfy*

$$A^2(z) - p(z)B^2(z) = (-1)^{nN} P_n^N(z).$$

Proof. The expression $4(A^2 - pB^2)$ equals

$$(f_1^N + f_2^N)^2 - (f_1^N - f_2^N)^2 = 4(f_1 f_2)^N$$

which is $4(-1)^{nN}P_n^N(z)$. This equation implies that any non-constant common factor of two of the polynomials A, pB and P_n is also a factor of the third. But then f_1 and f_2 would have a common complex zero α. This is a contradiction to one of the local exponents of α being 0. □

Theorem 6.7.8 *The rational map*

$$Q(z) := \frac{A^2(z)}{p(z)B^2(z)}$$

is of degree nN *and only ramifies above* 0, 1 *and* ∞. *Moreover, it is a Belyi-map with square-free polynomials* $A(z)$ *and* $pB(z)$.

Proof. The map $Q(z)$ has degree nN in z, since the maximum of the degrees of the coprime polynomials A^2 and pB^2 in z is nN. We consider the ramification indices of Q. Let us deal with nN even. The case of nN odd is similar.
If e_P denotes the ramification at a point $P \in \mathbb{P}^1$ of Q, then the Riemann-Hurwitz Formula (2.2) yields

$$2nN - 2 = \sum_{\{P:Q(P)\in\{0,1,\infty\}\}} (e_P - 1) + \sum_{\{P:Q(P)\notin\{0,1,\infty\}\}} (e_P - 1). \qquad (6.4)$$

It suffices to show

$$\sum_{\{P:Q(P)\in\{0,1,\infty\}\}} (e_P - 1) \geq 2nN - 2$$

for in that case there is no ramification above any point other than 0, 1 and ∞. One has $Q(\infty) = \infty$. The ramification index at $z = \infty$ then is

$$\begin{aligned} e_\infty &= nN - 2\deg_z(B) - 3 \\ &= nN - (nN - 2n - 4) - 3 \\ &= 2n + 1. \end{aligned}$$

It follows from

$$\frac{A^2(z)}{p(z)B^2(z)} - 1 = \frac{(-1)^{nN} P_n^N(z)}{p(z)B^2(z)}$$

that there are exactly n distinct points mapped to 1, as P_n is square-free. Each of these points have ramification index equal to N. We now derive

$$\begin{aligned} \sum_{\{P:Q(P)\in\{0,1,\infty\}\}} (e_P - 1) &= 2n + n(N-1) + \sum_{\{P:P\neq\infty,Q(P)\in\{0,\infty\}\}} (e_P - 1) \\ &\leq n + nN + \frac{nN}{2} + \frac{nN}{2} - n - 2 \\ &\leq 2nN - 2. \end{aligned}$$

Equality holds precisely when $A(z)$ and $pB(z)$ are square-free polynomials. It follows from Equation (6.4) that we are in this situation. □

Notice that $Q(z)$ does not ramify above points other than 0, 1 and ∞. It exactly satisfies the conditions of being a rational pull-back function $R(z)$ of the hypergeometric equation we have encountered before.

Theorem 6.7.9 *Let L_n denote a Lamé operator with given $n \in \mathbb{Z} \setminus \{0\}$ and group $M \cong D_N$. Then the number of algebraic Lamé operators L_n up to scalar equivalence is finite.*

Proof. We may suppose that M is given as $G(N, N, 2)$ with respect to a suitable basis (f_1, f_2) as above. Then there is a 1-1 correspondence between such bases (f_1, f_2) and the rational functions $Q(z)$. It is therefore sufficient to prove that the number of these functions $Q(z)$'s is finite up to scalar multiplications of z. The degree of the Belyi-map $Q(z)$ is nN. It is fixed, since n and N are. It follows from Theorem 1 of [Bir94] that the number of Belyi-maps up to projective linear transformations is finite. Consider a Belyi-map $R(z)$ that is a $Q(z)$ for a certain algebraic Lamé operator L_n as above. If analogously $R(az + b/cz + d)$ with $a, b, c, d \in \mathbb{C}$ and $ad - bc \neq 0$ belongs to a Lamé operator $\tilde{L}_n$, then $\phi := z \mapsto az + b/cz + d$ should satisfy $\phi(\infty) = \infty$. It follows that c is 0. We may assume

d to be 1. The finite singular points of $\tilde{L}_n$ are then $\{az_i + b : i = 1, 2, 3\}$. This yields $b = 0$, as the sum of the finite singular points is 0. □

We are now ready to describe the algorithm that determines all variables g_2 and g_3 of the Lamé equation for a given n, N and B, such the Lamé equation with these parameters is algebraic. The output contains the minimal polynomial $f_{\mathbb{Q}}^{g_2}(X)$ in X of g_2 over $\mathbb{Q}$ up to multiplication by constants. Analogously the polynomials $f_{\mathbb{Q}}^{g_3}(Y)$ and $f_{\mathbb{Q}(g_2)}^{g_3}(Y)$ in Y denote the minimal polynomial of g_3 over $\mathbb{Q}$ and $\mathbb{Q}(g_2)$, respectively.

Algorithm 4: lameDN(n, N, B)
Input: $n \in \mathbb{Z}_{>0}, N \in \mathbb{Z}_{\geq 3}, B \in \mathbb{C}$.
Output: The list L of arrays $[n, N, B, f_{\mathbb{Q}}^{g_2}, f_{\mathbb{Q}}^{g_3}, f_{\mathbb{Q}(g_2)}^{g_3}, A(z), B(z)]$ corresponding to the Lamé operators L_n having a basis of algebraic solutions with $M < D_N$ and parameter B.

1. Check the input on $n \in \mathbb{Z}_{\geq 1}$, $N \in \mathbb{Z}_{\geq 3}$ and $B \in \mathbb{C}$.
2. Compute s_1 and s_2 up to the orders $\max(\lceil nN/2 \rceil + 4, 2n+1)$ and $\lceil nN/2 + 7/2 \rceil - n$ in t, respectively.
3. Determine γ.
4. Determine $(f_1^N + f_2^N)(t)$ up to degree 4 in t.
5. Determine the first 4 coefficients C_1, C_2, C_3 and C_4 of the positive powers of t in $(f_1^N + f_2^N)(t)$.
6. Replace C_i by C_i^2 in the case of odd nN.
7. Factor out all occurring factors of γ in the C_i's. The resulting polynomials in $\mathbb{Q}[g_2, g_3, B]$ are E_1, E_2, E_3 and E_4.
8. Set $L := [\,]$.
9. Compute all $[f_{\mathbb{Q}}^{g_2}, f_{\mathbb{Q}}^{g_3}, f_{\mathbb{Q}(g_2)}^{g_3}]$ for the system $\{E_i = 0 | i = 1, 2, 3, 4\}$.
   Return the list S with operands $[f_{\mathbb{Q}}^{g_2}, f_{\mathbb{Q}}^{g_3}, f_{\mathbb{Q}(g_2)}^{g_3}]$.
10. **for** $[f_{\mathbb{Q}}^{g_2}, f_{\mathbb{Q}}^{g_3}, f_{\mathbb{Q}(g_2)}^{g_3}] \in S$ **do**
    - Let g_2 and g_3 be formal roots of $f_{\mathbb{Q}}^{g_2}$ and $f_{\mathbb{Q}(g_2)}^{g_3}$, respectively.
    - **if** the discriminant of $p(z)$ with respect to z is non-zero **then**
        - Compute $A(z)$ as the polynomial part of $(f_1^N + f_2^N)/2$ given as a series in $z = 1/t$.
        - Compute $B(z)$ from $A^2(z) - p(z)B^2(z) = (-1)^{nN} P_n^N(z)$ as a polynomial in $z = 1/t$.
        - **if** $L_n(A(z) \pm \sqrt{p(z)}B(z))^{1/N} = 0$ **then**
            - Add $[n, N, B, f_{\mathbb{Q}}^{g_2}, f_{\mathbb{Q}}^{g_3}, f_{\mathbb{Q}(g_2)}^{g_3}, A(z), B(z)]$ as an entry to L.
11. **Return** L.

Remark 6.7.10 Given n and N, it is necessary and sufficient to run the algorithm twice; once with $B = 0$ and once for a chosen $B \in \mathbb{C}^*$. Strategy 6.4.1 then describes how to obtain all Lamé equations L_n with monodromy group $M < D_N$.

We have implemented and run the Algorithm lameDN in Maple 5.3. We refer to Table A.4 of Appendix A for a few examples of the output. The algorithm shows that there is only one Lamé operator L_n for $n = 1$ and $N = 3$. It is

$$L_n = (4z^3 - g_3)\frac{d^2}{d^2z} + 6z^2\frac{d}{dz} - 2z$$

for every $g_3 \in \mathbb{C}^*$. This Lamé operator is scaled equivalent to one which is a rational pull-back of a hypergeometric equation by $f(z) = z^3$, see Section 5.5 and in particular Example 5.5.10.

6.8 Specific invariants for G_{12}, G_{13} and G_{22}

In Chapter 5 we determined all finite monodromy groups M that project to the octahedral or icosahedral groups. In the octahedral case M is either G_{12} or G_{13}. For an icosahedral projective group the monodromy group is G_{22}. In this section M is G_{12}, G_{13} or G_{22} unless stated otherwise. Table 5.1 and Remark 5.2.20 give a characterisation of M in terms of n. We are going to consider M with respect to the basis $(y_1(t), y_2(t))$. As usual we take

$$\begin{aligned} y_1(t) &= t^{-n/2}s_1(t) \\ y_2(t) &= t^{(n+1)/2}s_2(t), \end{aligned}$$

as in Definition 4.3.1. Notice that one has $-n/2 \leq (n+1)/2$ due to the assumption $n \geq -1/2$. We shall construct certain invariant polynomials of minimal degree for this particular representation. Analogously to the dihedral case the invariants are then used for the determination of all g_2, g_3 and B.

Assumption 6.8.1 *We assume M to be represented with respect to the ordered basis $(y_1(t), y_2(t))$ of the solution space of L_n..*

Lemma 6.8.2 *One has*

$$\gamma_\infty = \begin{pmatrix} \mathrm{e}^{-\frac{n}{2}(2\pi \mathrm{i})} & 0 \\ 0 & \mathrm{e}^{(\frac{n+1}{2})(2\pi \mathrm{i})} \end{pmatrix}.$$

Proof. The functions $y_1(t)$ and y_2 are multiplied by $\mathrm{e}^{-\frac{n}{2}(2\pi \mathrm{i})}$ and $\mathrm{e}^{(\frac{n+1}{2})(2\pi \mathrm{i})}$, respectively, after analytic continuation along a single closed path around $z = \infty$ that has no other singular points of L_n inside. □

The group M acts on the column vector $(X, Y)^T$ by left multiplication as usual. Let $J_m(X, Y)$ be a homogeneous polynomial of minimal degree m, that is invariant under M. So, m is 6, 8 and 12 for $M = G_{12}$, G_{13} and G_{22}, respectively. We write $J_m(X, Y)$ as

$$J_m(X, Y) := a_m X^m + a_{m-1} X^{m-1} Y + \cdots + a_0 Y^m,$$

for certain $a_0, a_1, \ldots, a_m \in \mathbb{C}$.

Remark 6.8.3 Notice that J_m is actually determined up to multiplication by a non-zero constant. So we may and shall assume one particular non-zero coefficient of J_m to be 1 whenever we want to.

Lemma 6.8.4 *The function $J_m(y_1, y_2)$ is a polynomial in $1/t$.*

Proof. The function $J_m(y_1, y_2)$ is a solution of the m-th symmetric power of L_n. It is invariant under the action of M and thus is a rational function in $z = 1/t$. Moreover, the function $J_m(y_1, y_2)$ is a polynomial in z, since $J_m(y_1, y_2)$ has non-negative exponents at any point in $\mathbb{C}$. □

Definition 6.8.5 The degree of $J_m(y_1, y_2)$ in $z = 1/t$ is denoted by d_m.

Lemma 6.8.6 *Let m and j be non-negative integers with $0 \le j \le m$. Then the coefficient a_j of $J_m(X, Y)$ is 0 when $\frac{m}{2}(n+1) - j(n + \frac{1}{2}) \notin \mathbb{Z}$.*

Proof. Let j be an integer with $0 \le j \le m$. The monodromy group acts invariant on $J_m(X, Y)$. In particular γ_∞ fixes $J_m(X, Y)$. Lemma 6.8.2 implies $\gamma_\infty \cdot X^j Y^{m-j} = \mathrm{e}^{2\pi \mathrm{i}\left(\frac{nj}{2} + (m-j)\frac{n+1}{2}\right)} X^j Y^{m-j}$. It follows that $\frac{nj}{2} + (m-j)\frac{n+1}{2}$ is an integer if a_j is non-zero. □

Proposition 6.8.7 *The following holds.*

(i) For $M = G_{12}$ one has

$$J_6(X, Y) = XY(X^4 + \gamma Y^4), \tag{6.5}$$

for a certain $\gamma \in \mathbb{C}$.

(ii) For $M = G_{13}$ we have

$$J_8(X,Y) = XY(X^6 + \beta X^3Y^3 + \gamma Y^6) \tag{6.6}$$

for certain $\beta, \gamma \in \mathbb{C}$.

(iii) If M is G_{22} and n is contained in $\{\pm 1/10, \pm 3/10\} + \mathbb{Z}$ then

$$J_{12}(X,Y) = XY(X^{10} + \beta X^5Y^5 + \gamma Y^{10}) \tag{6.7}$$

holds for certain $\beta, \gamma \in \mathbb{C}$.

(iv) If $M = G_{22}$ and n is $\pm 1/6 \bmod \mathbb{Z}$ then

$$J_{12}(X,Y) = X^{12} + a_9X^9Y^3 + a_6X^6Y^6 + a_3X^3Y^9 + a_0Y^{12} \tag{6.8}$$

holds for certain $a_0, a_3, a_6, a_9 \in \mathbb{C}$.

Proof. Let m be 6, 8 or 12 when M is G_{12}, G_{13} and G_{22}, respectively. Let i be an integer with $0 \leq i \leq m$. According to Lemma 6.8.6 the term X^iY^{m-i} can only appear in the polynomial $J_m(X,Y)$ if $\frac{m}{2}(n+1) - i(n+\frac{1}{2})$ is an integer. Except for $M = G_{22}$ and $n \in \pm\{1/6\} + \mathbb{Z}$ the index n can be written as $n = a/(m-2)$ with $a \in \mathbb{Z}$ and $\overline{a} \in (\mathbb{Z}/(m-2)\mathbb{Z})^*$, see Corollary 5.2.18 and Theorem 5.2.11. We consider this case first. From the fact that m is even we derive

$$\begin{aligned} \frac{m}{2}(n+1) - i(n+\frac{1}{2}) \in \mathbb{Z} \quad &\iff \quad \frac{ma}{2(m-2)} + \frac{-ia}{m-2} + \frac{1}{2}(m-i) \in \mathbb{Z} \\ &\iff \quad \frac{a(m-2i)}{2(m-2)} - \frac{i}{2} \in \mathbb{Z}. \end{aligned}$$

Recall that $\gcd(a, 2(m-2)) = 1$. This implies $m/2 - i \equiv 0 \bmod (m-2)\mathbb{Z}$ if i is even. The conditions $m > 4$ and $0 \leq i \leq m$ then lead to $i = m/2$. In particular, this only occurs for $m = 8$ and $m = 12$ as i was assumed to be even.
Any odd i satisfies

$$\begin{aligned} \frac{a(m-2i)}{2(m-2)} - \frac{i}{2} \in \mathbb{Z} \quad &\iff \quad \frac{a(m-2i)}{(m-2)} \equiv 1 \bmod 2\mathbb{Z} \\ &\iff \quad \frac{(m-2i)}{(m-2)} \equiv 1 \bmod 2\mathbb{Z} \\ &\iff \quad \frac{(-2i+2)}{(m-2)} \equiv 0 \bmod 2\mathbb{Z} \\ &\iff \quad 1 - i \equiv 0 \bmod (m-2)\mathbb{Z}. \end{aligned}$$

It follows that i is 1 or $m-1$. We conclude that J_m is of the form

$$J_m(X,Y) = \alpha_m X^{m-1}Y + \beta_m X^{m/2}Y^{m/2} + \gamma_m XY^{m-1},$$

with $\alpha_m, \beta_m, \gamma_m \in \mathbb{C}$ for all m and $\beta_6 = 0$. If α_m is 0 then on hand $J_m(y_1, y_2)$ is a polynomial in $1/t$, see Lemma 6.8.4. On the other hand it would only have positive powers of t in its Laurent expansion. This implies $J_m(y_1, y_2) \equiv 0$ and thus either $y_1 = 0$, $y_2 = 0$ or $\beta_m y_1^{m/2-1} + \gamma_m y_2^{m/2-1} = 0$. All of these possibilities give a contradiction. Hence the index α_m is non-zero. The invariants as described in the cases (i), (ii) and (iii) of the proposition are obtained after the dividing J_m by α_m.
This leaves us to the proof of item (iv) of the proposition. The invariant is J_{12} for $M = G_{22}$ and $n \equiv \pm 1/6 \bmod \mathbb{Z}$. As before one has $n = a/6$ for an integer a with $\overline{a} \in (\mathbb{Z}/6\mathbb{Z})^*$. This yields

$$\begin{aligned} \frac{12}{2}(n+1) - i(n+\frac{1}{2}) \in \mathbb{Z} \quad &\Longleftrightarrow \quad 6(\frac{a}{6}+1) - i(\frac{a}{6}+\frac{1}{2}) \in \mathbb{Z} \\ &\Longleftrightarrow \quad \frac{-i(a+3)}{6} \in \mathbb{Z} \\ &\Longleftrightarrow \quad i \equiv 0 \bmod 3. \end{aligned}$$

If a_{12} were to be 0, then J_{12} would have a triple root $(X, Y) = (1, 0)$. Then so would Klein's invariant of degree 12 of the icosahedral group. However, this invariant polynomial has 12 distinct roots, see Proposition 3.6.2. Hence, a_{12} is non-zero. Finally, Equation (6.8) follows from the condition $0 \le i \le 12$, if we assume a_{12} to be 1. □

Corollary 6.8.8 *Let m be 6, 8 or 12 in the case of M being G_{12}, G_{13} or G_{22}, respectively. Then*

(i) one has

$$d_m = \frac{mn-1}{2} - n$$

except for $M = G_{22}$ with $n \equiv \pm 1/6 \bmod \mathbb{Z}$.

(ii) For $M = G_{22}$ with $n \equiv \pm 1/6 \bmod \mathbb{Z}$ we have $d_{12} = 6n$.

Proof. We adopt the notation as in Proposition 6.8.7. We take m to be 6, 8 or 12 as stated in the corollary. Suppose that M is not G_{22} with $6n \in \mathbb{Z}$. Then the proposition implies that the lowest order term in t of $J_m(y_1, y_2)$ comes from $X^{m-1}Y$. Hence, its order in t is $-n(m-1)/2 + (n+1)/2$. The polynomial degree in z thus is $nm/2 - n - 1/2$, as had to be proved. For the remaining case the degree in t is $-12n/2$ as X^{12} yields the lowest order term of $J_{12}(y_1, y_2)$ in t. □

Theorem 6.8.9 *If the Lamé operator L_n has a finite monodromy group, then one has $n \notin [-1, 0]$.*

Proof. Suppose that L_n has a finite monodromy group. As usual we assume n to be at least $-1/2$. The Lamé operator L_n has an infinite monodromy group for $n = -1/2$, since $1/4$ is a double exponent at infinity. Due to Corollary 5.2.9, the lowest index that occurs for $M = G(4, 2, 2)$ is $n = 1/2$. For the dihedral monodromy groups the conclusion $n \neq 0$ is already covered by Corollary 6.7.5. So we have $n \geq 1$ in this case.
Let us consider $M = G_{22}$ with $6n \in \mathbb{Z}$. According to item (ii) of Corollary 6.8.8 the polynomial J_{12} is of degree $6n \geq 0$ in $1/t$. We even have $6n \geq 1$, since $n = 0$ belongs to dihedral groups. In particular $n \geq 1/6$ follows.
Finally, let M be G_{12}, G_{13} or G_{22}. On the latter we put the restriction $10n \in \mathbb{Z}$. We are in case (i) of Corollary 6.8.8. From $d_m \geq 0$ we deduce $n \geq 1/(m-2)$. As a consequence n must be positive. □

Definition 6.8.10 In the remainder of this chapter the polynomial $J_m(X, Y)$ denotes $J_6(X, Y)$, $J_8(X, Y)$ or $J_{12}(X, Y)$ as given in Proposition 6.8.7.

6.9 Algorithms for G_{12}, G_{13} and G_{22}

We adopt the notation and assumptions as in the previous section. To explicitly determine g_2, g_3 and B for given $M \in \{G_{12}, G_{13}, G_{22}\}$ we distinguish two cases. In the first case M is G_{12}, G_{13} or G_{22} with $10n \in \mathbb{Z}$. The second is the situation in which M is G_{22} and $6n$ is an integer. For each of these two cases we shall give an algorithm that computes amongst other things g_2 and g_3 for given M, n and B. The algorithms use the series expansion of $J_m(y_1, y_2)$ in t.

The function $J_m(y_1, y_2)$ is a polynomial in $1/t$. At the same time the series expansions of y_1 and y_2 in t yield $J_m(y_1, y_2)$ as a series in t. Formally it can be given as a Laurent series.

Definition 6.9.1 The Laurent series of $J_m(y_1, y_2)$ in t is denoted by

$$J_m(y_1, y_2) := \sum_{i=-d_m}^{\infty} c_i t^i \tag{6.9}$$

with $c_i \in \mathbb{Q}[g_2, g_2, B]$ for every i.

The fact that such an expansion with coefficients in $\mathbb{Q}[g_2, g_2, B]$ exists, follows from the following lemma.

Lemma 6.9.2 *Every coefficient c_i in* (6.9) *is a polynomial in* $\mathbb{Q}[g_2, g_2, B]$. *Moreover, for $i = 1, 2, \ldots$ one has $c_i = 0$.*

Proof. It follows from Corollary 6.2.2 that the series expansions of y_1 and y_2 in t have coefficients in $\mathbb{Q}[g_2, g_2, B]$. Then so does the Laurent expansion of $J_m(y_1, y_2)$. Each coefficient c_i that belongs to the positive power t^i must be 0, since $J_m(y_1, y_2)$ is a polynomial in $1/t$. This proves the lemma. □

Let us take a closer look at where the $c_i' s$ come from and what they look like. We first consider the situation in which $J_m(X, Y)$ is of the form

$$X^{m-1}Y + \beta X^{m/2}Y^{m/2} + \gamma XY^{m-1},$$

as in Equations (6.5), (6.6) and (6.7). Notice that β is 0 for J_6. The term $y_1^{m-1}y_2$ appears in $J_m(y_1, y_2)$. It is a Laurent series in t that will be written as

$$y_1^{m-1}y_2 := \sum_{i=-d_m}^{\infty} u_i t^i.$$

Each u_i is contained in $\mathbb{Q}[g_2, g_2, B]$, as the Puiseux expansions of y_1 and y_2 in t have coefficients in $\mathbb{Q}[g_2, g_2, B]$. The analogue is true for the series of $y_1 y_2^{m-1}$ and $y_1^{m/2}y_2^{m/2}$. We also fix notation for these expansions.

Definition 6.9.3 For $m = 8$ or 12 we denote the Laurent series $y_1^{m/2}y_2^{m/2}$ in t by

$$y_1^{m/2}y_2^{m/2} := \sum_{i=d_\beta}^{\infty} v_i t^i$$

with $v_i \in \mathbb{Q}[g_2, g_2, B]$ for each i.

Definition 6.9.4 The Laurent series $y_1 y_2^{m-1}$ in t is defined as

$$y_1 y_2^{m-1} := \sum_{i=d_\gamma}^{\infty} w_i t^i$$

with $w_i \in \mathbb{Q}[g_2, g_2, B]$ for each i.

Proposition 6.9.5 *Let*

$$J_m(X, Y) = X^{m-1}Y + \beta X^{m/2}Y^{m/2} + \gamma XY^{m-1}$$

be in accordance with Equations (6.5), (6.6) *and* (6.7). *Then one has*

$$d_\beta = m/4, \quad d_\gamma = (m-2)n/2 + (m-1)/2,$$

$v_{d_\beta} = 1$ *and* $w_{d_\gamma} = 1$. *In particular, we have*

$$c_i = \begin{cases} u_i & \text{for} \quad i = -d_m, -d_m + 1, \ldots, \frac{m}{4} - 1 \\ u_i + \beta & \text{for} \quad i = \frac{m}{4} \\ u_i + \beta v_i & \text{for} \quad i = \frac{m}{4} + 1, \frac{m}{4} + 2, \ldots, \frac{(m-2)n}{2} + \frac{(m-3)}{2} \\ u_i + \beta v_i + \gamma & \text{for} \quad i = \frac{(m-2)n}{2} + \frac{(m-1)}{2} \end{cases}$$

in which β *and each* βv_i *have to be taken* 0 *in the case* $m = 6$.

Proof. The lowest power of t that occurs in $y_1 y_2^{m-1}$ is of degree $-n/2 + (m-1)(n+1)/2$. A short calculation then gives $d_\gamma = (m-2)n/2 + (m-1)/2$. The index d_β is $-nm/4 + m/2(n+1)/2 = m/4$. We derive from $v_{d_\beta} = s_1^{m/4} s_2^{m/4}(0)$ and $w_{d_\gamma} = s_1 s_2^{m-1}(0)$ that v_{d_β} and w_{d_γ} are 1. The specific expressions for the c_i then follow from the identity

$$J_m(y_1, y_2) = \sum_{i=-d_m}^{\infty} u_i t^i + \beta \sum_{i=d_\beta}^{\infty} v_i t^i + \gamma \sum_{i=d_\gamma}^{\infty} w_i t^i,$$

in which the second summand has to be taken 0 for $m = 6$. □

Corollary 6.9.6 *For* $M = G_{12}$ *one has* $\gamma = -u_{2n+5/2}$. *We have* $\beta = -u_2$ *and* $\gamma = -u_{3n+7/2} - \beta v_{3n+7/2}$ *in the case when* $M = G_{13}$. *Finally, one has* $\beta = -u_3$ *and* $\gamma = -u_{5n+11/2} - \beta v_{5n+11/2}$ *for* $M = G_{22}$ *with* $10n \in \mathbb{Z}$.

Proof. According to Lemma 6.9.2 each c_i with $i \geq 1$ must be 0. For $M = G_{12}$, G_{13} or G_{22} the index m is 6, 8 or 12, respectively. The substitution of each value of m into the equations of $c_{m/4}$ and $c_{(m-2)n/2+(m-1)/2}$ yields the identities as required. □

The results obtained in this section are used in the algorithm lameG. This algorithm gives all Lamé equations with given monodromy group $M \in \{G_{12}, G_{13}\}$, index n and parameter B. It also works for $M = G_{22}$ with $10n \in \mathbb{Z}$.

Algorithm 5: lameG(M, n, B)
Input: $M \in \{G_{12}, G_{13}, G_{22}\}$, $n > 0$ with the condition $10n \in \mathbb{Z}$ for $M = G_{22}$, and $B \in \mathbb{C}$.
Output: The list L of arrays $[M, n, B, f_{\mathbb{Q}}^{g_2}, f_{\mathbb{Q}}^{g_3}, f_{\mathbb{Q}(g_2)}^{g_3}, \beta, \gamma, J_m(y_1, y_2)(t)]$ that correspond to the Lamé operators having monodromy group M, index n, parameter B and invariant J_m.

1. Check the input on the validity of M and n.
2. Check the input on the validity of B.

3. Let m be 6, 8 or 12 in case of $M = G_{12}$, G_{13} or G_{22}, respectively.
4. Define $d_m := nm/2 - n - 1/2$ and $d_\gamma := (m-2)n/2 + (m-1)/2$.
5. Compute s_1 and s_2 up to degree $d_m + 5$ in t.
6. Determine $su := y_1^{m-1} y_2$ up to degree 5 in t.
7. **for** $M \neq G_{12}$ **do**

   $sv := y_1^{m/2} y_2^{m/2}$ in t up to degree 5.
8. Set $L := [\,]$.
9. **if** $M = G_{12}$ **then**
   - Set $\beta := 0$.
   - Set $E := [u_1, u_2, \ldots, u_j]$ for $j = \min(d_\gamma - 1, 4)$.

   elif $M = G_{13}$ **then**
   - Set $\beta := -u_2$.
   - Set $E := [u_1, u_3 + \beta v_3, u_4 + \beta v_4, \ldots, u_j + \beta v_j]$ for $j = \min(d_\gamma - 1, 5)$.

   else
   - Set $\beta := -u_3$.
   - Set $E := [u_1, u_2, u_4 + \beta v_4, u_5 + \beta v_5]$.
10. Determine all $[f_{\mathbb{Q}}^{g_2}, f_{\mathbb{Q}}^{g_3}, f_{\mathbb{Q}(g_2)}^{g_3}]$ for the set of equations $\{E[i] = 0 | i = 1, 2, \ldots, |E|\}$.
    Return the list S with operands $[f_{\mathbb{Q}}^{g_2}, f_{\mathbb{Q}}^{g_3}, f_{\mathbb{Q}(g_2)}^{g_3}]$.
11. **for** $[f_{\mathbb{Q}}^{g_2}, f_{\mathbb{Q}}^{g_3}, f_{\mathbb{Q}(g_2)}^{g_3}] \in S$ **do**
    - Let g_2 and g_3 be formal roots of $f_{\mathbb{Q}}^{g_2}$ and $f_{\mathbb{Q}(g_2)}^{g_3}$, respectively.
    - **if** the discriminant of $p(z)$ with respect to z is non-zero **then**
      - Compute s_1 and s_2 up to degree $d_m + \max(d_\gamma, 3m/2 + 2)$ in t.
      - Compute su up to degree $\max(d_\gamma, 3m/2 + 2)$ in t.
      - Compute sv up to degree $\max(d_\gamma, 3m/2 + 2)$ in t when $M \neq G_{12}$.
      - Define $\gamma := -u_{d_\gamma} - \beta v_{d_\gamma}$.
      - Compute $J_m(y_1, y_2)$ up to degree $3m/2 + 2$ in t.
      - If $J_m(y_1, y_2)$ is a polynomial in $1/t$ **then**
        - Add $[M, n, B, f_{\mathbb{Q}}^{g_2}, f_{\mathbb{Q}}^{g_3}, f_{\mathbb{Q}(g_2)}^{g_3}, \beta, \gamma, J_m(y_1, y_2)(t)]$ as an operand to L.
12. **Return** L.

Some of the output that we have obtained from the algorithm is given in Tables $A.5$, $A.6$ and $A.7$ of Appendix A.

Notice that the algorithm lameG uses the series expansions of $J_8(y_1, y_2)$ in the case $M = G_{13}$. The group G_{13} has a semi-invariant homogeneous polynomial

$J'_6(X,Y)$ of degree 6, which is an invariant for G_{12}. Computations would be easier and analogous to the ones above if J'_6 were in fact an invariant for $\gamma_\infty \in G_{13}$. However, it turns out that γ_∞ does not fix J'_6.

Proposition 6.9.7 *If G_{13} is the monodromy group of the Lamé equation, then γ_∞ acts non-trivially on J'_6.*

Proof. Let M be G_{13}. The matrix γ_∞ is an element of $G_{13} \setminus G_{12}$ according to Theorem 5.2.17. It then follows from Proposition 3.6.1 that J'_6 is a semi-invariant for γ_∞. □

We move to the remaining case in which $M = G_{22}$ and $6n \in \mathbb{Z}$. The invariant polynomial $J_{12}(X,Y)$ is then of the form

$$X^{12} + a_9X^9Y^3 + a_6X^6Y^6 + a_3X^3Y^9 + a_0Y^{12},$$

for certain $a_0, a_3, a_6, a_9 \in \mathbb{C}$. These unknowns can be obtained in a similar way as for β and γ in Proposition. As before we use the series expansion of the monomials in y_1 and y_2 of $J_{12}(y_1, y_2)$ in t. We write

$$y_1^k y_2^{12-k} := \sum_{i=d_k}^{\infty} v_{k,i}t^i, \quad k = 0, 3, 6, 9, 12,$$

in which d_k denotes the lowest degree of $y_1^k y_2^{12-k}$ in t and each $v_{k,i}$ is contained in $\mathbb{Q}[g_2, g_3, B]$. Notice that v_{k,d_k} is 1 for all k, as it is equal to $s_1^k s_2^{12-k}(0)$.

Lemma 6.9.8 *Let M be G_{22} with $6n \in \mathbb{Z}_{>0}$. Then one has*

$$c_i = \begin{cases} v_{12,i} & , -6n \leq i \leq -3n + \frac{1}{2} \\ v_{12,i} + a_9 v_{9,i} & , -3n + \frac{3}{2} \leq i \leq 2 \\ v_{12,i} + a_9 v_{9,i} + a_6 v_{6,i} & , 3 \leq i \leq 3n + \frac{7}{2} \\ v_{12,i} + a_9 v_{9,i} + a_6 v_{6,i} + a_3 v_{3,i} & , 3n + \frac{9}{2} \leq i \leq 6n + 5 \\ v_{12,i} + a_9 v_{9,i} + a_6 v_{6,i} + a_3 v_{3,i} + a_0 v_{0,i} & , i \geq 6n + 6. \end{cases}$$

Proof. The lowest degree of the series $y_1^k y_2^{12-k}$ in t for $k \in \{0, 3, 6, 9, 12\}$ is $-kn/2 + (12-k)(n+1)/2$. The latter expression is equal to $-k(n+1/2)+6n+6$. It follows that the lemma is a direct consequence of the identity

$$J_{12}(y_1, y_2) = y_1^{12} + a_9 y_1^9 y_2^3 + a_6 y_1^6 y_2^6 + a_3 y_1^3 y_2^9 + a_0 y_2^{12}$$

after substitution of the series expansions of $J_m(y_1, y_2)$ and of each involving product $y_1^k y_2^{12-k}$. □

We would like to remark the following. If n is 1/6, then the lowest term of $J_{12}(y_1, y_2)$ in which a_9 appears is c_1. This yields $a_9 = -v_{12,1}$, since c_1 has to be 0. In the case $n \geq 5/6$ one has $-3n + \frac{3}{2} \leq -1$. Then there are two linear equations

$$\begin{aligned} v_{12,1} + a_9 v_{9,1} &= 0 \\ v_{12,2} + a_9 v_{9,2} &= 0 \end{aligned}$$

for a_9 with coefficients in $\mathbb{Q}[g_2, g_3, B]$. The unknowns a_0, a_3 and a_6 are easier to determine.

Lemma 6.9.9 *Let M be G_{22} with $6n \in \mathbb{Z}_{>0}$. Then we have*

$$\begin{aligned} a_6 &= -v_{12,3} - a_9 v_{9,3} \\ a_3 &= -v_{12,3n+9/2} - a_9 v_{9,3n+9/2} - a_6 v_{6.3n+9/2} \\ a_0 &= -v_{12,6n+6} - a_9 v_{9,6n+6} - a_6 v_{6,6n+6} - a_3 v_{3,6n+6} \end{aligned}$$

In addition, we have $a_9 = -v_{12,1}$ for $n = 1/6$.

Proof. We need to prove the identities for a_6, a_3 and a_0. Each coefficient c_i, $i > 0$, of $J_{12}(y_1, y_2)$ must be 0. In particular this is true for c_3, $c_{3n+9/2}$ and c_{6n+6}. The equations of a_6, a_3 and a_0 then follow from Lemma 6.9.8 as the coefficients $v_{6,3}$, $v_{3,3n+9/2}$ and $v_{0,6n+6}$ are 1. □

We have gained enough information to give an algorithm that is similar to the algorithm lameG. It is called lameG22(n, B) and finds all Lamé equations with $M = G_{22}$, given $n \in \{1/6, 5/6\} + \mathbb{Z}_{\geq 0}$ and parameter B. It also gives $J_{12}(X, Y)$ and the polynomial $J_{12}(y_1, y_2)$ in $1/t$.

Algorithm 6: lameG22(n, B)
Input: $n \in \{1/6, 5/6\} + \mathbb{Z}_{\geq 0}$ and B.
Output: The list L of arrays $[G_{22}, n, B, f_{\mathbb{Q}}^{g_2}, f_{\mathbb{Q}}^{g_3}, f_{\mathbb{Q}(g_2)}^{g_3}, a_9, a_6, a_3, a_0]$. Its entries correspond to the Lamé operators having monodromy group G_{22}, index n, parameter B and invariant J_{12}.

1. Check the input on the validness of n and B.
2. Compute s_1 up to the order $6n + 8$ in t
3. Compute s_2 up to the order $3n - 3/2 + 8$ in t.
4. Determine $su_k := y_1^k y_2^{12-k}$ up to degree 8 in t for $k = 0, 3, 6, 9, 12$.
5. Set $L := [\,]$.
6. **if** $n = 1/6$ **then**
   - Determine a_9, a_6, a_3 and a_0 as in Lemma 6.9.9.
   - Set $E := [c_2, c_4, c_6, c_8]$.

else

- Determine a_6 as in Lemma 6.9.9.
- **if** $n = 5/6$ **then**
  - Determine a_3 as in Lemma 6.9.9.
  - Set $C := [c_1, c_2, c_4, c_5, c_6, c_8]$.

  else
  - Set $C := [c_1, c_2, c_4, c_5, c_6, c_7]$
- Define $d_1 := \text{Resultant}(C[1], C[2], a_9)$, the resultant of $C[1]$ and $C[2]$ by eliminating a_9. Analogously define $d_2 := \text{Resultant}(C[3], C[4], a_9)$ and $d_3 := \text{Resultant}(C[5], C[6], a_9)$.
- Set $E := [d_1, d_2, d_3]$.

7. Determine $[f_{\mathbb{Q}}^{g_2}, f_{\mathbb{Q}}^{g_3}, f_{\mathbb{Q}(g_2)}^{g_3}]$ for the set of equations $\{E[i] = 0 | i = 1, 2, \ldots, |E|\}$.
   Return the list S with operands $[f_{\mathbb{Q}}^{g_2}, f_{\mathbb{Q}}^{g_3}, f_{\mathbb{Q}(g_2)}^{g_3}]$.
8. **for** $[f_{\mathbb{Q}}^{g_2}, f_{\mathbb{Q}}^{g_3}, f_{\mathbb{Q}(g_2)}^{g_3}] \in S$ **do**
   - Let g_2 and g_3 be formal roots of $f_{\mathbb{Q}}^{g_2}$ and $f_{\mathbb{Q}(g_2)}^{g_3}$, respectively.
   - **if** the discriminant of $p(z)$ with respect to z is non-zero **then**
     - Compute s_1 up to degree $6n + \max(6n + 6, 20)$ in t.
     - Compute s_2 up to degree $3n - 3/2 + \max(6n + 6, 20)$ in t.
     - Determine a_9, a_6, a_3 and a_0
     - Compute $J_{12}(y_1, y_2)$ up to degree 20 in t.
     - **if** $J_{12}(y_1, y_2)$ is a polynomial in $1/t$ **and** $J_{12}(y_1, y_2)$ is not a square **then**
       - Add $[G_{22}, n, B, f_{\mathbb{Q}}^{g_2}, f_{\mathbb{Q}}^{g_3}, f_{\mathbb{Q}(g_2)}^{g_3}, J_{12}(X, Y), J_{12}(y_1, y_2)]$ as an entry to L.
9. **Return** L.

An example of an algebraic Lamé equation obtained from the algorithm is given in Table A.7 of Appendix A.

Given finite M and index n we have described algorithms that will give all of their Lamé equations with algebraic solution spaces. Until now each valid combination of M and n, as in Table 5.1, leads to an algebraic Lamé equation. We have not yet encountered a single example in which there are no solutions.

All algebraic Lamé equations with $B = 0$ and $M \neq G(4, 2, 2)$ that we have obtained from the algorithms, are Lamé equations that are rational pull-backs by $f(z) = z^2$ or $g(z) = z^3$ as in Section 5.5. We conjecture that there are no other algebraic Lamé equations with accessory parameter $B = 0$.

Appendix A

Computational results

In this appendix we list the output that we obtained after several runs of the algorithms of Chapter 6. The tables we have put these data in are:

$A.1$: The invariant polynomial $R_n(X)$ of degree $n + 1/2$ for $0 < n < 10$.

$A.2$: The polynomial $P_n(z)$ of degree n for $0 \leq n \leq 7$.

$A.3$: The constant γ^2 of $P_n(z)$ for $0 \leq n \leq 7$.

$A.4$: All output of lameDN(n,N,B) with $M < D_N$ for $0 < nN \leq 12$, $N \leq 10$ and $B = 0, 1$.

$A.5$: Data of all Lamé equations with $M = G_{12}$, $0 < n \leq 9/4$ and $B = 0, 1$.

$A.6$: Data of all Lamé equations with $M = G_{13}$, $0 < n \leq 7/6$ and $B = 0, 1$.

$A.7$: Data of all Lamé equations with $M = G_{22}$, $0 < n \leq 7/10$ and $B = 0, 1$.

n	$R_n(X)$
$\frac{1}{2}$	X
$1\frac{1}{2}$	$X^2 - \frac{3}{4}g_2$
$2\frac{1}{2}$	$X^3 - 7g_2X + 20g_3$
$3\frac{1}{2}$	$X^4 - \frac{63}{2}g_2X^2 + 216g_3X + \frac{945}{16}g_2^2$
$4\frac{1}{2}$	$X^5 - 99g_2X^3 + 1188g_3X^2 + 1188g_2^2X - 11664g_2g_3$
$5\frac{1}{2}$	$X^6 - \frac{1001}{4}g_2X^4 + 4576g_3X^3 + \frac{172315}{16}g_2^2X^2 - 231400g_2g_3X - \frac{2338875}{64}g_2^3 + 616000g_3^2$
$6\frac{1}{2}$	$X^7 - 546g_2X^5 + 14040g_3X^4 + 63297g_2^2X^3 - 2232360g_2g_3X^2 - 1168668g_2^3X + 13122000g_3^2X + 24766560g_2^2g_3$
$7\frac{1}{2}$	$X^8 - 1071g_2X^6 + 36720g_3X^5 + \frac{2244051}{8}g_2^2X^4 - 14375880g_2g_3X^3 + 138801600g_3^2X^2 - \frac{265557663}{16}g_2^3X^2$ $+737817255g_2^2g_3X - 5401015200g_2g_3^2 + \frac{22347950625}{256}g_2^4$
$8\frac{1}{2}$	$X^9 - 1938g_2X^7 + 85272g_3X^6 + 1016481g_2^2X^5 - 70612968g_2g_3X^4 - 148922380g_2^3X^3 + 988364496g_3^2X^3$ $+10509076320g_2^2g_3X^2 + 3888129600g_2^4X - 164219462400g_2g_3^2X - 144441792000g_2^3g_3 + 426922496000g_3^3$
$9\frac{1}{2}$	$X^{10} - \frac{13167}{4}g_2X^8 + 180576g_3X^7 + \frac{25320141}{8}g_2^2X^6 - 284665752g_2g_3X^5 + 5381722944g_3^2X^4 - \frac{31382011143}{32}g_2^3X^4$ $+97881000210g_2^2g_3X^3 - 2445522365664g_2g_3^2X^2 + \frac{19627235976789}{256}g_2^4X^2 - \frac{11670300496953}{2}g_2^3g_3X$ $+13604889600000g_3^3X - \frac{584689432201875}{1024}g_2^5 + 82113138102996g_2^2g_3^2$

Table A.1: The invariant polynomial $R_n(X)$ of degree $n + 1/2$ for $0 < n < 10$.

n	P_n
0	1
1	$z - B$
2	$z^2 - \frac{1}{3}Bz + \frac{1}{9}B^2 - \frac{1}{4}g_2$
3	$z^3 - \frac{1}{5}Bz^2 + \left(\frac{2}{75}B^2 - \frac{1}{4}g_2\right) z - \frac{1}{225}B^3 + \frac{1}{15}Bg_2 - \frac{1}{4}g_3$
4	$z^4 - \frac{1}{7}Bz^3 + \left(\frac{3}{245}B^2 - \frac{3}{10}g_2\right) z^2 + \left(-\frac{2}{2205}B^3 + \frac{53}{1260}Bg_2 - \frac{2}{9}g_3\right) z + \frac{1}{11025}B^4 - \frac{113}{22050}B^2 g_2 + \frac{11}{252}Bg_3 + \frac{9}{400}g_2^2$
5	$z^5 - \frac{1}{9}Bz^4 + \left(\frac{4}{567}B^2 - \frac{5}{14}g_2\right) z^3 + \left(-\frac{1}{2835}B^3 + \frac{47}{1260}Bg_2 - \frac{1}{4}g_3\right) z^2 + \left(\frac{1}{59535}B^4 - \frac{191}{79380}B^2 g_2 + \frac{1}{36}Bg_3 + \frac{25}{784}g_2^2 \right) z$ $-\frac{1}{893025}B^5 + \frac{53}{297675}B^3 g_2 - \frac{29}{11340}B^2 g_3 - \frac{44}{11025}Bg_2^2 + \frac{27}{560}g_2 g_3$
6	$z^6 - \frac{1}{11}Bz^5 + \left(\frac{5}{1089}B^2 - \frac{5}{12}g_2\right) z^4 + \left(-\frac{4}{22869}B^3 + \frac{7}{198}Bg_2 - \frac{2}{7}g_3\right) z^3 + \left(\frac{2}{343035}B^4 - \frac{28}{16335}B^2 g_2 + \frac{19}{770}Bg_3 + \frac{7}{144}g_2^2\right) z^2$ $\left(-\frac{1}{5145525}B^5 + \frac{467}{6860700}B^3 g_2 - \frac{977}{762300}B^2 g_3 - \frac{91}{23760}Bg_2^2 + \frac{5}{84}g_2 g_3 \right) z + \frac{1}{108056025}B^6 - \frac{71}{20582100}B^4 g_2 + \frac{107}{1455300}B^3 g_3$ $+\frac{191}{784080}B^2 g_2^2 - \frac{173}{27720}Bg_2 g_3 - \frac{1}{576}g_2^3 + \frac{1}{49}g_3^2$
7	$z^7 - \frac{1}{13}Bz^6 + \left(\frac{6}{1859}B^2 - \frac{21}{44}g_2\right) z^5 + \left(-\frac{5}{50193}B^3 + \frac{265}{7722}Bg_2 - \frac{35}{108}g_3\right) z^4 + \left(\frac{10}{3864861}B^4 - \frac{1505}{1104246}B^2 g_2\right.$ $\left.+\frac{135}{1936}g_2^2 + \frac{115}{4914}Bg_3 \right) z^3 + \left(-\frac{2}{32207175}B^5 + \frac{21}{511225}B^3 g_2 - \frac{5557}{5855850}B^2 g_3 - \frac{25801}{5662800}Bg_2^2 + \frac{1673}{19800}g_2 g_3 \right) z^2$ $+\left(\frac{4}{2608781175}B^6 - \frac{5917}{5217562350}B^4 g_2 + \frac{7307}{237161925}B^3 g_3 + \frac{261769}{1490732100}B^2 g_2^2 - \frac{833611}{145945800}Bg_2 g_3 - \frac{25}{7744}g_2^3 + \frac{35}{1458}g_3^2 \right) z$ $-\frac{1}{18261468225}B^7 + \frac{2}{47432385}B^5 g_2 - \frac{45251}{36522936450}B^4 g_3 - \frac{19231}{2608781175}B^3 g_2^2 + \frac{3155839}{10435124700}B^2 g_2 g_3 + \frac{28}{117975}Bg_2^3$ $-\frac{15685}{7429968}Bg_3^2 - \frac{24389}{4065600}g_2^2 g_3$

Table A.2: The polynomial $P_n(z)$ of degree n for $0 \leq n \leq 7$.

n	γ^2
0	B
1	$\frac{1}{9}B^3 - \frac{1}{36}g_2B - \frac{1}{36}g_3$
2	$\frac{1}{8100}(B^2 - 3g_2)(4B^3 - 9g_2B + 27g_3)$
3	$\frac{1}{39690000}B(16B^6 - 504B^4g_2 + 2376B^3g_3 + 4185B^2g_2^2 - 36450Bg_2g_3 - 3375g_2^3 + 91125g_3^2)$
4	$\frac{1}{157529610000}(B^3 - 52g_2B + 560g_3)(16B^6 - 1016B^4g_2 + 8200B^3g_3 + 10297B^2g_2^2 - 41650Bg_2g_3 - 27783g_2^3$ $-42875g_3^2)$
5	$\frac{1}{6175790830440000}(B^2 - 27g_2)(64B^9 - 18864B^7g_2 + 308880B^6g_3 + 1637388B^5g_2^2 - 48901320B^4g_2g_3 + 382214700B^3g_3^2$ $-35102457B^3g_2^3 + 1073018745B^2g_2^2g_3 + 60142500Bg_2^4 - 9475888275Bg_2g_3^2 + 22785532875g_3^3 - 246037500g_2^3g_3)$
6	$\frac{1}{1262887466916675600000}(B^4 - 294B^2g_2 + 7776Bg_3 + 3465g_2^2)(64B^9 - 29232B^7g_2 + 552528B^6g_3 + 3688524B^5g_2^2$ $-102454632B^4g_2g_3 - 153514305B^3g_2^3 + 423341964B^3g_3^2 + 4988259045B^2g_2^2g_3 + 1008522900Bg_2^4$ $-36260514675Bg_2g_3^2 - 23772325500g_2^3g_3 + 88418496375g_3^3)$
7	$\frac{1}{1920851837180263587600000}(B^3 - 196g_2B + 2288g_3)(256B^{12} - 345856B^{10}g_2 + 11238656B^9g_3 + 155623776B^8g_2^2$ $-9171314880B^7g_2g_3 - 26255254480B^6g_2^3 + 134061879264B^6g_3^2 + 1882751576496B^5g_2^2g_3 + 1549816204321B^4g_2^4$ $-40720503277776B^4g_2g_3^2 - 101898240434564B^3g_2^3g_3 + 229583639450000B^3g_3^3 - 31239201699342B^2g_2^5$ $+1836081662875278B^2g_2^2g_3^2 + 1591804079535852Bg_2^4g_3 - 5030708845662500Bg_2g_3^3 + 83433026210625g_2^6$ $-1881028152005347 8g_2^3g_3^2 - 5375469208484375g_3^4)$

Table A.3: The constant γ^2 of $P_n(z)$ for $0 \leq n \leq 7$.

n, N, B	$f^{g_2}_{\mathbb{Q}}(X)$	g_3	$A(z)^\bullet$, $B(z)^\diamond$
$1, 3, 0$	X	g_3	$\bullet\ \frac{1}{2}\sqrt{-g_3}$ $\diamond\ \frac{1}{2}$
$1, 5, 1$	$X + 15$	$\frac{395}{8}$	$\bullet\ \frac{15}{8}\sqrt{-6}z - \frac{57}{16}\sqrt{-6}$ $\diamond\ \frac{1}{2}z - \frac{5}{4}$
$1, 6, 0^*$	X	g_3	$\bullet\ z^3 - \frac{1}{2}g_3$ $\diamond\ \frac{1}{2}\sqrt{-g_3}$
$1, 7, 1$	$25X^2 - 168X - 5229$	$\frac{77}{80} - \frac{161}{80}g_2$	$\bullet\ \frac{21}{40}\sqrt{15 + 5g_2}z^2 - \frac{609}{400}\sqrt{15 + 5g_2}z + \frac{3}{1600}\sqrt{15 + 5g_2}(15g_2 + 757)$ $\diamond\ \frac{1}{2}z^2 - \frac{7}{4}z + \frac{35}{16} + \frac{1}{16}g_2$
$1, 8, 1$	$25X + 564$	$\frac{18872}{125}$	$\bullet\ z^4 - 4z^3 + 6z^2 - \frac{14324}{125}z + \frac{194161}{625}$ $\diamond\ \frac{48}{25}\sqrt{-15}z - \frac{816}{125}\sqrt{-15}$
$1, 9, 0^*$	X	g_3	$\bullet\ \frac{3}{2}\sqrt{-g_3}z^3 - \frac{1}{2}\sqrt{-g_3}g_3$ $\diamond\ \frac{1}{2}z^3 - \frac{1}{2}g_3$
$1, 9, 1$	$25X^3 - 369X^2$ $-729X + 61209$	$-\frac{5}{672}g_2^2 - \frac{167}{112}g_2$ $+\frac{881}{224}$	$\bullet\ \frac{1}{112}\sqrt{\alpha}z^3 - \frac{39}{1120}\sqrt{\alpha}z^2 + \frac{1}{1128960}\sqrt{\alpha}(540g_2 + 56484)z$ $+\frac{1}{1128960}\sqrt{\alpha}(25g_2^2 - 780g_2 - 32697)^{**}$ $\diamond\ \frac{1}{2}z^3 - \frac{9}{4}z^2 + (\frac{63}{16} + \frac{1}{16}g_2)z + \frac{5}{1344}g_2^2 - \frac{1343}{448} - \frac{11}{112}g_2$

* Solution coincides that of $(n, N, B) = (1, 3, 0)$.

** $\alpha := 1890 + 13860g_2 + 210g_2^2$

Table A.4: All output of lameDN(n,N,B) with $M < D_N$ for $0 < nN \leq 12$, $N \leq 10$ and $B = 0, 1$.

n, N, B	$f_{\mathbb{Q}}^{g_2}(X)$	g_3	$A(z)^{\bullet}$, $B(z)^{\diamond}$
$1, 10, 1^*$	$X + 15$	$\frac{395}{8}$	$\bullet\ z^5 - 5z^4 + 10z^3 - \frac{835}{16}z^2 + \frac{2645}{16}z - \frac{9811}{64}$ $\diamond\ \frac{15}{8}\sqrt{-6}z^2 - \frac{33}{4}\sqrt{-6}z + \frac{285}{32}\sqrt{-6}$
$1, 10, 1$	$125X - 3228$	$-\frac{75848}{625}$	$\bullet\ z^5 - 5z^4 + 10z^3 + \frac{3206}{25}z^2 - \frac{65039}{125}z + \frac{3445963}{3125}$ $\diamond\ \frac{24}{5}\sqrt{3}z^2 - \frac{528}{25}\sqrt{3}z + \frac{7224}{125}\sqrt{3}$
$2, 5, 1$	$147X - 145$	$\frac{2105}{9261}$	$\bullet\ z^5 - \frac{5}{6}z^4 - \frac{115}{882}z^3 + \frac{835}{5292}z^2 + \frac{61685}{3111696}z - \frac{1356151}{130691232}$ $\diamond\ \frac{4}{343}\sqrt{-14}(z - \frac{25}{42})$
$2, 6, 1$	$147X - 109$	$\frac{1727}{9261}$	$\bullet\ z^6 - z^5 + \frac{65}{588}z^4 + \frac{295}{2646}z^3 - \frac{8515}{1037232}z^2 - \frac{260095}{21781872}z + \frac{16227685}{5489031744}$ $\diamond\ \frac{3}{343}\sqrt{-42}(z^2 - \frac{16}{21}z + \frac{211}{1764})$
$3, 3, 1$	$300X - 37$	$\frac{251}{27000}$	$\bullet\ \frac{1}{10000}\sqrt{-15}z - \frac{13}{900000}\sqrt{-15}$ $\diamond\ \frac{1}{2}z^3 - \frac{3}{20}z^2 + \frac{29}{2400}z - \frac{17}{216000}$
$4, 3, 0$	X	g_3	$\bullet\ z^6 - \frac{1}{3}g_3z^3 + \frac{1}{54}g_3^2$ $\diamond\ \frac{1}{54}g_3\sqrt{-g_3}$

* Solution coincides that of $(n, N, B) = (1, 5, 1)$.

Table $A.4$: All output of lameDN(n,N,B) with $M < D_N$ for $0 < nN \leq 12$, $N \leq 10$ and $B = 0, 1$.

n	B	$f_{\mathbb{Q}}^{g_2}(X)$	g_3	γ	$J_6(y_1, y_2)(t)$
$\frac{1}{4}$	0	X	g_3	$-\frac{1}{432}g_3$	1
$\frac{3}{4}$	1	$3X + 3584$	$\frac{318464}{27}$	9216	$t^{-1} - \frac{44}{9}$
$\frac{5}{4}$	1	$16471125X^2$ $-45350400X$ $+26476544$	$\frac{318464}{16335} - \frac{17942}{1815}g_2$	$-\frac{8604643488}{129687123005}g_2 + \frac{16455401472}{129687123005}$	$t^{-2} - \frac{52}{33}t^{-1} + \frac{64}{495} + \frac{41}{160}g_2$
$\frac{7}{4}$	0	X	g_3	$-\frac{2401}{80621568}g_3^3$	$t^{-3} - \frac{1}{112}g_3$
$\frac{7}{4}$	1	$7500386359375X^3$ $-21327465600000X^2$ $+20197638144000X$ -6367154798592	$+\frac{26624}{1445} - \frac{10864}{289}g_2$ $+\frac{1437345}{73984}g_2^2$	$\frac{13279493052114138}{4270468350473694687 5}g_2^2$ $-\frac{6486110101218710016}{1046264745866055198437 5}g_2$ $+\frac{5785373252154477772 8}{1868329903332241425781 25}$	$t^{-3} - \frac{12}{13}t^{-2} + (\frac{912}{1105} - \frac{405}{544}g_2)t^{-1}$ $-\frac{49024}{93925} + \frac{22923}{30056}g_2 - \frac{205335}{1183744}{g_2}^2$

Table A.5: Data of all Lamé equations with $M = G_{12}$, $0 < n \leq 9/4$ and $B = 0, 1$ from lameG.

n	B	$f_{\mathbb{Q}}^{g_2}(X)$	g_3	$\beta^\bullet$, $\gamma^\diamond$	$J_8(y_1,y_2)(t)$
$\frac{1}{6}$	0	0	0	$\bullet\ -\frac{7}{1024}g_2$ $\diamond\ -\frac{1}{131072}g_2^2$	1
$\frac{5}{6}$	0	0	0	$\bullet\ -\frac{875}{1048576}g_2^2$ $\diamond\ -\frac{15625}{137438953472}{g_2}^4$	$t^{-2}+\frac{1}{320}g_2$
$\frac{5}{6}$	1	$153125X^2-14823000X$ $+3201872976$	$-\frac{201933}{1120}-\frac{18621}{4480}g_2$	$\bullet\ -\frac{3830895}{7340032}g_2+\frac{1421958969}{45875200}$ $\diamond\ \frac{154320815962881}{157905814814720}g_2$ $+\frac{3814823856413247597}{4934556712960000}$	$t^{-2}-\frac{36}{7}t^{-1}$ $+\frac{5751}{560}+\frac{1}{320}g_2$
$\frac{7}{6}$	1	$459730632842047X^4$ $-3832495939139616X^3$ $+17997408825378318X^2$ $-42147037528833384X$ $+35644514774969547$	$+\frac{949887}{102740}-\frac{320607}{102740}g_2$ $-\frac{165389}{462330}g_2^2$	$\bullet\ -\frac{2779692923}{1249844428800}g_2^3+\frac{2690126019}{46290534400}g_2^2$ $-\frac{7760239263}{46290534400}g_2+\frac{5174641017}{46290534400}$ $\diamond\ -\frac{79997983240625724605029480 53}{7385781929905134603804134604800}{g_2}^3$ $+\frac{74911875453012742179815864043}{7385781929905134603804134604800}{g_2}^2$ $-\frac{20927187651618615335294826 3411}{7385781929905134603804134604800}g_2$ $+\frac{89650628027074492202819570 31351}{361903314565351595586402595635200}$	$t^{-3}-\frac{81}{32}t^{-2}$ $+(\frac{1215}{704}+\frac{131}{704}g_2)t^{-1}$ $-\frac{333153}{2630144}-\frac{351405}{1315072}g_2$ $+\frac{1677517}{71013888}{g_2}^2$

Table A.6: Data of all Lamé equations with $M=G_{13}$, $0<n\leq 7/6$ and $B=0,1$.

n	B	$f_{\mathbb{Q}}^{g_2}(X)$	g_3	$J_{12}(X,Y)^{\bullet}$, $J_{12}(y_1,y_2)(t)^{\diamond}$
$\frac{1}{10}$	0	X	g_3	$\bullet\ X^7Y - \frac{11}{6912}g_3X^4Y^4 - \frac{1}{47775744}g_3^2XY^7$ $\diamond\ 1$
$\frac{1}{6}$	1	$X - 2160$	19440	$\bullet\ X^{12} + \frac{1485}{32}X^9Y^3 - \frac{120285}{1024}X^6Y^6 - \frac{5412825}{32768}X^3Y^9$ $+\frac{13286025}{1048576}Y^{12}$ $\diamond\ t^{-1} + 9$
$\frac{3}{10}$	1	$\frac{10000}{3}$	$\frac{1250000}{27}$	$\bullet\ X^7Y - \frac{348046875}{65536}X^4Y^4 - \frac{1001129150390625}{4294967296}XY^7$ $\diamond\ t^{-1} + \frac{125}{9}$
$\frac{7}{10}$	0	X	g_3	$\bullet\ X^7Y + \frac{184877}{47775744}g_3^2X^4Y^4 - \frac{282475249}{2282521714753536}{g_3}^4XY^7$ $\diamond\ t^{-3} - \frac{16}{189}g_3$
$\frac{7}{10}$	1	$2033647X^2 - 1988542500X$ $+657675000000$	$-\frac{53575000}{26411} - \frac{235}{154}g_2$	$\bullet\ X^7Y + \left(-\frac{147185859375}{2791309312}g_2 + \frac{36577788857421875}{239354773504}\right)X^4Y^4$ $+\left(-\frac{895052004547472190856933359375}{97818454805734326198599 68}g_2\right.$ $\left.+\frac{4614449883289731967449188232421875}{8387932499591718471529922 56}\right)XY^7$ $\diamond\ t^{-3} - \frac{150}{11}t^{-2} + \left(\frac{463125}{5632} - \frac{93}{2048}g_2\right)t^{-1}$ $-\frac{10821503125}{94657024} + \frac{177945}{1103872}g_2$

Table A.7: Data of all Lamé equations with $M = G_{22}$, $0 < n \leq 7/10$ and $B = 0, 1$.

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Samenvatting

Lamé-vergelijkingen met eindige monodromie

In de eerste drie hoofdstukken van dit proefschrift geven we een inleiding in gewone lineaire differentiaalvergelijkingen met coëfficienten in $\mathbb{C}(z)$. We beperken ons hierbij tot de Fuchse differentiaalvergelijkingen. Een klassieke vraag voor een Fuchse differentiaalvergelijking is wanneer haar oplossingsruimte slechts uit algebraïsche functies bestaat. Dergelijke vergelijkingen noemen we algebraïsche Fuchse vergelijkingen. Een bekend vermoeden op dit gebied is Grothendieck's Vermoeden 1.5.4, dat nog steeds onbewezen is. Bij iedere Fuchse differentiaalvergelijking is een groep gedefinieerd, die men de monodromiegroep van de vergelijking noemt. Voor het bepalen van de algebraïciteit van een Fuchse vergelijking volstaat het de eindigheid van haar monodromiegroep aan te tonen.

Het is precies bekend en triviaal wanneer een gegeven Fuchse differentiaalvergelijking van de eerste orde een oplossingsruimte van algebraïsche functies heeft. De monodromiegroep is in dit geval cyclisch. Een stuk gecompliceerder zijn de Fuchse differentiaalvergelijkingen waarvan de orde minstens 2 is. Een klassiek voorbeeld van een tweede orde Fuchse vergelijking met drie singuliere punten is de hypergeometrische differentiaalvergelijking. H.A. Schwarz heeft in 1873 een volledige classificatie van de algebraïsche hypergeometrische vergelijkingen gegeven. Hierdoor werden tevens alle algemene algebraïsche Fuchse vergelijkingen met drie singulariteiten van orde 2 bepaald.

Een volgende categorie van Fuchse vergelijkingen van orde 2 bestaat uit de orde twee vergelijkingen met vier singuliere punten. Wat hun algebraïciteit betreft is relatief weinig bekend. Dit heeft onder andere te maken met het feit dat er een accessoire parameter in de vergelijkingen optreedt. In dit proefschrift bekijken we een specifieke vergelijking uit deze categorie, namelijk de Lamé-(differentiaal)vergelijking $L_n(y) = 0$, zie Definitie 4.1.1.

De Hoofdstukken 4, 5 en 6 staan in het teken van de vraag wanneer de Lamé-vergelijking algebraïsch is. In het werk van F. Baldassarri en B. Chiarellotto wordt deze vraag voor het eerst systematisch aangepakt. Voor een aantal van hun resultaten geven we in de Hoofdstukken 4 en 5 een vereenvoudigd bewijs. Hiernaast bevatten deze hoofdstukken een volledige classificatie en karakterisatie van alle eindige groepen, die als de monodromiegroep van een Lamé-vergelijking voor kunnen komen. Hiermee completeren wij het werk van Baldassarri en Chiarellotto.

In Hoofdstuk 6 presenteren we een algoritme waarmee een aftelling van de algebraïsche Lamé-vergelijkingen verkregen kan worden. De onderliggende ideeën hiervoor zijn afkomstig uit de invariantentheorie van spiegelingsgroepen. Het verband tussen (semi)-invarianten en het vinden van algebraïsche oplossingen van een Fuchse vergelijking is eerder beschreven en toegepast door onder anderen J.J. Kovacic, M.F. Singer, F. Ulmer, M. van Hoeij en J.-A. Weil. Tot slot geven we in Appendix A het begin van de aftelling van de algebraïsche Lamé-vergelijkingen in de Tabellen $A.4$ tot en met $A.7$.

Dankwoord

Allereerst wil ik mijn promotor Frits Beukers bedanken voor zijn begeleiding gedurende het onderzoek van de afgelopen jaren. Zijn enthousiasme en inzichten zijn een belangrijke inspiratiebron geweest. Ik dank hem voor zijn geduld en vele suggesties die hebben bijgedragen aan de totstandkoming van dit proefschrift.

Voorts ben ik de leescommissie, bestaande uit dr. W. Bosma, prof. dr. A.M. Cohen, prof. dr. A.H.M. Levelt en prof. dr. M. van der Put, zeer erkentelijk voor het zorgvuldig doornemen van het manuscript van dit proefschrift en voor haar commentaar op de inhoud.

De Nederlandse Organisatie voor Wetenschappelijk Onderzoek en de Universiteit Utrecht wil ik bedanken voor de gelegenheid die zij mij geboden hebben om promotieonderzoek te kunnen doen. In het bijzonder wil ik hierbij Martijn Dekker, Wim Aspers en Lex Zandee noemen.

Ik wil mijn collega's van het Mathematisch Instituut bedanken voor de prettige werksfeer en hun belangstelling. Met veel plezier denk ik terug aan de tijd in kamer 603 met Letterio Gatto, Joost Hermans, Michael Ruzhansky, Dario Gasbarra, Henrike Petzl, Giovanna Carnovale, Chris Stolk, Martijn Pistorius en Manuel Ballester Sanchez. Ook wil ik hier Lorna Booth en Martijn van Manen vermelden; Lorna voor het verbeteren van het engels en Martijn voor zijn hulp bij het draaien van de tabellen.

Bovendien wil ik mijn dank betuigen aan Karin Groothuis-Oudshoorn en mijn broer Rob voor hun bereidheid om als paranimf op te willen treden, en aan Nils Bruin voor de vele aanmoedigingen vanuit een ver land.

Tenslotte, maar zeker niet op de laatste plaats, komen mijn ouders. Ik ben hen zeer erkentelijk voor hun steun, belangstelling en voor alles wat ze voor mij gedaan hebben.

Alexa van der Waall

Curriculum vitae

De schrijfster van dit proefschrift is op 1 september 1970 te Nijmegen geboren. Vanaf 1982 bezocht zij het Christelijk College Stad en Lande te Huizen, alwaar ze in 1988 het diploma Voorbereidend Wetenschappelijk Onderwijs behaalde. Aansluitend hierop begon zij met de studie wiskunde aan de Universteit van Amsterdam en legde zij in juli 1989 het propedeutisch examen in de wiskunde cum laude af. Tijdens de daarop volgende jaren specialiseerde zij zich op het gebied van de algebraïsche getaltheorie. In augustus 1994 behaalde ze cum laude het doctoraaldiploma in de wiskunde aan de Universiteit van Amsterdam.

Vanaf februari 1995 kwam zij in dienst van de Nederlandse Organisatie voor Wetenschappelijk Onderzoek (NWO) om als onderzoeker in opleiding te werken aan de Universiteit Utrecht. Binnen het project Algoritmen in de algebra van NWO verrichtte zij wetenschappelijk onderzoek onder begeleiding van prof. dr. F. Beukers. De verkregen resultaten staan in dit proefschrift beschreven. In het kader van haar aanstelling volgde zij tevens cursussen van het Mathematical Research Institute op het gebied van modulaire krommen en computeralgebra en nam zij deel aan de Spring School en conferentie “Aspects of Differential Equations” in het voorjaar van 1999.