ASYMPTOTIC ERROR EXPANSIONS FOR STIFF EQUATIONS
PART 2: THE MILDLY STIFF CASE

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ASYMPTOTIC ERROR EXPANSIONS FOR STIFF EQUATIONS. PART 2: THE MILDLY STIFF CASE

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Abstract. We derive an asymptotic expansion for the global error of the implicit Euler scheme applied to stiff nonlinear systems of ordinary differential equations. Part 2, in particular, is devoted to the mildly stiff case. It is shown by a discrete singular perturbation analysis of the remainder term equation that the desired order $O(h^{s+1})$ of the remainder term, which inevitably breaks down at the first grid points, reappears at the subsequent grid points.

1. INTRODUCTION

The purpose of this paper (Part 1 & the present Part 2) is to discuss rigorously the structure of the discretization error of the implicit Euler scheme applied to stiff nonlinear initial value problems. Our aim is to derive an asymptotic error expansion **)

$$
\begin{align*}
\zeta_\nu - z(t_\nu) &= he_1(t_\nu) + \ldots + h^s e_s(t_\nu) + R_\nu , \\
\intertext{with}
R_\nu &= O(h^{s+1}) \quad \text{in the "B-sense"},
\end{align*}
$$

i.e. the $O$-constant is not allowed to be affected by the Lipschitz constant $L$ of the right hand side $f$ of the given differential equation but must only depend on the one-sided Lipschitz constant $m$ and on other problem-dependent parameters of moderate size.

The functions $e_\nu(t)$ are solutions of the V.E.'s (variational equations)

$$
\begin{align*}
e_1'(t) &= f_\nu(t, z(t)) e_1(t) + \frac{1}{2} z''(t) , \\
(1.3) \quad e_2'(t) &= f_\nu(t, z(t)) e_2(t) - \frac{1}{6} z'''(t) + \frac{1}{2} e_1''(t) + \frac{1}{2} f_{y_\nu}(t, z(t)) e_3'(t) , \\
&\vdots
\end{align*}
$$

and the remainder term $R_\nu$ satisfies a nonlinear difference equation:

$$
(1.4) \quad \frac{1}{h} (R_\nu - R_{\nu-1}) = \dot{J}(R_\nu) \cdot R_\nu + b_\nu - e_\nu
$$

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**) For notations not explained in Part 2, as for instance $h, \zeta_\nu, z(t), \ldots$ we refer to Part 1 [1]. Formulas from Part 1 will be quoted as $[1,(...)]$. 
(1.5) \[ b_\nu - c_\nu = O(h^{q+1}) \quad \text{for} \quad \varepsilon \to 0, \]

where \( \varepsilon > 0 \) is a small real parameter characterizing the stiffness (cf. [1, Section 4]). (1.5) means that the full order \( q + 1 \) is present for sufficiently small \( \varepsilon \ll h \) (strongly stiff case) at all grid points. From this the main result of [1], namely

(1.6) \[ R_\nu = O(h^{q+1}) \quad \text{for} \quad \varepsilon \to 0, \]

was easily concluded by estimates of \( B \)-type (cf. [1,2.19]). A structural analysis of the remainder term equation (1.4) was not necessary.

In the mildly stiff case \( \varepsilon \approx h \) (more precisely: \( \varepsilon \) not at the \( O(h^{q+1}) \)-level), the situation is different: Order reductions of the quantities \( b_\nu - c_\nu \) appear at the first grid points (cf. [1, Section 3]). The obvious consequence is an order reduction of \( R_\nu \) itself at the first grid points. Numerical experience, however, shows that at the later grid points the full order \( q + 1 \) of \( R_\nu \) reappears by and by, i.e. the behaviour of \( R_\nu \) is similar to that of \( b_\nu - c_\nu \). A proof of this phenomenon is not possible on the basis of \( B \)-type estimates since

i) the bound in [1,2.19] involves \( \max_{\mu=1,\ldots,m} ||b_\mu - c_\mu|| \),

ii) in the non-scalar case, \( m \) depends on non-stiff eigenvalues of the Jacobian \( f_\nu \), i.e. \( m \ll 0 \) is not true in general; hence the factor \( \frac{1}{m} [e^{mt} - 1] \) in [1,2.19] has no damping power.

However, the damping effect mentioned above, namely \( R_\nu = O(h^{q+1}) \) at the later grid points, can be proved rigorously. Our proof, which is given in the present paper, is based on an extension of singular perturbation techniques to the discrete equation (1.4). This extension is by no means straightforward. Recall (cf. [1]) that the functions \( \epsilon_i(t) \) are unbounded near \( t = 0 \): they are \( O(\varepsilon^2) \) for \( i \geq 3 \). This necessitated a singular perturbation ansatz involving negative powers of \( \varepsilon \). Such an ansatz is obviously not appropriate in the context of nonlinear equations: If, for instance, the coefficients \( x_1 \) in an ansatz \( x = \varepsilon^{-k} x_0 + \varepsilon^{-k+1} x_1 + \ldots \) would be defined by equating coefficients of \( \varepsilon^{-k}, \varepsilon^{-k+1}, \ldots \), and if \( x^2 \) would appear in the nonlinear operator, then \( x^2 = \varepsilon^{-2k} x_0^2 + \ldots \), which shows that the ansatz fails. (The occurrence of \( \varepsilon^{-2k} x_0^2 \), for instance, would necessitate to equate coefficients at \( \varepsilon^{-2k} \) - level, but then - instead of \( x = \varepsilon^{-k} x_0 + \varepsilon^{-k+1} x_1 + \ldots \) - an ansatz \( x = \varepsilon^{-2k} x_0 + \ldots \) would be appropriate with \( x^2 = \varepsilon^{-4k} x_0^2 + \ldots \) (!) and so on.) Thus - in contrast to the linear V.E.'s, for which an ansatz with negative powers of \( \varepsilon \) could be applied successfully (cf. [1]) - such an ansatz must be avoided for the nonlinear difference equation (1.4).

For our discrete singular perturbation analysis we distinguish two cases:

i) The case \( \varepsilon \leq C h \) (where \( C \) denotes some constant of moderate size) will be discussed in Section 3. In Subsection 3.1 we will show that the inhomogeneity \( b_\nu - c_\nu \) of (1.4) can be expanded into powers of \( h \) (of course, the leading term of this expansion is not \( O(h^{q+1}) \) as in the classical case but is at the reduced \( O(h^2) \)-level). It would therefore be unnatural to work with (negative) powers of \( \varepsilon \) in the singular perturbation ansatz for \( R_\nu \). Our ansatz will be in (positive) powers of \( h \), and the main result will be that all slowly varying outer solution
components of \( R \) at \( O(h^n) \)-level, \( i < q + 1 \), vanish, such that only strongly decaying inner solution terms are present in the domain of reduced powers of \( h \).

ii) The case \( h \leq C \varepsilon \) (\( C \) a moderate constant) will be considered in Section 4. In this case the expansion of \( b_\nu - c_\nu \) in powers of \( h \) (as discussed in Subsection 3.1) breaks down. Avoiding negative powers of \( \varepsilon \), we will do an analysis which is based on smooth solutions \( \varepsilon_i(t) \) of the V.E.'s i.e. on solutions with moderate \( t \)-derivatives up to a certain order (and not affected with negative powers of \( \varepsilon \)). We shall then obtain smooth inhomogenous terms \( b_\nu - c_\nu \) at \( O(h^{q+1}) \)-level. But, in contrast to the initial conditions \( \varepsilon_i(0) = 0 \) or \( \varepsilon_i(0) = (\frac{h}{k})^i \varepsilon_i(0) \) (cf. [1,(2.15),(2.17)]), the starting values \( \varepsilon_i(0) \) are now fixed by the requirement that the \( \varepsilon_i(t) \) are smooth. These starting values influence the starting value \( R_0 \) of the difference equation (1.4): Instead of \( R_0 = 0 \) (or \( R_0 = O(h^{q+1}) \)) we only have \( R_0 = O(h) \). Again it can be shown by singular perturbation techniques (cf. Section 4) that the effect of the \( O(h) \)-starting value is damped away and that \( R_\nu \) converges towards a \( O(h^{q+1}) \)-solution of (1.4) with increasing \( \nu \).

Summarizing the results of Section 3 (case \( \varepsilon \leq C \varepsilon \)) and Section 4 (case \( h \leq C \varepsilon \)), we will observe that our analysis covers the whole "\( \varepsilon - h \)-plane" in a satisfactory way.

2. THE TRANSFORMED REMAINDER TERM EQUATION

Due to [1,(2.7a)], the nonlinear term \( \dot{J}(R_\nu) \cdot R_\nu \) in (1.4) can be expanded into

\[
\dot{J}(R_\nu) \cdot R_\nu = f(t_\nu, z(t_\nu) + v_q(t_\nu) + R_\nu) - f(t_\nu, z(t_\nu) + v_q(t_\nu)) =
\]

\[
= f(t_\nu, z(t_\nu) + v_q(t_\nu)) + f_y(t_\nu, z(t_\nu) + v_q(t_\nu)) \cdot R_\nu +
\]

\[
+ \int_{0}^{1} f_{yy}(t_\nu, z(t_\nu) + v_q(t_\nu) + \sigma R_\nu) (1 - \sigma) d\sigma \cdot R_\nu^2 - f(t_\nu, z(t_\nu) + v_q(t_\nu)) =
\]

\[
= f_y(t_\nu, z(t_\nu)) \cdot R_\nu + \int_{0}^{1} f_{yy}(t_\nu, z(t_\nu) + \sigma v_q(t_\nu)) d\sigma \cdot v_q(t_\nu) \cdot R_\nu +
\]

\[
+ \int_{0}^{1} f_{yy}(t_\nu, z(t_\nu) + v_q(t_\nu) + \sigma R_\nu) (1 - \sigma) d\sigma \cdot R_\nu^2 ;
\]

here we have used the abbreviation \( v_q(t) := he_1(t) + \ldots + h^q e_q(t) \) - cf. [1,(4.46)]. Using the denotation

\[
G_\nu(R_\nu) \equiv G(t_\nu, R_\nu) := \int_{0}^{1} f_{yy}(t_\nu, z(t_\nu) + \sigma v_q(t_\nu)) d\sigma \cdot v_q(t_\nu) \cdot R_\nu +
\]

\[
+ \int_{0}^{1} f_{yy}(t_\nu, z(t_\nu) + v_q(t_\nu) + \sigma R_\nu) (1 - \sigma) d\sigma \cdot R_\nu^2
\]

and the representation [1,(4.49)] for the inhomogeneity \( b_\nu - c_\nu \), we rewrite (1.4):
Due to our smoothness assumptions concerning \( f_y \) (cf. [1, 4.2c]), \( G_v(R) \) is Lipschitz continuous with a moderate Lipschitz bound; let \( \text{Lip}_C \) denote a common Lipschitz bound for the \( G_v \) uniformly in \( v \). We have

\[
G_v(0) = 0, \quad \|G_v(R)\| \leq \text{Lip}_C\|R\|
\]

Now we transform (2.2) analogously as in [1, Section 4], using the transformation

\[
T_v := T(t_v),
\]

which diagonalizes \( f_y(t_v, z(t_v)) \) (cf. [1,4.2a,b]):

\[
f_y(t_v, z(t_v)) = T_v \Lambda_v T_v^{-1},
\]

with

\[
\Lambda_v := \Lambda(t_v) = \begin{pmatrix} c(t_v) & 0 \\ 0 & \xi(t_v) \end{pmatrix}.
\]

Denoting

\[
\rho_v := T_v^{-1}R_v, \quad \bar{v}_q(t_v) := T_v^{-1}v_q(t_v),
\]

\[
\Theta_v := -T_v^{-1} \frac{1}{h} (T_v - T_v^{-1}) = \frac{1}{h} (T_v^{-1} - T_v^{-1})T_v^{-1},
\]

\[
\delta_v := T_v^{-1}(b_v - c_v),
\]

and premultiplying all terms in (2.2) by \( T_v^{-1} \), we obtain

\[
T_v^{-1} \frac{1}{h} (R_v - R_{v-1}) = \frac{1}{h} \left[ T_v^{-1}R_v - T_v^{-1}R_{v-1} + (T_v^{-1} - T_v^{-1})R_{v-1} \right] = \frac{1}{h} (\rho_v - \rho_{v-1}) - \Theta_v \rho_v - 1,
\]

\[
T_v^{-1} f_y(t_v, z(t_v)) T_v T_v^{-1} R_v = \Lambda_v \rho_v,
\]

\[
G_v(T_v \rho_v) := T_v^{-1} \int_0^1 f_y(t_v, z(t_v)) + \sigma T_v \bar{v}_q(t_v)) d\sigma \cdot T_v \bar{v}_q(t_v) \cdot T_v \rho_v + \frac{1}{2} \int_0^1 f_y(t_v, z(t_v)) + T_v \bar{v}_q(t_v) + \sigma T_v \rho_v)(1 - \sigma)d\sigma \cdot (T_v \rho_v)^2.
\]
We end up with the transformed remainder term equation

\[ (2.7) \quad \frac{1}{h} (\rho_\nu - \rho_{\nu-1}) = \Lambda_\nu \rho_\nu + \Theta_\nu \rho_{\nu-1} + \Gamma_\nu (\rho_\nu) + \delta_\nu \, . \]

Note the analogy between (2.7) and the transformed V.E.'s (cf.,[1,(4.4)):

\[ \epsilon'(t) = \Lambda(t)\epsilon(t) + A(t)\epsilon(t) + \bar{g}(t) \]

(\text{where } A(t) = -T^{-1}(t)T'(t)). In contrast to the V.E.'s, the difference equation (2.7) is nonlinear.

Using the denotation

\[ \rho_\nu = \begin{pmatrix} x_\nu \\ y_\nu \end{pmatrix}, \quad \delta_\nu = \begin{pmatrix} \delta_{\nu,1} \\ \delta_{\nu,2} \end{pmatrix}, \]

\[ \Theta_\nu = \begin{pmatrix} \vartheta_{1,1}(t_\nu) \\ \vartheta_{2,1}(t_\nu) \end{pmatrix}, \quad \Theta_\nu = \begin{pmatrix} \vartheta_{1,2}(t_\nu) \\ \vartheta_{2,2}(t_\nu) \end{pmatrix}, \quad \Gamma_\nu(\rho_\nu) = \begin{pmatrix} \gamma_{1}(t_\nu, x_\nu, y_\nu) \\ \gamma_{2}(t_\nu, x_\nu, y_\nu) \end{pmatrix}, \]

and introducing the "\textit{stretched stepsize}"

\[ (2.9) \quad \omega := \frac{h}{\varepsilon}, \]

we obtain from (2.7) after multiplying the second component by \( \varepsilon = \frac{h}{\omega} \):

\[ \frac{1}{h} (x_\nu - x_{\nu-1}) = \]
\[ = c_1(t_\nu)x_\nu + \vartheta_{1,1}(t_\nu)x_{\nu-1} + \vartheta_{1,2}(t_\nu)y_{\nu-1} + \gamma_{1}(t_\nu, x_\nu, y_\nu) + \delta_{\nu,1}, \]

\[ \frac{1}{\omega} (y_\nu - y_{\nu-1}) = \]
\[ = -c_2(t_\nu)y_\nu + \frac{h}{\omega} \vartheta_{2,1}(t_\nu)x_{\nu-1} + \frac{h}{\omega} \vartheta_{2,2}(t_\nu)y_{\nu-1} + \frac{h}{\omega} \gamma_{2}(t_\nu, x_\nu, y_\nu) + \frac{h}{\omega} \delta_{\nu,2}. \]

3. DISCRETE SINGULAR PERTURBATION ANALYSIS:
THE CASE \( \varepsilon \leq Ch \)

In accordance with part 1 of this paper, where the V.E.'s were considered in detail up to the fourth order, we will now (Section 3) restrict our considerations to the case \( q = 4 \) (i.e., the remainder term \( R_\nu \) is of the order \( q+1 = 5 \)). The extension to a higher order is straightforward but technically nontrivial.
3.1 $h$ - Expansion of the Inhomogeneity $\delta_v$

First of all, the inhomogeneity $\delta_v$ of the transformed remainder term equation (2.7) is expanded into powers of $h$ on the basis of the singular perturbation expansion of the $\hat{e}_i(t)$. Due to [1,(4.49)], (2.4) and (2.5), $\delta_v$ reads for $q = 4$:

$$
\delta_v = T_{\nu}^{-1}[z'(t_\nu) - \frac{1}{h}(z(t_\nu) - z(t_{\nu - 1}))] + \\
+ \Lambda_v \hat{v}_4(t_\nu) + \Theta_v \hat{v}_4(t_{\nu - 1}) - \frac{1}{h}(\hat{v}_4(t_\nu) - \hat{v}_4(t_{\nu - 1})) + \\
+ \frac{1}{2} T_{\nu}^{-1} f_{yy}(t_\nu, z(t_\nu))(T_{\nu} \hat{v}_4(t_\nu))^2 + \ldots + IF_v(\hat{v}_4(t_\nu))(T_{\nu} \hat{v}_4(t_\nu))^5,
$$

where the nonlinear operator $IF_v(v)$ is defined by

$$
IF_v(v) := \frac{1}{4!} T_{\nu}^{-1} \int_0^{1} \frac{\varepsilon^5}{\varepsilon^5} f(t_\nu, z(t_\nu) + \sigma T_{\nu} v)(1 - \sigma)^4 d\sigma.
$$

$\delta_v$ splits into smooth terms (depending on $t_\nu$) and stretched-variable terms (depending on $\tau_\nu = \frac{t_\nu}{\varepsilon}$):

With

$$
\hat{v}_4(t_\nu) = \sum_{i=1}^{4} h^i \hat{e}_i(t_\nu) = \sum_{i=1}^{4} h^i \hat{E}_i(t_\nu) + \sum_{i=1}^{4} h^i \hat{S}_i(t_\nu) =: \hat{E}(t_\nu) + \hat{S}(t_\nu),
$$

(where $\hat{E}_i(t)$ and $\hat{S}_i(t)$ denote the outer and inner solution components of the $\hat{e}_i(t)$ (cf. [1,(4.38)]), we have

$$
\delta_v = \delta_v^{sm} + \delta_v^{str},
$$

where $\delta_v^{sm}$ is the smooth part *)

$$
\delta_v^{sm} := T_{\nu}^{-1}[z'(t_\nu) - \frac{1}{h}(z(t_\nu) - z(t_{\nu - 1}))] + \\
+ \Lambda_v \hat{E}(t_\nu) + \Theta_v \hat{E}(t_{\nu - 1}) - \frac{1}{h}(\hat{E}(t_\nu) - \hat{E}(t_{\nu - 1})) + \\
+ \frac{1}{2} T_{\nu}^{-1} f_{yy}(t_\nu, z(t_\nu))(T_{\nu} \hat{E}(t_\nu))^2 + \ldots + IF_v(\hat{E}(t_\nu))(T_{\nu} \hat{E}(t_\nu))^5,
$$

and

$$
\delta_v^{str} := \delta_v - \delta_v^{sm}
$$

reads

$$
\delta_v^{str} = \Lambda_v \hat{S}(\tau_\nu) + \Theta_v \hat{S}(\tau_{\nu - 1}) - \frac{1}{h}(\hat{S}(\tau_\nu) - \hat{S}(\tau_{\nu - 1})) + \\
+ \frac{1}{2} T_{\nu}^{-1} f_{yy}(t_\nu, z(t_\nu)) [(T_{\nu} \hat{E}(t_\nu) + T_{\nu} \hat{S}(\tau_\nu))^2 - (T_{\nu} \hat{E}(t_\nu))^2] + \ldots + \\
+ IF_v(\hat{E}(t_\nu) + \hat{S}(\tau_\nu))(T_{\nu} \hat{E}(t_\nu) + T_{\nu} \hat{S}(\tau_\nu))^5 - IF_v(\hat{E}(t_\nu))(T_{\nu} \hat{E}(t_\nu))^5 =
$$

$$
\delta_v^{str} = \Lambda_v \hat{S}(\tau_\nu) + \Theta_v \hat{S}(\tau_{\nu - 1}) - \frac{1}{h}(\hat{S}(\tau_\nu) - \hat{S}(\tau_{\nu - 1})) + \\
+ B(t_\nu) \left[\hat{E}(t_\nu) \hat{S}(\tau_\nu) + \frac{1}{2} \hat{S}^2(\tau_\nu)\right] + \ldots + \\
+ [IF_v(\hat{E}(t_\nu) + \hat{S}(\tau_\nu)) - IF_v(\hat{E}(t_\nu))](T_{\nu} \hat{E}(t_\nu) + T_{\nu} \hat{S}(\tau_\nu))^5 + \\
+ IF_v(\hat{E}(t_\nu))(T_{\nu} \hat{E}(t_\nu) + T_{\nu} \hat{S}(\tau_\nu))^5 - (T_{\nu} \hat{E}(t_\nu))^5.
$$

*) $\Lambda_v \hat{E}(t_\nu)$ is a smooth term (despite $\Lambda_v = O(\varepsilon^{-1})$) as has already been shown in Part 1 (cf. [1,(4.50)]).
The bilinear operator $B(t_\nu)$ is the same as in [1,(4.69)]. Note that
\begin{equation}
(3.4) \quad \xi_\nu^{en} = O(h^5),
\end{equation}
as has already been shown in [1, Subsection 4.3]. On the contrary, $\xi_\nu^{str}$ shows order reductions at the first grid points.

We shall now expand $\xi_\nu^{str}$ into powers of $h$, proceeding from the following $h$ - expansion of the inner solution part $S(\tau)$ of $\bar{v}_4(t)$ (cf. (3.2)):
\begin{equation}
S(\tau) = \sum_{i=1}^{4} h^i \bar{S}_i(\tau) = \left( \sum_{i=1}^{4} h^i \varepsilon^{3-i} m_{0,i}(\tau) + \sum_{i=1}^{4} h^i \varepsilon^{4-i} m_{1,i}(\tau) + \ldots \right) = \left( \sum_{i=1}^{4} h^i \varepsilon^{2-i} n_{0,i}(\tau) + \sum_{i=1}^{4} h^i \varepsilon^{3-i} n_{1,i}(\tau) + \ldots \right)
\end{equation}
\begin{equation}
= \left( h^3 \sum_{i=1}^{4} \omega^{i-3} m_{0,i}(\tau) + h^4 \sum_{i=1}^{4} \omega^{i-4} m_{1,i}(\tau) + \ldots \right)
\end{equation}
\begin{equation}
= \left( h^2 \sum_{i=1}^{4} \omega^{i-2} n_{0,i}(\tau) + h^3 \sum_{i=1}^{4} \omega^{i-3} n_{1,i}(\tau) + \ldots \right)
\end{equation}

Here we have used the $\varepsilon$ - expansion of the $\bar{e}_i(t)$ (cf. [1,(4.29a)]); recall that $\omega = \frac{h}{\varepsilon}$ (cf. (2.9)). With the abbreviations
\begin{equation}
(3.6a) \quad M_i(\tau) := \sum_{i=1}^{4} \omega^{i-3-i} m_{i,i}(\tau), \quad N_i(\tau) := \sum_{i=1}^{4} \omega^{i-2-i} n_{i,i}(\tau), \quad l = 0, 1, 2, \ldots,
\end{equation}
(3.5) reads
\begin{equation}
S(\tau) = \left( h^3 M_0(\tau) + h^4 M_1(\tau) + \ldots \right) \left( \sum_{i=1}^{4} h^i \varepsilon^{2-i} N_0(\tau) + \sum_{i=1}^{4} h^i \varepsilon^{3-i} N_1(\tau) + \ldots \right).
\end{equation}

We shall need the usual expansions around $t = 0$: *)
\begin{equation}
(3.7a) \quad \Lambda_\nu = \Lambda(t_\nu) = \Lambda(0) + t_\nu \Lambda'(0) + \ldots = \Lambda(0) + h \mu \Lambda'(0) + \ldots,
\end{equation}
\begin{equation}
\Theta_\nu = \Theta(t_\nu) = -\frac{1}{h} T^{-1}(t_\nu) T(t_\nu) - T(t_\nu) + h \frac{1}{2} T^{-1}(t_\nu) T''(t_\nu) + \ldots =
\end{equation}
\begin{equation}
= A(t_\nu) + h A'(t_\nu) + \ldots = A(0) + t_\nu A'(0) + \ldots + h A(0) + \ldots
\end{equation}
\begin{equation}
(3.7b) \quad = A(0) + h(\nu A'(0) + \tilde{A}(0)) + \ldots
\end{equation}
(see [1,(4.64)] for the denotation $A(t)$, $\tilde{A}(t)$),
\begin{equation}
(3.7c) \quad B(t_\nu) = B(0) + h \mu B'(0) + \ldots,
\end{equation}
\begin{equation}
\bar{E}(t_\nu) = h \bar{E}_1(t_\nu) + O(h^2) = h \left( X_{0,1}(t_\nu) + \varepsilon X_{1,1}(t_\nu) + \ldots \right) + O(h^2) =
\end{equation}
\begin{equation}
= h \left( X_{0,1}(0) + h \nu X_{0,1}'(0) + \ldots + h \omega^{-1} X_{1,1}(0) + h \nu X_{1,1}'(0) + \ldots \right) + O(h^2) =
\end{equation}
\begin{equation}
= h^2 \left[ \omega^{-1} X_{0,1}(0) + \omega^{-1} X_{1,1}(0) + \ldots \right] + O(h^2).
\end{equation}

*) It will turn out that the terms of these expansions always appear in conjunction with exponentially decaying inner solution components; since terms of the form "$\nu$ - exponentially decaying term" are bounded uniformly in $\nu$ (at $O(1)$-level), terms like "$h^1 \nu$ - exponentially decaying term" are indeed at $O(h^3)$-level. The situation is analogous as in Part 1 where "$t'$, inner solution term" = $\varepsilon^t \cdot t'$, inner solution term" was considered $O(\varepsilon^t)$.
(Cf. [1, (4.29a)] for the definition of the $X_{1,i}(t), Y_{1,i}(t)$.)

(3.6b) and the expansions (3.7) are now inserted into (2.3c), and the second component is multiplied by $\varepsilon = h\omega^{-1}$. Rearranging in powers of $h$ we obtain the $h$-expansion for $\delta_{\nu}^{str}$:

$$
(3.8a) \quad \left( \frac{\delta_{\nu}^{str}}{h^{\omega - 1} \varepsilon_{\nu}^{str}} \right) = \left( \begin{array}{c}
h^2 l_{\nu,1}^{(0)} + h^3 l_{\nu,1}^{(1)} + \ldots \\
h^2 \omega - 1 l_{\nu,2}^{(0)} + h^3 \omega - 1 l_{\nu,2}^{(1)} + \ldots \end{array} \right),
$$

where the leading terms read

$$
(3.8b) \quad l_{\nu,1}^{(0)} = -[M_0(\tau_\nu) - M_0(\tau_{\nu-1})] + a_{1,2}(0) N_0(\tau_{\nu-1}),
$$

$$
(3.8c) \quad l_{\nu,2}^{(0)} = -[N_0(\tau_\nu) - N_0(\tau_{\nu-1})] - \omega c_2(0) N_0(\tau_\nu),
$$

and

$$
(3.8c) \quad l_{\nu,1}^{(1)} = -[M_1(\tau_\nu) - M_1(\tau_{\nu-1})] + a_{1,1}(0) M_0(\tau_{\nu-1}) + c_1(0) M_0(\tau_\nu) + a_{1,1}(0) M_0(\tau_{\nu-1}) + + c_1(0) M_0(\tau_\nu) + a_{1,1}(0) M_0(\tau_{\nu-1}) + + \omega c_1(0) N_0(\tau_{\nu-1}) + a_{1,2}(0) N_0(\tau_{\nu-1}) + b(0) N_0(\tau_\nu),
$$

$$
(3.8c) \quad l_{\nu,2}^{(1)} = -[N_1(\tau_\nu) - N_1(\tau_{\nu-1})] - \omega c_2(0) N_0(\tau_\nu) - + \omega c_2(0) N_0(\tau_\nu) + a_{2,2}(0) N_0(\tau_{\nu-1}.
$$

For the term originating from $B(\tau_{\nu})$ we have used the abbreviation

$$
(3.8d) \quad b(0) := b_{1,2}(0) X_{0,1}(0)
$$

in accordance with [1, (4.70)]. For the present purpose (discussion of remainder term $R_{\nu}$ of fifth order) we shall need the explicit representations of the leading terms $l_{\nu,1}^{(0)}, l_{\nu,1}^{(1)}$ (cf. (3.8b,c)) but not of the $l_{\nu,1}^{(j)}, j \geq 2$.

It should be pointed out that the expansion (3.8a) is not to be understood as a pure asymptotic expansion in powers of $h$; the coefficients $l_{\nu,1}^{(j)}$ are themselves $h$-dependent (via the parameter $\omega = \frac{h}{\varepsilon}$).

Powers of $\omega$ appear in the $l_{\nu,1}^{(j)}$ explicitly (cf. (3.8b,c)) as well as implicitly within the $M_i(\tau)$ and $N_i(\tau)$ (cf. (3.6a)). Thus it is not a priori obvious that our arrangement in powers of $h$ (cf. (3.8a)) is really justified. (For instance, positive powers of $\omega$ become unbounded with increasing stiffness, i.e. with $\varepsilon \to 0$.) Let us discuss this point in more detail:

i) Due to assumption $\varepsilon \leq Ch$, any negative power of $\omega$ is harmless because $\omega^{-k} \leq C^k$. The positive powers of $\omega$ which appear in the $l_{\nu,1}^{(j)}$ are harmless, too, for the following reasons: Each term $\omega^k m_{i,i}(\tau_\nu)$ or $\omega^k n_{i,i}(\tau_\nu)$ (where $m_{i,i}(\tau_\nu), n_{i,i}(\tau_\nu)$ are inner solution terms of the form $p(\tau_\nu)e^{-c_2(0)\tau_\nu}$) is uniformly bounded for $\varepsilon \to 0$ at all grid points $\nu \geq 1$ due to the exponential damping (cf. [1, (3.5)]). However, a problem arises for $\nu = 1$ because $M_1(\tau_{\nu-1}) = M_1(0), N_1(\tau_{\nu-1}) = N_1(0)$ appear (for which $e^{-c_2(0)\tau_0} = e^0 = 1$ has no damping power). But it can easily be seen that those quantities $m_{i,i}(0), n_{i,i}(0)$ which are affected with positive powers of $\omega$ vanish due to the "$e^\omega$-property" which has been proved in Part 1 of this paper (cf. [1, Subsections 4.3, 4.4]). Hence (3.8) is a reasonable expansion for $\varepsilon \leq Ch$.

ii) For $h \leq C\varepsilon$ (C some moderate constant) the situation is contrary. In particular, negative powers of $\omega$ blow up for $h \ll \varepsilon$, the consequence of which is that only

$$
(3.9) \quad \left( \frac{\delta_{\nu}^{str}}{h^{\omega - 1} \varepsilon_{\nu}^{str}} \right) = O(h)
$$
can immediately be concluded from the above representation. This shows that our representation (3.1), (3.3), (3.8) for $\delta_\nu$ is not suitable for the subdomain $h \leq C\varepsilon$ of the "$\varepsilon - h$ - plane". (In the classical case - $\varepsilon$ not small - for instance, it is well known that $\delta_\nu^{sm}$ as well as $\delta_\nu^{tr}$ are at $O(h^5)$-level.) In Section 4 the case $h \leq C\varepsilon$ will be treated by a different approach.

For $\delta_\nu^{sm}$ we will use the expansion

$$
\begin{pmatrix}
\delta_\nu^{sm}
\end{pmatrix} = \begin{pmatrix}
h^{5/2}k_\nu^{(0)} + \ldots \\
h^{3/2}k_\nu^{(0)} + \ldots
\end{pmatrix}.
$$

3.2 Ansatz for $\rho_\nu$ and Equating Coefficients in the Remainder Term Equation

For $\rho_\nu = \begin{pmatrix} x_\nu \\ y_\nu \end{pmatrix}$ we make a discrete singular perturbation ansatz in powers of $h$:

$$
\begin{align*}
x_\nu &= h^2 x_\nu^{(0)} + h^4 x_\nu^{(2)} + h^5 x_\nu^{(3)} + h^3 \xi_\nu^{(2)} + h^4 \xi_\nu^{(2)} + x_\nu^{(R)} \\
y_\nu &= h^2 y_\nu^{(0)} + h^4 y_\nu^{(2)} + h^5 y_\nu^{(3)} + h^2 \eta_\nu^{(0)} + h^4 \eta_\nu^{(2)} + h^5 \eta_\nu^{(3)} + y_\nu^{(R)}
\end{align*}
$$

$x_\nu^{(i)}$, $y_\nu^{(i)}$ are discrete analogs of the smooth outer solution terms, and $\xi_\nu^{(i)}$, $\eta_\nu^{(i)}$ will be understood as discrete, rapidly decaying inner solution terms. For the remainder terms $x_\nu^{(R)}$, $y_\nu^{(R)}$ of the expansion (3.11) one would expect that estimates $x_\nu^{(R)} = O(h^6)$, $y_\nu^{(R)} = O(h^6)$ can be derived. As the discussion below will show (cf. Subsection 3.4), only $x_\nu^{(R)} = O(h^5)$, $y_\nu^{(R)} = O(h^5)$ holds, which is of course sufficient for our purpose (recall that our aim is to show that $\rho_\nu = O(h^5)$ with increasing $\nu$). (This situation - that one power of $h$ is lost" within the (straightforward B-type-) estimation of $x_\nu^{(R)}$, $y_\nu^{(R)}$ - is quite analogous to Part 1 where one power of $\varepsilon$ was lost within the estimation of the remainder terms of the $\varepsilon$ - expansions of the $\varepsilon_i(t)$; cf. [1,(4.24)].)

Up to now we have not fixed the starting values $x_0$ and $y_0$. Recall that our difference equation is considered in an interval $[0, T]$ which is interpreted as a subinterval (with constant stepsize $h$) of the whole integration interval (cf. [1,Section 2]). The starting value for the remainder term $R_\nu$ is inductively defined by the remainder term on the preceding interval (cf. [1,(2.17)]). Only for the first of these subintervals the starting value is 0. Anticipating the main result of the present paper - that at the end of each subinterval all order reduction effects are damped away - we inductively assume $R_\nu = O(h^5)$; thus,

$$
\rho_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = O(h^5),
$$

i.e., the starting conditions for the $X_\nu^{(i)}$, $Y_\nu^{(i)}$, $\xi_\nu^{(i)}$, $\eta_\nu^{(i)}$, $x_\nu^{(R)}$, $y_\nu^{(R)}$ of our ansatz (3.11) read

$$
\begin{align*}
X_0^{(i)} &= 0, & Y_0^{(i)} + \eta_0^{(i)} &= 0 & \ldots & h^2 \text{- level} \\
x_0^{(1)} + \xi_0^{(0)} &= 0, & Y_0^{(1)} + \eta_0^{(1)} &= 0 & \ldots & h^3 \text{- level} \\
x_0^{(2)} + \xi_0^{(2)} &= 0, & Y_0^{(2)} + \eta_0^{(2)} &= 0 & \ldots & h^4 \text{- level} \\
(X_0^{(3)} + \xi_0^{(2)}) + x_0^{(R)} &= x_0, & (Y_0^{(3)} + \eta_0^{(3)}) + y_0^{(R)} &= y_0 & \ldots & h^5 \text{- level}
\end{align*}
$$

Concerning the freedom at the $O(h^6)$-level (last line of (3.13a)) it will turn out to be appropriate (for technical reasons) to choose $^*$

$$
\begin{align*}
X_0^{(3)} + \xi_0^{(2)} &= 0, & Y_0^{(3)} + \eta_0^{(3)} &= 0, & x_0^{(R)} &= x_0, & y_0^{(R)} &= y_0.
\end{align*}
$$

$^*$ As mentioned above, it will turn out that $x_\nu^{(R)}$ and $y_\nu^{(R)}$ are only $O(h^5)$ but not $O(h^6)$. It is therefore natural to choose $x_0^{(R)}$ and $y_0^{(R)}$ at $O(h^5)$-level.
Now we proceed as usual in the singular perturbation theory: The ansatz (3.11) and the expansion for the inhomogeneity \( \xi_\nu \) (cf. (3.3a), (3.8a) and (3.10)) will be inserted into the difference equation (2.10). Our main goal is to show that all order reduction effects disappear with increasing \( \nu \), which means that the quantities \( X_\nu^{(j)} \) and \( Y_\nu^{(j)} \) must vanish for \( j \leq 2 \), such that smooth outer solution components appear at \( O(h^5) \)-level only. Our proof of this damping property is based on algebraic investigations of the leading inner solution terms. The "kernel" of our argumentation will be that the usual requirement \( \lim_{\nu \to \infty} \xi_\nu^{(j)} = 0 \) must be compatible with a starting condition \( \xi_\nu^{(j)} = 0 \) for \( j = 0, 1 \). From \( \xi_\nu^{(0)} = 0 \) and \( \xi_\nu^{(1)} = 0 \) and (3.13) the conclusion \( X_\nu^{(0)} = 0 \), \( X_\nu^{(1)} = 0 \) and \( X_\nu^{(2)} = 0 \) can be drawn; then it will be easy to derive the desired property \( X_\nu^{(j)} = Y_\nu^{(j)} = 0 \) for \( j \leq 2 \).

There are several technical points to be discussed. Let us consider in detail the nonlinear terms \( \gamma_i(t_\nu, x_\nu, y_\nu) \) of (2.10). With the denotations

\[
X_\nu := h^2 Y_\nu^{(0)} + h^3 Y_\nu^{(1)} + h^4 Y_\nu^{(2)} + h^5 Y_\nu^{(3)},
Y_\nu := h^2 Y_\nu^{(0)} + h^3 Y_\nu^{(1)} + h^4 Y_\nu^{(2)} + h^5 Y_\nu^{(3)},
\]

\[
\xi_\nu := h^5 \xi_\nu^{(0)} + h^4 \xi_\nu^{(1)} + h^5 \xi_\nu^{(2)},
\eta_\nu := h^5 \eta_\nu^{(0)} + h^5 \eta_\nu^{(1)} + h^5 \eta_\nu^{(2)} + h^5 \eta_\nu^{(3)},
\]

and

\[
\tilde{\rho}_\nu := \left( \frac{X_\nu + \xi_\nu}{Y_\nu + \eta_\nu} \right),
\tilde{\rho}_\nu^{\text{str}} := \left( \frac{X_\nu}{Y_\nu} \right),
\tilde{\rho}_\nu^{\text{str}} := \left( \frac{\xi_\nu}{\eta_\nu} \right),
\n(3.14b)
\]

these nonlinear terms read

\[
(3.15)
\gamma_i(t_\nu, \tilde{\rho}_\nu) = \gamma_i(t_\nu, \tilde{\rho}_\nu + \rho_\nu^{(R)}),
= \left[ \gamma_i(t_\nu, \tilde{\rho}_\nu + \rho_\nu^{(R)}) - \gamma_i(t_\nu, \tilde{\rho}_\nu) \right] + \left[ \gamma_i(t_\nu, \tilde{\rho}_\nu^{\text{str}}) - \gamma_i(t_\nu, \tilde{\rho}_\nu^{\text{str}}) \right] + \gamma_i(t_\nu, \tilde{\rho}_\nu^{\text{str}}) + \gamma_i(t_\nu, \tilde{\rho}_\nu^{\text{str}}) .
\]

According to the first bracket on the right hand side of (3.15) we define the nonlinear function

\[
(3.16a)
V_\nu(\rho) \equiv \left( \frac{V_{\nu,1}(\rho)}{V_{\nu,2}(\rho)} \right) := \left( \frac{\gamma_1(t_\nu, \tilde{\rho}_\nu + \rho) - \gamma_1(t_\nu, \tilde{\rho}_\nu)}{\gamma_2(t_\nu, \tilde{\rho}_\nu + \rho) - \gamma_2(t_\nu, \tilde{\rho}_\nu)} \right).
\]

The properties of \( V_\nu(\rho) \) will be essential for the estimation of the remainder term \( \rho_\nu^{(R)} \) of the ansatz (3.11). (When \( \rho_\nu^{(R)} \) is to be discussed, \( \tilde{\rho}_\nu \) is already determined due to the recursive structure of the singular perturbation technique, i.e. \( V_\nu(\rho) \) depending on \( \tilde{\rho}_\nu \) - is then a well-defined function.) Due to the definition of the \( \gamma_i(t_\nu, x, y) \) (cf. (2.6c) and (2.8)) and due to our smoothness assumptions with respect to \( f \) (cf. [1,(4.2c)]), the functions \( V_\nu(\rho) \) are Lipschitz continuous for all \( \nu \). Let \( L_V \) denote a Lipschitz bound which holds uniformly in \( \nu \):

\[
(3.16b)
\| V_\nu(\rho) - V_\nu(\rho) \| \leq L_V \| \rho - \tilde{\rho} \|.
\]

In particular, due to the definition of \( V_\nu(\rho) \) (cf. (3.16a)):

\[
(3.16c)
V_\nu(0) = 0, \quad \| V_\nu(\rho) \| \leq L_V \| \rho \| .
\]

Our next step is to expand the last term \( \gamma_i(t_\nu, \tilde{\rho}_\nu^{\text{str}}) \) on the right hand side of (3.15):
\[ \gamma_i(t, \tilde{\rho}^{zm}) = \gamma_i(t, h^2 X^{(0)} + \ldots h^2 X^{(2)} + h^2 Y^{(0)} + \ldots h^2 Y^{(2)}) = \\
= \gamma_i(t, 0,0) + \gamma_{iz}(t, 0,0)(h^2 X^{(0)} + \ldots h^2 X^{(2)}) + \gamma_{iy}(t, 0,0)(h^2 Y^{(0)} + \ldots h^2 Y^{(2)}) + \\
+ h^4 \frac{1}{2} \gamma_{ixx}(t, 0,0)(X^{(0)})^2 + h^4 \gamma_{ixy}(t, 0,0)X^{(0)}Y^{(0)} + \\
+ h^4 \frac{1}{2} \gamma_{iyy}(t, 0,0)(Y^{(0)})^2 + h^4 \gamma_{ixx}(t, 0,0)Y^{(0)}X^{(0)} + \\
+ h^4 \gamma_{ixy}(t, 0,0)(X^{(0)}Y^{(1)} + X^{(1)}Y^{(0)}) + h^4 \gamma_{iyy}(t, 0,0)Y^{(0)}Y^{(1)} + O(h^6) . \\
\] (3.17a)

From (2.3) and (2.6c) we have immediately *)

\[ \gamma_i(t, 0,0) = 0 , \]

(3.17b) \[ \gamma_{iz}(t, 0,0) = O(h) , \quad \gamma_{iy}(t, 0,0) = O(h) , \]

(3.17c) \[ \gamma_{ixx}(t, 0,0) = O(1) , \quad \gamma_{ixy}(t, 0,0) = O(1) , \quad \gamma_{iyy}(t, 0,0) = O(1) , \]

(3.17d) All higher derivatives are \( O(1) \), too.

From (3.17) we can conclude that no difficulties arise from the nonlinear terms \( \gamma_i(t, \tilde{\rho}^{zm}) \): The quantities \( X^{(j)} \) and \( Y^{(j)} \), which are defined by equating coefficients of \( h^{j+2} \), appear at least at \( O(h^{j+3}) \)-level in the expansion (3.17a) (due to (3.17c)). Hence, due to the recursive nature of the singular perturbation technique, these terms do not influence the definition of the \( X^{(j)} \) and \( Y^{(j)} \) themselves; they are only \textit{inhomogeneous} terms in the respective difference equations defining the \( X^{(j)} \), \( Y^{(j)} \), \( l > j \). (Many of these terms will actually vanish since we shall show \( X^{(j)} = Y^{(j)} = 0 \) for \( j \leq 2 \).)

One point which is of minor importance for our purpose has to be mentioned briefly: The terms \( \gamma_{iz}(t, 0,0), \gamma_{ixx}(t, 0,0), \ldots \) which occur in the expansion (3.17a) in conjunction with non-vanishing quantities \( X^{(j)}, Y^{(j)} \) have to be split into smooth and stretched variable terms in the usual way - for instance \( \gamma_{ix}(t, 0,0) = \gamma_{iz}(t, 0,0) + \gamma_{ixx}(t, 0,0) - \) according to the dependence of the \( \gamma_i's \) on \( \tilde{\rho}_4(t) = E(t_\nu) + S(t_\nu) \) (cf. for instance the splitting of \( \delta_\nu \) in (3.3)). Terms originating from the smooth components of this splitting - as for instance \( \gamma_{iz}(t, 0,0)X^{(j)}, j > 2 \) - will appear in the inhomogeneities of the difference equations which define the outer solution terms \( X^{(j)}, Y^{(j)} \), and terms originating from the rapidly decaying components - as for instance \( \gamma_{ixx}(t, 0,0)X^{(j)}, j > 2 \) - are involved in the inhomogeneities of the difference equations defining the inner solution terms \( \xi^{(j)}, \eta^{(j)} \).

Summarizing, we observe that \textit{none} of the terms of the expansion (3.17a) has an influence on the definition of the \( X^{(j)}, Y^{(j)} \) and the \( \xi^{(j)}, \eta^{(j)} \) themselves - they only appear as recursively

*) With the notation \( W \equiv \begin{pmatrix} w_{1,1} & w_{1,2} \\ w_{2,1} & w_{2,2} \end{pmatrix} := T_\nu^{-1} \int_0^1 f_{yy}(t, x(t_\nu)) \sigma T_\nu \tilde{\rho}_4(t_\nu) d\sigma \cdot T_\nu \tilde{\rho}_4(t_\nu) \cdot T_\nu \) for the matrix appearing in the first term of (2.6c), we have \( \frac{\partial}{\partial x}(W \cdot \rho) = \frac{\partial}{\partial x}(W \cdot \begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} w_{1,1} \\ w_{2,1} \end{pmatrix} \) and \( \frac{\partial}{\partial y}(W \cdot \rho) = \frac{\partial}{\partial y}(W \cdot \begin{pmatrix} x \\ y \end{pmatrix}) = \begin{pmatrix} w_{1,2} \\ w_{2,2} \end{pmatrix} ; \) note that \( W = O(h) \) due to the factor \( \tilde{\rho}_4(t_\nu) = O(h) \). The partial derivatives of the second term of (2.6c) (with the bilinear operator) obviously vanish for \( \rho = 0 \). Thus (3.17c) holds. (3.17d,e) is an immediate consequence of our smoothness assumptions w.r.t \( f \) (cf. [1,(4.2c)]).
defined inhomogenous terms in the respective difference equations. Recall that our aim is the explicit determination - with algebraic manipulations - of the leading inner solution terms and to show that the starting condition \( \xi^{(j)}_0 = 0, \ j = 0,1 \), is compatible with \( \lim_{\nu \to \infty} \xi^{(j)}_\nu = 0 \). To this end we shall explicitly determine the stretched-variable coefficients at the \( h^2 \)- and \( h^3 \)-level. Since the second equation in (2.10) is multiplied by \( \varepsilon = h \omega^{-1} \), there exists only one stretched-variable term of the expansion (3.17a) at \( O(h^3) \)-level, namely \( \gamma_{1z}(t_\nu,0,0) \cdot h^2 X^{(0)}_\nu = O(h) \cdot h^2 X^{(0)}_\nu \). However, when equating the coefficients of \( h^2 \) we shall show that \( X^{(0)}_\nu \equiv 0 \), hence \( \gamma_{1z}(t_\nu,0,0)h^2 X^{(0)}_\nu \) will vanish and there do not exist terms of the expansion (3.17a) which influence the leading terms \( \xi^{(j)}_\nu, \eta^{(j)}_\nu, \ j = 0,1 \).

As a final remark concerning the expansion (3.17a) we note that a simple stability consideration will show that the non-vanishing terms \( X^{(j)}_\nu, Y^{(j)}_\nu \) (which depend on \( h \) in a harmless way) are \( O(1) \) uniformly in \( h \). This will serve as a justification for the fact that we have written \( O(h^6) \) for the remainder term of the expansion (3.17a).

Let us now continue the discussion of the splitting (3.15). The second bracket on the right hand side of (3.15), which is of purely stretched-variable type, is treated in a similar way as \( \gamma_{i}(t_\nu, \hat{\eta}^{(m)}_\nu) \):

\[
\begin{align*}
\gamma_{1z}(t_\nu, X^{(0)}_\nu, Y^{(0)}_\nu) & \equiv \gamma_{1z}(t_\nu, X^{(0)}_\nu, Y^{(0)}_\nu) = \gamma_{1z}(t_\nu, X^{(0)}_\nu, Y^{(0)}_\nu) = \\
+ & \gamma_{i}(t_\nu, X^{(0)}_\nu, Y^{(0)}_\nu) \xi^{(0)}_\nu + \gamma_{iy}(t_\nu, X^{(0)}_\nu, Y^{(0)}_\nu) \eta^{(0)}_\nu + \\
+ & \frac{1}{2} \gamma_{ixy}(t_\nu, X^{(0)}_\nu, Y^{(0)}_\nu) \xi^{(0)}_\nu \eta^{(0)}_\nu + \frac{1}{2} \gamma_{iyy}(t_\nu, X^{(0)}_\nu, Y^{(0)}_\nu) \eta^{(0)}_\nu^2 + \ldots = \gamma_{1z}(t_\nu, X^{(0)}_\nu, Y^{(0)}_\nu) = \gamma_{1z}(t_\nu, X^{(0)}_\nu, Y^{(0)}_\nu).
\end{align*}
\]

For expansion (3.18) essentially the same remarks apply as for (3.17a): None of the terms in (3.18) do influence the definition of the \( \xi^{(j)}_\nu, \eta^{(j)}_\nu \) themselves; they only appear as recursively defined inhomogeneous terms in the difference equations defining the \( \xi^{(l)}_\nu, \eta^{(l)}_\nu, l > j \). All terms in (3.18) are of course assumed to be expanded around \( x = 0, y = 0 \), as for instance

\[
\begin{align*}
h^2 \gamma_{i}(t_\nu, X^{(0)}_\nu, Y^{(0)}_\nu) \eta^{(0)}_\nu & = \\
= & h^2 \gamma_{i}(t_\nu,0,0) + h^2 \gamma_{i}(t_\nu,0,0)X^{(0)}_\nu + h^2 \gamma_{i}(t_\nu,0,0)Y^{(0)}_\nu + \ldots \cdot \eta^{(0)}_\nu = \\
= & h^2 \gamma_{i}(t_\nu,0,0) + h^2 \gamma_{i}(t_\nu,0,0)X^{(0)}_\nu + h^2 \gamma_{i}(t_\nu,0,0)Y^{(0)}_\nu + \ldots = \\
= & h^2 \gamma_{i}(t_\nu,0,0) \eta^{(0)}_\nu + h^2 \gamma_{i}(t_\nu,0,0) \eta^{(0)}_\nu + O(h^6).
\end{align*}
\]

(The leading terms \( h^2 \gamma_{i}(t_\nu,0,0) \eta^{(0)}_\nu \) and \( h^2 \gamma_{i}(t_\nu,0,0) \eta^{(0)}_\nu \) are actually \( O(h^5) \) due to (3.17c); all other non-vanishing terms are at least at \( O(h^6) \)-level since \( X^{(j)}_\nu, Y^{(j)}_\nu \equiv 0 \) for \( j \leq 2 \) which will be shown below.)

There remains one point w.r.t. the expansions (3.17a) and (3.19) to be mentioned: All smooth coefficient functions appearing in the equations for the inner solution terms are to be expanded around \( t = 0 \) (cf. for instance (3.7) or [1,(4.11)]). Hence, for instance, \( \gamma_{i}(t_\nu,0,0) \) is to be expanded into

\[
\begin{align*}
\gamma_{i}(t_\nu,0,0) & = \gamma_{i}(0,0,0) + \nu \gamma_{i}(0,0,0) + h^2 \gamma_{i}(0,0,0) + \ldots.
\end{align*}
\]

\(^{a)}\) The expansion (3.19) contains, among others, a non-vanishing term \( h^2 h^5 \gamma_{i}(t_\nu,0,0)X^{(2)}_\nu \eta^{(0)}_\nu \), and
It is obvious that only smooth data functions are to be expanded around \( t = 0 \). It would not be reasonable to expand stretched - exponentially decaying - terms (as for instance \( \gamma_{\nu}^{(3)}(\nu,0,0) \) or \( \xi_{\nu}^{(0)} \)) around \( t = 0 \) due to their lack of smoothness.

Recall that, for our purpose, the explicit form of the stretched-variable terms in the expansions (3.17a) and (3.19) is only needed at \( O(h^2) \) and \( O(h^3) \)-level (since we will explicitly determine \( \xi_{\nu}^{(0)} , \eta_{\nu}^{(0)} \) and \( \xi_{\nu}^{(1)} , \eta_{\nu}^{(1)} \) - cf. Subsection 3.3 below). Taking into account that the second equation in (2.10) is multiplied by \( \varepsilon = h^3 \), we see that there is only one such term at \( O(h^3) \)-level, originating from \( h^2 \gamma_{1y}(0,0,0) \eta_{\nu}^{(0)} \) in (3.19). (Note that the \( O(h^2) \)-level is not influenced by the nonlinearity.) Using the definition (2.6c) of \( \Gamma_{\nu}(\rho_{\nu}) \) and taking into account *) that \( \bar{v}_{4}(0) = hX_{0,1}(0) + O(h^2) \), a simple calculation shows that the only term at \( h^3 \)-level originating from \( h^2 \gamma_{1y}(0,0,0) \eta_{\nu}^{(0)} \) reads

\[
(3.21) \quad h^3 b(0) \eta_{\nu}^{(0)}
\]

(for the definition of \( b(0) = b_{1,2}^{(4)}(0)X_{0,1}(0) \) see (3.8d)). (3.21) will appear in the equation for \( \xi_{\nu}^{(1)} \).

For the collection of all other terms of the above expansions (cf. (3.17) - (3.20)) at \( O(h^2) \) - level, which do not influence our algebraic manipulations, we simply write

\[
(3.22) \quad h^3 (I_{\nu_{1}}^{(j-3)sm} + I_{\nu_{2}}^{(j-3)str}) ;
\]

here the subscript \( i \) \( (i = 1,2) \) denotes the first or second component, respectively. Hence the \( \gamma_{i}(t_{\nu},\rho_{\nu}) \) from (3.15) have an expansion (for the definition of \( V_{\nu_{i}}(\rho) \) cf. (3.16))

\[
\begin{align*}
(3.23a) \quad & \gamma_{1}(t_{\nu},\rho_{\nu}) = h^3 \gamma_{\nu_{1}}^{(0) sm} + h^3 b(0) \eta_{\nu}^{(0)} + h^4 I_{\nu_{1}}^{(1) sm} + h^4 I_{\nu_{1}}^{(1) str} + \\
& + h^5 I_{\nu_{1}}^{(2) sm} + h^5 I_{\nu_{1}}^{(2) str} + V_{\nu_{1}}(\rho_{\nu})(R) ,
\end{align*}
\]

\[
\begin{align*}
(3.23b) \quad & \gamma_{2}(t_{\nu},\rho_{\nu}) = h^3 I_{\nu_{2}}^{(0) sm} + h^3 I_{\nu_{2}}^{(1) str} + h^4 I_{\nu_{2}}^{(1) sm} + h^4 I_{\nu_{2}}^{(1) str} + \\
& + h^5 I_{\nu_{2}}^{(2) sm} + h^5 I_{\nu_{2}}^{(2) str} + V_{\nu_{2}}(\rho_{\nu})(R) .
\end{align*}
\]

The terms \( I_{\nu_{i}}^{(j) sm} \) \( j = 0,1,2 \), will turn out to be zero since \( X_{\nu_{i}}^{(j)} = Y_{\nu_{i}}^{(j)} = 0 \) for \( j = 0,1,2 \).

With these preliminaries, the transformed remainder term equation (2.10) reads:

\[
\begin{align*}
& h^2 \frac{1}{h} [X_{\nu}^{(0)} - X_{\nu}^{(i-1)}] + \ldots + h^2 \xi_{\nu}^{(0)} - \xi_{\nu}^{(i-1)} + \ldots = \\
& = c_{1}(t_{\nu}) \cdot \left[ h^2 X_{\nu}^{(0)} + \ldots \right] + |c_{1}(0) + h\nu c_{1}(0) + \ldots | \cdot \left[ h^2 \xi_{\nu}^{(0)} + \ldots \right] + \\
& + g_{1,1}(t_{\nu}) \cdot \left[ h^2 X_{\nu}^{(0)} + \ldots \right] + |c_{1,1}(0) + h\nu c_{1,1}(0) + \ldots | \cdot \left[ h^2 \xi_{\nu}^{(1)} + \ldots \right] + \\
& + g_{1,2}(t_{\nu}) \cdot \left[ h^2 Y_{\nu}^{(0)} + \ldots \right] + |a_{1,2}(0) + h(\nu a_{1,2}(0) + \bar{a}_{1,2}(0)) + \ldots | \cdot \left[ h^2 \eta_{\nu}^{(0)} + \ldots \right] + \\
& + h^2 I_{\nu_{1}}^{(0) sm} + h^2 I_{\nu_{1}}^{(1) str} + h^3 b(0) \eta_{\nu}^{(0)} + h^4 I_{\nu_{1}}^{(1) sm} + h^4 I_{\nu_{1}}^{(1) str} + \\
& + h^5 I_{\nu_{1}}^{(2) sm} + h^5 I_{\nu_{1}}^{(2) str} + V_{\nu_{1}}(\rho_{\nu})(R) + \\
& + \text{terms which are at least at } O(h^5)-\text{level} ,
\end{align*}
\]

therefore it seems to be necessary to expand \( X_{\nu}^{(3)} \) around \( t = 0 \): \( X_{\nu}^{(3)} = X_{\nu}^{(3)} + h \cdot \nu X_{\nu}^{(2)} - X_{\nu}^{(5)} + \ldots \); such a "discrete Taylor expansion" does indeed exist with a remainder term which is uniformly bounded at a certain \( h^R \)-level. Nevertheless, such an expansion is not required here because all respective (non-vanishing) terms are at least at \( O(h^5) \) (remainder term) - level. (With the equation for the remainder term \( \rho_{\nu}^{(R)} \) no expansions around \( t = 0 \) will be used.)

*) Cf. (3.7d) and recall that the inner solution terms of \( \bar{v}_{4}(t) \) appear with higher powers of \( h \): For instance, \( h^2 \omega_{m,1}(\frac{\tau}{\omega}) = h^5 \omega_{m,1}(\frac{\tau}{\omega}) \).
Equating coefficients in (3.24) and using (3.13) for the definition of the starting values we obtain:

**Coefficients of \( h^2 \):**

\[
\frac{1}{h} [X^{(0)}_{\nu} - X^{(0)}_{\nu-1}] = c_1(t_\nu) X^{(0)}_{\nu} + \vartheta_{1,1}(t_\nu) X^{(0)}_{\nu-1} + \vartheta_{1,2}(t_\nu) Y^{(0)}_{\nu-1},
\]
\[
0 = -c_2(t_\nu) Y^{(0)}_{\nu}.
\]

Thus, \( Y^{(0)}_{\nu} \equiv 0 \). From (3.13) we have \( X^{(0)}_0 \equiv 0 \), which yields \( X^{(0)}_{\nu} \equiv 0 \).

\[
\frac{1}{\omega} [\xi^{(0)}_{\nu} - \xi^{(0)}_{\nu-1}] = a_{1,2}(0) \eta^{(0)}_{\nu-1} + I^{(0)}_{\nu-1},
\]
\[
\frac{1}{\omega} [\eta^{(0)}_{\nu} - \eta^{(0)}_{\nu-1}] = -c_2(0) \eta^{(0)}_{\nu} + \frac{1}{\omega} l^{(0)}_{\nu-1}.
\]

(Note the interrelation between the coefficients in (3.25b) and (3.8b).) From \( Y^{(0)}_\nu \equiv 0 \) and (3.13) we have \( \eta^{(0)}_0 \equiv 0 \). \( \eta^{(0)}_\nu \) will be explicitly determined in Subsection 3.3. As usual in the singular perturbation theory we shall fix the starting value \( \xi^{(0)}_0 \) such that \( \lim_{\nu \to \infty} \xi^{(0)}_\nu = 0 \). It will turn out in Subsection 3.3 that \( \xi^{(0)}_0 = 0 \). Due to (3.13) this entails \( X^{(1)}_0 = 0 \).

**Coefficients of \( h^3 \):**

From \( X^{(0)}_\nu \equiv Y^{(0)}_\nu \equiv 0 \) and (3.17a-c) we immediately obtain \( I^{(0)}_{\nu-1} = 0 \), \( \frac{1}{\omega} \vartheta_{1,1}(t_\nu) X^{(0)}_{\nu-1} = 0 \), \( \frac{1}{\omega} \vartheta_{2,2}(t_\nu) Y^{(0)}_{\nu-1} = 0 \), \( \frac{1}{\omega} Y^{(0)}_{\nu} - Y^{(0)}_{\nu-1} = 0 \). Therefore:

\[
\frac{1}{h} [X^{(1)}_{\nu} - X^{(1)}_{\nu-1}] = c_1(t_\nu) X^{(1)}_{\nu} + \vartheta_{1,1}(t_\nu) X^{(1)}_{\nu-1} + \vartheta_{1,2}(t_\nu) Y^{(1)}_{\nu-1},
\]
\[
0 = -c_2(t_\nu) Y^{(1)}_{\nu}.
\]
Again, \( Y^{(1)}_\nu \equiv 0 \). From \( X^{(1)}_0 = 0 \) we conclude \( X^{(1)}_\nu \equiv 0 \).

\[
|\xi^{(1)}_\nu - \xi^{(1)}_{\nu-1}| = a_{1,2}(0)\eta^{(1)}_{\nu-1} + c_1(0)\xi^{(0)}_{\nu} + a_{1,1}(0)\xi^{(0)}_{\nu-1} + |\nu a'_{1,2}(0) + \bar{a}_{1,2}(0)|\eta^{(0)}_{\nu-1} + b(0)\eta^{(0)}_\nu + f^{(1)}_{\nu-1},
\]

(3.26b)

\[
\frac{1}{\omega}|\eta^{(1)}_\nu - \eta^{(1)}_{\nu-1}| = -c_2(0)\eta^{(0)}_{\nu} - \nu c'_2(0)\eta^{(0)}_{\nu} + \frac{1}{\omega}a_{2,2}(0)\eta^{(0)}_{\nu-1} + \frac{1}{\omega}f^{(1)}_{\nu},
\]

(Note the interrelation between the coefficients in (3.26b) and (3.8c).) From \( Y^{(1)}_\nu \equiv 0 \) and (3.13) we have \( \xi^{(1)}_0 = 0 \). Again we shall see in Subsection 3.3 that the condition \( \lim_{\nu \to \infty} \xi^{(1)}_\nu = 0 \) yields \( \xi^{(1)}_0 = 0 \). Thus, \( X^{(2)}_0 = 0 \).

**Coefficients of \( h^4 \):**

From \( X^{(0)}_\nu \equiv Y^{(0)}_\nu \equiv X^{(1)}_\nu \equiv Y^{(1)}_\nu \equiv 0 \) and (3.17a-c) immediately \( f^{(1)\text{str}}_{\nu;1} \equiv 0 \), \( \frac{1}{\omega}\vartheta_{2,2}(t_\nu)Y^{(1)}_{\nu-1} \equiv 0 \), \( \frac{1}{\omega}\kappa[Y^{(1)}_\nu - Y^{(1)}_{\nu-1}] \equiv 0 \). Therefore:

(3.27a)

\[
\frac{1}{\kappa}[X^{(2)}_\nu - X^{(2)}_{\nu-1}] = c_1(t_\nu)X^{(2)}_\nu + \vartheta_{1,1}(t_\nu)X^{(2)}_{\nu-1} + \vartheta_{1,2}(t_\nu)Y^{(2)}_{\nu-1} ,
\]

\[
0 = -c_2(t_\nu)Y^{(2)}_\nu .
\]

Again, \( Y^{(2)}_\nu \equiv 0 \). From \( X^{(2)}_0 = 0 \) we conclude \( X^{(2)}_\nu \equiv 0 \).

\[
|\xi^{(2)}_\nu - \xi^{(2)}_{\nu-1}| = a_{1,2}(0)\eta^{(2)}_{\nu-1} + p_0(\nu)\xi^{(1)}_\nu + p_1(\nu)\xi^{(0)}_\nu + p_1(\nu)\xi^{(1)}_{\nu-1} + p_1(\nu)\eta^{(1)}_{\nu-1} + p_2(\nu)\eta^{(0)}_{\nu-1} + f^{(1)\text{str}}_{\nu;1} + f^{(1)}_{\nu},
\]

(3.27b)

\[
\frac{1}{\omega}|\eta^{(2)}_\nu - \eta^{(2)}_{\nu-1}| = -c_2(0)\eta^{(2)}_\nu + p_1(\nu)\eta^{(1)}_\nu + p_2(\nu)\eta^{(0)}_\nu + \frac{1}{\omega}p_0(\nu)\xi^{(0)}_{\nu-1} + \frac{1}{\omega}p_0(\nu)\eta^{(0)}_{\nu-1} + \frac{1}{\omega}p_1(\nu)\eta^{(0)}_{\nu-1} + \frac{1}{\omega}f^{(0)\text{str}}_{\nu;2} + \frac{1}{\omega}f^{(2)}_{\nu;2} .
\]

\( p_i(\nu) \) is used as a generic denotation for polynomials of degree \( \leq i \) in \( \nu \). E.g., \( p_1(\nu) \) within the term \( p_1(\nu)\xi^{(0)}_\nu \) (first component of (3.27b)) denotes the quantity \( \nu c'_1(0) \); \( p_1(\nu) \) within the term \( \frac{1}{\omega}p_1(\nu)\eta^{(0)}_{\nu-1} \) (second component of (3.27b)) denotes the quantity \( (\nu a'_{2,2}(0) + \bar{a}_{2,2}(0)) \).

The starting value for \( \eta^{(2)}_\nu \) is \( \eta^{(2)}_0 = 0 \) (cf. (3.13)) because \( Y^{(2)}_0 = 0 \); \( \xi^{(2)}_\nu \) is again determined by the condition \( \lim_{\nu \to \infty} \xi^{(2)}_\nu = 0 \). In Subsection 3.4 we shall see that \( \xi^{(2)}_\nu \) and \( \eta^{(2)}_\nu \) are uniformly bounded and decay like \( p_1(\nu)(1 + \omega c_2(0))^{-\nu} \).

Furthermore, the starting value for \( X^{(3)}_\nu \) is determined by \( X^{(3)}_0 = 0 - \xi^{(2)}_0 \neq 0 \) (cf. (3.13)).

**Coefficients of \( h^5 \):**

[15]
The equations defining $X^{[3]}_{\nu}$ and $X^{[\nu]}_{\nu}$ read (again we have $f^{(2)\nu,\nu}_{\nu,\nu} \equiv 0$ due to $X^{[j]}_{\nu} \equiv Y^{[j]}_{\nu} \equiv 0$ for $j = 0, 1, 2$):

$$
\frac{1}{h} [X^{[3]}_{\nu} - X^{[3]}_{\nu - 1}] = c_1(t_{\nu})X^{[3]}_{\nu} + \theta_{1,1}(t_{\nu})X^{[3]}_{\nu - 1} + \theta_{1,2}(t_{\nu})Y^{[3]}_{\nu - 1} + k^{[0]}_{\nu,1},
$$

$$
0 = -c_2(t_{\nu})Y^{[8]}_{\nu}.
$$

Again, $Y^{[3]}_{\nu} \equiv 0$. Due to $k^{[0]}_{\nu,1} \not\equiv 0$ and $X^{[0]}_{\nu} = 0 - \xi^{[2]}_{0} \not\equiv 0$, $X^{[8]}_{\nu}$ does not vanish. But the boundedness of $X^{[8]}_{\nu}$ (uniformly in $h$) will become clear by a simple stability argument; it is essential that $d_{\nu}^{2}$ is bounded. (Cf. Subsection 3.4 for details.)

For the inner solution at $h^{5}$-level there is only an equation defining $\eta^{[3]}_{\nu}$ ($\xi^{[2]}_{\nu}$ is already defined by (3.27b), and a quantity $\xi^{[3]}_{\nu}$ at $h^{6}$-level does not appear in our ansatz (3.11)):

$$
\frac{1}{\omega} [\eta^{[3]}_{\nu} - \eta^{[3]}_{\nu - 1}] = -c_2(0)\eta^{[3]}_{\nu} + p_1(\nu)\eta^{[2]}_{\nu} + p_2(\nu)\eta^{[1]}_{\nu} + p_3(\nu)\eta^{[0]}_{\nu} +
$$

$$
+ \frac{1}{\omega} p_0(\nu)\xi^{(1)}_{\nu - 1} + \frac{1}{\omega} p_1(\nu)\xi^{(0)}_{\nu - 1} + \frac{1}{\omega} p_0(\nu)\eta^{(2)}_{\nu - 1} + \frac{1}{\omega} p_1(\nu)\eta^{(1)}_{\nu - 1} + \frac{1}{\omega} p_2(\nu)\eta^{(0)}_{\nu - 1} +
$$

$$
+ \frac{1}{\omega} f^{(1)}_{\nu,2} + \frac{1}{\omega} I^{(3)}_{\nu,1}.
$$

The starting value for (3.28b) is $\eta^{[3]}_{0} = 0 - Y^{[3]}_{0} = 0$. In Subsection 3.4 we shall again see that $\eta^{[3]}_{\nu}$ is uniformly bounded and decays like $p_3(\nu)(1 + \omega c_2(0))^{-\nu}$.

**Remainder term $\rho^{[R]}_{\nu}$ of ansatz (3.11):**

To obtain the desired estimate $\rho^{[R]}_{\nu} = O(h^{5})$, the difference equation for $\rho^{[R]}_{\nu}$ is re-scaled to its original form; in particular, the factor $\epsilon = \frac{h}{\omega}$ by which the second component is multiplied (cf. (2.10)) is now omitted. This results in a difference equation of essentially the same type as (2.7):

$$
\frac{1}{h} [\rho^{[R]}_{\nu} - \rho^{[R]}_{\nu - 1}] = A_{\nu}\rho^{[R]}_{\nu} + \Theta_{\nu}\rho^{[R]}_{\nu - 1} + V_{\nu}(\rho^{[R]}_{\nu}) + \delta^{[R]}_{\nu}.
$$

The inhomogeneity $\delta^{[R]}_{\nu}$ consists of all terms which do not depend on $\rho^{[R]}_{\nu}$ and have not yet appeared in the equations (3.25), (3.26), (3.27) and (3.28). In Subsection 3.4 we shall show

$$
\rho^{[R]}_{\nu} = O(h^{5})
$$

this will follow from

$$
\delta^{[R]}_{\nu} = O(h^{5}) \text{ for } \epsilon \leq C h
$$

by a standard B-convergence estimate. To prove (3.31) it will be essential that nonnegative powers of $\omega = \frac{h}{\epsilon}$ appear only together with exponentially decaying inner solution terms in $\delta^{[R]}_{\nu}$. But note that $\delta^{[R]}_{\nu} = O(h^{6})$ is not true because the second component is re-scaled by the factor $\frac{h}{\omega}$. (For instance, $\delta^{[R]}_{\nu,2}$ contains the term $h^{5}\theta_{2,1}(t_{\nu})X^{[3]}_{\nu - 1} \not\equiv 0$ which originates from $h^{\frac{h}{\omega}}\theta_{2,1}(t_{\nu})h^{5}X^{[3]}_{\nu - 1}$ in (3.24b).)

Let us summarize: Once it is proved that $\lim_{\nu \to \infty} \xi^{(j)}_{\nu} = 0$, $j = 0, 1$, is compatible with $\xi^{(j)}_{0} = 0$, $j = 0, 1$, the slowly varying terms $X^{(j)}_{\nu}$, $Y^{(j)}_{\nu}$ of the ansatz (3.11) turn out to vanish for $j \leq 2$ (moreover, $Y^{(3)}_{\nu} \equiv 0$). Hence (3.11) reduces to

$$
\rho^{(j)}_{\nu} = \left( h^{5}X^{(3)}_{\nu} + h^{3}\xi^{(0)}_{\nu} + h^{3}\xi^{(1)}_{\nu} + h^{3}\xi^{(2)}_{\nu} + x^{(R)}_{\nu} 
\right)
$$

$$
\left( h^{2}\eta^{(0)}_{\nu} + h^{2}\eta^{(1)}_{\nu} + h^{2}\eta^{(2)}_{\nu} + h^{2}\eta^{(3)}_{\nu} + y^{(R)}_{\nu} \right).
$$
From this the desired structure of $\rho_\nu$ immediately follows if the non-vanishing inner solution terms $\xi_\nu^{(j)}$, $\eta_\nu^{(j)}$ are proved to be of the exponentially decaying form $p_\nu(\nu)(1 + \omega c_2(0)^{-\nu})$, if the non-vanishing outer solution term $X_\nu^{(3)}$ is shown to be bounded uniformly in $\nu$, and if (3.30) is verified. All these points are discussed in detail in Subsections 3.3 and 3.4 below.

3.3 Explicit Determination of the Leading Inner Solution Terms

We shall now give explicit representations for the solutions $\xi_\nu^{(0)}$, $\eta_\nu^{(0)}$ and $\xi_\nu^{(1)}$, $\eta_\nu^{(1)}$ of the equations (3.25b) and (3.26b). In particular, we will show that $\xi_\nu^{(0)} = \xi_\nu^{(1)} = 0$ is compatible with $\lim_{\nu \to \infty} \xi_\nu^{(0)} = \lim_{\nu \to \infty} \xi_\nu^{(1)} = 0$. As we have seen in Subsection 3.2, this "damping property", which is in a certain sense analogous to the "$e^0$-property" of the $e_\nu(t)$ (cf. [1, Section 4]), is the crucial point in our analysis.

In the sequel we will use the abbreviation

$$Q := 1 + \omega c_2(0).$$

Note that $|Q| > 1$ due to our assumption $\text{Re}(c_2(t)) > 0$ (cf. [1, Section 4]).

**Determination of $\xi_\nu^{(0)}$, $\eta_\nu^{(0)}$:**

With (3.8b), the equations (3.25b) read

$$\begin{align*}
\xi_\nu^{(0)} - \xi_{\nu-1}^{(0)} &= a_{1,2}(0)\eta_{\nu-1}^{(0)} - [M_0(\nu) - M_0(\nu-1)] + a_{1,2}(0)N_0(\nu-1), \\
1/\omega[ \eta_\nu^{(0)} - \eta_{\nu-1}^{(0)} ] &= -c_2(0)\eta_\nu^{(0)} - 1/\omega[N_0(\nu) - N_0(\nu-1)] - c_2(0)N_0(\nu).
\end{align*}$$

With

$$\begin{align*}
\xi_{\nu-1}^{(0)} := &\xi_\nu^{(0)} + M_0(\nu), \\
\eta_{\nu-1}^{(0)} := &\eta_\nu^{(0)} + N_0(\nu),
\end{align*}$$

(3.33) is equivalent to

$$\begin{align*}
\xi_{\nu}^{(0)} - \xi_{\nu-1}^{(0)} &= a_{1,2}(0)\eta_{\nu-1}^{(0)}, \\
\eta_{\nu}^{(0)} - \eta_{\nu-1}^{(0)} &= -\omega c_2(0)\eta_{\nu}^{(0)}.
\end{align*}$$

Due to $\eta_0^{(0)} = 0$, we obtain (cf. (3.34)):

$$\begin{align*}
\eta_\nu^{(0)} &= Q^{-\nu}N_0(0), \\
\eta_{\nu}^{(0)} &= Q^{-\nu}N_0(0) - N_0(\nu).
\end{align*}$$

By linear combination of the equations in (3.35) (where the second equation is multiplied by $a_{1,2}(0)/\omega c_2(0)$) and with the definition

$$\begin{align*}
\xi_{\nu}^{(0)} := &\xi_\nu^{(0)} + a_{1,2}(0)\omega c_2(0)\eta_{\nu}^{(0)}, \\
\xi_{\nu}^{(0)} - \xi_{\nu-1}^{(0)} &= -a_{1,2}(0)[\eta_\nu^{(0)} - \eta_{\nu-1}^{(0)}].
\end{align*}$$

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The general solution of (3.38) is

\[ \xi^{(0)}(O) = -a_{1,2}(0)\eta^{(0)}(O) + C. \]

Thus, due to (3.34), (3.36) and (3.37),

\[ \xi^{(0)}(O) = -\frac{a_{1,2}(0)}{\omega c_2(O)}(1 + \omega c_2(O))\eta^{(0)}(O) + C = -\frac{a_{1,2}(0)}{\omega c_2(O)}Q^{-\nu+1}N_0(O) + C, \]

\[ \xi^{(0)}(O) = -\frac{a_{1,2}(0)}{\omega c_2(O)}Q^{-\nu+1}N_0(O) - M_0(\tau_\nu) + C. \]

Since \( \lim_{\nu \to \infty} Q^{-\nu} = \lim_{\nu \to \infty} M_0(\tau_\nu) = 0 \), the condition \( \lim_{\nu \to \infty} \xi^{(0)}(O) = 0 \) requires \( C = 0 \). Hence \( \xi^{(0)}(O) \) is fixed; the starting value is

\[ (3.41a) \quad \xi^{(0)}(O) = -M_0(0) - \frac{a_{1,2}(0)}{\omega c_2(O)}(1 + \omega c_2(O))N_0(O). \]

With (3.6a) this results in

\[ \xi(0) - \sum_{i=1}^{4} \left[ \omega^{i-3}m_{0,i}(0) + \frac{a_{1,2}(0)}{\omega c_2(O)}\omega^{i-2}n_{0,i}(0) + a_{1,2}(0)\omega^{i-2}n_{0,i}(0) \right] = \]

\[ (3.41b) \quad \xi(0) = -\sum_{i=1}^{4} \omega^{i-3}\left[ m_{0,i}(0) + \frac{a_{1,2}(0)}{c_2(O)}n_{0,i}(0) \right] - \sum_{i=2}^{5} \omega^{i-3}a_{1,2}(0)n_{0,i-1}(0). \]

Now we may write

\[ (3.42) \quad \sum_{i=2}^{5} \omega^{i-3}a_{1,2}(0)n_{0,i-1}(0) = \sum_{i=1}^{4} \omega^{i-3}a_{1,2}(0)n_{0,i-1}(0), \]

because \( n_{0,0}(0) = 0 \) by definition (in accordance with \([1, (4.72)]\)) and \( n_{0,4}(0) = 0 \) which holds due to the \( \epsilon_\nu \)-property (cf. \([1, (4.93b)]\)). Hence we obtain

\[ (3.43) \quad \xi(0) = -\sum_{i=1}^{4} \omega^{i-3}\left[ m_{0,i}(0) + \frac{a_{1,2}(0)}{c_2(O)}n_{0,i}(0) + c_2(0)n_{0,i-1}(0) \right] = 0, \]

because \( m_{0,i}(0) + \frac{a_{1,2}(0)}{c_2(O)}(n_{0,i}(0) + c_2(0)n_{0,i-1}(0)) = 0 \) for all \( i \geq 1 \) as has been shown in Part 1 of this paper (cf. \([1,(4.78)]\) and \([1,(4.73b)]\)).

\textbf{Determination of} \( \xi^{(1)}, \eta^{(1)}; \)

With (3.8c), the equations (3.26b) read

\[ \xi^{(1)}(O) - \xi^{(1)}(\nu - 1) = a_{1,2}(0)\eta^{(1)}(\nu - 1) + \]

\[ + \left[ c_1(0)\xi^{(0)}(O) + a_{1,1}(0)\xi^{(0)}(O) + [\nu a_{1,2}(0) + \bar{a}_{1,2}(0)]\eta^{(0)} + b(0)\eta^{(0)} - \right] \]

\[ - \left[ M_1(\tau_\nu) - M_1(\tau_{\nu - 1}) \right] + a_{1,2}(0)N_1(\tau_{\nu - 1}) + \]

\[ + \left[ c_1(0)M_0(\tau_\nu) + a_{1,1}(0)M_0(\tau_{\nu - 1}) + [\nu a_{1,2}(0) + \bar{a}_{1,2}(0)]N_0(\tau_{\nu - 1}) + b(0)N_0(\tau_\nu) \right], \]

\[ \frac{1}{\omega} \eta^{(1)}(\nu - 1) = -c_2(0)\eta^{(1)}(\nu) - \nu c'_2(0)\eta^{(0)} + \frac{1}{\omega}a_{2,2}(0)n^{(0)}(\nu - 1) - \]

\[ - \frac{1}{\omega} \left[ N_1(\tau_\nu) - N_1(\tau_{\nu - 1}) \right] - c_2(0)N_1(\tau_\nu) - \nu c'_2(0)N_0(\tau_\nu) + \frac{1}{\omega}a_{2,2}(0)N_0(\tau_{\nu - 1}). \]
With (3.34) and
\[ \xi^{(1)}_\nu := \xi^{(1)}_\nu + M_1(\tau) \]  
(3.45)  
\[ \eta^{(1)}_\nu := \eta^{(1)}_\nu + N_1(\tau) \]  
(3.44) is equivalent to
\[ \xi^{(1)}_\nu - \xi^{(1)}_{\nu - 1} = a_{1,2}(0)\eta^{(1)}_\nu + \]  
(3.46)  
\[ + c_1(0)\xi^{(0)}_\nu + a_{1,1}(0)\xi^{(0)}_{\nu - 1} + [\nu a_{1,2}(0) + \bar{a}_{1,2}(0)]\eta^{(0)}_{\nu - 1} + b(0)\eta^{(0)}_\nu , \]  
\[ \eta^{(1)}_\nu - \eta^{(1)}_{\nu - 1} = -\nu c_2(0)\eta^{(0)}_\nu - \omega c'_2(0)\eta^{(0)}_{\nu - 1} + a_{2,2}(0)\eta^{(0)}_\nu . \]  

Since the solution of a difference equation of the type
\[ \eta_\nu - \eta_{\nu - 1} = -\omega c_2(0)\eta_\nu + j_\nu \]  
(3.47)  
is
\[ \eta_\nu = Q^{-\nu}\eta_0 + \sum_{k=1}^{\nu} Q^{k-1-\nu}j_k , \]  
(3.48)  
we obtain together with the starting value \( \eta^{(1)}_0 = N_1(0) \) (which is a consequence of \( \eta^{(1)}_0 = 0 \)) and the representation (3.36) for \( \eta^{(0)}_\nu \):
\[ \eta^{(1)}_\nu = Q^{-\nu}N_1(0) + \sum_{k=1}^{\nu} Q^{k-1-\nu} \left[ a_{2,2}(0)\eta^{(0)}_{k-1} - \omega kc'_2(0)\eta^{(0)}_k \right] = \]  
(3.49)  
\[ = Q^{-\nu}N_1(0) + \nu Q^{-\nu}a_{2,2}(0)N_0(0) - \omega c'_2(0)\frac{\nu(\nu + 1)}{2}Q^{-\nu - 1}N_0(0) , \]  
 \[ \eta^{(1)}_\nu = [N_1(0) + a_{2,2}(0)N_0(0)\nu - \omega Q^{-1}c'_2(0)N_0(0)\frac{\nu(\nu + 1)}{2}]Q^{-\nu} - N_1(\tau) . \]  

By linear combination of the equations in (3.46) (where the second equation is multiplied by \( a_{1,2}(0) \))
and with the definition
\[ \xi^{(1)}_\nu := \xi^{(1)}_\nu + \frac{a_{1,2}(0)}{\omega c_2(0)}\eta^{(1)}_\nu \]  
(3.50)  
we obtain
\[ \xi^{(1)}_\nu - \xi^{(1)}_{\nu - 1} = -a_{1,2}(0)[\eta^{(1)}_\nu - \eta^{(1)}_{\nu - 1}] + c_1(0)\xi^{(0)}_\nu + a_{1,1}(0)\xi^{(0)}_{\nu - 1} + \]  
(3.51)  
\[ + [\nu a_{1,2}(0) + \bar{a}_{1,2}(0)]\eta^{(0)}_{\nu - 1} + b(0)\eta^{(0)}_\nu - \frac{a_{1,2}(0)c'_2(0)}{c_2(0)}\nu\eta^{(0)}_\nu + \frac{a_{1,2}(0)a_{2,2}(0)}{\omega c_2(0)}\eta^{(0)}_{\nu - 1} . \]  

Together with the representations (3.40) for \( \xi^{(0)}_\nu \) and (3.36) for \( \eta^{(0)}_\nu \) we obtain the general solution of (3.51):
\[ \xi^{(1)}_\nu = -a_{1,2}(0)\eta^{(1)}_\nu - \]  
(3.52)  
\[ - c_1(0)\frac{a_{1,2}(0)}{\omega c_2(0)}Q N_0(0)\sum_{k=1}^{\nu} Q^{-k} - a_{1,1}(0)\frac{a_{1,2}(0)}{\omega c_2(0)}Q^2 N_0(0)\sum_{k=1}^{\nu} Q^{-k} + \]  
\[ + b(0)N_0(0)\sum_{k=1}^{\nu} Q^{-k} - \frac{a_{1,2}(0)c'_2(0)}{c_2(0)}N_0(0)\sum_{k=1}^{\nu} kQ^{-k} + \]  
\[ + \left[ \bar{a}_{1,2}(0) + \frac{a_{1,2}(0)a_{2,2}(0)}{\omega c_2(0)} \right]Q N_0(0)\sum_{k=1}^{\nu} Q^{-k} + a'_{1,2}(0)Q N_0(0)\sum_{k=1}^{\nu} kQ^{-k} + C . \]
With
\[
\sum_{k=1}^{\nu} Q^{-k} = \frac{1}{\omega c_2(0)} \left[ 1 - Q^{-\nu} \right],
\]
\[
\sum_{k=1}^{\nu} kQ^{-k} = \frac{1}{\omega c_2(0)} \left[ \frac{Q}{\omega c_2(0)} (1 - Q^{-\nu}) - \nu Q^{-\nu} \right],
\]
we end up with the following representation for \( \xi_\nu^{(1)} \) (due to (3.45), (3.50) and using the representation (3.49) for \( \eta_\nu^{(1)} \)):
\[
\xi_\nu^{(1)} = \xi_\nu^{(1)} - M_1(\tau_\nu) = \xi_\nu^{(1)} - \frac{a_{1,2}(0)}{\omega c_2(0)} \eta_\nu^{(1)} - M_1(\tau_\nu) =
\]
\[
= -\frac{a_{1,2}(0)}{\omega c_2(0)} (1 + \omega c_2(0)) \left[ N_1(0) + a_{2,2}(0)N_0(0)Q - \omega Q^{-1}c_2'(0)N_0(0)\frac{\nu(\nu + 1)}{2} \right] Q^{-\nu} -
\]
\[
- \frac{a_{1,2}(0)c_1(0)}{\omega^2 c_2^2(0)} Q N_0(0) \left[ 1 - Q^{-\nu} \right] - \frac{a_{1,2}(0)a_{1,1}(0)}{\omega^2 c_2^2(0)} Q^2 N_0(0) \left[ 1 - Q^{-\nu} \right] +
\]
\[
+ \frac{b(0)}{\omega c_2(0)} N_0(0) \left[ 1 - Q^{-\nu} \right] - \frac{a_{1,2}(0)c_2'(0)}{\omega^2 c_2^2(0)} N_0(0) \left[ \frac{Q}{\omega c_2(0)} (1 - Q^{-\nu}) - \nu Q^{-\nu} \right] +
\]
\[
\quad + \frac{\tilde{a}_{1,2}(0)}{\omega c_2(0)} \frac{a_{1,2}(0)a_{2,2}(0)}{\omega^2 c_2^2(0)} Q N_0(0) \left[ 1 - Q^{-\nu} \right] +
\]
\[
+ \frac{a_{1,2}(0)}{\omega c_2(0)} Q N_0(0) \left[ \frac{Q}{\omega c_2(0)} (1 - Q^{-\nu}) - \nu Q^{-\nu} \right] - M_1(\tau_\nu) + C.
\]
\[
(3.54)
\]
Since \( \lim_{\nu \to \infty} \nu^\nu Q^{-\nu} = \lim_{\nu \to \infty} M_1(\tau_\nu) = 0 \), the condition \( \lim_{\nu \to \infty} \xi_\nu^{(1)} = 0 \) requires
\[
0 = -\frac{a_{1,2}(0)c_1(0)}{c_2^2(0)} Q\frac{Q}{\omega^2} N_0(0) - \frac{a_{1,2}(0)a_{1,1}(0)}{c_2^2(0)} Q^2 \frac{Q}{\omega^2} N_0(0) + \frac{b(0)}{c_2(0)} N_0(0) -
\]
\[
- \frac{a_{1,2}(0)c_2'(0)}{c_2^2(0)} Q\frac{Q}{\omega^2} N_0(0) + \frac{\tilde{a}_{1,2}(0)}{c_2(0)} Q\frac{Q}{\omega^2} N_0(0) + \frac{a_{1,2}(0)a_{2,2}(0)}{c_2^2(0)} Q\frac{Q}{\omega^2} N_0(0) +
\]
\[
+ \frac{a_{1,2}(0)Q}{c_2^2(0)} \frac{Q^2}{\omega^2} N_0(0) + C.
\]
\[
(3.55)
\]
Hence \( \xi_\nu^{(1)} \) is fixed; (3.54) and (3.55) yield for \( \nu = 0 \) (due to \( Q = 1 + \omega c_2(0), 1 - Q^{-0} = 0 \)):
\[
\xi_0^{(1)} = -M_1(0) - \frac{a_{1,2}(0)}{\omega c_2(0)} (1 + \omega c_2(0)) N_1(0) +
\]
\[
+ \frac{a_{1,2}(0)c_1(0)}{c_2^2(0)} \frac{1}{\omega^2} (1 + \omega c_2(0)) N_0(0) - \frac{a_{1,2}(0)a_{1,1}(0)}{c_2^2(0)} \frac{1}{\omega^2} (1 + \omega c_2(0))^2 N_0(0) -
\]
\[
\quad - \frac{b(0)}{c_2(0)} N_0(0) + \frac{a_{1,2}(0)c_2'(0)}{c_2^3(0)} \frac{1}{\omega^2} (1 + \omega c_2(0)) N_0(0) -
\]
\[
\quad - \frac{\tilde{a}_{1,2}(0)}{c_2(0)} \frac{1}{\omega^2} (1 + \omega c_2(0)) N_0(0) - \frac{a_{1,2}(0)a_{2,2}(0)}{c_2^2(0)} \frac{1}{\omega^2} (1 + \omega c_2(0)) N_0(0) -
\]
\[
\quad - \frac{a_{1,2}(0)}{c_2^2(0)} \frac{1}{\omega^2} (1 + \omega c_2(0))^2 N_0(0).
\]
\[
(3.56a)
\]
With (3.6a) and after some simple manipulations and rearrangement ( \( (1 + \omega c_2(0))^2 \) is split into \( (1 + \omega c_2(0)) + \omega c_2(0)(1 + \omega c_2(0)) \)) this results in:
\( \xi^{(1)}_0 = - \sum_{i=1}^{4} \left[ \omega^{i-4} m_{1,i}(0) + \frac{a_{1,2}(0)}{\omega c_2(0)} \omega^{i-3} n_{1,i}(0) + a_{1,2}(0) \omega^{i-3} n_{1,i}(0) \right] + \\
+ \sum_{i=1}^{4} \left[ \frac{a_{1,2}(0) a_{2,2}(0)}{c_2^2(0)} + \frac{a_{1,2}(0) c_1(0) + a_{1,1}(0)}{c_2^2(0)} - \frac{a_{1,2}(0)}{c_2^2(0)} + \frac{a_{1,2}(0) c_2(0)}{c_2^2(0)} \right] \cdot \omega^{i-4} (1 + \omega c_2(0)) n_{0,i}(0) + \\
+ \sum_{i=1}^{4} \left[ \frac{a_{1,2}(0) a_{1,1}(0)}{c_2(0)} - \frac{a_{1,2}(0)}{c_2(0)} - \frac{a_{1,2}(0)}{c_2(0)} \right] \omega^{i-3} (1 + \omega c_2(0)) n_{0,i}(0) - \\
- \sum_{i=1}^{4} \frac{b(0)}{c_2(0)} \omega^{i-3} n_{0,i}(0). \\
(3.56b)
\]

Now we may write (similarly as in (3.42) above)

\[(3.57) \sum_{i=1}^{4} a_{1,2}(0) \omega^{i-3} n_{1,i}(0) = \sum_{i=1}^{4} a_{1,2}(0) \omega^{i-4} n_{1,i-1}(0),
\]

because \( n_{1,0} \equiv 0 \) by definition (in accordance with \([1,(4.72)]\)) and \( n_{1,1}(0) = 0 \) due to the \( \epsilon^0 \)-property (cf. \([1,(4.93b)]\)). Furthermore,

\[(3.58) \sum_{i=1}^{4} [\cdots] \omega^{i-4} (1 + \omega c_2(0)) n_{0,i}(0) = \sum_{i=1}^{4} [\cdots] \omega^{i-4} (n_{0,i}(0) + c_2(0) n_{0,i-1}(0)),
\]

\[\sum_{i=1}^{4} \frac{b(0)}{c_2(0)} \omega^{i-3} n_{0,i}(0) = \sum_{i=1}^{4} \frac{b(0)}{c_2(0)} \omega^{i-4} n_{0,i-1}(0)
\]

hold due to definition \([1,(4.72)]\) and due to the \( \epsilon^0 \)-property (cf. \([1,(4.93b)]\)). Hence we obtain

\[\xi^{(1)}_0 = - \sum_{i=1}^{4} \omega^{i-4} \left[ m_{1,i}(0) + \frac{a_{1,2}(0)}{c_2(0)} (n_{1,i}(0) + c_2(0) n_{1,i-1}(0)) \right] + \\
+ \sum_{i=1}^{4} \omega^{i-4} \left[ - \frac{a_{1,2}(0) a_{2,2}(0)}{c_2^2(0)} + \frac{a_{1,2}(0) c_1(0) + a_{1,1}(0)}{c_2^2(0)} - \frac{a_{1,2}(0)}{c_2^2(0)} + \frac{a_{1,2}(0) c_2(0)}{c_2^2(0)} \right] \cdot (n_{0,i}(0) + c_2(0) n_{0,i-1}(0)) + \\
+ \sum_{i=1}^{4} \omega^{i-4} \left[ \frac{a_{1,2}(0) a_{1,1}(0)}{c_2(0)} - \frac{a_{1,2}(0)}{c_2(0)} - \frac{a_{1,2}(0)}{c_2(0)} \right] (n_{0,i-1}(0) + c_2(0) n_{0,i-2}(0)) - \\
- \sum_{i=1}^{4} \omega^{i-4} \frac{b(0)}{c_2(0)} n_{0,i-1}(0) = 0;
\]

the fact that \( \xi^{(1)}_0 = 0 \) follows immediately from the representation \([1,(4.92)]\) which holds for all \( i \geq 1 \) (cf. also \([1,(4.73b)]\)).
(3.43) and (3.59) show that, indeed,

\[
\begin{align*}
X^{(0)}_{\nu} &= X^{(1)}_{\nu} = X^{(2)}_{\nu} \\ Y^{(0)}_{\nu} &= Y^{(1)}_{\nu} = Y^{(2)}_{\nu} 
\end{align*}
\]

due to our argumentation in Subsection 3.2. I.e., \( \rho_{\nu} \) (and therefore \( R_{\nu} = T_{\nu} \rho_{\nu} \)) contains no smooth outer solution components of reduced order \( O(h^i) \), \( i < 5 \). (Moreover, \( Y_{\nu}^{(3)} \equiv 0 \) at \( O(h^5) \)-level.) Thus only inner solution terms are present at the level of reduced powers of \( h \). From the above representations (cf. (3.36), (3.40), (3.49), (3.54)) and the fact that

\[
(3.61) \quad M_l(0) = O(\omega^{-1}) , \quad N_l(0) = O(\omega^{-1}) , \quad l = 0,1,\ldots ,
\]

which is a consequence of the \( \varepsilon^0 \)-property, it follows that the leading inner solution terms are of the form

\[
\begin{align*}
\xi^{(0)}_{\nu} &= \omega^{-1} p_0(\nu) Q^{-\nu} + p_3(\tau_{\nu}) e^{-c_2(0) \tau_{\nu}} , \\ \eta^{(0)}_{\nu} &= \omega^{-1} p_0(\nu) Q^{-\nu} + p_3(\tau_{\nu}) e^{-c_2(0) \tau_{\nu}} ,
\end{align*}
\]

and

\[
\begin{align*}
\xi^{(1)}_{\nu} &= \omega^{-1} p_2(\nu) Q^{-\nu} + p_5(\tau_{\nu}) e^{-c_2(0) \tau_{\nu}} , \\ \eta^{(1)}_{\nu} &= \omega^{-1} p_2(\nu) Q^{-\nu} + p_5(\tau_{\nu}) e^{-c_2(0) \tau_{\nu}} .
\end{align*}
\]

Here, \( p_i(\nu) \) denotes some (generic) polynomial of degree \( \leq i \) in \( \nu \) whose coefficients may depend in a harmless way on \( \omega \), i.e. on negative powers of \( \omega \) or on terms like \( (\frac{h}{c})^k \) or \( (\frac{c^2}{h})^k \) which are uniformly bounded for \( \varepsilon \leq C h \). \( p_i(\tau) \) is a generic polynomial of degree \( \leq i \) in \( \tau \) whose coefficients are affected with certain (positive as well as negative) powers of \( \omega \). *) Note that due to the \( \varepsilon^0 \)-property the constant coefficients \( p_{i,\omega}(0) \) of each \( p_{i,\omega}(\tau) \) within \((3.62a,b)\) satisfy (cf. (3.36), (3.40), (3.49), (3.54) and (3.61)):

\[
(3.62c) \quad p_{i,\omega}(0) = O(\omega^{-1}) \quad \text{for} \quad \varepsilon \leq C h .
\]

(3.62) shows that the inner solution components consist of two types of decaying terms: There are "discretely exponentially" decaying terms of the type \( \omega^{-1} p_i(\nu) Q^{-\nu} \) and exponentially decaying terms \( p_{i,\omega}(\tau_{\nu}) e^{-c_2(0) \tau_{\nu}} . \) It is easy to see that \( p_i(\nu) Q^{-\nu} \) is uniformly bounded. \( p_{i,\omega}(\tau_{\nu}) e^{-c_2(0) \tau_{\nu}} \) is uniformly bounded, too, for all \( \nu \geq 1 \) in spite of the presence of positive powers of \( \omega \) (cf. [1,(3.5)])]. Since the decaying behaviour of \( p_i(\nu) Q^{-\nu} \) is weaker than that of \( p_{i,\omega}(\tau_{\nu}) e^{-c_2(0) \tau_{\nu}} \), the damping of the order reduction effects with increasing \( \nu \) is characterized by the discretely exponentially decaying terms.

*) The fact that polynomials \( p_{3,\omega}(\tau) \) of degree 3 and \( p_{5,\omega}(\tau) \) of degree 5 appear in (3.62a) and (3.62b), resp., follows easily from the singular perturbation analysis of the V.E.'s of Part 1; nevertheless, the degree of these polynomials is irrelevant for our purpose.
3.4 Estimation of the Remaining Terms in the h-Expansion of \( p_{\nu} \)

Up to now we have determined the leading terms \( X^j_{\nu} \equiv 0, j = 0, 1, 2, Y^j_{\nu} \equiv 0, j = 0, 1, 2, 3 \) and \( \xi^{(0)}_{\nu}, \eta^{(0)}_{\nu}, \xi^{(1)}_{\nu}, \eta^{(1)}_{\nu} \). We shall now derive estimates for the remaining terms in the ansatz (3.11).

Some technical preliminaries are required. Consider

\[
\sum_{j}^{(\nu)} := \sum_{k=1}^{\nu} k^j \alpha^k,
\]

where \( \alpha \) is some complex number. For \( \sum_{j}^{(\nu)} \), the following recursive representation holds:

\[
\begin{align*}
\sum_{0}^{(\nu)} &= \frac{\alpha}{\alpha - 1} (\alpha^{\nu} - 1), \\
\sum_{j}^{(\nu)} &= \frac{\alpha}{\alpha - 1} \nu^j \alpha^\nu + \frac{1}{\alpha - 1} \sum_{i=0}^{j-1} (-1)^{j-i} \binom{j}{i} \sum_{i}^{(\nu)} , \quad j \geq 1.
\end{align*}
\]

(Proof by partial summation.) A simple consequence of (3.63b) is

\[
\sum_{j}^{(\nu)} = \alpha [p_j(\nu) \alpha^\nu + C], \quad j \geq 0,
\]

where the constant \( C \) and the coefficients of the polynomial \( p_j(\nu) \) (degree \( \leq j \)), which depend on \( \alpha \), are of moderate size provided that

\[
|\alpha - 1| \geq \text{const.} > 0.
\]

**Coefficients of \( h^4 \):**

Here, only the solution \( \xi^{(2)}_{\nu}, \eta^{(2)}_{\nu} \) of equation (3.27b) remains to be studied. To this end we rewrite (3.27b), but now the second equation is multiplied by \( \omega \):

\[
\begin{align*}
\xi^{(2)}_{\nu} - \xi^{(2)}_{\nu-1} &= a_{1,2}(0) \eta^{(2)}_{\nu-1} + p_0(\nu) \xi^{(1)}_{\nu} + p_1(\nu) \xi^{(0)}_{\nu} + \\
\xi^{(1)}_{\nu-1} + p_1(\nu) \xi^{(0)}_{\nu} + p_1(\nu) \eta^{(1)}_{\nu-1} + p_2(\nu) \eta^{(0)}_{\nu-1} + \\
+ f^{(1)}_{\nu+1} + f^{(2)}_{\nu}.
\end{align*}
\]

\[
\begin{align*}
\eta^{(2)}_{\nu} - \eta^{(2)}_{\nu-1} &= -\omega c_2(0) \eta^{(2)}_{\nu} + \omega p_1(\nu) \eta^{(1)}_{\nu} + \omega p_2(\nu) \eta^{(0)}_{\nu} + \\
+ p_0(\nu) \xi^{(0)}_{\nu-1} + p_0(\nu) \eta^{(1)}_{\nu-1} + p_1(\nu) \eta^{(0)}_{\nu-1} + \\
+ f^{(0)}_{\nu+2} + f^{(2)}_{\nu+1}.
\end{align*}
\]

The starting value for \( \eta^{(2)}_{\nu} \) is \( \eta^{(0)}_{\nu} = 0 \). Due to (3.62) and due to the structure of \( \eta^{(2)}_{\nu-1} \) (cf. (3.3c), (3.6), (3.8)) it can be seen that the inhomogeneity of (3.64b) consists of terms of the type *)

\[
\begin{align*}
\nu^{4} Q^{\nu}, \quad \omega^{-1} \nu^{4} Q^{\nu},
\end{align*}
\]

*) Recall that the terms \( p_i(\nu) Q^{\nu} \) in (3.62) appear in conjunction with a factor \( \omega^{-1} \); notice further that the factor \( \omega \) in the inhomogeneity of (3.64b) appears only with \( \eta^{(j)}_{\nu} \) but not with \( \eta^{(j)}_{\nu-1} \).
The terms in (3.66) originate from $M_1(r_\nu)$ and $N_1(r_\nu)$ (via $i_{\nu_2}^{(2)}$ and indirectly via $\xi^{(j)}, \eta^{(j)}, j = 0, 1$); the fact that $\omega^{-1}$ appears in (3.66c) is a consequence of the $e^0$-property (cf. (3.62c)) and of the fact that $\xi^{(j)}$ and $\eta^{(j)}$ within (3.64b) are not affected with $\omega$.

Due to (3.48), the respective solution components in $\eta^{(2)}_\nu$ read as follows. From (3.65a),

\[(3.67a) \quad \sum_{k=1}^\nu Q^{k-1-\nu} k^i Q^{-k} = \omega^{-1} \omega Q^{-1} Q^{-\nu} \sum_{k=1}^\nu k^i = \omega^{-1} p_{i+1}(\nu) Q^{-\nu},\]

because $\omega Q^{-1}$ is uniformly bounded and $\sum_{k=1}^\nu k^i = p_{i+1}(\nu)$, where again $p_{i+1}(\nu)$ denotes a polynomial in $\nu$ of degree $\leq i + 1$. (The second term in (3.65a) yields even a factor $\omega^{-2}$.) From (3.65b),

\[(3.67b) \quad \sum_{k=1}^\nu Q^{k-1-\nu} (k-1)^i Q^{-(k-1)} = \omega^{-1} Q^{-\nu} \sum_{k=1}^\nu (k-1)^i = \omega^{-1} p_{i+1}(\nu) Q^{-\nu}.\]

From (3.67a,b) we see that the discretely exponentially decaying terms in $\eta^{(2)}_\nu$ (originating from (3.65)) are again multiplied by a factor $\omega^{-1}$.

From (3.66a) and due to (3.63c) (with $\alpha = Qe^{-\omega c_2(0)}$), *

\[(3.68a) \quad \sum_{k=1}^\nu Q^{k-1-\nu} \omega^i r_k^j e^{-c_2(0)\tau_k} = \omega^{i+j} Q^{-1} Q^{-\nu} \sum_{k=1}^\nu k^j (Qe^{-\omega c_2(0)})^k =
\]

\[= \omega^{i+j} Q^{-\nu} (Qe^{-\omega c_2(0)}) |p_j(\nu)(Qe^{-\omega c_2(0)})^\nu + C| =
\]

\[= \omega^{-1} [\omega^{i+j+1} e^{-\omega c_2(0)}] \cdot |p_j(\nu)e^{-\omega c_2(0)\tau_\nu} + const \cdot Q^{-\nu}].\]

Here it is essential that $|\alpha| = |(1 + \omega c_2(0))/e^{\omega c_2(0)}| \leq const. < 1$ (due to our assumption $\epsilon \leq C h$) such that (3.63d) holds; notice further that $\omega^{i+j+1} e^{-\omega c_2(0)}$ is uniformly bounded.

The terms originating from (3.66b), namely

\[(3.68b) \quad \sum_{k=1}^\nu Q^{k-1-\nu} \omega^i r_{k-1}^j e^{-c_2(0)\tau_{k-1}} = \sum_{k=1}^{\nu-1} Q^{k-1-(\nu-1)} \omega^i r_k^j e^{-c_2(0)\tau_k} =
\]

\[= Q \sum_{k=1}^\nu Q^{k-1-\nu} \omega^i r_k^j e^{-c_2(0)\tau_k} - \omega^i r_{\nu+1}^j e^{-c_2(0)\tau_\nu},\]

*) Note that $\tau_\nu = \frac{t_\nu}{\epsilon} = \nu \frac{h}{\epsilon} = \nu \omega$. 

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can be treated in the same way as (3.68a).

From (3.66c) we have
\[ \sum_{k=1}^{\nu} Q^{k-1} \omega^{-1} e^{-c_2(0)t_k-1} = \omega^{-1} Q^{-\nu} \sum_{k=1}^{\nu} [Q e^{-\omega c_2(0)t_k}]^{k-1} = \]
\[ \omega^{-1} Q^{-\nu} + \omega^{-1} Q^{-\nu} \sum_{k=1}^{\nu-1} [Q e^{-\omega c_2(0)t_k}]^{k}. \]

(3.68c)

Here, the \( Q^{-\nu} \)-term has again a factor \( \omega^{-1} \); the second term is analogous to (3.68a,b).

Summarizing, we see that the structure of \( \eta_{\nu}^{(2)} \) (in particular w.r.t. powers of \( \omega \)) is essentially the same as that of \( \eta_{\nu}^{(0)}, \eta_{\nu}^{(1)} \) (cf. (3.62)): The terms decaying like \( \nu^\alpha Q^{-\nu} \) have always a factor \( \omega^{-1} \); the same is true for the terms \( \tau_{\nu}^{(v)} e^{-c_2(0)t_\nu} \). Any powers of \( \omega \) appearing with \( \tau_{\nu}^{(v)} e^{-c_2(0)t_\nu}, j \geq 1 \), do not matter.

The first component \( \xi_{\nu}^{(2)} \) is treated as follows: As usual, the starting value \( \xi_{\nu}^{(2)} \) is chosen such that \( \lim_{\nu \to \infty} \xi_{\nu}^{(2)} = 0 \). Analogous considerations as for \( \eta_{\nu}^{(2)} \) above show that \( \xi_{\nu}^{(2)} \) has essentially the same structure as \( \eta_{\nu}^{(2)} \); moreover, the starting value \( \xi_{\nu}^{(2)} \) turns out to be \( O(\omega^{-1}) \). *(This can be shown by using (3.63) with \( \alpha = Q^{-1} \) and \( \alpha = e^{-t_\nu c_2(0)} \), resp., and again taking account of the \( \varepsilon^0 \)-property.)

**Coefficients of \( h^5 \):**

From (3.28a) we have \( Y_{\nu}^{(3)} \equiv 0 \). \( X_{\nu}^{(3)} \) has the starting value (cf. (3.13)):
\[ X_0^{(3)} = 0 - \xi_0^{(2)} = -\xi_0^{(2)} = O(\omega^{-1}); \]
\[ X_0^{(3)} \] is the solution of a simple difference equation with smooth coefficient functions ***) and a smooth inhomogeneous term \( k_{\nu;1}^{(1)} = O(1) \). By the usual classical stability estimates we immediately obtain
\[ X_0^{(3)} = O(1). \]

(3.70) \( X_0^{(3)} = O(1). \)

For the solution \( \eta_{\nu}^{(3)} \) of equation (3.28b) it can be shown by the same inductive arguments as for \( \eta_{\nu}^{(2)} \) above that it has the same structure as \( \eta_{\nu}^{(3)}, j \leq 2 \) (note that, according to (3.13b), \( \eta_0^{(3)} = 0 - Y_0^{(3)} = 0 \)).

**Remainder term \( \rho_{\nu}^{(R)} \) of the \( h \)-expansion of \( \rho_{\nu} \):**

The remainder term \( \rho_{\nu}^{(R)} \) (cf. (3.14c)) in the expansion (3.11) for \( \rho_{\nu} \) satisfies the difference equation (3.29); according to (3.13) the starting value is at \( O(h^5) \)-level.

*) Recall that we are discussing the case \( q = 4 \), i.e. the \( O(h^{q+1}) = O(h^5) \) remainder term. If we had in mind a remainder term of higher order \( (q > 4) \), i.e. if we had discussed more (and not only the leading two) inner solution terms of the \( \xi(t) \)'s \( (i \leq q) \) in part 1 and if we had expanded all up to a higher level, then it could be shown that not only \( \xi_0^{(2)} = O(\omega^{-1}) \) holds, but that, again, \( \xi_0^{(2)} = 0 \) is compatible with \( \lim_{\nu \to \infty} \xi_{\nu}^{(2)} = 0 \).

***) The coefficient functions \( \theta_{i,j}(t_\nu) \) in (3.28a) depend on \( h \), but in a harmless way (cf. (2.8),(3.7b)), which causes no problems for the stability estimate.
In order to estimate $\rho^{(R)}(\nu)$ in the spirit of the B-theory (cf. for instance [3]), we introduce the nonlinear function

$$\Lambda^*_\nu(\rho) := \Lambda_\nu \cdot \rho + V_\nu(\rho),$$

which is one-sided Lipschitz continuous with the (moderately sized) one-sided Lipschitz constant

$$m^* = \log\text{norm}(\Lambda_\nu) + L_\nu$$

(cf. (3.16)). Moreover (again due to (3.16))

$$\Lambda^*_\nu(0) = 0.$$  

According to the smoothness assumptions with respect to $T(t)$ we can assume that $\Theta_\nu$ is bounded:

$$||\Theta_\nu|| \leq \vartheta \quad \text{for all } \nu.$$  

The inhomogeneity $\delta^{(R)}_\nu$ of (3.29) originates from the collection of all terms which have not yet appeared in the equations (3.25) - (3.28). Recall that, when equating coefficients of $h^5$, we omitted the inner solution term in the first component (cf. (3.28b)) such that the respective $h^5$-terms now appear in $\delta^{(R)}_{\nu,1}$ (besides various terms of higher order). Recall further that $\delta^{(R)}_{\nu,2}$ originates from $h^6$-terms but contains an additional factor $\varepsilon^{-1} = \frac{\omega}{h}$ due to the re-scaling of the second equation.

Hence all terms in $\delta^{(R)}_\nu$ have a factor $h^5$ in common. However, $\delta^{(R)}_\nu$ depends also on various powers of the parameter $\omega = \frac{h}{\varepsilon}$. For $\varepsilon \leq Ch$, positive powers of $\omega$ are critical. But positive powers of $\omega$ appear within $\delta^{(R)}_\nu$ only in conjunction with $\tau_j^2 e^{-c_2(0)r_\nu}$, $j \geq 1$. This follows from the fact that all terms (originating from the recursively known quantities $\xi^{(j)}_\nu, \eta^{(j)}_\nu$) which decay like $\nu^2 Q^{-\nu}$ or like $\tau_j^0 e^{-c_2(0)r_\nu}$ contain a factor $\omega^{-1}$ (cf. the above discussion of the structure of the $\xi^{(j)}_\nu$ and $\eta^{(j)}_\nu$) such that after re-scaling of the second component, i.e. after multiplication by $\frac{\omega}{h}$, these terms are not affected with positive powers of $\omega$. Hence it is guaranteed that

$$||\delta^{(R)}_\nu|| \leq \delta = O(h^5) \quad \text{uniformly in } \nu \quad \text{for } \varepsilon \leq Ch.$$  

The B-convergence estimate for $\rho^{(R)}_\nu$ is now straightforward: Consider the step

$$\rho^{(R)}_{\nu-1} \rightarrow \rho^{(R)}_\nu$$

via

$$\frac{1}{h} [\rho^{(R)}_\nu - \rho^{(R)}_{\nu-1}] = \Lambda^*_\nu(\rho^{(R)}_{\nu-1}) + \Theta_\nu \rho^{(R)}_{\nu-1} + \delta^{(R)}_\nu,$$  

and the analogous step of the homogenous difference equation:

$$\rho^{(R)}_{\nu-1} \rightarrow \hat{\rho}^{(R)}_\nu$$

via

$$\frac{1}{h} [\hat{\rho}^{(R)}_\nu - \rho^{(R)}_{\nu-1}] = \Lambda^*_\nu(\hat{\rho}^{(R)}_{\nu-1}) + \Theta_\nu \rho^{(R)}_{\nu-1}.$$  

Subtraction of (3.77) from (3.76),

$$\rho^{(R)}_\nu - \hat{\rho}^{(R)}_\nu = h \Lambda^*_\nu(\rho^{(R)}_\nu) - h \Lambda^*_\nu(\hat{\rho}^{(R)}_\nu) + h \delta^{(R)}_\nu,$$

*) Recall that $\omega^j \tau_j^2 e^{-c_2(0)r_\nu}$ is uniformly bounded at all grid points. This is important because B-convergence estimates provide no insight into damping effects such that it is indeed required that the inhomogeneity has the full order for all $\nu$. 

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and scalar multiplication by \( \rho^{(R)}_{\nu} - \hat{\rho}^{(R)}_{\nu} \) leads to (cf. (3.72))

\[
\| \rho^{(R)}_{\nu} - \hat{\rho}^{(R)}_{\nu} \|^2 = h \left( \Lambda^{*}_{\nu}(\rho^{(R)}_{\nu}) - \Lambda^{*}_{\nu}(\hat{\rho}^{(R)}_{\nu}) \right) + h \left( \hat{\delta}^{(R)}_{\nu}, \rho^{(R)}_{\nu} - \hat{\rho}^{(R)}_{\nu} \right) \leq \leq h m^{*} \| \rho^{(R)}_{\nu} - \hat{\rho}^{(R)}_{\nu} \|^2 + h \| \hat{\delta}^{(R)}_{\nu} \| \| \rho^{(R)}_{\nu} - \hat{\rho}^{(R)}_{\nu} \|,
\]

from which immediately (cf. 3.75)

\[
(3.80) \quad \| \rho^{(R)}_{\nu} - \hat{\rho}^{(R)}_{\nu} \| \leq \frac{h \delta}{1 - h m^{*}}.
\]

Furthermore, \( \| \hat{\rho}^{(R)}_{\nu} \| \) can easily be estimated: From (3.77) we have

\[
(3.81) \quad \hat{\rho}^{(R)}_{\nu} = h \Lambda^{*}_{\nu}(\hat{\rho}^{(R)}_{\nu}) + (I + h \Theta_{\nu})\rho^{(R)}_{\nu-1},
\]

hence (cf. (3.73), (3.74))

\[
(3.82) \quad \| \hat{\rho}^{(R)}_{\nu} \|^2 = h \left( \Lambda^{*}_{\nu}(\hat{\rho}^{(R)}_{\nu}), \hat{\rho}^{(R)}_{\nu} \right) + (I + h \Theta_{\nu}) \left( \rho^{(R)}_{\nu-1}, \hat{\rho}^{(R)}_{\nu} \right) \leq \leq h m^{*} \| \hat{\rho}^{(R)}_{\nu} \|^2 + (1 + h \delta) \| \rho^{(R)}_{\nu-1} \| \| \hat{\rho}^{(R)}_{\nu} \|.
\]

This implies

\[
(3.83) \quad \| \hat{\rho}^{(R)}_{\nu} \| \leq \frac{1 + h \delta}{1 - h m^{*}} \| \rho^{(R)}_{\nu-1} \|.
\]

Thus we have

\[
(3.84) \quad \| \rho^{(R)}_{\nu} \| \leq \| \rho^{(R)}_{\nu} - \hat{\rho}^{(R)}_{\nu} \| + \| \hat{\rho}^{(R)}_{\nu} \| \leq \frac{h \delta}{1 - h m^{*}} + \frac{1 + h \delta}{1 - h m^{*}} \| \rho^{(R)}_{\nu-1} \|.
\]

In (3.84), \( \| \rho^{(R)}_{\nu} \| \) is estimated by \( \| \rho^{(R)}_{\nu-1} \| \) and \( h \delta = O(h^6) \). Now the induction \( \| \rho^{(R)}_{\nu} \| \to \rho^{(R)}_{r} \to \cdots \to \rho^{(R)}_{0} \) which is usual in the B-theory (cf. for instance [3]) immediately leads to

\[
(3.85) \quad \rho^{(R)}_{\nu} = O(h^5).
\]

As a final remark in Section 3 we note that the result of [1] for the strongly stiff case - \( R_{\nu} = O(h^5) \) as \( \varepsilon \to 0 \) for all \( \nu \geq 1 \) - can also be concluded immediately from the above discussion: according to the structure of the \( \xi^{(j)}_{\nu}, \eta^{(j)}_{\nu} \) (cf. (3.62) and the discussion of (3.64) - (3.68)) it is clear that \( \xi^{(j)}_{\nu} \to 0 \) and \( \eta^{(j)}_{\nu} \to 0 \) as \( \varepsilon \to 0 \) at all grid points.

This concludes our analysis for \( \varepsilon \leq C h \).
4. DISCRETE SINGULAR PERTURBATION ANALYSIS: 
THE CASE $h \leq C\varepsilon$

In Section 3 above, the assumption $\varepsilon \leq C h$ was essential. In the lower part of the "$\varepsilon - h$ - plane", i.e. for $h \leq C\varepsilon$ ($C$ some moderate constant), the various expansions w.r.t. powers of $h$ (cf. for instance (3.5)) are not reasonable: For $h \ll \varepsilon$ negative powers of $\omega = \frac{h}{\varepsilon}$ become arbitrarily large and are not compensated (at least not at the first grid points) by the factors $e^{-\varepsilon z^{(0)} r_{\nu}} = e^{-\varepsilon z^{(0)} \omega_{\nu}} \approx 1$ within the $m_{i,i}(r_{\nu})$ and $n_{i,i}(r_{\nu})$.

To motivate the following, let us have a look at the $\varepsilon - h$ - plane which divides into several subdomains representing the various cases considered so far (cf. Fig. 1).

Fig. 1

A... subdomain of the $\varepsilon - h$ - plane for which the problem is not stiff (classical case: $\varepsilon$ is not a small parameter)

B... subdomain of the $\varepsilon - h$ - plane which is covered by Part 1 of this paper ($\varepsilon$ is so small that the inhomogeneity $b_{\nu} - c_{\nu}$ of (1.4) is at $O(h^{q+1})$-level)

C... subdomain of the $\varepsilon - h$ - plane for which $\varepsilon \leq C h$ (covered by Section 3)

Notice that we have characterized the classical case by "$\varepsilon$ not small". This appears not to be in accordance with usual definitions of stiffness distinguishing between "non-stiff" and "stiff" via $h L \ll 1$ and "$h L$ not small", resp. ($L$ is the conventional Lipschitz constant, i.e. $L = O(\frac{1}{\varepsilon})$). For such a definition of stiffness, the case $h \leq C\varepsilon$ or $\frac{h}{\varepsilon} \approx h \cdot L \leq C$ would indeed be the "classical case", and one could expect that no further analysis has to be done. For the present purpose, however (a-priori - estimate of $R_{\nu}$ in the B-sense), we have to go back to our definition of "classical case". To realize this, consider the classical and B-type a-priori convergence estimates for the implicit Euler scheme (cf. [1,(2.19)]):

(4.1) $\|z_{\nu} - z(t_{\nu})\| \leq \left\{ \begin{array}{ll} \frac{e^{Lt_{\nu}} - 1}{L} M_{2} h & \text{(classical estimate),} \\
\frac{e^{Lt_{\nu}} - 1}{m} M_{2} h, & m \leq 0 \text{(B-estimate).} \end{array} \right.$

($M_{2}$ is a bound for $\|z''(t)\|$, $t \in [0,T]$.) From (4.1) it can be seen that, even for $h L \ll 1$, the quantity $(e^{Lt_{\nu}} - 1)/L$ is much larger than $(e^{Mt_{\nu}} - 1)/m$ (if $L \gg 0$ and $m$ is of moderate size). The same situation arises for estimates of $R_{\nu}$. Thus we see that we must indeed analyze the case of a small $\varepsilon$ and $h \leq C\varepsilon$ to cover the whole $\varepsilon - h$ - plane.
As mentioned at the beginning of this Section, an expansion in powers of $h$ is useless for $h \leq C\epsilon$. It will turn out that a discrete singular perturbation expansion w.r.t. the parameter $\epsilon$ is appropriate in this case. But we have to be careful: Recall that, due to the nonlinearity of the remainder term equation (1.4), negative powers of $\epsilon$ must be avoided (cf. the remark at the end of Section 1). To this end it is prompting to strive for smooth solutions of the variational equations. This necessitates that, in contrast to [1,(2.15)] and [1,(2.17)], starting values for the $\tilde{e}_i(t)$ have to be chosen such that no inner solution components appear. In the rest of this Section we will sketch an analysis based on this idea (without going into all of the details). It will turn out that this particular approach works successfully for $h \leq C\epsilon$ but fails for $\epsilon \leq C h$. Thus, both types of singular perturbation analysis - with an ansatz in powers of $h$ as presented in Section 3 and with an ansatz in powers of $\epsilon$ based on smooth solutions - are indeed required to establish the desired result for the whole $\epsilon-h$-plane.

4.1 Smooth Solutions of the Variational Equations

To establish smooth solutions of the V.E.'s, special starting values have to be chosen for the $\tilde{e}_i(t) = T^{-1}(t) e_i(t)$. Due to the identity

$$ (\text{recall that } \zeta_0 - z(0) \text{ is the accumulated error from the preceding integration intervals})$$

the starting value $\rho_0$ for $\rho_\nu = T^{-1}_\nu R_\nu$ is fixed by

$$ (4.2b) \quad \rho_0 = T^{-1}_0(\zeta_0 - z(0)) - h\tilde{e}_1(0) - \ldots - h^q \tilde{e}_q(0) = T^{-1}_0(\zeta_0 - z(0)) - \tilde{v}_q(0). $$

We immediately see what the central problem is: Since the starting value of $\tilde{v}_q(t)$ is defined by the requirement of smooth solutions $\tilde{e}_i(t)$, there is no interrelation between the accumulated error $\zeta_0 - z(0)$ and $\tilde{v}_q(0)$; hence only $\rho_0 = O(h)$ but not $O(h^{q+1})$. Thus it has to be shown that the $O(h^{q+1})$-level is achieved with increasing $\nu$. A proof of this damping property will be sketched in Subsection 4.2. In the rest of the present Subsection we give a precise characterization of smooth solutions in the singular perturbation context.

By "smooth" solutions we mean solutions of the V.E.'s for which no inner solution components are present up to a certain $\epsilon^k$-level and for which the starting values for the outer solution components $X_{l,i}(t), Y_{l,i}(t)$ ($l = 0, \ldots, k - 1$) and for all further components (at $\epsilon^l$-level, $l \geq k$) are $O(1)$. It is obvious that $k$ can be chosen such that the respective terms appearing in the inhomogeneity $b_\nu - c_\nu$ of (1.4) are $O(h^{q+1})$ (derivatives w.r.t. $t$ up to the $k$-th order of inner solution terms at $\epsilon^k$-level remain bounded).

Smooth solutions $\tilde{e}_i(t)$ in the above sense can easily be constructed by a slight modification of the singular perturbation technique used in [1,Section 4]. In contrast to [1], where the $\tilde{e}_i(t)$ were fit to prescribed starting values, we now determine starting values such that no inner solution components appear up to a certain $\epsilon^k$-level. For these starting values we make the ansatz

$$ (4.3) \quad \tilde{e}_i(0) = \left( \begin{array}{c} x_{0,i} + \epsilon x_{1,i} + \epsilon^2 x_{2,i} + \ldots \\ \epsilon^2 y_{1,i} + \epsilon^2 y_{2,i} + \ldots \end{array} \right). $$

The point is that we can choose the $x_{l,i}$ arbitrarily (of course $O(1)$) but that the terms in the second component are then determined recursively (to define a smooth solution $\tilde{e}_i(t)$), i.e. $y_{l,i}$ is fixed by
This can easily be seen by making the ansatz (*)

\[ (4.4) \quad \tilde{e}_i(t) = \left( X_{i,0}(t) + \epsilon X_{1,i}(t) + \epsilon^2 m_{1,i}(\frac{t}{\epsilon}) + \epsilon^3 m_{1,i}(\frac{t}{\epsilon}) + \ldots \right) \]

and equating coefficients of \( \epsilon^i \) (cf. for instance [1,(4.13)] or [1,(4.15)]):

\[ (4.5a) \quad X_{i,1}'(t) = (c_1(t) + a_{1,1}(t))X_{i,1}(t) + a_{1,2}(t)Y_{1,i}(t) + r_{1,i}(t) , \]

\[ 0 = -c_2(t)Y_{1,i}(t) + a_{2,1}(t)X_{i-1,1}(t) + a_{2,2}(t)Y_{i-1,i}(t) + s_{i-1,i}(t) , \]

(with certain smooth inhomogeneities \( r_{i,i}(t), s_{i-1,i}(t) \) depending recursively on the smooth functions \( \tilde{e}_j(t), j < i \)),

\[ (4.5b) \quad \frac{d}{dt} m_{i,i}(\tau) = a_{1,2}(0)n_{i,i}(\tau) , \]

\[ \frac{d}{dt} n_{i,i}(\tau) = -c_2(0)n_{i,i}(\tau) . \]

Here we assume inductively that the inner solution terms \( m_{j,i}(\tau), n_{j,i}(\tau), k < l \), vanish. Then the initial condition \( n_{i,i}(0) = 0 \) implies \( m_{i,i}(\tau) \equiv 0 \), and from \( \lim_{\tau \to \infty} m_{i,i}(\tau) = 0 \) we obtain \( m_{i,i}(\tau) \equiv 0 \). Hence \( y_{i,i} \) is fixed by

\[ (4.6) \quad y_{i,i} = Y_{i,i}(0) + n_{i,i}(0) = Y_{i,i}(0) = \frac{1}{c_2(0)} \left[ a_{2,1}(0)X_{i-1,1}(0) + a_{2,2}(0)Y_{i-1,i}(0) + s_{i-1,i}(0) \right] = \]

\[ = \frac{1}{c_2(0)} \left[ a_{2,1}(0)x_{i-1,1} + a_{2,2}(0)y_{i-1,i} + s_{i-1,i}(0) \right] , \]

but again the starting value \( x_{i,i} \) for \( X_{i,i}(t) \) can be chosen arbitrarily at \( O(1) \) - level.

From (4.2b) and from the above considerations it follows that the starting value \( \rho_0 \) is fixed in its second component by the requirement of smooth solutions \( \tilde{e}_i(t) \) (because the \( y_{i,i}, i = 1, \ldots, k \), are fixed) but that the first component can be chosen arbitrarily. This results in an \( \epsilon \)-expansion for \( \rho_0 \),

\[ (4.7a) \quad \rho_0 = \left( \begin{array}{c} x_0 \\ y_0 \end{array} \right) = \left( \begin{array}{c} x_0^{(0)} + \epsilon x_0^{(1)} + \cdots + \epsilon^{k-1} x_0^{(k-1)} + x_0^{(R)} \\ \epsilon y_0^{(1)} + \cdots + \epsilon^{k-1} y_0^{(k-1)} + y_0^{(R)} \end{array} \right) , \]

where **)

\[ (4.7b) \quad x_0^{(i)} = O(\epsilon h), \quad y_0^{(i)} = O(\epsilon h), \quad i = 1, \ldots, k-1 , \]

\[ (4.7c) \quad x_0^{(R)} = O(\epsilon^k h), \quad y_0^{(R)} = O(\epsilon^k h) , \]

and where only the \( y_0^{(i)}, i = 1, \ldots, k-1 \), are fixed. The degree of freedom in the first component will now be utilized in the discrete singular perturbation analysis for \( \rho_0 \) to establish that the outer solution components of \( \rho_0 \) are \( O(h^{q+1}) \).

*) For the moment, the ansatz (4.4) contains inner solution terms \( \epsilon m_{0,i}(\tau), \epsilon^2 m_{1,i}(\tau), \ldots, n_{0,i}(\tau), \ldots \) (of course, no inner solution components with negative powers of \( \epsilon \) are reasonable under the assumption of smooth \( \tilde{e}_j(t) \) ); the starting values in the expansion are set in such a way that all these inner solution terms recursively turn out to vanish up to to the \( \epsilon^k \)-level.

**) \( \tilde{e}_0(0) = h\tilde{e}_1(0) + \ldots + h^q \tilde{e}_q(0) \) is of course \( O(h) \); the accumulated error from the preceding intervals satisfies \( \tilde{z}(0) = O(h) \) due to B-convergence; hence \( \rho_0 = O(h) \).
4.2 $\epsilon$ - Expansion of $\rho_\nu$

It is now fundamental (in contrast to Section 3) that the inhomogeneity $\delta_\nu$ is $O(h^{q+1})$ and is generated by smooth functions in the sense of Subsection 4.1. Thus $\delta_\nu$ has an $\epsilon$-expansion

\begin{equation}
\delta_\nu = \left( \delta^{(0)}_{\nu;1} + \epsilon \delta^{(1)}_{\nu;1} + \ldots \right)
\end{equation}

where

\begin{equation}
\delta^{(i)}_{\nu;1} = O(h^{q+1}) \quad \delta^{(i)}_{\nu;2} = O(h^{q+1}), \quad i = 0, 1, \ldots
\end{equation}

With $\omega = \frac{h}{\epsilon}$, (2.10) reads

\begin{align}
\frac{1}{h}(x_\nu - x_{\nu-1}) &= c_1(t_\nu)x_\nu + \theta_{1,1}(t_\nu)x_{\nu-1} + \theta_{1,2}(t_\nu)y_{\nu-1} + \gamma_1(t_\nu, x_\nu, y_\nu) + \delta_{\nu;1}, \\
\frac{\epsilon}{h}(y_\nu - y_{\nu-1}) &= -c_2(t_\nu)y_\nu + \epsilon \theta_{2,1}(t_\nu)x_{\nu-1} + \epsilon \theta_{2,2}(t_\nu)y_{\nu-1} + \epsilon \gamma_2(t_\nu, x_\nu, y_\nu) + \epsilon \delta_{\nu;2}.
\end{align}

For $\rho_\nu$ we make the ansatz *)

\begin{equation}
\begin{aligned}
x_\nu &= X^{(0)}_\nu + \epsilon X^{(1)}_\nu + \ldots + \epsilon^k X^{(k)}_\nu + \xi^{(0)}_\nu + \ldots + \epsilon^k \xi^{(k-1)}_\nu + x^{(R)}_\nu, \\
y_\nu &= Y^{(0)}_\nu + \epsilon Y^{(1)}_\nu + \ldots + \epsilon^k Y^{(k)}_\nu + \eta^{(0)}_\nu + \ldots + \epsilon^k \eta^{(k)}_\nu + y^{(R)}_\nu.
\end{aligned}
\end{equation}

Again we proceed along the usual lines of the singular perturbation theory; i.e., $x_\nu$ and $y_\nu$ in (4.9) are expressed by (4.10) and the equations for $X^{(0)}_\nu, Y^{(0)}_\nu, \xi^{(0)}_\nu, \eta^{(0)}_\nu, \ldots$ are determined by equating coefficients of $\epsilon^i$, $i = 0, 1, \ldots$. For the sake of shortness we do not go into details here but will only sketch the essential ideas.

As usual (cf. for instance O'Malley [4]), the nonlinear functions $\gamma_i(t_\nu, x_\nu, y_\nu)$ are expanded into powers of $\epsilon$:

\begin{equation}
\gamma_i(t_\nu, x_\nu, y_\nu) = \gamma_i(t_\nu, X^{(0)}_\nu, Y^{(0)}_\nu + \eta^{(0)}_\nu) + \epsilon \gamma_{ix}(t_\nu, X^{(0)}_\nu, Y^{(0)}_\nu + \eta^{(0)}_\nu)(X^{(1)}_\nu + \xi^{(0)}_\nu) + \\
+ \epsilon \gamma_{iy}(t_\nu, X^{(0)}_\nu, Y^{(0)}_\nu + \eta^{(0)}_\nu)(Y^{(1)}_\nu + \eta^{(1)}_\nu) + \ldots.
\end{equation}

To obtain the equations at $\epsilon^0$-level, (4.11) is rewritten as

\begin{equation}
\gamma_i(t_\nu, x_\nu, y_\nu) = \gamma_i(t_\nu, X^{(0)}_\nu, Y^{(0)}_\nu) + [\gamma_i(t_\nu, X^{(0)}_\nu, Y^{(0)}_\nu + \eta^{(0)}_\nu) - \gamma_i(t_\nu, X^{(0)}_\nu, Y^{(0)}_\nu)] + O(\epsilon).
\end{equation}

Equating coefficients of $\epsilon^0$ we obtain

\begin{equation}
\frac{1}{h}(X^{(0)}_\nu - X^{(0)}_{\nu-1}) = c_1(t_\nu)X^{(0)}_\nu + \theta_{1,1}(t_\nu)X^{(0)}_{\nu-1} + \theta_{1,2}(t_\nu)Y^{(0)}_{\nu-1} + \gamma_1(t_\nu, X^{(0)}_\nu, Y^{(0)}_\nu) + \delta_{\nu;1}, \\
0 = -c_2(t_\nu)Y^{(0)}_\nu,
\end{equation}

*) Again it will turn out that $x^{(R)}_\nu$ and $y^{(R)}_\nu$ are $O(\epsilon^k)$ but not $O(\epsilon^{k+1})$. 31
and

\[ \frac{\varepsilon}{h} (\xi^{(0)}_{\nu} - \xi^{(0)}_{\nu-1}) = \theta_{1,2}(t_{\nu}) \eta^{(0)}_{\nu-1} + [\gamma_{11}(t_{\nu}, X^{(0)}_{\nu}, Y^{(0)}_{\nu}) + \eta^{(0)}_{\nu}] - \gamma_{11}(t_{\nu}, X^{(0)}_{\nu}, Y^{(0)}_{\nu}) \]  
(4.13b)

\[ \frac{\varepsilon}{h} (\eta^{(0)}_{\nu} - \eta^{(0)}_{\nu-1}) = -c_{2}(t_{\nu}) \eta^{(0)}_{\nu} . \]

From (4.13a) we have \( Y^{(0)}_{\nu} = 0 \). If the starting value \( X^{(0)}_{\nu} \) for \( X^{(0)}_{\nu} \) is \( O(h^{q+1}) \), a conventional stability consideration immediately yields \( X^{(0)}_{\nu} = O(h^{q+1}) \) because \( \delta^{(0)}_{\nu,1} = O(h^{q+1}) \) (cf. (4.8b)). Now the essential point is that the degree of freedom which remains in the first components of the starting values for the \( \xi(t) \) enables us to compensate the first component of \( T_{\nu}^{-1}(\xi - z(0)) \) which appears in (4.2b), such that \( x^{(0)} \) vanishes (or is at least \( O(h^{q+1}) \)). Since there is no \( \xi \)-component at \( \varepsilon^{0} \)-level, we obtain \( X^{(0)}_{\nu} = x^{(0)} = 0 \), establishing \( X^{(0)}_{\nu} = O(h^{q+1}) \).

Furthermore, the starting condition (cf. (4.7a))

\[ \eta^{(0)}_{0} = 0 - Y^{(0)}_{0} = 0 \]

implies \( \eta^{(0)}_{0} \equiv 0 \) (cf. (4.13b)). Hence the nonlinear \( \cdots \)-term in the first equation of (4.13b) vanishes. *) The condition \( \lim_{\nu \to \infty} \xi^{(0)}_{\nu} = 0 \) yields \( \xi^{(0)}_{\nu} \equiv 0 \).

The equations at \( \varepsilon^{1} \)-level, \( l \geq 1 \), can be treated in much the same way. These are of the type

\[ \frac{1}{h}(X^{(l)}_{\nu} - X^{(l)}_{\nu-1}) = c_{1}(t_{\nu})X^{(l)}_{\nu} + \theta_{1,1}(t_{\nu})X^{(l)}_{\nu-1} + \theta_{1,2}(t_{\nu})Y^{(l)}_{\nu-1} + \]

\[ + \gamma_{11}(t_{\nu}, X^{(0)}_{\nu}, 0)X^{(l)}_{\nu} + \gamma_{11}(t_{\nu}, X^{(0)}_{\nu}, 0)Y^{(l)}_{\nu} + \delta^{(l-1)}_{\nu,1} + \]

\[ + \text{outer solution terms (defined recursively)}, \]

\[ 0 = -c_{2}(t_{\nu})Y^{(l)}_{\nu} + \delta^{(l-1)}_{\nu,1} + \]

\[ + \text{outer solution terms (defined recursively)}, \]

and

\[ \frac{\varepsilon}{h}(\xi^{(l)}_{\nu} - \xi^{(l)}_{\nu-1}) = \theta_{1,2}(t_{\nu}) \eta^{(l)}_{\nu-1} + \]

\[ + \text{inner solution terms (defined recursively)}, \]

\[ \frac{\varepsilon}{h}(\eta^{(l)}_{\nu} - \eta^{(l)}_{\nu-1}) = -c_{2}(t_{\nu}) \eta^{(l)}_{\nu} + \]

\[ + \text{inner solution terms (defined recursively)}. \]

In contrast to the usual procedure (cf. [4] and [1,Section 4]) we do not expand the data functions \( c_{2}(t_{\nu}), \theta_{1,2}(t_{\nu}), \ldots \) (appearing in (4.15b)) around \( t = 0 \). This is due to technical reasons, namely to enable a sufficiently sharp estimate for the \( \varepsilon^{k} \)-remainder term (cf. the respective discussion below). The consequence is that the structure of the inner solution terms is slightly modified: Instead of terms of the form \( h p(\nu)Q^{-\nu} \) we obtain inner solutions \( \xi^{(l)}_{\nu}, \eta^{(l)}_{\nu} \) which decay like **)

\[ h p(\nu) \prod_{j=1}^{\nu} Q_{j}^{-1} , \]

*) Note that, in general, the singular perturbation analysis of nonlinear problems yields nonlinear equations for the leading inner solution terms. Thus our situation - that the leading equations turn out to be linear - is a special one.

**) The presence of the factor \( h \) in (4.16) and (4.17) is a consequence of the starting condition for \( \eta^{(1)}_{0} \): \( \eta^{(1)}_{0} = y^{(1)}_{0} - Y^{(1)}_{0} = O(h) \) due to (4.7b) and \( Y^{(1)}_{\nu} = O(h^{q+1}) \) (which is shown below).
where \( p(\nu) \) is some polynomial in \( \nu \) with moderate coefficients and \( Q_j := 1 + c_2(\nu_j)h^\varepsilon \). This can be proved in an inductive way: If the inhomogenous terms in (4.15b) are of the type (4.16), then the solutions \( \xi^{(i)} \), \( \eta^{(i)} \) of (4.15b) are again of the same type (with a polynomial of a higher degree). The starting value \( \xi^{(i)}_0 \), which is defined as usual by the condition \( \lim_{\nu \to -\infty} \xi^{(i)}_\nu = 0 \), turns out to be of the form

\[
(4.17) \quad \xi^{(i)}_0 = h p(h^\varepsilon), \quad p \text{ some polynomial}.
\]

Concerning the outer solution terms \( X^{(i)}_\nu, Y^{(i)}_\nu \) (defined by (4.15a)) our aim is to show that these terms are \( O(h^{q+1}) \). For \( Y^{(i)}_\nu \) this follows immediately from \( \delta^{(i,-1)}_\nu = O(h^{q+1}) \) and from the recursive assumption that all quantities \( X^{(j)}_\nu, Y^{(j)}_\nu, j < l \), are \( O(h^{q+1}) \). (Note that the second equation in (4.15a) involves, among other terms, the difference quotient \( h(Y^{(i)}_\nu - Y^{(i,-1)}_\nu) \), the \( O(h^{q+1}) \)-level of which can also be concluded inductively from our smoothness assumptions for the problem data.) \( X^{(i)}_\nu = O(h^{q+1}) \) follows again by a standard stability argument once that \( X^{(i)}_0 = O(h^{q+1}) \). To ensure the latter we notice that (cf. (4.7a))

\[
(4.18) \quad X^{(i)}_0 = x^{(i)}_0 - \xi^{(i-1)}_0.
\]

The degree of freedom in \( x^{(i)}_0 \) can now be used to compensate \( \xi^{(i-1)}_0 \) such that \( X^{(i)}_0 = 0 \) (or at least \( O(h^{q+1}) \)). This is possible because \( \xi^{(i-1)}_0 = O(h) \) due to (4.17) and our assumption \( h \leq C\varepsilon \). But note that this argumentation breaks down if \( h \leq C\varepsilon \) is violated: In this case, \( \xi^{(i-1)}_0 = O(h) \) is not true (since, then, \( p(h^\varepsilon) \) in (4.17) is not of moderate size) and therefore \( \xi^{(i-1)}_0 \) cannot be compensated by any choice of starting values \( \xi(t) \) at \( O(1) \)-level - which is required to ensure that \( \xi(t) \) is smooth (in the sense of Subsection 4.1) and that \( \delta_\nu = O(h^{q+1}) \).

**Estimation of the \( \varepsilon^k \)-remainder term:**

It now remains to be shown that the \( \varepsilon^k \)-remainder term

\[
(4.19) \quad \rho^{(R)}_\nu := \left( \begin{array}{c} x^{(R)}_\nu \\ y^{(R)}_\nu \end{array} \right)
\]

is \( O(\varepsilon^k h^{q+1}) \) with increasing \( \nu \). (Recall that order reductions are unavoidable at the first grid points.) \( \rho^{(R)}_\nu \) is the solution of a difference equation of the same type as (4.9) with an inhomogeneity at \( \varepsilon^k \)-level. It would therefore be straightforward to show (by a standard B-convergence argument) that \( \rho^{(R)}_\nu = O(\varepsilon^k) \). But from this the desired result \( \rho_\nu = O(h^{q+1}) \) with increasing \( \nu \) cannot be concluded because \( O(\varepsilon^k) \) is not \( O(h^{q+1}) \) for \( h \leq C\varepsilon \); we need the sharper estimate \( \rho^{(R)}_\nu = O(\varepsilon^k h^{q+1}) \) for which a more refined argumentation has to be used. Consider the difference equation for \( \rho^{(R)}_\nu \):

\[
(4.20) \quad \frac{1}{h}(\rho^{(R)}_\nu - \rho^{(R)}_{\nu-1}) = \Lambda_\nu \rho^{(R)}_\nu + \Theta_\nu \rho^{(R)}_{\nu-1} + V_\nu(\rho^{(R)}_\nu) + \delta^{(R)}_\nu + \Delta^{(R)}_\nu.
\]

Here, \( V_\nu(\rho) \) is a certain nonlinear Lipschitz continuous function with a moderate Lipschitz bound and satisfying \( V_\nu(0) = 0 \). The inhomogenous terms \( \delta^{(R)}_\nu, \Delta^{(R)}_\nu \) are defined by collection of all terms which are at least at \( \varepsilon^k \)-level. In particular, \( \delta^{(R)}_\nu \) denotes the collection of outer solution terms and is therefore \( O(\varepsilon^k h^{q+1}) \). \( \Delta^{(R)}_\nu \) originates from inner solution terms and is therefore only \( O(\varepsilon^k h^{q+1}) \) but shows a decaying behaviour like (4.16).

Let \( \tilde{\rho}^{(R)}_\nu \) denote the solution of

\[
(4.21) \quad \frac{1}{h}((\tilde{\rho}^{(R)}_\nu - \tilde{\rho}^{(R)}_{\nu-1}) = \Lambda_\nu \tilde{\rho}^{(R)}_\nu + \Theta_\nu \tilde{\rho}^{(R)}_{\nu-1} + V_\nu(\tilde{\rho}^{(R)}_\nu) + \delta^{(R)}_\nu
\]
(where we have omitted $\Delta^{(R)}_v$) with some starting value $\rho^{(R)}_0 = O(\varepsilon^k h^{q+1})$. Analogously as at the end of Subsection 3.4, we can use a B-convergence argument to show

$$\rho^{(R)}_\nu = O(\varepsilon^k h^{q+1}).$$

It can now be shown that there exists a solution $\rho^{(R)}_\nu$ of equation (4.20) with a starting value $\rho^{(R)}_0 = O(\varepsilon^k h)$ which tends towards $\rho^{(R)}_\nu$ with increasing $\nu$ such that

$$\|\rho^{(R)}_\nu - \rho^{(R)}_\nu\| = O(h p(\nu) \prod_{j=1}^\nu Q_j^{-1}).$$

(cf. (4.16)). The idea of the proof is the following: At the endpoint $t_N = T$ of the actual integration interval we choose $\rho^{(R)}_N := \rho^{(R)}_N$; now the difference equation (4.20) is solved in the backward direction. If we would omit the inhomogenous term $\Delta^{(R)}_v$, i.e. if we would solve (4.21) in the backward direction, we would reproduce the $O(\varepsilon^k h^{q+1})$-solution $\rho^{(R)}_\nu$. The effects of $\Delta^{(R)}_v$, interpreted as a perturbation of (4.21) with a very special decaying behaviour (i.e. an increasing behaviour for $\nu \rightarrow \nu - 1 \ldots \rightarrow 0$), can now easily be estimated by standard stability arguments; due to the special structure of $\Delta^{(R)}_v$ it can be guaranteed that the effects of this perturbation increase only in a moderate way (in spite of $\|\Delta^{(R)}_v\| = O(1)$) such that we end up with a starting value $\rho^{(R)}_\nu = O(\varepsilon^k h^{q+1})$. For this argumentation it is essential that we have modified the usual singular perturbation technique in the sense of a more precise description of the decaying behaviour of the inner solution terms (cf. (4.16) above), i.e. that we have not expanded the data functions $c_2(t), \ldots$ around $t = 0$ within the equations defining the inner solution terms; only this enables a sufficiently sharp estimate of the effects of $\Delta^{(R)}_v$.

Having shown the existence of a solution $\rho^{(R)}_\nu$ with $\rho^{(R)}_0 = O(\varepsilon^k h)$ which tends to $O(\varepsilon^k h^{q+1})$ with increasing $\nu$, we have found a starting value $\rho^{(R)}_\nu = (z^{(R)}_\nu, y^{(R)}_\nu)$ for (4.7a) which satisfies (4.7c). Since - due to our definition of smoothness of the $\tilde{e}_\nu(t)$ - the starting values at the $\varepsilon^k$-level in the $\varepsilon$-expansion of the $\tilde{e}_\nu(t)$ can be chosen arbitrarily at $O(1)$-level (not only in the first but also in the second component), $z^{(R)}_0 = O(\varepsilon^k h)$, $y^{(R)}_0 = O(\varepsilon^k h)$ is indeed compatible with smooth solutions $\tilde{e}_\nu(t)$ required for $\delta_\nu = O(h^{q+1})$.

This concludes our analysis for $h \leq C\varepsilon$.

5. CONCLUSION

In Sections 3 and 4, resp., the subdomains $\varepsilon \leq C h$ and $h \leq C\varepsilon$ of the $\varepsilon - h$ - plane are covered successfully. The considerations in Section 3 are of course of major practical relevance. Only if for problems with large Lipschitz constants the accuracy requirements are so strong that stepsizes $h \ll \varepsilon$ are necessary, our considerations from Section 4 have to be applied. For stiff problems ($\varepsilon$ small) such restrictive accuracy requirements hardly occur in practice. It might be expected that the considerations in Section 4 are important for the analysis of the error structure in the transients, where the solution of the original problem is not smooth and therefore very small stepsizes $h$ (and even $h \ll \varepsilon$) must be used to satisfy certain accuracy requirements.

Unfortunately, Section 4 does not apply here: In the transient phase, smooth solutions $\varepsilon(t)$ of the variational equations do not exist in general. *) This seems, at first sight, to destroy our inductive

*) The inhomogeneities of the V.E.'s contain certain derivatives of $\varepsilon(t)$ which cannot be assumed to be smooth; i.e., there are non-vanishing inhomogeneous terms in the equations defining the inner solution terms $m_{0,1}(r), n_{0,1}(r)$: \[ \frac{d}{dr} m_{0,1}(r) = \alpha_{1,2}(0)n_{0,1}(r) + r(r), \quad \frac{d}{dr} n_{0,1}(r) = -c_{2}(0)n_{0,1}(r) + s(r). \] Hence no starting value $n_{0,1}(0)$ exists such that $n_{0,1}(r) \equiv 0$. 

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argument for the starting values: Recall that we have throughout assumed that the starting values for the $\xi_i(t)$ are of the form

$$
(5.1) \quad \xi_i(0) = \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \ldots \\ \varepsilon y_1 + \varepsilon^2 y_2 + \ldots \end{pmatrix}
$$

(cf. for instance (4.3) or [1,(4.10)]), justified by the inductive assumption that on the preceding subinterval the outer solution term $\varepsilon^0 Y_0,i(t)$ vanishes and that all inner solution terms are already damped away at the end of that subinterval, such that there remains no term at $\varepsilon^0$-level in the second component (cf. [1,(2.17)]). This inductive assumption seems not to be correct if a stepsize $h \ll \varepsilon$ is used in the transient phase, because we cannot use the argumentation from Section 4. However, (5.1) can be justified after the transients by a simple B-convergence argument: The length of the transient phase is always $O(\varepsilon)$, which can easily be shown by application of the singular perturbation theory to the original problem. The $(O(h))$-B-convergence estimate for the implicit Euler scheme, evaluated at a point $t = \alpha \varepsilon$, shows a factor $\varepsilon$ due to $\varepsilon^{m-1} = O(\varepsilon)$. Thus it is justified to assume that all solutions of the V.E.'s are at $O(\varepsilon)$ - level immediately after the transients; this shows that (5.1) is always justified.

It should also be noted that our recursion over subintervals with constant stepsizes (with starting values given by (5.1)) may be of a "mixed" type; i.e. the relation between $h$ and $\varepsilon$ may vary such that different considerations (Section 3 in conjunction with [1] or Section 4, resp.) have to be applied in different subintervals.

Let us, in this concluding Section, briefly remark another point concerning nonequidistant grids: We have always made the assumption of coherent grid sequences and the question arises whether this assumption is too restrictive. To our opinion this is not the case as the following arguments should show:

i) Coherent grid sequences can be perfectly adjusted to the local smoothness of the solution or of the problem data. Only in the transients, where the local smoothness of the solution varies very quickly, it might happen that, if for a certain accuracy requirement a certain grid would be optimal (w.r.t. efficiency), none of the coherent refinements of that grid would be ideal under much stronger accuracy requirements.

ii) Every particular grid which is generated by a certain code with a certain stepsize control, applied to a particular problem, can be interpreted as a member of a coherent grid sequence (at least as the coarsest grid defining that sequence).

In the present paper we have extensively discussed the asymptotic expansion of the discretization error of the implicit Euler method. The question arises whether similar results can also be proved for other methods. In a forthcoming paper we will discuss asymptotic error expansions for the implicit midpoint rule and the implicit trapezoidal rule, trying to extend the results of Dahlquist and Lindberg [2]. In contrast to the strongly stable implicit Euler scheme, there is no damping in the difference equations for the midpoint and trapezoidal rules. If, therefore, the order of the remainder term breaks down at the first grid points, no damping effects are to be expected; hence the reduced order remains present in the whole integration interval. There is, however, some hope that discrete singular perturbation techniques can be applied to establish a systematically oscillating behaviour of the dominant components of the remainder term. Therefore we expect that certain numerical procedures for error estimation, stepsize control, convergence acceleration etc. can be shown to work successfully if the information at odd or even grid points only is used. Another question is whether the full asymptotic expansion exists in the strongly stiff case $\varepsilon \ll h$ (as it is the case for the implicit Euler scheme - cf. [1]). Simple models show that this is not the case in general but only for certain classes of problems which we will try to characterize. The trapezoidal scheme seems to be more "robust" w.r.t. this point.
REFERENCES


