Path Monotonicity and Truthful Implementation

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joint work with
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Introduction

Some questions in multi-dimensional MD

- When is monotonicity of the allocation rule sufficient for truthful implementation?
- If not sufficient, what do we have to add?
- How much freedom do we have to set incentive compatible transfers?

This paper

- Link between network approach and analytical approaches.
- Finite sets of alternatives: (path-)monotonicity is equivalent with implementability.

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Setting

Single Agent (w.l.o.g)

- A set of outcomes
- $T \subset \mathbb{R}^k$ set of types, private information
- $v : T \times A \to \mathbb{R}$
- $f : T \to A$ allocation rule

truthful implementation

f is called *truthfully implementable* if there exist transfer $\pi : T \to \mathbb{R}$ such that for all $s, t \in T$

$$v(f(s), s) + \pi(s) \geq v(f(t), s) + \pi(t)$$

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Network approach

Type-graph

- Define a complete, directed graph T_f with node set T.
- Define either of the two arc lengths (see Archer and Kleinberg (2008))

$$\begin{aligned} l_p(s,t) &:= v(f(s),s) - v(f(t),s), \\ l_u(s,t) &:= v(f(t),t) - v(f(t),s). \end{aligned}$$

- Truthful transfers are node potentials with respect to p-length:

$$\pi(t) \leq \pi(s) + v(f(s),s) - v(f(t),s) = \pi(s) + l_p(s,t)$$

- Equilibrium utility is node potential with respect to $l_u(s,t)$

$$v(f(t),t) + \pi(t) \leq \pi(s) + v(f(s),s) + l_u(s,t)$$

Rochet's characterization

Theorem (Rochet (1987),. . . ,Archer and Kleinberg (2008))

The following are equivalent:

1. f is truthfully implementable
2. T_f does not have a negative cycle with respect to p-length
3. T_f does not have a negative cycle with respect to u-length

Proof: (2) $\Leftrightarrow$ (3)

$l_u(s,t) = l_p(s,t) + v(f(t),t) - v(f(s),s).$

(2) $\Rightarrow$ (1)

Fix some node x. Define $\pi(t) = \inf(l_p(P) \mid P$ is path from t to $x)$. Observe

$$\pi(t) \leq +l_p(t,s) + \pi(s).$$

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Monotonicity

Definition

An allocation rule is called *monotone* if for all $s, t \in T$:
$v(f(t), t) - v(f(s), t) \geq v(f(t), s) - v(f(s), s)$

Observations

- Monotonicity is equivalent to no-negative 2-cycle: $l_p(s, t) + l_p(t, s) \geq 0$.
- Monotonicity is necessary for f being implementable.
- Suppose $v(a, t) = a \cdot t$, that is for every a $v(a, .)$ is *linear* in t, then $v(f(t), t) - v(f(s), t) + v(f(s), s) - v(f(t), s) = (f(t) - f(s))(t - s)$.
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Jehiel, Moldovanu, Stacchetti (1999), M., Peréa, Wolf (2007)

T convex, $v(a,t)$ linear in t, then an allocation rule $f : T \to A$ is implementable if and only if it is monotone and path-integrals of the vector field "induced by v and f" are equal to 0.

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Path-monotonicity

Revenue Equivalence literature

Krishna and Maenner (2001) and Milgrom and Segal (2002): with sufficient analytical assumptions we have for every implementable f, $s, t \in T$:

$$v(f(t), t) + \pi(t) = v(f(s), s) + \pi(s) + \int_\sigma \Psi \cdot \sigma \; d\sigma$$

for any "smooth" path σ from s to t, "appropriate" vector field Ψ.

Path-integral

Assume $\sigma : [0,1] \to T$, then

$$\int_\sigma \Psi \cdot \sigma \; d\sigma \; := \int_0^1 \Psi(\sigma(x)) \cdot \sigma'(x) \; dx.$$

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Appropriate Ψ

If $v(a,.)$ is convex for all $a \in A$:

$$\Psi = \text{sub-differential of } v(f(t), s) \text{ at } s = t.$$

If $v(a,.)$ is differentiable for all $a \in A$:

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Path-monotonicity

Path-integrals of Ψ exist and

$$\int_\sigma \Psi \cdot \sigma d\sigma \leq v(f(t), t) - v(f(t), s) = l_u(s, t).$$

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Path-monotonicity

Observation

For "RE settings" path monotonicity necessary for implementation.

What to add?

Suppose path-integrals of Ψ on closed paths are equal to 0.
Fix type x, and define transfer $\pi(t) = \int_\sigma \Psi \cdot \sigma \, d\sigma - v(f(t), t)$.

$$\begin{aligned}
\pi(t) - \pi(s) &= \int_x^t \Psi \cdot \sigma d\sigma - v(f(t), t) - \int_x^s \Psi \cdot \sigma d\sigma + v(f(s), s) \\
&= \int_s^t \Psi \cdot \sigma d\sigma - v(f(t), t) + v(f(s), s) \\
&\leq l_u(s, t) - v(f(t), t) + v(f(s), s) \\
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Characterization differentiable valuations

Theorem

Let $T \subset \mathbb{R}^k$ be path-connected, for all a let $v(a,.)$ be *differentiable* in T, and let the norms of the gradients of the functions $v(f(t),.)$ be bounded, then an allocation rule $f : T \to A$ is implementable if and only if

- path-monotonicity holds,
- path-integrals on closed paths are equal to 0.

Here, $\Psi(t)$ is the gradient of $v(f(t),.)$ at t.

One direction resembles Milgrom and Segal (2002).

Characterization convex valuations

Theorem

Let $T \subset \mathbb{R}^k$ be convex, $v(a,.)$ *convex* in t, with non-empty subdifferential at all $t \in T$, then an allocation rule $f : T \to A$ is implementable if and only if

- path-monotonicity holds on line-segments,
- path-integrals on triangles are equal to 0.

Here, $\Psi(t)$ is any vector chosen from the sub-differential of $v(f(t),.)$ at t.

Elementary, direct proof is given in the paper.

Local implementability

Local implementation

f is called *locally implementable*, if for every $t \in T$ there exists some neighborhood U of t such that f is implementable on $T \cap U$.

Characterization

Let T be path-connected, $v(a,.)$ convex for all a, and let f be path-monotone. Then f is implementable if and only if it is locally implementable. (Similar for convex T and convex valuations).

Proof

By Lebesgue Number Lemma (as in Archer and Kleinberg (2008)).

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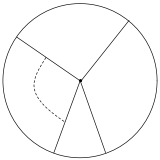
Finite A

Lemma

Let $T \subseteq \mathbb{R}^d$. Assume that for every $a \in A$ the function $v(a, .) : T \to \mathbb{R}$ is continuous and f is onto. For all $a \in A$ let

$$D_a := \overline{f^{-1}(a)}$$

If $f : T \to A$ is monotone and $T \cap \bigcap_{a \in A} D_a \neq \emptyset$ then f is implementable.



Characterization for finite A

Differentiable Case

Let $T \subseteq \mathbb{R}^d$ be simply path connected, $v(a, .)$ differentiable for all $a \in A$ and A be finite. $f : T \to A$ is implementable if and only if f is path monotone.

Convex case

Let $T \subseteq \mathbb{R}^d$ convex, $v(a, .)$ convex for all $a \in A$ and A be finite. $f : T \to A$ is implementable if and only if it is path-monotone on line segments.

Proof

Using previous Lemma we show that f is locally implementable. Then apply characterization based on local implementability.

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Monotonicity versus path-monotonicity

Assume path-integrals with respect to Ψ exist

- Path-monotonicity implies monotonicity.
- v satisfies *monotone differences* if for all $a, b \in A$, $v(a, .) - v(b, .)$ is monotone (in the classical sense) when restricted to line segments.
- Note: monotone differences holds if $v(a, .)$ is linear.
- monotonicity + monotone differences implies implementability (no analytical assumptions needed).
- For convex valuations $v(a, .)$, we provide $T \subset \mathbb{R}$ and f that is monotone but *not* implementable.

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Conclusions

- By replacing monotonicity by path-monotonicity we get characterizations for convex and differentiable valuation functions similar to those known for linear valuation functions.
- For finite A, path-monotonicity is equivalent to implementability in those settings.
- Monotonicity implies path-monotonicity, if we assume monotone differences.