Approximation and Mechanism Design

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Mechanism Design: how can a social planner / optimizer achieve objective when participant preferences are private.

Challenge: designer does not know participant preferences, participants may strategize when reporting preference!

Goals for Theory

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- *Descriptive:* predict/affirm mechanisms arising in practice.
- *Prescriptive:* suggest how good mechanisms can be designed.
- *Conclusive:* pinpoint salient characteristics of good mechanisms.

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Informal Thesis: *approximately optimality* is often descriptive, prescriptive, and conclusive.

Example 1: Gambler's Stopping Game

A Gambler's *Stopping Game*:

- *sequence* of n games,
- *prize* of game i is distributed from F_i,
- *prior-knowledge* of distributions.

On day i, gambler plays game i:

- *realizes* prize $v_i \sim F_i$,
- chooses to keep prize and *stop*, or
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Question: How should our gambler play?

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Discussion:

- *Complicated:* n different, unrelated thresholds.
- *Inconclusive:* what are properties of good strategies?
- *Non-robust:* what if order changes? what if distribution changes?
- *Non-general:* what do we learn about variants of Stopping Game?

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$$\mathbf{E}[\text{prize for strategy } t] \geq \mathbf{E}[\max_i v_i]/2.$$

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Discussion:

- *Simple:* one number t.
- *Conclusive:* trade-off "stopping early" with "never stopping".
- *Robust:* change order? change distribution above or below t?
- *General:* same solution works for similar games: invariant of "tie-breaking rule"

Prophet Inequality Proof

0. Notation:

- $q_i = \mathbf{Pr}[v_i < t]$.
- $x = \mathbf{Pr}[\text{never stops}] = \prod_i q_i$.

1. Upper Bound on $\mathbf{E}[\max]$:

2. Lower Bound on $\mathbf{E}[\text{prize}]$:

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- Must make tradeoff between understanding and optimality.

	$(1+\epsilon)$	constant	super-constant
Performance	great	ok	bad
Understanding	little	lots	some

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  - no, if mech without X is constant approx
  - yes, otherwise.
- Seller can always try ad hoc improvements on approximation.

Overview

1. Single-dimensional Bayesian settings.

   (e.g., single-item auctions)

2. Multi-dimensional Bayesian settings.

   (e.g., multi-item auctions)

3. Prior-free settings.

Part I: Approximation for single-dimensional Bayesian mechanism design

(where agent preferences are given by a private value for service, zero for no service; preferences are drawn from a distribution)

Example 2: Single-item auction

Problem: Bayesian Single-item Auction Problem

- a single item for sale,
- n buyers, and
- a dist. $\mathbf{F} = F_1 \times \cdots \times F_n$ from which the consumers' values for the item are drawn.

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Question: What is optimal auction?

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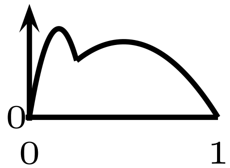

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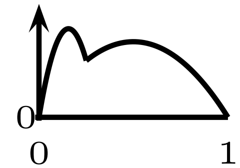

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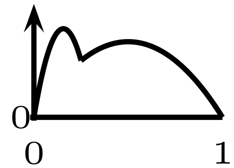

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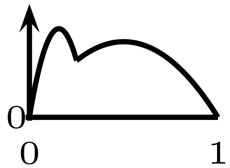

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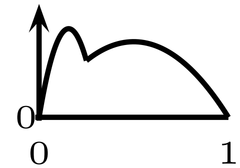

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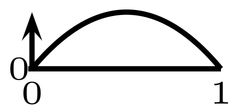

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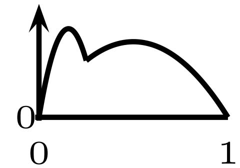

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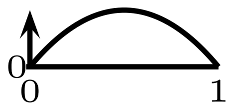

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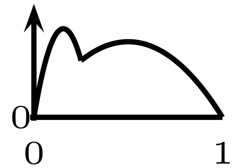

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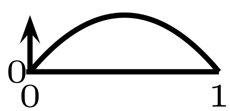

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7. **Cor:** for iid, regular dists, optimal auction is *Vickrey with monopoly reserve price* $\varphi^{-1}(0)$.

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Discussion:

- iid, regular case: seems very special.
- general case: nobody runs optimal auction (too complicated?).

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Discussion:

- constant virtual price $\Rightarrow$ bidder-specific reserves.
- *simple:* reserve prices natural, practical, and easy to find.
- *robust:* posted pricing with arbitrary tie-breaking works fine, collusion fine, etc.

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Discussion:

- theorem is not tight, actual bound is in $[2, 4]$.
- justifies wide prevalence.
- approximation good for *platform design*.

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Proof technique:

- optimal mechanism is a virtual surplus maximizer.
- reserve-price mechanisms are virtual surplus approximators.

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Proof technique:

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Basic Open Question: to what extent to simple mechanisms approximate (well understood but complex) optimal ones?

Challenges: non-downward-closed settings, negative virtual values.

Part II: Approximation for multi-dimensional Bayesian mechanism design

(where agent preferences are given by values for each available service, zero for no service; preferences drawn from distribution)

Example 3: unit-demand pricing

Problem: Bayesian Unit-Demand Pricing

- a single, unit-demand consumer.
- n items for sale.
- a dist. $\mathbf{F} = F_1 \times \cdots \times F_n$ from which the consumer's values for each item are drawn.

Goal: seller optimal *item-pricing* for $\mathbf{F}$.

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Discussion:

- little conceptual insight and
- not generally tractable.

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Problem: Bayesian Unit-demand Pricing (a.k.a., MD-PRICING)

- a single, *unit-demand* buyer,
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Problem: Bayesian Single-item Auction (a.k.a., SD-AUCTION)

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- n buyers, and
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Goal: seller opt. item-pricing for $\mathbf{F}$.	**Goal:** seller opt. auction for $\mathbf{F}$.

Note: Same informational structure.

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Proof: prophet inequality (tie-break by $v_i - p_i$). [Chawla,H,Malec,Sivan'10]

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4. *Instantiation:* SD-PRICING $\geq \frac{1}{\beta}$SD-AUCTION

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- *robust* to agent ordering, collusion, etc.
- *conclusive*: competition not important for approximation.
- *practical*: posted pricings widely prevalent. (e.g., eBay)
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Open Question: identify upper bounds beyond unit-demand settings that are

- conceptually tractable and
- approximable.

Part III: Approximation for prior-free mechanism design.

(mechanisms should be good for any set of agent preferences, not just given distributional assumptions)

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Question: can we design good auctions without knowledge of prior-distribution?

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- *non-generic*: e.g., for k-unit auctions, need k additional bidders.

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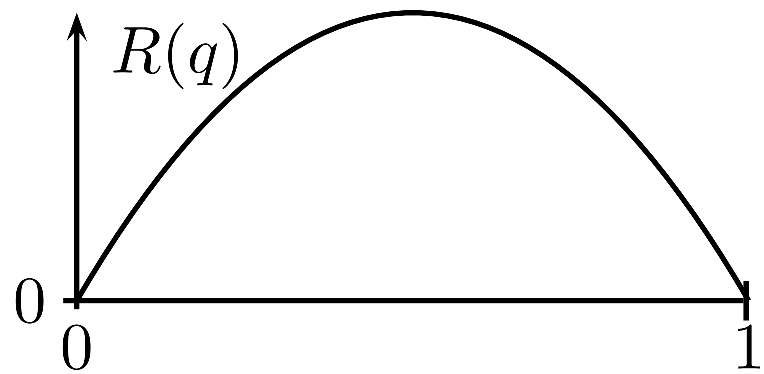

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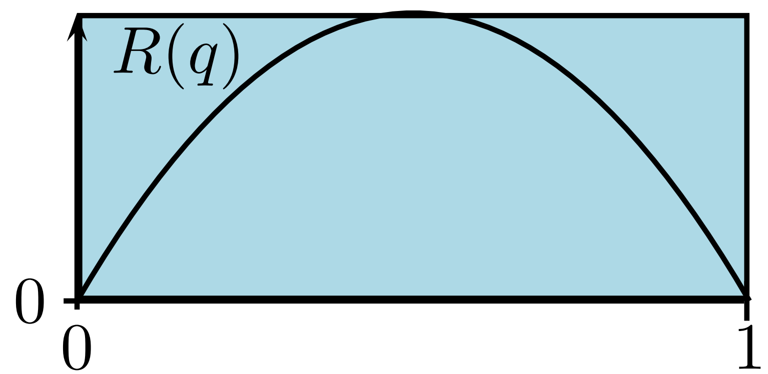

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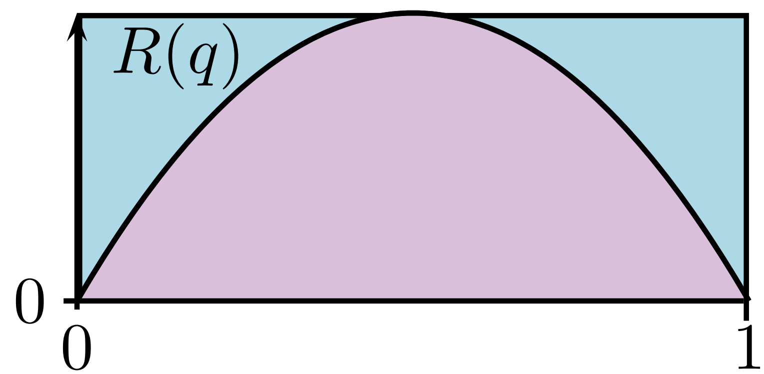

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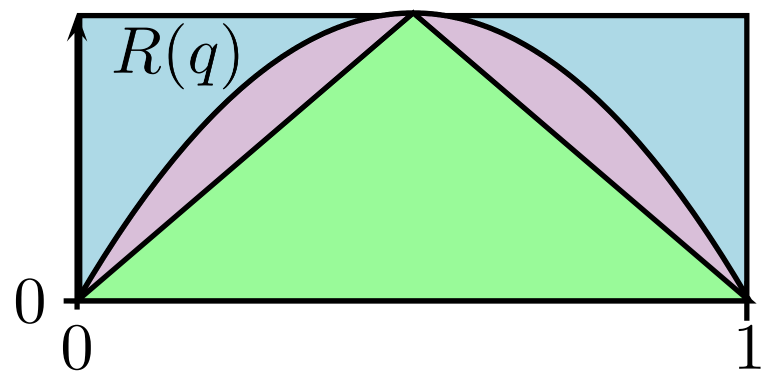

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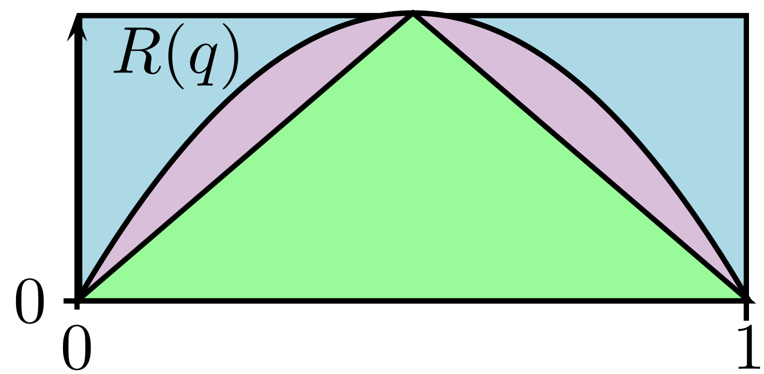

- So Vickrey with two bidders $\geq$ optimal revenue from one bidder.

Example 4: digital goods

Question: how should a profit-maximizing seller sell a *digital good* (n bidder, n copies of item)?

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Discussion:

- optimal,
- simple, but
- not prior-free

Approximation via Single Sample

Single-Sample Auction: (for digital goods)

[Dhangwatnotai, Roughgarden, Yan ’10]

1. pick random agent i as sample.

2. offer all other agents price v_i.

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Discussion:

- *prior-free*.
- *conclusive*, don't need precise distribution, only need single sample for approximation. (more samples can improve approximation factor.)
- *generic*, applies to general settings.

Average-case vs Worst-case

Note: prior-free auction cannot be optimal in every setting.

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Average Case Approximation: $\exists \mathcal{A}, \forall \mathbf{F} \in$ IID,

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Notes:

- worst-case approximation implies average-case approximation.
- $\sup_{\mathbf{F}\in\text{IID}} \text{OPT}_{\mathbf{F}}(\mathbf{v})$ is *prior-free performance benchmark*.
- for digital goods, prior-free benchmark $=$ optimal posted price revenue.

Approximation via Random Sampling

Random Sampling Auction: (for digital goods)

[Goldberg, H, Wright '01]

1. Randomly partition agents into two sets.

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Discussion:

- *conclusive*, market analysis can be done "on the fly"
- *worst-case* is for $n = 2$.
- *practical*, bounds approach 1 in limit with n.
- *generic*, analysis extends beyond digital goods.

Extensions

Prior-free results extend to limited supply, downward-closed settings, non-identical distributions, other objectives, etc.

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Open Questions:

- non-downward-closed settings?
- multi-dimensional settings?
- beyond the *revelation principle*?

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Basic Open Question: attack economic impossibility w. approximation.