Pure Nash Equilibria in Weighted Congestion Games

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Congestion Model $\mathcal{M} = (N, F, X, (c_f)_{f \in F})$

- $N = \{1, \ldots, n\}$ finite set of players
- $F = \{1, \ldots, m\}$ finite set of facilities
- $X = \bigtimes_{i \in N} X_i$ set of strategy profiles with $X_i \subseteq 2^F$
- Set of cost functions $(c_f)_{f \in F}$ where $c_f : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$

Weighted Congestion Game

Congestion model $\mathcal{M} = (N, F, X, (c_f)_{f \in F})$
Vector of demands $d = (d_i)_{i \in N}$, $d_i \in \mathbb{R}_{>0}$

Weighted Congestion Game

$G = (N, X, (\pi_i)_{i \in N})$

- private cost functions $\pi_i(x) = \sum_{f \in x_i} d_i c_f(\ell_f(x))$
- load $\ell_f(x) = \sum_{i \in N : f \in x_i} d_i$

Unweighted congestion game $\Leftrightarrow$ $d_i = 1$ for all $i \in N$
Singleton Congestion Game $\Leftrightarrow$ $|x_i| = 1$ for all $i \in N$, $x_i \in X_i$

Definition: Pure Nash Equilibrium (PNE)

Definition

As strategy profile x is a pure Nash equilibrium (PNE) if no player has an incentive to unilaterally change her decision:

$\pi_i(x_i, x_{-i}) \leq \pi_i(y_i, x_{-i})$ for all $i \in N, x_i, y_i \in X_i,$ and $x_{-i} \in X_{-i}$

Unweighted Congestion Games ($d_i = 1$)

- PNE exists (via exact potential) [Rosenthal, IJGT '73]

Previous Work

Unweighted Congestion Games ($d_i = 1$)

- PNE exists (via exact potential) [Rosenthal, IJGT '73]

Singleton Weighted Congestion Games ($|x_i| = 1$)

- non-decreasing cost functions, non-increasing cost functions
- PNE exists (via potential) [Fotakis et al., ICALP '02] [Even-Dar et al., ICALP '03] [Fabrikant et al., STOC '04], [Rozenfeld & Tennenholz, WINE '06]

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Matroid Weighted Congestion Games

- non-decreasing functions
  - PNE exists [Ackermann et al., WINE '06]

Arbitrary Strategy Spaces

Affine Costs

- PNE exists (via exact potential) [Fotakis et al., ICALP '05]

Exponential costs

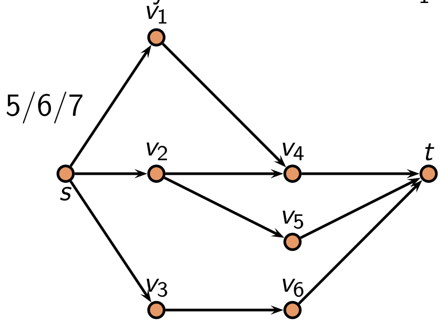
- PNE exist for $c_f(x) = \exp(x)$ for all $f \in F$ [Spirakis and Panagopoulou, JEA '06]
- PNE exist for $c_f(x) = a_f \exp(\phi x) + b_f$ for all $f \in F$ [H, Klimm, Möhring, SAGT '09]

Counterexamples

- No PNE (2-players with demands $d_1 = 1, d_2 = 2$) [Libman & Orda, TS '01] [Fotakis et al., ICALP '05] [Goemanns et al., FOCS '05]

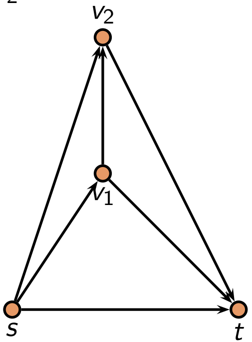
Counterexamples in Single-Commodity Networks

Two Players with demands $d_1 = 1, d_2 = 2$



Two-wise linear costs

[Fotakis et al., ICALP '05]



quadratic and linear costs

[Goemanns et al., FOCS '05]

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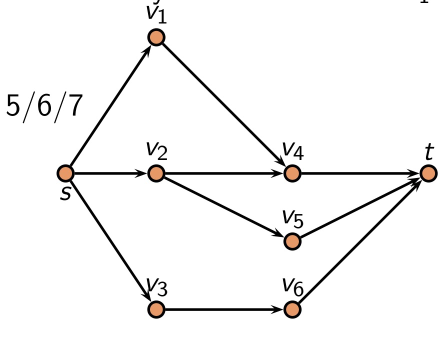

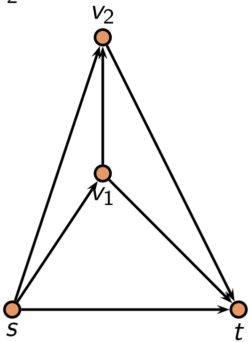

Two-wise linear costs

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2-player Shapley cost sharing games ($c_e(x) = k_e/\ell_e(x)$) always have a PNE [Anshelevich et al., SICOMP '08]

Definition

A set $\mathcal{C}$ of cost functions is consistent if every weighted congestion game to a congestion model $\mathcal{M} = (N, F, X, (c_f)_{f \in F})$ with $c_f \in \mathcal{C}$ for all $f \in F$ admits a PNE.

Definition

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- Examples of consistent cost functions:
  - $\mathcal{C} = \{c : \mathbb{R}_{\geq 0} \to \mathbb{R} : c(\ell) = a\ell + b, a, b \in \mathbb{R}\}$
  - $\mathcal{C}_\phi = \{c : \mathbb{R}_{\geq 0} \to \mathbb{R} : c(\ell) = ae^{\phi\ell} + b, a, b \in \mathbb{R}\}$

Necessary Condition: Monotonicity Lemma

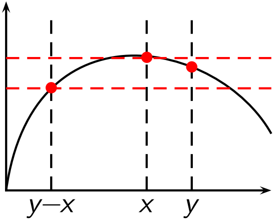
Lemma (Monotonicity Lemma)

Let $\mathcal{C}$ be a set of continuous cost functions. If $\mathcal{C}$ is consistent, then $\mathcal{C}$ contains only monotonic functions.

Even valid under the following restrictions

- games with 2 players
- games with 2 facilities
- singleton games
- games with identical cost functions on all facilities
- symmetric games

Proof of the Monotonicity Lemma



- Consider game with $N = \{1, 2\}$, $F = \{f, g\}$, $d_1 = y - x$ and $d_2 = x$
- Player 1 prefers to be alone
- Player 2 prefers to share a facility with Player 1

$\Rightarrow$ no PNE

Game without PNE

1 \ 2	$\{f\}$	$\{g\}$
$\{f\}$	$(y-x)c(y)$; $xc(y)$	$(y-x)c(y-x)$; $xc(x)$
$\{g\}$	$(y-x)c(y-x)$; $xc(x)$	$(y-x)c(y)$; $xc(y)$

Towards an Extended Monotonicity Lemma

Model

1 \ 2	$\{f\}$	$\{g\}$
$\{f\}$	f	
$\{g\}$		g

Game

1 \ 2	$\{f\}$	$\{g\}$
$\{f\}$	$x\,c(y)$ $(y-x)\,c(y)$	$x\,c(x)$ $(y-x)\,c(y-x)$
$\{g\}$	$x\,c(x)$ $(y-x)\,c(y-x)$	$x\,c(y)$ $(y-x)\,c(y)$

Towards an Extended Monotonicity Lemma

Model

1 \ 2	$\{f, j\}$	$\{h, g\}$
$\{f, h\}$	f c_1	h c_2
$\{j, g\}$	j c_2	g c_1

Game

1 \ 2	$\{f, j\}$	$\{h, g\}$
$\{f, h\}$	$x(c_1(y) + c_2(x))$ $(y-x)(c_1(y) + c_2(y-x))$	$x(c_1(x) + c_2(y))$ $(y-x)(c_1(y-x) + c_2(y))$
$\{j, g\}$	$x(c_1(x) + c_2(y))$ $(y-x)(c_1(y-x) + c_2(y))$	$x(c_1(y) + c_2(x))$ $(y-x)(c_1(y) + c_2(y-x))$

Towards an Extended Monotonicity Lemma

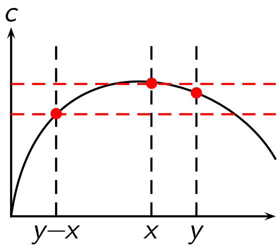
Model

1 \ 2	$F \cup J$	$H \cup G$
$F \cup H$	F — c_1, $\lvert F \rvert = a_1$	H — c_2, $\lvert H \rvert = a_2$
$J \cup G$	J — c_2, $\lvert J \rvert = a_2$	G — c_1, $\lvert G \rvert = a_1$

Game

1 \ 2	$F \cup J$	$H \cup G$
$F \cup H$	$x(a_1 c_1(y) + a_2 c_2(x))$ $(y-x)(a_1 c_1(y) + a_2 c_2(y-x))$	$x(a_1 c_1(x) + a_2 c_2(y))$ $(y-x)(a_1 c_1(y-x) + a_2 c_2(y))$
$J \cup G$	$x(a_1 c_1(x) + a_2 c_2(y))$ $(y-x)(a_1 c_1(y-x) + a_2 c_2(y))$	$x(a_1 c_1(y) + a_2 c_2(x))$ $(y-x)(a_1 c_1(y) + a_2 c_2(y-x))$

Towards an Extended Monotonicity Lemma



- Consider cost function $c(\ell) = a_1 c_1(\ell) - a_2 c_2(\ell)$ and chose $d_1 = y - x$, $d_2 = x$
- Player 1 prefers to be alone
- Player 2 prefers to share

$\Rightarrow$ no PNE

Game

1 \ 2	$F \cup J$	$H \cup G$
$F \cup H$	$x(a_1 c_1(y) + a_2 c_2(x))$ $(y-x)(a_1 c_1(y) + a_2 c_2(y-x))$	$x(a_1 c_1(x) + a_2 c_2(y))$ $(y-x)(a_1 c_1(y-x) + a_2 c_2(y))$
$J \cup G$	$x(a_1 c_1(x) + a_2 c_2(y))$ $(y-x)(a_1 c_1(y-x) + a_2 c_2(y))$	$x(a_1 c_1(y) + a_2 c_2(x))$ $(y-x)(a_1 c_1(y) + a_2 c_2(y-x))$

Definition (Integer 2-hull)

For a set $\mathcal{C}$ of cost functions we call

$$\mathcal{L}^2_{\mathbb{N}}(\mathcal{C}) = \{c : \mathbb{R}_{\geq 0} \to \mathbb{R} \mid c(x) = a_1\, c_1(x) - a_2\, c_2(x), \\ a_1, a_2 \in \mathbb{N}, c_1, c_2 \in \mathcal{C}\}.$$

the integer 2-hull of $\mathcal{C}$.

Lemma (Extended Monotonicity Lemma 1)

Let $\mathcal{C}$ be a set of continous cost functions. If $\mathcal{C}$ is consistent then $\mathcal{L}^2_{\mathbb{N}}(\mathcal{C})$ contains only monotonic functions.

- Even valid for games with 2 players.

Generalizing Extended Monotonicity Lemma 1

Model

1 \ 2	$F \cup J$	$H \cup G$
$F \cup H$	F c_1, $\lvert F \rvert = a_1$	H c_2, $\lvert H \rvert = a_2$
$J \cup G$	J c_2, $\lvert J \rvert = a_2$	G c_1, $\lvert G \rvert = a_1$

- Introduce a third player with demand $d_3 = b$ and a single strategy $X_3 = \{J \cup H\}$
- Take $c_1 = c_2$
- effective cost on F and G equals $c_1(x)$
- effective cost on H and J equals $c_1(x + b)$

Definition (Integer 3-hull)

For a set $\mathcal{C}$ of cost functions we call

$$\mathcal{L}^3_{\mathbb{N}}(\mathcal{C}) = \{c : \mathbb{R}_{\geq 0} \to \mathbb{R} \mid c(x) = a_1\, c_1(x) - a_2\, c_1(x+b), \\ a_1, a_2, b \in \mathbb{N}, c_1 \in \mathcal{C}\}.$$

the integer 3-hull of $\mathcal{C}$.

Lemma (Extended Monotonicity Lemma 1)

Let $\mathcal{C}$ be a set of continous cost functions. If $\mathcal{C}$ is consistent then $\mathcal{L}^3_{\mathbb{N}}(\mathcal{C})$ contains only monotonic functions.

- Even valid for games with 3 players.

Our Results so far

Singleton 2-player weighted congestion games

$\mathcal{C}$ is consistent $\Rightarrow$ $\mathcal{C}$ contains only monotonic functions

2-player weighted congestion games

$\mathcal{C}$ is consistent $\Rightarrow \mathcal{L}^2_{\mathbb{N}}(\mathcal{C}) = \{c \mid c(x) = a_1 c_1(x) - a_2 c_2(x)\}$ contains only monotonic functions

3-player weighted congestion games

$\mathcal{C}$ is consistent $\Rightarrow \mathcal{L}^3_{\mathbb{N}}(\mathcal{C}) = \{c \mid c(x) = a_1 c_1(x+b) - a_2 c_1(x)\}$ contains only monotonic functions

Characterizing monotonicity of $\mathcal{L}^2_{\mathbb{N}}(\mathcal{C})$

$$\mathcal{L}^2_{\mathbb{N}}(\mathcal{C}) = \{c \mid c(x) = a_1 c_1(x) - a_2 c_2(x),\ a_1, a_2 \in \mathbb{N}, c_1, c_2 \in \mathcal{C}\}$$

Lemma

Let $\mathcal{C}$ be a set of twice continuously differentiable and monotonic functions. Then, $\mathcal{L}^2_{\mathbb{N}}(\mathcal{C})$ contains only monotonic increasing or decreasing functions iff for all $c_1, c_2 \in \mathcal{C}$ there are $a, b \in \mathbb{R}$ such that $c_2(x) = a c_1(x) + b$ for all $x \geq 0$.

Proof.

" $\Leftarrow$ "

$$\begin{aligned} \tilde{c}(x) &= a_1 c_1(x) - a_2 c_2(x) \\ \tilde{c}(x) &= a_1 c_1(x) - a\, a_2\, c_1(x) - ab \\ \tilde{c}'(x) &= (a_1 - a\, a_2) c_1'(x) \end{aligned}$$

No change in sign of $\tilde{c}'$ $\Rightarrow$ $\tilde{c}$ is monotonic

Characterizing monotonicity of $\mathcal{L}^2_{\mathbb{N}}(\mathcal{C})$

Proof. (cont'd)

1. Show $D(x) := \det \begin{pmatrix} c_1'(x) & c_2'(x) \\ c_1''(x) & c_2''(x) \end{pmatrix} = 0 \quad$ for all $x \geq 0$.

   - $D(x_0) \neq 0 \Rightarrow D(x) \neq 0$ for all $x \in (x_0 - \epsilon, x_0 + \epsilon)$
   - Non-trivial solution of

   $$\begin{pmatrix} c_1'(x) & c_2'(x) \\ c_1''(x) & c_2''(x) \end{pmatrix} \begin{pmatrix} a_1(x) \\ a_2(x) \end{pmatrix} = \begin{pmatrix} 0 \\ -\frac{D(x)}{c_2'(x)} \end{pmatrix}.$$

   - $a_1(x) = 1$, $a_2(x)$ continous $\Rightarrow$ find $x \in (x_0 - \epsilon, x_0 + \epsilon)$ with $p \,/\, q = a_2(x) \in \mathbb{Q}$
   - $c = qc_1 - pc_2 \in \mathcal{L}^2_{\mathbb{N}}(\mathcal{C})$ has strict local extremum in x

Characterizing monotonicity of $\mathcal{L}^2_{\mathbb{N}}(\mathcal{C})$

Proof. (cont'd).

2. Show $D(x) = 0$ for all $x \geq 0 \quad \Rightarrow \quad c_2(x) = ac_1 + b$
   - If $c_1' \neq 0$ note that

$$D(x) = 0 \qquad \Rightarrow \qquad \left(\frac{c_2'(x)}{c_1'(x)}\right)' = 0$$

   - Integration delivers $c_2(x) = ac_1 + b$ for $a, b \in \mathbb{R}$
   - Glueing togesther intervals with $c_1' = 0$ and $c_1' \neq 0$ delivers the result

□

Characterizing 2-player Weighted Congestion Games

[H, Klimm, ICALP '10]

Theorem

Let $\mathcal{C}$ be a set of twice continously differentiable functions. Then $\mathcal{C}$ is consistent w.r.t. 2-player weighted congestion games iff the following holds

1. *$\mathcal{C}$ contains only monotonic functions*
2. *for all $c_1, c_2 \in \mathcal{C}$ there are constants $a, b \in \mathbb{R}$ such that $c_2 = ac_1 + b$*

The if-part follows from a generalization of [H, Klimm, Möhring, SAGT '09].

Characterizing monotonicity of $\mathcal{L}^3_{\mathbb{N}}(\mathcal{C})$

$$\mathcal{L}^3_{\mathbb{N}}(\mathcal{C}) = \{c \mid c(x) = a_1 c_1(x+b) - a_2 c_1(x),\ a_1, a_2, b \in \mathbb{N}, c_1 \in \mathcal{C}\}$$

[H, Klimm, ICALP '10]

Theorem

Let $\mathcal{C}$ be a set of twice continously differentiable functions. Then, $\mathcal{L}^3_{\mathbb{N}}(\mathcal{C})$ contians only monotonic functions iff one of the following holds

1. *$\mathcal{C}$ contains only affine functions*
2. *$\mathcal{C}$ contains only functions of type $c(x) = a_c e^{\phi x} + b_c$ where $a_c, b_c \in \mathbb{R}$ may depend on x while ϕ is independent of c.*

The if-part follows from [Fotakis et al., ICALP '05], [H, Klimm, Möhring, SAGT '09], [Spirakis and Panagopoulou, JEA '06].

Necessary conditions on consistency of costs

Strategy	2-player	3-player
Singleton	monotonic	monotonic
Arbitrary	aff. transformations	affine or exponential

If cost functions are strictly increasing and positive

Strategy	2-player	3-player
Single-commodity	aff. transformations	[FIP $\Leftrightarrow$ aff. or exp.]
Multi-commodity	aff. transformations	affine or exponential

Red conditions are tight