Pricing Lotteries

Patrick Briest

Shuchi Chawla Robert Kleinberg Matt Weinberg

Workshop on Advances in Algorithmic Game Theory.

CWI, Amsterdam, 09/03/2010.

Randomness is a useful resource ...

... in algorithm design.

... in market design.

But how useful is it?

Unit-Demand Envy-Free Pricing

- Seller (monopolist) has unlimited supply of n types of goods.
- Consumer i wants to buy a single good, has value v_{ij} for good j. (i=1,...,m).
- Seller posts price vector $(p_j)_{j=1,\ldots,n}$.
- Each consumer chooses one good (or none) to maximize $v_{ij} - p_j$ ("utility"). Consumer pays p_j.
- Compute profit-maximizing prices.

Distributional Version

Economist's version of the problem: instead of a discrete set of m consumer types, one is given a distribution over consumers.

University of Paderborn

Computational Results

- There is an efficient algorithm with approximation ratio n, e.g. single-price algorithm. [Guruswami et al. '05]
- For product distributions (i.e. components of the valuation vector are independent) there is a polynomial-time *3*-approximation. [Chawla, Hartline, Kleinberg '07]
- $\Omega(n^{\varepsilon})$ -hardness of approx. if $\exists\delta$ *s.t.* $\mathrm{NP} \not\subseteq \mathrm{BPTIME}(2^{\mathcal{O}(n^{\delta})})$. [B. '08]

Lotteries

- Change the pricing problem: instead of just selling items, you can also sell "lotteries": distributions over items.
- Lottery is given by $(p, \lambda_1, \ldots, \lambda_n)$, p=price, λ =vector of probabilities. ($\sum \lambda_i \leq 1$).
- Consumer's utility is $\lambda \cdot v - p$. (Expected value of random sample, minus price.)

Do lotteries help?

Riley & Zeckhauser (1983): With *just one item type*, randomization doesn't help. Can always maximize profit using one fixed offer.

University of Paderborn

The Single-Item Case

Given $\{(p_i, \lambda_i)\}$ with $0 = p_0 < p_1 < \cdots < p_k$, a consumer prefers (p_i, λ_i) to its predecessor and successor only if

$$\lambda_i v - p_i \geq \lambda_{i-1} v - p_{i-1}$$
$$\lambda_i v - p_i \geq \lambda_{i+1} v - p_{i+1}$$

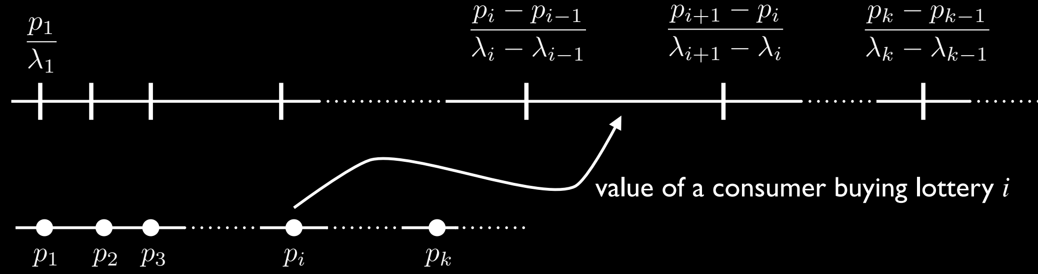
or, rearranging:

$$v \in \left[\frac{p_i - p_{i-1}}{\lambda_i - \lambda_{i-1}}, \frac{p_{i+1} - p_i}{\lambda_{i+1} - \lambda_i}\right]$$

Fix a consumer buying lottery i at price p_i.

University of Paderborn

The Single-Item Case



Setting price $\frac{p_j - p_{j-1}}{\lambda_j - \lambda_{j-1}}$ with probability $(\lambda_j - \lambda_{j-1})$ yields an expected payment of:

$$\sum_{j<i} (\lambda_j - \lambda_{j-1}) \frac{p_j - p_{j-1}}{\lambda_j - \lambda_{j-1}} = \sum_{j<i} (p_j - p_{j-1}) = p_i$$

Do lotteries help?

For multiple item types, some similar argument should work...

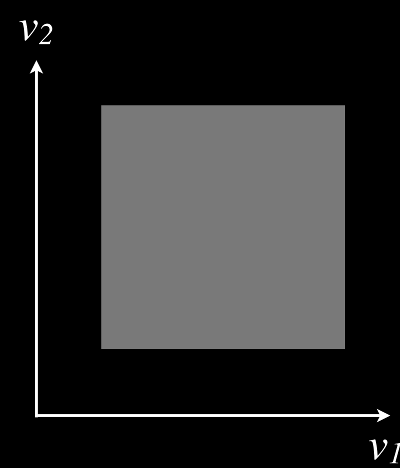
Do lotteries help?

- If goods are substitutes, lotteries can improve profit. [Thanassoulis, 2004; Manelli & Vincent, 2006]
- Thanassoulis example:

  v_1 = value for Hilton

  v_2 = value for Hyatt

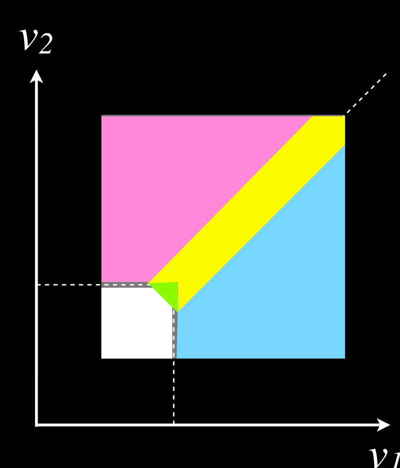
  Independent, uniformly distributed on $[200,250]$.



University of Paderborn

Do lotteries help?

- Price vector (p_1,p_2) divides the type space into 3 regions.
- Optimizing over (p_1,p_2) we find that profit is maximized at $p_1=p_2=w$.
- Now introduce lottery $(w\text{-}\delta, 0.5, 0.5)$.
- For small enough δ, profit increases.



University of Paderborn

Do lotteries help?

Proposition 9. *Suppose consumers are uniformly distributed on a square* $[a, a+1]^2$ *with* $a > 1$ *and the seller has two symmetric substitutable goods to sell with marginal costs normalised to* 0. *The fully optimal selling strategy is to use take it or leave it prices in combination with the lottery* $\left(\frac{1}{2}, \frac{1}{2}\right)$ *only.*

Numerical Proof. This proposition is substantiated through a large number of numerical optimisations with different grid sizes and different supports, $[a, a+1]^2$. I have run this experiment for $a \in \{0, 1, 2, 3, 5, 10, 20, 50\}$ and have found that the proposition holds in all of these cases (Fig. 3). □

We note that the above proof is numerical and so does not constitute an analytical proof.[21] Such proofs would be hard to come by due to the large number of constraints active on candidate surplus functions $v(\cdot)$. It is worth mentioning that the profit gain from the fully optimal sales strategy as compared to the best tioli prices is very modest: at best of the order of a single percentage point.

(From Thanassoulis, J. Economic Theory, 2004)

How big can the percentage gain be?

Is it computationally hard to find optimal lotteries?

University of Paderborn

Quantifying the gain

When $n=2$, any system of lotteries (each with total probability 1 of allocating an item) is 3-approximated by pure item pricing.

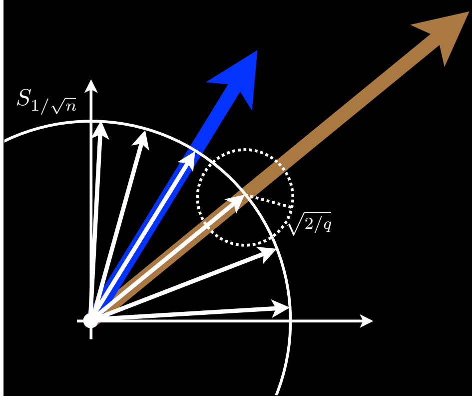
Randomized rounding + geometric arguments.

For multiple item types, some similar argument should work...

University of Paderborn

Quantifying the gain

For any $n \geq 4$, the gap between the optimal item pricing and lottery pricing revenue cannot be bounded in terms of the number of item types.



Geometric argument: vector packing.

University of Paderborn

Our guiding questions

How big a percentage gain can one get from using lottery pricing?

Unbounded when selling at least 4 item types.

Is it computationally hard to find optimal lotteries?

How is the input specified?

University of Paderborn

Computing Lotteries

- Assume input specifies a distribution μ over m type vectors.
- Let (p_i, q_{ij}) be the preferred lottery of consumer type i.
- The LP to the right describes the optimal system of lotteries.

$$
\begin{array}{lll}
\text{max.} & \sum_{i=1}^{m} \mu_i p_i & \\
\text{s.t.} & \sum_{j=1}^{n} q_{ij} \leq 1 & \forall i \\
 & \sum_{j=1}^{n} q_{ij} v_{ij} - p_i \geq 0 & \forall i \\
 & \sum_{j=1}^{n} q_{ij} v_{ij} - p_i \geq \sum_{j=1}^{n} q_{kj} v_{ij} - p_k & \forall i, k
\end{array}
$$

Our guiding questions

How big a percentage gain can one get from using lottery pricing?

Unbounded when selling at least 4 item types.

Is it computationally hard to find optimal lotteries?

No, it's easy, unlike the case of item pricing.

EXCEPT...

... for this

- A single item, a single consumer with value $v=1$.
- Offer two lotteries:

$$\begin{array}{llll} L_1: & \lambda = 1 & p = 1/2 \\ L_2: & \lambda = 1/2 & p = \varepsilon \end{array}$$

- What to buy if disposal is free...?

$$\text{util}(k \times L_2) = (1 - 2^{-k}) - k\varepsilon$$

University of Paderborn

The "Buy Many" Model

Theorem: In the "buy many" model:

1. The optimal item pricing approximates the optimal lottery pricing within $O(\log n)$.
2. There exist consumer distributions for which this factor cannot be improved.
3. Optimal lottery pricing inherits the same approximation hardness as envy-free item pricing, up to a $O(\log n)$ factor.

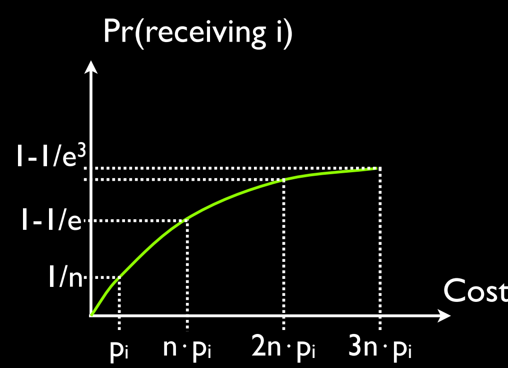
University of Paderborn

The uniform case

First assume everyone has:

- A set S of desired items.
- A value v for items in S.

Suppose some lottery with price p_i offers item i with probability at least $1/n$.



Key insight: Someone who wants i and spends $k{\cdot}n{\cdot}p_i$ has value ~$e^k{\cdot}n{\cdot}p_i$.

How to round lotteries

- Let Λ_{opt} be the optimal set of lotteries.
- Let b_i = price of cheapest lottery with $Pr(i)>1/(2n)$

  = "base price of item i"
- Output price vector $r{\cdot}(b_1,\ldots,b_n)$
- r = random power of e.
- First $ln(n)$ powers have probability $\propto 1/ln(n)$.
- $r = e^{ln(n)+k} \sim e^k{\cdot}n$ has probability $\propto e^{-k}$.

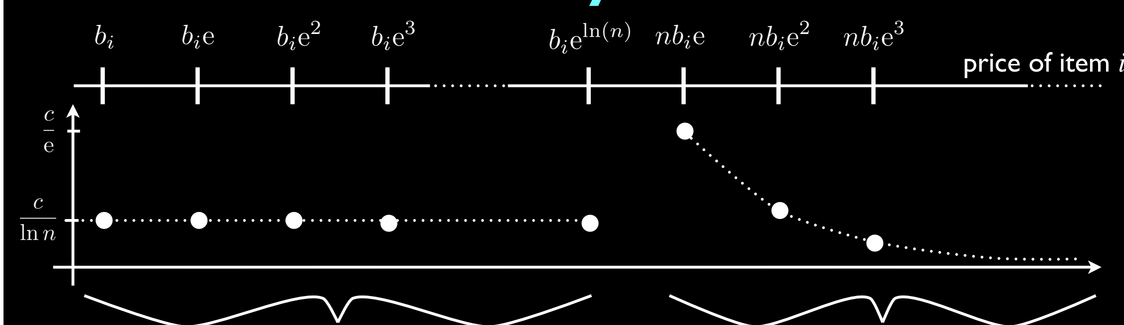
University of Paderborn

Analysis

- Fix a consumer of type *(v,S)*. Assume she buys a lottery with *Pr(S) > 1/2* and cost *q*.
- Let i be the item in S with cheapest base price.
- Given item prices $r(b_1, \ldots, b_n)$ for any value of r, this consumer will purchase item i or nothing at all!

University of Paderborn

Analysis



If $b_i \leq q \leq nb_i$, we charge her within factor e of her full value with probability *c/ln(n)*.

If $q \approx knb_i$, her value is around $\mathrm{e}^k nb_i$, which yields expected payment

$$\sum_{j \leq k} (c\mathrm{e}^{-j})(\mathrm{e}^j nb_i) = cknb_i\,.$$

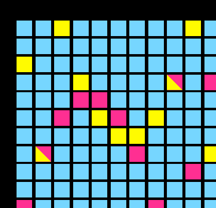
In general...

- Consumers may have different nonzero values for different items.
- Analysis is too lengthy for this talk.
- Main problem: A consumer chooses to buy different items as you scale up the prices.
- Solution: Carefully organize these different choices into a telescoping sum.

O(log n) is optimal

To find consumer distribution where lottery pricing improves item pricing by $\Omega(\log n)$:

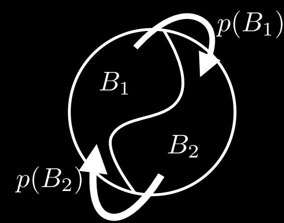
- Consumers must prefer their "intended lottery" to all bundles of cheaper ones. Type vectors must be nearly orthogonal.
- Geometry to the rescue once again, but this time the geometry of degree-2 curves in the affine plane over a finite field.



- Bounding item pricing revenue is tricky. Can prove existence of bad instance via the probabilistic method.

Applications & Open Problems

- Revenue-maximizing auction mechanisms via random sampling [Balcan,Blum,Hartline,Mansour '05]:



- *Bundle-pricing* as done on, e.g., hotwire.com: lottery tickets' "probability distributions" are non-public. [B,Röglin '10]
- Randomness in Multi-Dimensional Mechanism Design

Thank you!

Questions?