Quantum Computing Exercises

September 15, 2000

1. What is the probability of measuring $|1\rangle$ in state $H|0\rangle$?
   And in state $H|1\rangle$?
   And in $H(\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle)$?
   What is strange about this?

2. Is the controlled-NOT operation C Hermitian?
   Determine C^{-1}.

3. Compute the result of applying a Hadamard transform to both qubits of $|0\rangle \otimes |1\rangle$ in two ways, and show that the two results are equal:
$$H|0\rangle \otimes H|1\rangle = (H \otimes H)(|0\rangle \otimes |1\rangle).$$

4. A matrix A is *inner product-preserving* if the inner product between $A|v\rangle$ and $A|w\rangle$ equals $\langle v|w\rangle$, for all $|v\rangle, |w\rangle$.
   (a) Prove that A is norm-preserving iff A is inner product-preserving.
   (b) Prove that A is inner product-preserving iff $A^\dagger A = AA^\dagger = I$.
   (c) Conclude that A is norm-preserving iff A is unitary.

Solution to exercise 4:

Proposition 1 *A is norm-preserving iff A is inner product-preserving.*

Proof Since the norm $\| |v\rangle \|$ is defined as $\sqrt{\langle v|v\rangle}$, it is clear that if A preserves inner product, then it also preserves the norm.

Conversely, suppose A is norm-preserving. Note that, for all $|v\rangle, |w\rangle$, we can write out (using $\langle u|v+w\rangle = \langle u|v\rangle + \langle u|w\rangle$, $\langle v|cw\rangle = c\langle v|w\rangle$, $\langle cv|w\rangle = \overline{c}\langle v|w\rangle$):
$$\begin{aligned} \| |v\rangle + |w\rangle \|^2 &= \| |v\rangle \|^2 + \| |w\rangle \|^2 + \langle v|w\rangle + \langle w|v\rangle \\ \| |v\rangle + i|w\rangle \|^2 &= \| |v\rangle \|^2 + \| |w\rangle \|^2 + i\langle v|w\rangle - i\langle w|v\rangle \end{aligned}$$
The first equation $- i$ times the second equation gives
$$\| |v\rangle + |w\rangle \|^2 - i \, \| |v\rangle + i|w\rangle \|^2 = (1-i) \, \| |v\rangle \|^2 + (1-i) \, \| |w\rangle \|^2 + 2\langle v|w\rangle$$
From this we can obtain a formula for $\langle v|w\rangle$ which only depends on norms of various vectors. Since A preserves norms, it follows from the formula that A preserves $\langle v|w\rangle$ as well. □

Proposition 2 *A is inner product-preserving iff $A^{-1} = A^\dagger$ (i.e. A is unitary).*

Proof Suppose A preserves inner product. If $|v\rangle$ is some unit vector, then we have
$$1 = \langle v|v\rangle = (A|v\rangle, A|v\rangle) = \langle v|(A^\dagger A|v\rangle).$$
Since $\langle v|w\rangle = 1$ iff $|v\rangle = |w\rangle$, it follows that $A^\dagger A|v\rangle = |v\rangle$whenever $|v\rangle$ is a unit vector. This implies that $A^\dagger A$ is the identity, hence $A^\dagger = A^{-1}$.

Conversely, suppose $A^{-1} = A^\dagger$. Then for all $|v\rangle, |w\rangle$ we have $(A|v\rangle, A|w\rangle) = (A|v\rangle)^\dagger A|w\rangle = \langle v|A^\dagger A|w\rangle = \langle v|I|w\rangle = \langle v|w\rangle$, so A preserves inner product. □