

Objectively combining AR5 instrumental period and paleoclimate climate sensitivity evidence

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Abstract

Combining instrumental period evidence regarding equilibrium climate sensitivity with largely independent paleoclimate proxy evidence should enable a more constrained sensitivity estimate to be obtained. Previous, subjective Bayesian approaches involved selection of a prior probability distribution reflecting the investigators' beliefs about climate sensitivity. Here a recently developed approach employing two different statistical methods – objective Bayesian and frequentist likelihood-ratio – is used to combine instrumental period and paleoclimate evidence based on data presented and assessments made in the IPCC Fifth Assessment Report. Probabilistic estimates from each source of evidence are represented by posterior probability density functions (PDFs) of physically-appropriate form that can be uniquely factored into a likelihood function and a noninformative prior distribution. The three-parameter form is shown accurately to fit a wide range of estimated climate sensitivity PDFs. The likelihood functions relating to the probabilistic estimates from the two sources are multiplicatively combined and a prior is derived that is noninformative for inference from the combined evidence. A posterior PDF that incorporates the evidence from both sources is produced using a single-step approach, which avoids the order-dependency that would arise if Bayesian updating were used. Results are compared with an alternative approach using the frequentist signed root likelihood ratio method. Results from these two methods are effectively identical, and provide a 5-95% range for climate sensitivity of 1.1–4.05 K (median 1.87 K).

1. Introduction

Equilibrium climate sensitivity (ECS) – the rise in global mean surface temperature (GMST) resulting from a doubling of atmospheric carbon dioxide (CO₂) concentration, once the ocean has reached equilibrium – is a key climate system parameter. Effective radiative forcing (ERF) provides a metric, designed to be a good indicator of their impact on GMST, of the effects on the Earth's radiative imbalance of changes in CO₂, in other radiatively-active gases and in other drivers of climate change. For changes between equilibrium states, ECS may then be estimated as the ratio of the change in GMST (ΔT) to the change (ΔF_n) in ERF normalised by division by the ERF from a doubling of CO₂ concentration: $ECS = \Delta T / \Delta F_n$.

It has proved difficult to narrow uncertainty about ECS, given observational data limitations and internal climate system variability. It may be possible better to constrain ECS by combining probabilistic evidence from different sources. Several authors have sought to do so (Hegerl et al. 2006; Annan and Hargreaves 2006; Aldrin et al. 2012; Stevens et al. 2016) using subjective Bayesian methods, however their results were significantly influenced by the initial prior distributions they chose. Here we apply a recently established method of combining evidence with differing characteristics to generate an ECS estimate based on instrumental and paleoclimate evidence. That evidence, formulated as probability density functions (PDFs), is derived from estimates presented in the IPCC Fifth Assessment Working Group 1 Report (AR5). The results themselves are considered to be of some importance since they are intended to reflect carefully considered assessments in AR5, but an equally important motivation is bringing a more objective method of combining evidence regarding ECS and similar parameters to the attention of the climate science community.

Before going further, let us clarify the scope of our study. We study ECS, and not Earth system sensitivity (ESS) – the GMST increase from a doubling of CO₂ concentration once slow climate system components such as the cryosphere and vegetation have equilibrated as well as the atmosphere and ocean. Recently, using proxy data for the past 800 kyr, Snyder (2016) estimated a 7–13 K range for ESS. However, using data from the same period Palaeosens Project Members (2012) estimated ESS to be 3.4 times as high as ECS. Since ESS is by definition a quite different notion of climate sensitivity from ECS we will not consider it here.

We use both “objective Bayesian” and frequentist likelihood-ratio statistical methods, and show that these approaches yield results which are almost identical to each other, but very different from those obtained with a subjective Bayesian method when employing a uniform prior; this commonly used prior is in this instance unintentionally highly informative (inference using it is not dominated by the data involved). Objective Bayesian methods (Berger 2006, Bernardo 2009), which use 'noninformative' priors, are an integral and thriving part of Bayesian statistics (a substantial fraction of papers at the main Bayesian (ISBA) meetings are dedicated to ‘objective Bayes’). Nevertheless, they are not often employed within the climate science community and since some sort of subjectivity is inevitable in any statistical analysis, if only regarding the chosen model, some discussion is warranted as to what is precisely meant by ‘objective’ and 'noninformative' here; we defer this discussion to Section 2.2.

A key attraction of combining a single instrumental period and a single paleoclimate evidence source regarding ECS is that, unlike when combining evidence from, e.g., different sources from within the instrumental period, sources of error should be largely independent; this greatly simplifies the statistical inference. Our results thus combine a single instrumental period source with a single paleoclimate source. In our main result, the instrumental period source is taken from Lewis and Curry (2015); for comparison, we provide a second result in which the instrumental period source is taken from Otto et al. (2013), a study in which many AR5 lead authors were involved. The paleoclimate

source is the same in both results, representing a summary of the expert assessment made in AR5 and of studies cited in it. Below we first describe our methodology and then the sources of evidence in more detail.

1.1 Methodology and Overview

Standard objective Bayesian methodology, reviewed in Section 2, requires the evidence to be represented as a likelihood, i.e. the probability density of observed data as a function of parameters in the assumed statistical model (a "statistical model" meaning a family of probability distributions). In our case, however, the evidence is of a different form: the two instrumental period sources provide a probability density function (PDF) for the single parameter θ (ECS) of interest, while the paleoclimate source provides a partial specification of such a PDF. From a Bayesian perspective we may view these PDFs as 'posteriors' arising from some data (likelihood) and prior. We somehow need to construct the required likelihoods based on the given PDFs. To this end, we assume a ratio-normal model, which (Section 3.1) is a physically appropriate model to use in this case, both for instrumental and paleoclimate evidence. Combining the ratio-normal model with its corresponding objective Bayesian prior leads to a posterior distribution on ECS that is severely constrained in form: it is fully determined by three parameters. Raftery and Schweder (1993; henceforth RS 93) provided a remarkably accurate analytic approximation of the posterior for a parameter that is the ratio of two other parameters whose joint posterior is bivariate normal, which (Section 3.1) corresponds to both of them being location parameters estimated with normal error and using a noninformative uniform prior. We henceforth call this the 'RS93 posterior'. (A location parameter is one where the probability density of an observed datum depends only on the error – the difference between the datum and parameter values.) The two location parameters may be scaled equally without affecting the distribution of their ratio; it is convenient to scale so that the denominator distribution has a unit mean. The distribution of their ratio is therefore specified by three parameters, provided the numerator and denominator errors are independent: the mean of the scaled numerator distribution and the standard deviations of the scaled numerator and denominator distributions. The same three parameters specify the approximating RS93 posterior. By adjusting these three parameters, the RS93 posterior can be made to fit published PDFs for ECS very well: not just the two instrumental-period PDFs used for our main results, but, as we show in Section 3.2, also a wide range of variously shaped PDFs obtained by other authors. While the original PDFs were not directly given in the RS93 form, we do take this as evidence that the ratio-normal model is indeed appropriate, and that sound published PDFs for ECS can be legitimately viewed as (the RS 93 approximation to) posteriors arising from data and an objective (noninformative) prior for the ratio-normal model. As shown by Lewis (in press), the RS93 posterior induces a unique factorization into a likelihood and (noninformative) prior, giving us the desired likelihood needed for our objective Bayesian approach.

While in the paleoclimate case there is no published PDF that appears to be representative of the evidence in AR5, combining the range assessed in AR5 with a median estimate from studies it cites provides sufficient information to uniquely fit an RS93 posterior, again leading to the desired likelihood. Below we explain how this fitting process faithfully represents paleoclimate evidence.

Both our results thus combine two likelihoods, induced by RS93 posterior PDFs, using an objective Bayesian approach. The obvious but incorrect way to do this is to copy the subjective Bayesian updating method (using the posterior based on the first source as the prior for the second). However, as explained in section 2.2, this leads to anomalies such as order dependence (Seidenfeld 1979, Kass and Wasserman 1996), since the noninformative prior for the first likelihood is in general different from the noninformative prior for the second likelihood; moreover, as shown by Lewis (in press), it has quite bad frequentist calibration properties. The 'correct' combination or updating method, without such anomalies, was proposed by Lewis (2013b), and is also the method of choice according to the information-theoretic *Minimum Description Length* method (Grünwald, 2007). The

details of applying such a method to RS93 ratio-normal approximation likelihoods were recently worked out by Lewis (in press), who also shows that it enjoys much better frequency properties – we review the method in Section 4.1 and 4.2. As a sanity check (Section 4.3), we also use a purely frequentist combination method and find that it gives near-identical results, in the sense that it gives rise to confidence intervals that almost precisely match our objective-Bayesian credible intervals. Section 5 then presents these results, which are summarized in Figure 3.

In the remainder of this introduction, we shall further motivate our particular choice for the two instrumental period sources as well as our single paleoclimate source, itself a summary of various sources represented in the AR5 expert assessment.

1.2 Instrumental Period Evidence

AR5 cited a large number of instrumental period ECS estimates that gave widely varying best estimates and uncertainty ranges, but expressed [Section 12.5.3] doubts about estimates based on short timescales or non-greenhouse gas forcings. The AR5 authors evidently considered that estimates based on longer-term greenhouse gas-dominated warming over the instrumental period should be more reliable: AR5 attributes its reduction in the bottom of the likely range for ECS to estimates using multidecadal data from the instrumental period, stating [Box 12.2] that the reduction "reflects the evidence from new studies of observed temperature change, using the extended records in atmosphere and ocean". Here, rather than using some average of instrumental period ECS estimates cited in AR5, we focus on estimation that conforms with the assessment in AR5 of the relative merits of different approaches and reflects AR5 values for radiative forcings and heat uptake, its reassessment of uncertain aerosol forcing being particularly relevant. One important such estimate was provided by Otto et al. (2013, hereafter Oa13), a study in which many AR5 lead authors were involved. Oa13 was an energy-budget study that reflected estimated changes in global surface temperature, forcing and climate system heat uptake over the instrumental period, using a multimodel atmosphere-ocean general circulation model (AOGCM)-derived estimate of forcing changes (with aerosol forcing adjusted). Subsequent to the release of AR5, another energy budget ECS study (Lewis and Curry 2015, hereafter LC15) used the updated estimates of forcing series from AR5, not available at the time of the Oa13 study, which differ from and have wider uncertainty ranges than those derived from AOGCMs. Like Oa13, LC15 used the HadCRUT4 v2 global temperature dataset (Morice et al. 2012), AR5's observationally-based estimate of recent heat accumulation in the climate system and a model-based estimate of the small heat uptake early in the instrumental period. Its "preferred results" ECS estimate is based on changes from 1859-1882 to 1995-2011, two periods with low volcanic activity and which appear to have similar influences from internal variability. The distribution of ECS from those LC15 results is used as the most representative instrumental period evidence for this present study. Comparative results using the best-constrained Oa13 ECS estimate (based on changes from 1860-1879 to 2000-2009) are also presented. That estimate has a slightly higher median than LC15's, principally reflecting a greater estimate of the change in heat accumulation rate (LC15), but it is better constrained, reflecting the narrower AOGCM-derived forcing uncertainty range.

1.3 Paleoclimate Evidence

Chapter 10 of AR5 concluded that uncertainties in paleoclimate estimates of ECS were likely to be larger than for estimates from the instrumental record, since, as well as data uncertainties, feedbacks could change between different climatic states.

Unlike for the instrumental evidence case, no single PDF has been published that fully represents the evidence in and assessments of AR5. Instead, AR5 gives a probability range, based on studies from various authors (Chylek and Lohmann 2008; Hargreaves and Annan 2012; Holden et al. 2010; Köhler et al. 2010; Palaeosens Project Members 2012; Schmittner et al. 2011; Schneider von

Deimling et al. 2006). To use our Bayesian combination method we first need to summarize this information into a single RS93 posterior PDF, as explained above.

For this, we first recall AR5's conclusion that paleoclimate evidence implied ECS was very likely (90%+ probability) greater than 1 K and very likely less than 6 K, without giving any best estimate. We treat that, conservatively, as implying a 10–90% (80%) range of 1–6 K. Whilst just a range is sufficient to fit a symmetrical Gaussian probability distribution, it is insufficient to specify a distribution in which the central tendency, dispersion and skewness are independent of each other. ECS estimates are generally skewed, mainly reflecting uncertainty in the estimates of changes in forcings, and their central tendency, dispersion and degree of skewness will vary from one estimate to another. A RS93 posterior has three free parameters and provides independent control of these three characteristics. To fit it to a given partially specified distribution thus requires, at a minimum, three percentile points. Therefore, we first derive a median (50% percentile) paleoclimate estimate from all the eight median estimates shown in the paleoclimate section of AR5 Figure 10.20b. Additionally, for Chylek and Lohmann (2008) and Schneider von Deimling et al. (2006) (where only 5–95% ranges are given) we take their midpoint as the median, there being no evidence of skewness in those studies' results. We exclude Annan et al. (2005) as it provides no ECS lower limit. The mean and the median of the ten median estimates are both marginally below 2.75 K. We round up and use 2.75 K as the median ECS estimate from paleoclimate evidence.

Almost all the paleoclimate studies featured in AR5 Figure 10.20b focused on the well-studied transition from Last Glacial Maximum (LGM) to preindustrial conditions. All but one of these LGM-based studies estimated ECS for preindustrial conditions. They allowed for possible state dependence of ECS by using climate models in which ECS was an emergent property that could vary with climate state; the relationship between ECS for the LGM transition and ECS in preindustrial conditions varies among climate models (Hargreaves et al. 2007; Otto-Bliesner et al., 2009; Colman et al. 2009). Recently, by regressing temperature against forcing over the warmest parts of glacial cycles during the last 784 kyr, Friedrich et al. (2016) estimated ECS applying in preindustrial conditions as 4.9 K, well above all the median estimates in AR5 Figure 10.20b. Their estimated 5.75 K warming and 6.5 Wm⁻² forcing change for the LGM transition implies a lower energy-budget ECS estimate of 3.3K, in line with the 3.2 K they derive from regressing temperature against forcing over complete glacial cycles during the 784 kyr period. The high Friedrich et al. (2016) warm-climate ECS estimate may be more a reflection of their temperature and forcing change estimates than of unusually strong estimated climate-state dependence. Carefully constructed modern best estimates of the LGM transition temperature change (4.0 K; Annan and Hargreaves 2013) and forcing change (9.5 Wm⁻²; Köhler et al. 2010) imply an energy budget LGM-transition ECS estimate of 1.6 K, half the Friedrich et al. (2016) value. Moreover, regression plots of temperature against forcing in Martinez-Boti et al. (2015), using data for both the 0-800 kyr and the late Plio-Pleistocene (2,300–3,300 kyr) periods, show little evidence of higher sensitivity during the warmest parts of interglacials in the last 800 kyr or during the even warmer late Plio-Pleistocene period, relative to that over the entire 0–800 kyr period. Accordingly, we see no reason to revise upwards the 2.75 K median paleoclimate evidence ECS estimate derived from paleoclimate studies shown in AR5 Figure 10.20b.

Figure 1 compares PDFs, medians and uncertainty ranges shown in AR5 Figure 10.20b, for each of the paleoclimate studies it features, with the RS93 posterior which is based on a physically appropriate model (see Section 3 for details), fitted to 10th, 50th and 90th percentiles of 1, 2.75 and 6 K. This distribution (black line), although not itself an IPCC assessment, is presented as fairly representing the AR5 assessment from paleoclimate evidence of a 1–6 K uncertainty range for ECS, treated conservatively as 10-90%, with a median consistent with those from studies featured in AR5. The 17–83% range for the fitted RS93 posterior is 1.4–4.85 K. The fact that that range is only slightly wider than the overall 'likely' (at least 66% probability) range of 1.5–4.5 K that AR5 derives in Chapter

10 from assessing all lines of observational evidence combined supports treating the 1–6 K range as covering only the central 80% probability. So also does the lack of a high confidence appraisal being given to the paleoclimate range in AR5.

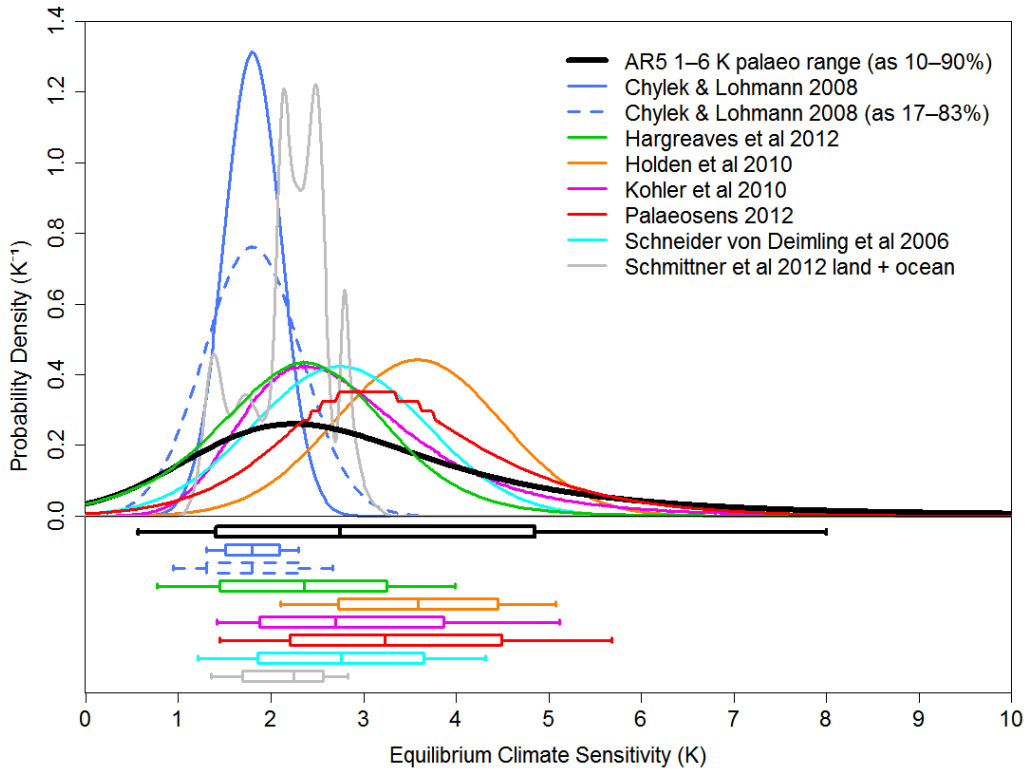


Figure 1. Estimated PDFs for individual paleoclimate studies compared with the distribution fitted to the AR5 paleoclimate range. The thick black line shows the distribution used to represent the AR5 1–6 K paleoclimate range, treated as representing 10–90% probability and as having a median of 2.75 K. The thin colored lines show PDFs for the paleoclimate studies featured in Figure 10.20b of AR5, fitted by Gaussian distributions to their ranges in the case of Chylek and Lohmann (2008) (blue line) and Schneider von Deimling et al (2006) (cyan line) and otherwise as plotted in AR5 Figure 10.20b. Although Chylek and Lohmann's estimate is a 95% range, since AR5 assesses its true uncertainty as likely larger, the range is treated here as being 5–95% and a PDF based on treating it as a 17–83% range is also shown (dashed blue line). Secondary estimates (shown dashed or dashed-dotted in Figure 10.20b of AR5) are not plotted but were included when determining the median of the distribution used to represent the AR5 paleoclimate range,. The box plots indicate boundaries for the percentiles 5–95 (vertical bar at ends), 17–83 (box-ends), and 50 (vertical bar in box).

The rest of this paper is organized as follows. Section 2 considers statistical parameter inference methods. Section 3 deals with selecting a suitable form of parameterized standard distribution to match ECS estimates and with fitting the form selected. Section 4 deals with likelihoods and noninformative priors corresponding to the fitted PDFs and the combined evidence that they provide. Section 5 presents results, and Section 6 discusses issues raised.

2. Statistical inference methodology

2.1 Subjective Bayesian inference and calibration of Bayesian probabilities

In the continuous case, from Bayes theorem (Bayes 1763) we can state conventionally that the (posterior) PDF, $p_{\theta}(\theta | \mathbf{y})$, for a parameter (vector) θ on which observed data \mathbf{y} depend, is proportional to the probability density of the data $p_{\mathbf{y}}(\mathbf{y} | \theta)$ (called the likelihood function, or just the

likelihood, denoted by $L(\theta)$, when considered as a function of θ , with \mathbf{y} fixed) multiplied by the density of a prior distribution (prior) for θ , $p_\theta(\theta)$:

$$p_\theta(\theta | \mathbf{y}) \propto p_y(\mathbf{y} | \theta) p_\theta(\theta) \quad (1)$$

(the subscripts indicating which variable density is for). The constant of proportionality is ascertainable since the posterior PDF must integrate to unit probability. Under subjective Bayesian interpretations, the prior (and hence the posterior) represent the investigator's degree of belief regarding possible parameter values; the prior and posterior reflect respectively relevant prior knowledge and that knowledge updated by the observed data. Both prior and posterior are personal to the investigator.

Bayesian credible intervals – i.e. uncertainty intervals taken from percentile points of the posterior cumulative distribution function (CDF) – and frequentist confidence intervals involve conceptually different kinds of probability, respectively epistemic (relating to knowledge) and aleatory (random) probabilities. However, to avoid Bayesian inference giving misleading results it is necessary that a prior be used that provides correct calibration of posterior probabilities to frequencies, at least approximately (Fraser et al. 2010). To explain this in more detail, consider a research center where every month a new experimental study is performed both by a Bayesian and a frequentist statistician. Each month t ($t = 1, 2, \dots, m$) they are given a new data set, unrelated to any previous data set, involving a parameter θ_t , and they agree on an appropriate statistical model for the data set. (To simplify the picture, we could also imagine, less realistically, that all the data sets refer to the same statistical model and the same parameter θ_t – the story below is valid irrespective of whether the data sets come from the same physical source or not.) They then analyze the data separately. The Bayesian outputs a 5-95% *credible interval* $[a_t, b_t]$. This means she believes that the true parameter value, θ_t , for the t -th data set is contained in $[a_t, b_t]$ with (epistemic) probability 90%; more specifically, it is smaller than a_t with probability 5%, and larger than b_t with probability 5%. The frequentist outputs a 5-95% *confidence interval* $[c_t, d_t]$. Such confidence intervals are constructed such that, no matter what the true parameter value θ_t is, the probability that it will be contained in $[c_t, d_t]$ is 90%, with a 5% probability of it being below c_t and a 5% probability of it being above d_t . These are aleatory probabilities, deriving from randomness in the data values. As a result, the frequentist is guaranteed to be (long-run) *calibrated*: in the long run (for large m), about 90% of the intervals he outputs will contain the true parameter value. On the other hand, whether the Bayesian's credible interval in month t is really a confidence interval in the frequentist sense depends on the prior she uses in month t . If she uses an informative prior, then this expresses a belief that certain parameter values are more likely than others, and if these beliefs are wrong then her credible interval in month t will not be a confidence interval; and if this happens at many t then she will not be calibrated. We say that *coverage* holds for Bayesian estimation (for a single experiment, e.g. at a particular t) if the credible interval it produces is also (approximately) a confidence interval. For most common single-parameter statistical models, there exist priors which guarantee this: use of a suitably noninformative prior will result in posterior PDFs and related credible intervals that at least approximately equate to confidence intervals (Bernardo 2009) and hence satisfy coverage. If a Bayesian always (at each t) uses priors guaranteeing coverage, then she is guaranteed to be calibrated in the long run.

As an example, consider a simplified case where in each month t , the statisticians obtain a sample of length n_t from a normal distribution with unknown mean θ_t and variance 1; denote the sample mean by $\hat{\theta}_t$. The standard 5-95% confidence interval for this model is the interval

$[\hat{\theta}_t - f_t, \hat{\theta}_t + f_t]$, where $f_t = 1.96/\sqrt{n_t}$. A Bayesian who uses a uniform prior on θ_t will output exactly

the same 5-95% credible interval, hence coverage is guaranteed – and indeed, a uniform prior is the standard noninformative prior for the normal family with fixed variance. However, a Bayesian may also use an informative prior for θ_t , for example, a normal with mean 0 and small variance; in that case the 5-95% interval will be much narrower and coverage cannot be guaranteed. If the Bayesian's prior beliefs are realistic and at many t 's we have a θ_t close to 0, then she will still be calibrated in the long run, but not otherwise. Of course, a Bayesian could use different informative priors at different t , and again, we would have no guarantee of calibration.

Priors with good coverage properties are generally viewed as desirable, and determining such priors is still a major topic in Bayesian inference; see for example Szabó et al. (2015). If, however, a Bayesian uses (whether intentionally or not) an informative prior, that is not overwhelmed by the data, then the Bayesian posterior density is unlikely to generate a credible interval that is also a confidence interval. Since the available data are insufficient to constrain ECS narrowly, informative priors generally exert a strong influence on ECS estimation, and one would like to use 'noninformative' priors wherever possible (we shall be more precise on what we mean by 'noninformative' below). Some subjective Bayesian studies probe sensitivity of results to choice of prior. However, whilst useful in revealing the extent of dependence on the prior, such exploration does not reveal which, if any, prior is suitably noninformative. Moreover, where the form of a noninformative prior differs greatly from beliefs regarding the relative probabilities of different parameter values, which is often the case, it may well not be included in the exploration.

2.2 Objective Bayesian inference and noninformative prior distributions

Unlike subjective Bayesian interpretations, objective Bayesian approaches deal primarily with inference in the absence of existing knowledge about parameter values, and do not involve a prior representing belief regarding them. The aim is for the results to be a function only of the data from which they are derived and the assumed statistical model, as for frequentist results. In order to satisfy this requirement using Bayesian methods, a "noninformative" prior, which is a weight function mathematically derived from the assumed statistical model and has no probabilistic interpretation, must be used (Bernardo 2009; Kass and Wasserman 1996). A few studies have estimated ECS using explicit objective Bayesian methods with noninformative priors (e.g., Lewis 2013a; Lewis 2014). In addition, where studies have used statistical methods that are not explicitly Bayesian, such as sampling or regression (e.g., Andronova and Schlesinger 2001; Gregory et al. 2002; Forster and Gregory 2006), typically they implicitly use a noninformative prior when viewed from a Bayesian perspective. Now, different 'noninformative', 'reference', 'objective' or 'default' priors often exist for the same given experimental situation, and one of the main criticisms of objective Bayesian inference is that it may well not be clear what 'noninformative' really means or whether a 'truly noninformative prior' really exists. Like many authors, we call priors 'noninformative' if they have good *coverage* properties: Bayesian credible intervals should be close to (or even indistinguishable from) frequentist confidence intervals whenever the data are actually sampled from any (arbitrary) distribution in the contemplated statistical model. A prototypical example noninformative prior is the uniform prior for the normal location statistical model mentioned above, where coverage is perfect: credible intervals precisely correspond to confidence intervals. For other statistical models there often exist priors with adequate, though usually not perfect, coverage. We thus use 'noninformative' simply as a (somewhat informal) definition and make no claims as to whether such priors are really noninformative or objective in any universal, more general sense.

In order to achieve good coverage, even with relatively weak data, a noninformative prior must vary depending on the problem-specific relationships the data values have with the parameters and on the data-error characteristics, and hence on the form of the likelihood function. To return to our previous example, if each data value y represents a measurement of a univariate parameter θ with

random error ε having a normal distribution with mean 0 and variance 1 (so that the data value itself is normally distributed with mean θ and variance 1) then the likelihood is proportional to $\exp[-(y - \theta)^2 / 2]$. As we saw before, in this case a uniform prior is noninformative for inference about θ . On the other hand, if the data value represents a measurement of the cube of θ , then the likelihood function is proportional to $\exp[-(y - \theta^3)^2 / 2]$, and it turns out that to be noninformative the prior must be proportional to θ^2 . Noninformative priors for parameters therefore vary with the experiment involved. Accordingly, if two studies estimating the same parameter using data from different experiments involve different likelihood functions, as is likely, they will give rise to different noninformative priors. This might appear to militate against using objective Bayesian methods to combine evidence in such cases. Using the appropriate, individually noninformative, prior, standard Bayesian updating would, inconsistently, produce a different result according to the order in which Bayes' theorem was applied to the two datasets (Kass and Wasserman 1996). While this fact has been used to criticize objective Bayes methods (Seidenfeld 1979), the issue can be avoided by applying Bayes theorem once only, to the joint likelihood function for the combination of the instrumental period and paleoclimate estimates, with a single noninformative prior being computed for inference therefrom, as was proposed by Lewis (2013b). It is also the proper way to proceed according to the *Minimum Description Length Principle*, an alternative approach to statistics that has its roots in information theory and data compression (Grünwald 2007, Chapter 11, Section 11.4.2), and it is the method that will be followed in this paper.

The first general method for computing noninformative priors (Jeffreys 1946) was devised using invariance arguments. Jeffreys prior is the square root of the (expected) Fisher information, or of its determinant where the parameter is a vector. Fisher information – the expected value of the negative second derivative of the log-likelihood function with respect to the parameters upon which the probability of the data depend – measures the amount of information that the data carries, averaged across possible data values, about unknown parameters at varying values thereof. Inference using a Jeffreys prior is invariant under transformations of parameter and/or data variables, a highly desirable characteristic. In the normal location family example above, Jeffreys' prior coincides with the uniform prior.

Reference analysis (Bernardo 1979; Berger and Bernardo 1992), which involves maximizing the (expected value of) missing information about the parameters of interest, is usually regarded as the gold-standard for objective Bayesian inference, as reference priors have the least influence on inference relative to the data, where 'influence' is measured in an information-theoretic sense. Reference analysis may be argued to provide an "objective" Bayesian solution to statistical inference problems in just the same sense that conventional statistical methods claim to be "objective": in that the solutions only depend on model assumptions and observed data (Bernardo 2009). In the univariate parameter continuous case, as here, Jeffreys prior is normally the reference prior, and it provides credible intervals that are closer to confidence intervals than with any other prior (Welch and Peers 1963; Hartigan 1965) – thus being noninformative in exactly the sense we desire.

2.3 Likelihood-ratio inference

Another way of objectively combining probabilistic evidence regarding ECS from two independent sources is to use frequentist likelihood-ratio methods, which directly provide estimated confidence intervals. A profile likelihood may be used when there are nuisance parameters to eliminate. The likelihood ratio is the ratio, at each parameter value, of the likelihood there to the maximum likelihood at any parameter value. Equivalently, it is the likelihood normalised to a maximum value of one. Likelihood-ratio methods are based on asymptotic normal approximations and produce frequentist confidence intervals that are exact only when the underlying distributions are normals or normals after transform, but typically provide reasonable approximations in other cases (Pawitan, 2001).

Independent likelihoods may be multiplicatively combined and a likelihood-ratio method applied to their product.

The standard basic signed root (log) likelihood ratio (SRLR) method is used here. The ECS range given in Allen et al. (2009) was based on this method, as were the graphical confidence intervals in Otto et al. (2013). Defining the log of the likelihood function $L(\theta)$ as

$l(\theta) \equiv \log[L(\theta)] \equiv \log[p_y(y|\theta)]$ with the datum y fixed, and $\hat{\theta}$ as the value which maximises $L(\theta)$, so that the log of the likelihood ratio $[L(\theta)/L(\hat{\theta})]$ can be written as $-[l(\hat{\theta})-l(\theta)]$, the SRLR statistic

$$r(\theta) = \text{sign}(\hat{\theta} - \theta)(2[l(\hat{\theta}) - l(\theta)])^{1/2}$$

is, asymptotically, distributed as $N(0, 1)$. Moreover, $-r(\theta)$ is a strictly increasing monotonic function of θ . It follows that an α upper confidence bound for θ is found by finding the value at which $\Phi[-r(\theta)] = \alpha$, and hence any two-sided (α_1, α_2) confidence interval constructed.

2.4 ECS inference methods in AR4 and AR5

Most estimated PDFs for ECS in AR5 and its predecessor report (AR4) were produced using subjective Bayesian methods, using either a prior for ECS that is intentionally informative (normally an 'expert' prior) or a uniform-in-ECS prior that cannot be interpreted as a noninformative prior in the standard sense of the word – it is typically informative, albeit unintentionally so. We will discuss issues with the uniform-in-ECS prior, since it is a popular choice, recommended, for example, by Frame et al. (2005).

Roe and Baker (2007), focussing on feedback analysis, showed that a uniform-prior-in-ECS is inconsistent with the common assumption that estimation errors in the total climate feedbacks f are normally distributed: under this normality assumption estimated ECS (which is proportional to $1/(1-f)$, with fractional uncertainty in the constant of proportionality being small relative to that in $(1-f)$) will have a skewed PDF with an upper tail that, although long, asymptotically declines with ECS^{-2} at high ECS. Roe and Baker (2007) implicitly used a uniform prior for the variable $(1-f)$ when estimating its PDF, which was noninformative given their assumption of normally distributed errors in f and thus a normal location statistical model. This contrasts with directly estimating ECS on the same assumptions using a Bayesian approach with a uniform prior for ECS, which would produce a PDF whose upper tail is asymptotically constant at high ECS. Use of a uniform prior was further criticized by Annan and Hargreaves (2011), who concluded that "the popular choice of a uniform prior has unacceptable properties and cannot be reasonably considered to generate meaningful and usable results"; see also the critical assessment of Frame et al. (2005) by Lewis (2014).

AR4 and AR5 discussed subjective Bayesian attempts to combine evidence as to ECS from different studies, each such study involving separate priors for each parameter being estimated. The investigator undertaking the combination (possibly as an exercise within the latest such study involved) chose an initial, subjective, prior for ECS, typically adopting that used in one of those studies. Priors for any other uncertain parameters common to two or more of the studies were treated likewise. Priors for any parameters specific to individual studies were as specified in the study concerned. Applying Bayes theorem, the data-derived joint likelihood function for one study was then multiplied by the joint prior, formed by multiplying all the individual priors relevant to that study, to produce a joint estimated posterior PDF. The estimated marginal PDF for ECS resulting from integrating out all other parameters was then used as the ECS prior for the next study, multiplied by the priors for any other parameters estimated in that study, and Bayes theorem applied again ("standard Bayesian updating"). The updating was repeated until a final marginal PDF for ECS was obtained. Any common parameters other than ECS were treated jointly with ECS until the final integration. Since multiplication and the integrating

out of unwanted parameters are commutative, this process is insensitive to the order in which studies are selected. However, as the likelihood functions are wide, results are sensitive to the choice of prior. In the absence of other, independent, observational evidence being available there are no rules to guide the choice of prior.

2.5 ECS inference methods in this study

The main objective of this study is to present practical and objective methods for combining, using alternative objective Bayesian and frequentist statistical frameworks, observationally-based estimates of ECS in the simple case where they are based on independent evidence and no other common parameter is being estimated. The selected ECS estimates based respectively on evidence from the instrumental period and on paleoclimate evidence satisfy these requirements. Both an objective Bayesian method based on Jeffreys prior and the SRLR likelihood-based frequentist method are used to combine these estimates.

The statistical methods employed in this study require that suitable parameterized standard form distributions be fitted to the actual PDFs (or representative percentile points) for each of the available independent ECS estimates. The standard form used – which need not necessarily be the same for each ECS estimate – is selected for its ability to meet three requirements:- first, that it can provide an accurate approximation to the actual PDF with a suitable choice of a small number of parameters; secondly, that its functional form is compatible with the source of uncertainties in the physical problem being considered; and thirdly, that the fitted PDF can be conveniently represented as the product of an identifiable likelihood function and a prior distribution for ECS that is noninformative, given that likelihood function. This last characteristic of “separability” is a prerequisite for the present application.

3. Selection and fitting of a parameterized distribution to match ECS estimates

3.1 Selection of a suitable parameterized distribution

The concept of ECS relates changes in GMST to changes in radiative forcing. Effective radiative forcing, the change in the Earth's (top-of-atmosphere) net downward radiative flux imbalance caused by an external driver of climate change, once rapid adjustments are complete, is a measure of forcing closely related to its impact on GMST. In reality CO₂ is never the sole driver of climate change, but as explained (Section 1), ECS may be estimated as the ratio of the change in GMST (ΔT) to the change (ΔF_n) in aggregate ERF normalised (as indicated by the subscript n) by division by $F_{2\times\text{CO}_2}$, the ERF from a doubling of CO₂ concentration. Since fractional uncertainty in $F_{2\times\text{CO}_2}$ is in practice generally small relative to that in the change in ERF, it can usually be ignored or incorporated within the uncertainty estimate for ΔF_n . Unlike for paleoclimate ECS estimates relating to changes in equilibrium state, ECS estimates based on disequilibrium instrumental period data involve, in some form or other, an adjustment to the forcing change to reflect the change in the Earth's radiative imbalance or its counterpart, the rate of accumulation of heat in the climate system (ΔQ). The ratio $\Delta T / \Delta F_n$ used to estimate ECS from equilibrium changes becomes, for non-equilibrium changes, $\Delta T / (\Delta F - \Delta Q)_n$. The changes in the numerator and denominator factors in these ratios are uncertain, and errors in the estimates thereof are in most cases assumed to be independent. It turns out that making the common assumption that they are Gaussian distributed generally works well. It is standard to use uniform priors to infer posterior distributions for variables that are estimated with errors that are normally distributed or otherwise independent of the value of the variable; these are 'location parameter' situations, for which all principles agree that a uniform prior is noninformative (Lindley 1958, Lindley 1972 section 12.5, Gelman et al. 2004 page 64). The foregoing considerations suggest that it is reasonable to think of estimated PDFs for ECS as posterior distributions of the ratio of two normally-distributed variables. Assuming the data consists of single observations, each subject to a normally distributed independent

error, of the two unknown variables, and assuming noninformative (Jeffreys) uniform priors for estimating the 'true' (error free) values of these variables, we end up with a posterior for ECS of a particular constrained form that is determined by just four numbers – the standard deviations of the two normals (assumed fixed) and the observed values of the variables. Since the denominator distribution may without loss of generality be taken to have a unit mean, only three parameters are required to specify such a ratio-normal posterior distribution. The simple analytical approximation to such ratio-normal posterior distributions provided by RS 93 is both very accurate, provided the denominator normal distribution has minimal probability at values below zero, and has a form which can readily be factored into an identifiable likelihood function and an appropriate noninformative prior. RS 93 used their ratio-normal approximation posterior quite successfully in another practical application (whale censusing).

Formally, our *ratio-normal model* assumes that θ (here ECS) can be represented probabilistically as $\theta = \psi_1 / \psi_2$ where ψ_1 and ψ_2 are two parameters estimated with errors that are independent and normally-distributed with known standard deviations σ_1 and σ_2 , and having observed values $\tilde{\psi}_1$ and $\tilde{\psi}_2$. Define $V = \sigma_1^2 + \theta^2 \sigma_2^2$ and a univariate pivot variable

$$z(\theta | \tilde{\psi}_1, \tilde{\psi}_2) = -\frac{\tilde{\psi}_1 - \theta \tilde{\psi}_2}{\sqrt{V(\theta)}}$$

RS 93 show, implicitly using an objective Bayesian approach (with uniform, i.e. Jeffreys priors for ψ_1 and ψ_2), that on this basis the posterior distribution of $z \sim N(0,1)$ to a good approximation, provided that $\tilde{\psi}_2 / \sigma_2$ is sufficiently large for the normal distribution to assign negligible probability to negative values of ψ_2 (and hence of θ), so that the RS93 approximation is appropriate. That being the case, the non-monotonicity in the relationship between z and θ that arises when θ becomes sufficiently negative is inconsequential. On that basis (which is assumed hereafter unless otherwise stated) RS 93 derive an approximate posterior density for θ by changing variable, giving:

$$p_\theta(\theta | D) = \frac{(\tilde{\psi}_2 \sigma_1^2 + \tilde{\psi}_1 \sigma_2^2 \theta)}{V(\theta)^{3/2}} \phi \left[-\frac{(\tilde{\psi}_1 - \theta \tilde{\psi}_2)}{\sqrt{V(\theta)}} \right] \quad (2)$$

where $\phi[\cdot]$ is the standard normal density function. Note that the exact posterior distribution approximated by (2) is that for a parameter that is the ratio of two other parameters, (ψ_1, ψ_2) , whose joint posterior is bivariate normal with zero correlation.

It is shown in Lewis (in press) that the posterior distribution (2) can be uniquely factored into a likelihood function and a prior that is noninformative for inference under the RS93 approximation, being respectively:

$$L(\theta) = \phi \left[-\frac{(\tilde{\psi}_1 - \theta \tilde{\psi}_2)}{\sqrt{V(\theta)}} \right] \quad (3)$$

and

$$\pi_\theta^{JP}(\theta) = \frac{(\tilde{\psi}_2 \sigma_1^2 + \tilde{\psi}_1 \sigma_2^2 \theta)}{V(\theta)^{3/2}} \quad (4)$$

Although not Jeffreys prior for an exact ratio-normal model, the prior (4) is Jeffreys prior for inference based on the pivotal z -space used in the RS 93 approximation.

Frequentist analysis is also performed using the SRLR method on the likelihood function (3). Provided the RS 93 approximation is appropriate, the RS 93 likelihood will be a transformed normal, for which the SRLR method provides exact confidence intervals.

3.2 Validation of the ability of the chosen distribution to match ECS estimates

In order to validate the ability of the RS93 posterior accurately to match observationally-based estimated posterior PDFs for ECS, we have fitted parameterized RS93 posterior PDFs to the PDFs for eight of the ECS estimates represented in Figure 10.20b of AR5. The selected cases provide coverage across a range of data and representative methodologies, primarily involving approaches that explicitly or implicitly use noninformative priors but also including subjective Bayesian approaches. Results and relevant information are shown in Figure 2. Taking for convenience $\tilde{\psi}_2 = 1$, the fitting process seeks values for $(\tilde{\psi}_1, \sigma_1^2, \sigma_2^2)$ by minimizing a measure of the fitting error. Fitting using just the medians and 17–83%, 5–95% and 2.5–97.5% ranges of the ECS estimates was generally found to be satisfactory. Where the amount of probability lying in the tail of the distribution above an ECS of 10 K removed by the truncation and normalization employed in AR5 Figure 10.20b could be identified, the Figure 10.20b PDF was rescaled to its pre-normalization level before fitting was undertaken. This applies to Forster and Gregory 2006 (uniform prior in ECS case) and Gregory et al. 2002; in other cases published information does not enable ready quantification, but very little probability appears to have been removed.

Figure 2 shows that in all but one case the fitted ratio-normal approximation PDF very closely approximates the original PDF, both visually and in terms of the respective 5%, 17%, 50%, 83% and 95% points of the corresponding CDFs, shown by the box plots. Moreover, the frequentist confidence intervals obtained, from the deduced likelihood functions, using the SRLR method almost exactly match the credible intervals derived from the fitted PDFs, confirming that the way in which the fitted PDF has been factored into an inferred likelihood function and a noninformative prior is correct. The fitted ratio-normal PDF, and the box plots, are noticeably different from the originals only in the case of the Forster and Gregory (2006) PDF in panel c, as used in AR4. In AR4, the PDF implied by that paper's published results (which equated to one from use of a noninformative prior) was multiplied by a factor of ECS^2 , to convert it from a uniform-in-feedback prior to a subjective Bayesian uniform-in-ECS prior basis. The AR4 version of the PDF was too far removed from a PDF corresponding to the uncertainties in the study for it to be closely matched by a ratio-normal distribution, and the RS93 implied likelihood and noninformative prior are unreliable. However, an excellent fit is obtained to the original, uniform-in-feedback prior basis, PDF (panel d).

The point of this exercise is to show that RS93 ratio-normal approximation PDFs can provide close fits to PDFs for ECS obtained in a wide range of studies, supporting the appropriateness of this parametric form. It is not suggested that, where the original PDF was based on a prior that was informative, a good fit by an RS93 PDF that equates to the product of a likelihood function and a prior that is noninformative in relation thereto would enable objective inference to be obtained or that the likelihood function implicit in the fitted RS93 PDF will necessarily approximate that in the original study. In some cases a likelihood function and noninformative prior used in a study may be available as well as a PDF for ECS, enabling the information from the study to be combined objectively with other information without fitting an RS93 PDF. However, such cases appear rare: out of the studies considered Forster and Gregory (2006) and Lewis (2013a) appear to be the only ones for which a likelihood function and noninformative prior for ECS can be obtained. The RS93 fit to the Forster and Gregory (2006) PDF for ECS well matches the shape of the original likelihood function and noninformative prior as well as that of the PDF. For Lewis (2013a) such a comparison is difficult, as its likelihood function and prior were trivariate.

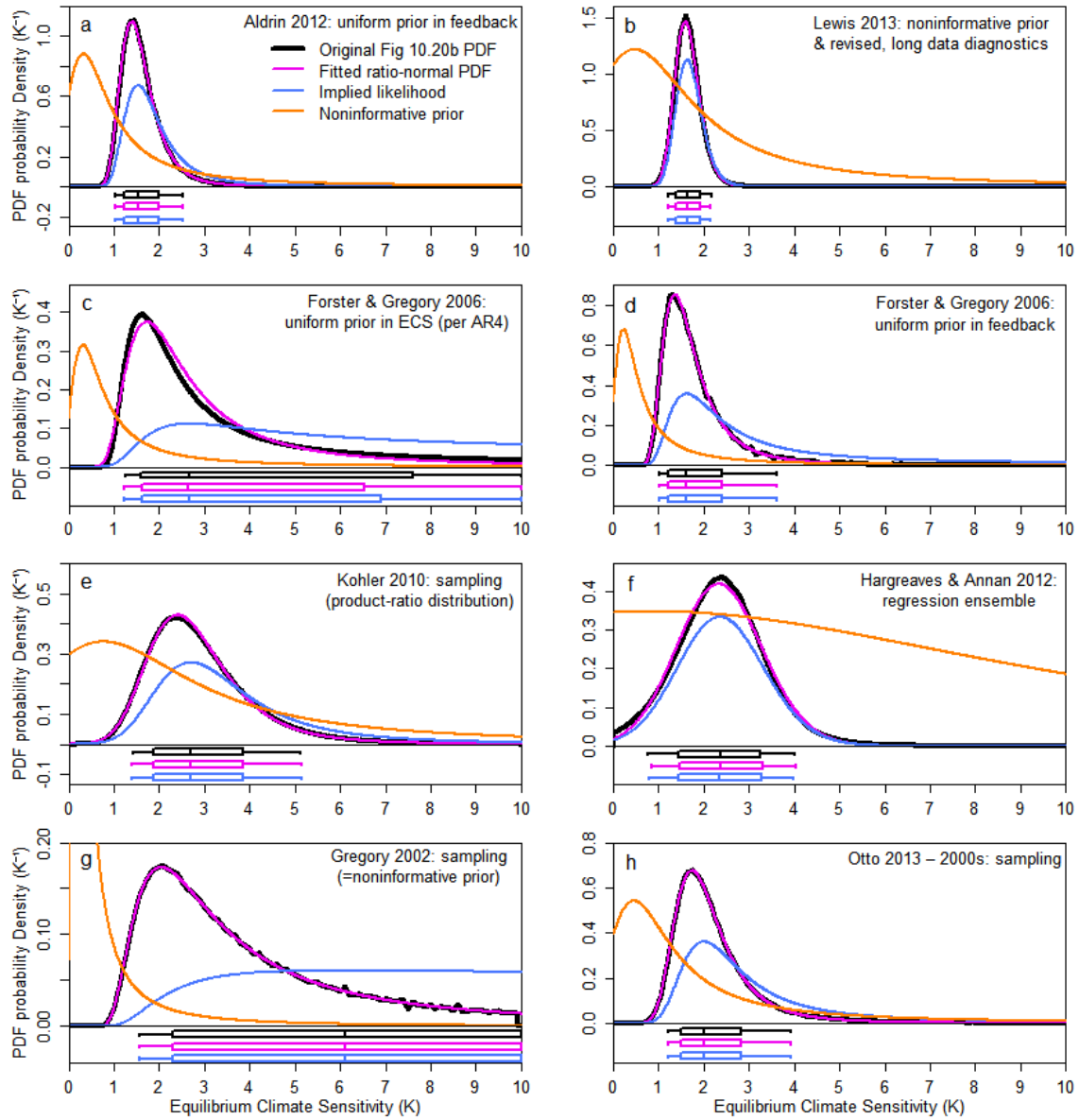


Figure 2. PDFs for eight of the observationally-based ECS estimates featured in Figure 10.20b of AR5 (black lines – original; magenta lines – fitted RS93 posterior (ratio-normal), along with the corresponding implied likelihood (blue lines) and noninformative prior (orange lines). The panels show: (a) Aldrin et al. (2012) using a uniform in feedback prior (a $1/\text{ECS}^2$ prior for ECS); (b) Lewis (2013a) objective Bayesian results using a noninformative Jeffreys prior and revised diagnostics; (c) Forster and Gregory (2006) using a uniform in ECS prior, being the basis on which the results per that paper were restated in AR4; (d) Forster and Gregory (2006) using a uniform in feedback prior (a $1/\text{ECS}^2$ prior for ECS) as per the regression-derived results in that paper; (e) Köhler et al. (2010), based on sampling the ratio of the product of two normals to a third normal (effectively using a noninformative prior); (f) Hargreaves and Annan (2012), estimated from an ensemble of regressions; (g) Gregory et al. (2002), based on sampling the ratio of two normals (effectively using a noninformative prior); (h) Otto et al (2013), based on data from the 2000s and a complex sampling basis that effectively uses a noninformative prior. The scaling of the likelihoods and priors is arbitrary. The box plots indicate boundaries for the percentiles 5–95 (vertical bar at ends), 17–83 (box-ends), and 50 (vertical bar in box). The black and magenta box plots show Bayesian credible intervals calculated from respectively the original posterior PDFs (downscaled in panels c and g to allow for significant off-graph probability that lies above an ECS of 10 K) and the fitted ratio-normal PDF. The blue box plots show confidence intervals derived, using the SRLR method, from the inferred likelihood functions.

3.3 Fitting parameterised PDFs to the instrumental and paleoclimate ECS estimates

Having demonstrated the ability of the RS93 approximation to a ratio-normal PDF accurately to represent estimated PDFs for ECS over a range of examples, we fit it to the preferred results ECS estimate from LC15, the primary selected representative instrumental period study, using the full LC15 PDF. The fitted PDF is not normalised to unit probability over the fitted range; since a small amount of probability lies above that range doing so would distort the derived CDF percentile point ECS values. The fitting process compensates for the AR5 forcing distribution being slightly skewed (which means the denominator distribution normality assumption in RS 93 is not completely accurate) by assigning almost all uncertainty to the denominator distribution, which increases the skewness of the fitted distribution. Figure 3a shows the fitted PDF to be extremely close to the original. Table 1 shows that the ECS values for the fitted distribution match those of the original within ~ 0.05 K at 5%, 17%, 50%, 83% and 95% CDF percentiles. Moreover, the amount of probability lying above 10 K is only marginally smaller for the fitted PDF than the original: 1.0% vs 1.4%. Accordingly, the fit may be regarded as very satisfactory.

For Oa13, the RS93 fit is as obtained in Section 3. That PDF is an almost perfect fit (all ECS values match within 0.01 K up to the 95% CDF point), as shown in Figure 3a by its exact coincidence with the original Oa13 PDF.

Whilst negative ECS values can reasonably be ruled out on the grounds that they imply cooling in response to a positive forcing and are inconsistent with understanding of the energy balance of the climate system, observationally-based paleoclimate ECS estimates will correctly show some probability lying below 0 K where the fractional uncertainty in their temperature change estimate is substantial. This is evident, for instance, with Hargreaves and Annan (2012), where their ECS estimate PDF remains significantly positive at 0 K. It is also the case for the AR5 paleoclimate range (as 10–90%) based estimate used here. In contrast, the AR5 instrumental period energy budget based ECS estimates have essentially zero probability below 0 K, as the fractional uncertainty in the temperature change involved is relatively low, even allowing for internal variability. Therefore, when the energy budget and paleoclimate estimates are combined negligible probability lies below 0 K.

Fitting an RS93 posterior distribution to the AR5 paleoclimate evidence based 2.75 K median estimate and 1–6 K range, treated as 10–90%, is straightforward, since with three points and three free parameters the fit is unique. The fitted PDF has very small but non-negligible probability situated below 0 K and above 10 K. In order to accommodate probability at negative and very high ECS values, the fitted PDF is computed over an ECS range of -2 K to 100 K, with a 0.05% probability mass added at -2 K to adjust for probability below that ECS level, but is not normalized to unit probability. The fitted probability at ECS values between -2 K and 0 K arises almost entirely as a result of the numerator normal distribution taking negative values and is therefore correctly attributable to ECS values below zero. Figure 3a shows the fitted distribution's PDF. The three fitted CDF percentile points are within 0.01 K of the specified values.

4. Priors and likelihoods for fitted distributions and combined evidence

4.1 Deriving a noninformative prior for inference from the combined evidence

The priors for the separate-evidence cases – LC15 and Oa13 AR5 instrumental period energy budget based estimates, and AR5 paleoclimate evidence 1–6 K range based estimate using a 2.75 K median – are Jeffreys priors for the fitted PDFs, derived using (4). Jeffreys prior equals the square root of Fisher information, and the Fisher information for two independent sets of data is the sum of that for the individual data sets. Accordingly, Jeffreys prior for inference from two independent sources of observationally-based evidence combined is the sum in quadrature of the Jeffreys priors relating to

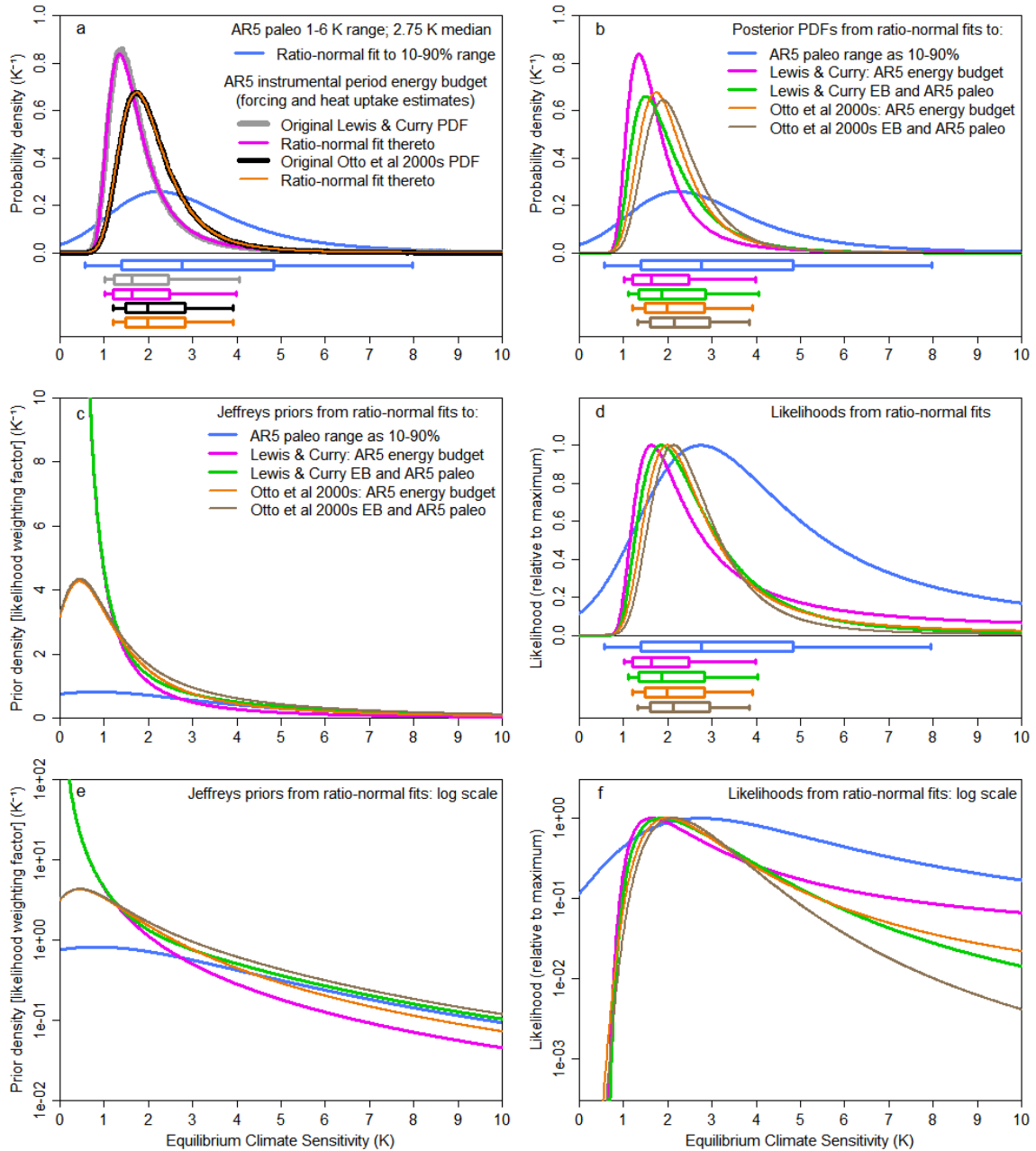


Figure 3. Inference based on PDFs for ECS selected to represent AR5 paleoclimate evidence and evidence derived from the AR5 instrumental period forcing and heat uptake estimates using an energy budget approach. The paleoclimate PDF is an RS93 posterior (ratio-normal approximation) fit to the AR5 paleoclimate 1–6 K range, taken as 10-90%, with a 2.75 K median. The instrumental period energy budget PDFs are RS93 posterior fits to the LC15 and, alternatively, the Oa13 best-constrained (2000s) results. The panels show: (a) RS93 posterior fits to the AR5 paleoclimate range (blue) and to the two AR5 energy budget PDFs: LC15 (magenta; original PDF in grey) and Oa13 (orange; original PDF in black); (b) posterior PDFs for the combined energy budget and paleoclimate evidence (green line – based on LC15; brown line – based on Oa13) being the normalized products of the corresponding likelihoods and Jeffreys priors, and, for comparison, the fitted posterior PDFs shown in (a); (c) the noninformative Jeffreys priors pertaining to each of the posterior PDFs in (b); (d) likelihood functions pertaining to each of the posterior PDFs in (b); (e) as for (c) but with a logarithmic y-axis scale; and (f) as for (d) but with a logarithmic y-axis having the same scale as in (e). The box plots indicate boundaries, to the nearest grid value, for the percentiles 5–95 (vertical bar at ends), 17–83 (box-ends), and 50 (vertical bar in box). The box plots in panels (a) and (b) show Bayesian credible intervals calculated from posterior PDFs and allow for probability that lies outside the x -axis ECS range. The box plots in panel (d) show confidence intervals derived from the corresponding likelihood functions using the SRLR method.

each of them separately. Denoting those evidence sources A and B, where the RS93 ratio-normal approximation is used the combined-inference prior is thus:

$$\pi_{\theta_{AB}}^{JP}(\theta) = \sqrt{\frac{(\tilde{\psi}_{A2}\sigma_{A1}^2 + \tilde{\psi}_{A1}\sigma_{A2}^2\theta)^2}{(\sigma_{A1}^2 + \theta^2\sigma_{A2}^2)^3} + \frac{(\tilde{\psi}_{B2}\sigma_{B1}^2 + \tilde{\psi}_{B1}\sigma_{B2}^2\theta)^2}{(\sigma_{B1}^2 + \theta^2\sigma_{B2}^2)^3}} \quad (5)$$

It was shown in Lewis (in press) that combining on this basis priors of the form in (4) relating to two independent sources of evidence, A and B, each represented by a RS93 posterior PDF, produces inference with excellent probability matching properties. That is, when tested by repeated sampling from the two pairs of normal distributions involved, Bayesian credible intervals derived from using such a prior with the two likelihood functions of the form in (3) multiplicatively-combined, prove to be almost exact confidence intervals. Lewis (in press) also shows that the (standard, but generally incorrect) method of Bayesian updating, simply using a noninformative prior for one of the two likelihoods, has substantially worse probability matching properties compared to the method described above; see Section 6 for more discussion.

4.2 Characteristics of the different priors

Figure 3c shows the prior relating to each of the three separate-evidence fitted RS93 posterior PDFs and for the two combined-evidence cases. The Jeffreys priors for the instrumental period and paleoclimate RS93 posterior fits have different shapes, more easily seen when plotted on a log scale (Figure 3e). At very high ECS values, in all cases the prior declines as ECS^{-2} . Very high ECS values are far more likely to arise from the denominator normal being very small – close to zero – than from the numerator normal being extremely large. Small changes in the denominator normal as it approaches zero translate into very large changes in ECS, so the data is increasingly uninformative about ECS, even if the likelihood is still significant. Accordingly, the Jeffreys prior declines as ECS increases, down-weighting the likelihood. Put another way, because of their reciprocal relationship, a small region of the denominator normal CDF near zero (representing near zero ΔF_n or $(\Delta F - \Delta Q)_n$ values) maps into a large range of high values for ECS. That requires the prior to decline with ECS^{-2} (the same as the Jacobian factor used to convert PDFs when a variable is changed to its reciprocal) in order to preserve matching of probability between data space and parameter space. As explained in section 3.3, the optimized ratio-normal approximation fit to the LC15 instrumental period ECS estimate gives almost zero standard deviation to the numerator normal distribution (which represents ΔT). As a result, in this case at low ECS levels the prior continues to rise nearly with ECS^{-2} as ECS declines, before peaking when ECS is just above zero. Since the likelihood function declines much more rapidly as ECS approaches zero than the prior increases, the fitted posterior PDF nevertheless tracks the sharply declining original.

The Jeffreys prior for the paleoclimate-evidence based fitted PDF declines much more gently with ECS at moderate ECS values than do the instrumental period fitted PDFs, reflecting the relatively greater informativeness of paleoclimate evidence over this range.

At low ECS levels, the priors for inference from the combined instrumental period and paleoclimate evidence are dominated by the high Jeffreys priors from the instrumental period evidence, but as ECS rises and those priors declines rapidly – reflecting increments in ECS corresponding to increasingly smaller areas in data-space – the combined-evidence priors deviate towards the paleoclimate prior, in each case asymptoting to a modest multiple of it at high ECS values.

4.3 Likelihoods

Figure 3d shows (normalized to unit maximum) the likelihood functions given by (3) that correspond, using the relevant Jeffreys prior, to the fitted RS93 posterior distributions for the LC15 and Oa13 instrumental period energy budget estimates and for the RS93 posterior distribution that matches the

PDF concerned and relevant characteristic	RS93 numerator normal distribution		RS93 denominator normal distribution		ECS at 5% CDF or CI point (K)	ECS at 17% CDF or CI point (K)	ECS at 50% CDF or CI point (K)	ECS at 83% CDF or CI point (K)	ECS at 95% CDF or CI point (K)
	Mean	Standard deviation	Mean	Standard deviation					
Original LC15	-	-	-	-	1.05	1.25	1.64	2.45	4.05
Instrumental period: fitted to LC15 PDF	1.635	0.000	1 (fixed)	0.359	1.05 <i>1.05</i>	1.2 <i>1.2</i>	1.64 <i>1.64</i>	2.5 <i>2.5</i>	4.0 <i>4.0</i>
Fitted LC15 adjusted for time-varying E_{ffCS}	1.787	0.193	1 (fixed)	0.359	1.1 <i>1.1</i>	1.3 <i>1.3</i>	1.79 <i>1.79</i>	2.75 <i>2.75</i>	4.40 <i>4.40</i>
Instrumental period: fitted to Oa13 PDF	1.991	0.321	1 (fixed)	0.288	1.2 <i>1.2</i>	1.5 <i>1.5</i>	1.99 <i>1.99</i>	2.85 <i>2.85</i>	3.9 <i>3.9</i>
Paleo: fitted to range as 10-90% & median	2.750	1.317	1 (fixed)	0.361	0.55 <i>0.55</i>	1.4 <i>1.4</i>	2.75 <i>2.75</i>	4.85 <i>4.85</i>	7.95 <i>7.95</i>
Combined fitted LC15 + paleo 10-90%	-	-	-	-	1.1 <i>1.1</i>	1.35 <i>1.35</i>	1.88 <i>1.86</i>	2.85 <i>2.85</i>	4.05 <i>4.05</i>
Combined fitted LC15 adjusted for varying E_{ffCS} + paleo 10-90%	-	-	-	-	1.2 <i>1.2</i>	1.45 <i>1.45</i>	2.02 <i>2.01</i>	3.0 <i>3.0</i>	4.2 <i>4.2</i>
Combined fitted Oa13 + paleo 10-90%	-	-	-	-	1.3 <i>1.3</i>	1.6 <i>1.6</i>	2.14 <i>2.13</i>	2.95 <i>2.95</i>	3.85 <i>3.85</i>
<i>Bayesian update PDFs</i>									
LC15 PDF fit updated by paleo fit likelihood					1.1	1.3	1.77	2.6	3.6
Paleo PDF fit updated by LC15 fit likelihood					1.25	1.55	2.20	3.3	4.65

Table 1. Characteristics of all the PDFs in terms of one-sided credible intervals (CDF points) derived from fitted separate-evidence and estimated combined-evidence posterior PDFs and (in *italics*) one-sided confidence intervals (CIs) derived using the SRLR method from the likelihoods pertaining to those PDFs, along with the optimized parameter values for the fitted PDFs. The ECS values for all percentage points other than the median (50%) have been rounded to the nearest 0.05 K. Median values given in the text are, if their 50% points differ, averaged across the two methods.

AR5 paleoclimate evidence based 1–6 K range, treated as 10–90%, using a 2.75 K median. Since the errors involved in the instrumental period and paleoclimate ECS estimates are assumed to be independent, the likelihood functions for their combined evidence is given by the product of their separate likelihood functions. These multiplicatively-combined likelihood functions are shown in Figure 3d, the LC15 and Oa13 instrumental period likelihoods being separately combined with, in each case, the AR5 paleoclimate 1–6 K range based likelihood. The box plots in Figure 3d show the median and estimated 17–83% and 5–95% confidence intervals derived from each likelihood using the SRLR method. For the non-combined fitted PDFs, all the SRLR likelihood-ratio derived confidence interval points agree closely with the corresponding posterior CDF credible interval points shown in Figure 3a, confirming that the correct likelihoods have been inferred. For the two instrumental period estimates,

where there is no probability at below 0 K to degrade the RS93 fit, the match is effectively exact: at all percentage points the credible interval and confidence interval ECS values are identical to three decimal places. For the AR5 paleoclimate range based distribution, the match is within 0.02 K up to the 95% point (7.95 K).

Figure 3f shows the likelihoods plotted on the same log scale as for the priors in Figure 3e. Since Bayes theorem, (1), involves addition rather than multiplication when the terms are replaced by their logarithms, the relative influences of the prior and likelihood on each of the posterior PDFs may be appraised by comparing the lines of the relevant color in Figure 3e and 3f.

5. Results

Figure 3b shows the posterior PDFs for the combined instrumental period and paleoclimate evidence cases, derived from the product of the corresponding Jeffreys priors and likelihoods. These PDFs have been normalized to unit probability over the fitted range, which captures essentially all probability. They produce a 5–95% range for climate sensitivity of 1.1–4.05 K, median 1.87 K, based on the primary, LC15, fitted distribution. Using the alternative Oa13 fitted distribution gives a 5–95% range of 1.3–3.85 K, with a median of 2.14 K.

Figure 3b also replicates, for comparison, the fitted posterior PDFs shown in Figure 3a for the separate ECS estimates derived from the LC15 and Oa13 AR5 instrumental period energy budget and the AR5 paleoclimate evidence based estimates. These PDFs equate to the normalised product of the corresponding Jeffreys priors and likelihoods shown in Figures 3c and 3d.

The combined-evidence PDFs are largely dominated by the region of overlap of the two input datasets, and may be viewed as either an update of the paleoclimate evidence using instrumental period energy budget data or vice versa. Viewed from the first perspective, the instrumental data greatly sharpens the paleoclimate data, raising the 5% point but bringing down the median, 83% and 95% points. Viewed from the second perspective, the paleoclimate evidence increases the median and 83% point of the instrumental evidence – more so for LC15 than Oa13 – reflecting the paleoclimate estimate having a much higher median and upper tail. That uncertainty can increase despite additional knowledge is well-established (Hannart et al. 2013). The left hand tail of the combined-evidence PDFs is largely controlled by the extremely sharp cutoff below 1 K observable in both energy-budget study fitted (and actual) PDFs. The combined-evidence 95% point is a little higher than for the instrumental period energy-budget estimate alone for LC15, but a little lower for Oa13, in both cases representing a substantial reduction from the estimate based on the paleoclimate data alone. For both LC15 and Oa13, the upper tails of the combined evidence distributions beyond the 95% point decline much faster than that for the instrumental period energy budget estimate alone.

It is also relevant to compare, for the combined-evidence cases, the confidence interval percentage points from using the SRLR method on their multiplicatively-combined likelihood functions with the credible interval percentage points derived from the Bayesian posterior PDFs. The box plots in Figure 3b and 3d shows the ECS values at 5%, 17%, 50%, 83% and 95% points for the respective methods. It can be seen that agreement between results from the two fundamentally different statistical approaches is very close at all five percentile points. Table 1 gives the ECS values at the 5%, 17%, 50%, 83% and 95% CDF points (tops of one-sided credible intervals) for all the PDFs, and for the corresponding one-sided confidence intervals derived using the SRLR method for all the fitted and combined likelihoods, along with the optimized parameter values for the fitted PDFs.

The forcing and heat uptake estimates given in AR5, used in LC15, ended in 2011. Strong warming has occurred since then, and ocean heat uptake has increased. We have updated the LC15 main results by extending the final period to 1995–2015, using primarily observational data for post-2011 changes in forcing and heat uptake and version 4 of the HadCRUT4 GMST dataset

(supplementary material: S1). Although the updated transient climate response estimate (median 1.34 K, 5–95% range 0.9–2.45 K) is virtually unchanged, the updated ECS estimate has a slightly higher median of 1.75 K, reflecting greater estimated ocean heat uptake, and is worse constrained above (5–95% range 1.1–4.5 K). Using this updated ECS estimate, the median of the resulting combined-evidence posterior distribution is 0.1 K higher than when based on the non-updated LC15 fitted PDF, whilst the 95% point is 0.15 K higher. The latter increase is much less than the 0.4 K increase in the LC15 fitted PDF 95% point caused by the updating, reflecting the fact that the paleoclimate evidence contributes more to constraining the upper tail of the combined ECS estimate when the upper tail of the instrumental evidence is less well constrained.

Paleoclimate ECS estimates, except where based on changes during the Holocene, normally reflect differences in equilibrium climate states. Energy budget ECS estimates, however, reflect climate feedback strength over the instrumental period, and thus strictly are of effective climate sensitivity (E_{effCS}) rather than ECS. In many AOGCMs, E_{effCS} is time-varying and starts off lower than ECS. To investigate sensitivity of the combined-evidence ECS estimates to the possibility that ECS similarly exceeds E_{effCS} in the real world, the LC15 fitted distribution is divided by a $N(0.925, 0.10)$ distribution and a revised RS93 distribution fitted. A $N(0.925, 0.10)$ distribution is considered to be representative of the relationship in current generation AOGCMs between E_{effCS} derivable from the instrumental-period forcing profile and ECS as usually diagnosed (supplementary material: S2). The median and 95% point of the resulting combined-evidence posterior distribution are both 0.15 K higher than when based on the unadjusted LC15 fitted PDF.

The supplementary material includes (S3) a comparison of the results obtained here with those obtained using a shifted log- t distribution as the functional form to fit the PDFs, instead of the RS93 posterior based on the ratio-normal model. This is done (a) to underline the important point that the concepts demonstrated in this paper are not restricted to the use of a single functional form, (b) to demonstrate the superiority of the RS93 ratio-normal approximation in this particular instance and (c) to provide a direct comparison of the final results against a distribution that has been used to represent ECS estimates and other skewed PDFs by previous authors.

6. Discussion

We have combined independent evidence about ECS with different uncertainty characteristics, each representable by a ratio-normal distribution, using an objective Bayesian method that has been shown elsewhere to provide accurate probability matching when combining such evidence. Previous studies have used subjective Bayesian approaches, which almost certainly involved priors that led to results different from those implied by the observational data used. If, for this present study, inference from the combined evidence had been performed using a uniform rather than noninformative prior for ECS, but with the same likelihood functions derived from the fitted ratio-normal approximation RS93 posteriors, the 95% uncertainty bound (CDF point) for ECS from the combined evidence would have been far higher, at 13.45 K. The median estimate would also have been substantially higher, at 2.58 K. This offers a clear example of a subjective informative prior greatly distorting inference.

Using standard Bayesian updating to combine instrumental and paleoclimate evidence produces results that vary according to ordering (Table 1, bottom section) and disagree with those obtained using the objective Bayesian method. Updating the LC15 fitted RS93 posterior by multiplying it by the likelihood function inherent in the RS93 fit to the paleoclimate evidence gives a median ECS estimate of 1.77 K and 95% bound of 3.6 K. Reversing the ordering, taking the PDF from the paleoclimate evidence and updating it with the likelihood pertaining to the LC15 fitted distribution, increases the median to 2.20 K and the 95% bound to 4.65 K. In neither case is inference close to that from the objective method (1.88 K median, 4.05 K 95% bound).

The probability matching of posterior PDFs derived by combining observationally-based evidence involving two different likelihood functions using a prior that is the sum in quadrature of the separate-evidence Jeffreys priors, has also been tested and found to be accurate in cases involving very different probability distributions from those considered here (Lewis 2013a, Supplemental Information; Lewis 2013b).

Both the objective Bayesian and likelihood-ratio methods used here require likelihood functions pertaining to each source of evidence. It is therefore highly desirable for probabilistic results presented by scientific studies to include full (joint) likelihood functions, as well as joint posterior PDFs. In this case, despite likelihood functions being unavailable, it proved possible to estimate them by fitting ratio-normal approximation distributions to the posterior PDFs or percentile points available, but often such a route would not be available.

The approach used here requires that the uncertainties involved in the two sources of evidence are independent. Combining different evidence for ECS from the instrumental period is more challenging because, given the influence of climate system internal variability, errors in different lines of evidence are unlikely to be fully independent. Some instrumental period studies that use multiple data variables (Forest et al. 2006, Lewis 2013a) have explicitly sought to address independence issues, using optimal fingerprint techniques based on covariances estimated from unforced control runs of AOGCMs to transform data variables into an approximately conditionally-independent set of variables, despite imperfect representation of internal variability. In suitable cases information derived from such AOGCM control runs might likewise enable correlated variables from different studies to be so transformed, and thus widen the scope for combining evidence regarding ECS. However, even if successful, the observed changes over the instrumental period only represent a single realisation of internal climate variability, and some principal uncertainties likely affect all aspects of the resulting ECS estimate.

Where other uncertain parameters have been estimated alongside ECS by different sets of data, it is necessary to combine evidence for all common parameters jointly and only seek a marginal distribution for ECS at the final stage. That is potentially relevant where combining instrumental and non-equilibrium recent paleoclimate evidence (Hegerl et al. 2006), adding complexity, but when the paleoclimate period concerned is more distant and changes are quasi-equilibrium ECS is typically the only common constrained parameter.

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