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Semidefinite Programming Bounds for Stable Sets and Coloring

Monique Laurent, CWI, Amsterdam

Joint work with

Nebojša Gvozdenović, Univ. Novi Sad

and partly **F. Vallentin, CWI**

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The stability number $\alpha(G)$ and the chromatic number $\chi(G)$

$G=(V,E)$ graph; $S\subseteq V$ is **stable** if S contains no edge

$\alpha(G)$:= maximum cardinality of a stable set

$\chi(G)$:= minimum number of colors needed to properly color G
= minimum number of stable sets needed to cover V

$$\alpha(G) = \max \sum_{i\in V} x_i \ \text{ s.t. } x_ix_j = 0 \ (ij\in E),\ x\in\{0,1\}^V$$

$$\chi(G) = \min \sum_{S\subseteq V,\ \text{stable}} \lambda_S \ \text{ s.t. } \sum_S \lambda_S \chi^S = \mathbf{1},\ \lambda_S \in \{0,1\}$$

$\alpha(G)$, $\chi(G)$ are hard to compute (and approximate)

SDP bounds via the theta tumber $\vartheta(G)$ of Lovász [1979]

The **theta number**

can be computed in polynomial time (to any precision) via SDP:

$$\vartheta(G) := \max\langle J, X\rangle \text{ s.t. } \mathrm{Tr}(X) = 1,\ X_{ij} = 0\ (ij \in E), X \succeq 0$$

The **'sandwich theorem'**:

$$\alpha(G) \leq \vartheta(G) \leq \bar{\chi}^*(G) := \chi^*(\bar{G}) \leq \bar{\chi}(G) := \chi(\bar{G})$$

with equality if G is a perfect graph

$$\chi^*(G) := \min \sum_{S\subseteq V,\ \text{stable}} \lambda_S \text{ s.t. } \sum_S \lambda_S \chi^S = \mathbf{1},\ \lambda_S \geq 0$$

is the **fractional chromatic number** of G

How to improve the theta number toward $\alpha(G)$ and $\chi(G)$?

- Toward $\alpha(G)$: *Add nonnegativity* [McEliece, Rodemich, Rumsey 1978], [Schrijver 1979]

$$\vartheta'(G) := \max\langle J, X\rangle \text{ s.t. } \mathrm{Tr}(X) = 1, X_{ij} = 0\ (ij \in E), X \succeq 0, X \geq 0$$

- Toward $\bar{\chi}(G)$: *Relax the edge conditions* [Szegedy 1994]

$$\vartheta^{+}(G) := \max\langle J, X\rangle \text{ s.t. } \mathrm{Tr}(X) = 1,\ X_{ij} \leq 0\ (ij \in E), X \succeq 0$$

$$\alpha(G) \leq \vartheta'(G) \leq \vartheta(G) \leq \vartheta^{+}(G) \leq \bar{\chi}^{*}(G) \leq \bar{\chi}(G)$$

How to get further improved bounds toward $\alpha(G)$?

Several constructions exist producing bounds for $\alpha(G)$ s.t.
- the t-th step bound can be computed in *poly-time for fixed* t
- *finite convergence to* $\alpha(G)$ *in* $\alpha(G)$ *steps* [for • (SDP), •]

• (LP) lift-and-project method [Balas-Ceria-Cornuéjols 1993], RLT method [Sherali-Adams 1990]

• (LP/SDP) matrix-cut method [Lovász-Schrijver 1991]

• (SDP) method [Lasserre 2001] (based on moment theory)

• (SDP) method [de Klerk-Pasechnik 2002] (based on SOS relaxations for the copositive cone)
Conjecture: finite convergence in $\alpha(G)$ steps ?

Note: (Las) $\leq$ (SA) $\leq$ (LS) $\leq$ (BCC) [L 03]
(Las) $\leq$ (dKP) [GL 08]

What about $\chi(G)$?

Much less known, at the start of our work ...

• Meurdesoif [2005] strengthens $\bar{\vartheta}^{+}(G)$ towards $\chi(G)$ by *adding triangle inequalities*

• Dukanovic-Rendl [2006] introduced a hierarchy of SDP bounds (based on SOS relaxations for the copositive cone) converging asymptotically to $\chi^{*}(G)$

Plan of the talk

Two basic ideas for constructing SDP bounds:
• Use moment matrices and the $0/1$ constraints
• Use SOS relaxations for the copositive cone
$\rightsquigarrow$ hierarchies of bounds for $\alpha(G)$ and $\chi^*(G)$

(1) How to get more compact SDP programs ?
(2) How to go beyond $\chi^*(G)$?

(1) Exploit structure/symmetry to **block-diagonalize matrices** in the SDP
• Design (weaker) block-diagonal hierarchies
• Exploit the symmetry of the graph G (e.g. Hamming, Kneser graphs)

(2) Recipe: Convert any upper bound β on α to a lower bound Ψ_β on χ

First basic idea for SDP bounds

$$x \in \{0,1\}^n \rightsquigarrow y := (1\, x_1 \ldots x_n) \rightsquigarrow Y := yy^T$$

$$Y = \begin{pmatrix} 1 & x_1 & \ldots & x_n \\ x_1 & x_1 & & \\ \vdots & & \ddots & \\ x_n & & & x_n \end{pmatrix} \text{ satisfies: } \begin{cases} Y \succeq 0 \\ Y_{\mathbf{0},\mathbf{0}} = 1 \\ Y_{i,i} = Y_{\mathbf{0},i} \quad \forall i \end{cases}$$

Linear conditions: $Ax \leq b$

$\rightsquigarrow x_i(b - Ax) \geq 0, \;\; (1 - x_i)(b - Ax) \geq 0$

$\rightsquigarrow$ Linear conditions on Y

Stable set problem: Edge condition: $x_i x_j = 0 \rightsquigarrow Y_{i,j} = 0$

$\rightsquigarrow$ Theta number $\vartheta(G)$

SDP relaxations of higher order t

$$x \in \{0,1\}^n \rightsquigarrow y := \left(\prod_{i\in I} x_i\right)_{I\in\mathcal{P}_t(V)} \rightsquigarrow Y := yy^T$$

$$\mathcal{P}_t(V) := \{I \subseteq V \mid |I| \leq t\}$$

Ex: $y = (1,\ x_1, \cdots, x_n,\ x_1x_2,\ \cdots, x_1x_2x_3, \cdots)$

$$\begin{cases} Y \succeq 0 \\ Y_{\mathbf{0},\mathbf{0}} = 1 \\ Y_{I,J} \text{ depends only on the union } I \cup J \\ + \text{ LP (SDP) } \textit{localizing} \text{ conditions corresponding to } Ax \leq b \end{cases}$$

Notation: $M_t(y) := (y_{I\cup J})_{I,J\in\mathcal{P}_t(V)}$ is the **moment matrix** of order t of $y \in \mathbb{R}^{\mathcal{P}_{2t}(V)}$

Get SDP/LP formulation of the original $0/1$-problem at order n

For $y \in \mathbb{R}^{\mathcal{P}(V)}$

$$M_n(y) = (y_{I\cup J})_{I,J\subseteq V} \succeq 0 \iff \sum_{S'\supseteq S}(-1)^{|S'\setminus S|} y_{S'} \geq 0 \ \ \forall S \subseteq V$$
$$\iff y \in \mathbb{R}_+(y^S \mid S \subseteq V)$$

where $y^S := (\prod_{i\in I} x_i)_{I\subseteq V}$, x:= incidence vector of $S \subseteq V$

$$\begin{array}{c} \\ \mathbf{0} \\ 1 \\ 2 \\ 12 \end{array} \begin{array}{c} \begin{array}{cccc} \mathbf{0} & 1 & 2 & 12 \end{array} \\ \begin{pmatrix} y_{\mathbf{0}} & y_1 & y_2 & y_{12} \\ y_1 & y_1 & y_{12} & y_{12} \\ y_2 & y_{12} & y_2 & y_{12} \\ y_{12} & y_{12} & y_{12} & y_{12} \end{pmatrix} \end{array} \succeq 0 \iff \begin{cases} y_{\mathbf{0}} - y_1 - y_2 + y_{12} \geq 0 \\ y_1 - y_{12} \geq 0 \\ y_2 - y_{12} \geq 0 \\ y_{12} \geq 0 \end{cases}$$

$$\iff y \in \mathbb{R}_+(y^{\mathbf{0}}, y^{\{1\}}, y^{\{2\}}, y^{\{1,2\}})$$

The ‘Lasserre type’ hierarchies for $\alpha(G)$, $\chi^*(G)$

$$\text{las}^{(t)}(G) := \max \sum_{i\in V} y_i \ \text{s.t.}\ M_t(y) \succeq 0,\ y_{\mathbf{0}} = 1,\ y_{ij} = 0\ (ij \in E)$$

$$\psi_{\text{las}}^{(t)}(G) := \min\ y_{\mathbf{0}}\ \text{s.t.}\ M_t(y) \succeq 0,\ y_i = 1\ (i \in V),\ y_{ij} = 0\ (ij \in E)$$

If $\mathbf{1} = \sum_S \lambda_S \chi^S$ *(*$\lambda_S \geq 0$*) is a fractional coloring, then* $\sum_S \lambda_S (y^S)(y^S)^T =: M_t(y)$ *is feasible with* $y_{\mathbf{0}} = \sum_S \lambda_S$

- Bounds $\text{las}^{(t)}$, $\psi_{\text{las}}^{(t)}$ for α, χ^*, with **equality** if $t = \alpha(G)$
- $\text{las}^{(t)}$, $\psi_{\text{las}}^{(t)}$ are computable by a SDP of matrix size $O(n^t)$, thus in time polynomial in n for *fixed* t (to any precision)
- For $t = 1$, $\text{las}^{(1)} = \vartheta$, $\text{las}_+^{(1)} = \vartheta'$, $\psi_{\text{las}}^{(1)} = \bar{\vartheta}$, $\psi_{\text{las},+}^{(1)} = \bar{\vartheta}^+$

‘Reciprocity’ between the two hierarchies $\text{las}^{(t)}(G)$, $\psi_{\text{las}}^{(t)}(G)$

(α, χ^*) form a ‘**reciprocal pair**’:

$$\alpha(G)\chi^*(G) \geq |V|, \text{ with equality if } G \text{ is vertex-transitive}$$

The same holds for the following pairs:

- $(\vartheta, \bar{\vartheta})$ [Lovász 1979]
- $(\vartheta', \bar{\vartheta}^+)$ [Szegedy 1994]
- $(\text{las}^{(t)}, \psi_{\text{las}}^{(t)})$
- $(\text{las}_+^{(t)}, \psi_{\text{las},+}^{(t)})$

Second basic idea for SDP bounds: Relax matrix copositivity by sums of squares of polynomials

$\mathcal{C}$, $\mathcal{C}^*$: cones of copositive / completely positive matrices

M **copositive** if $x^T M x \geq 0 \;\; \forall x \in \mathbb{R}^n_+$

i.e., if $p_M(x) := \sum_{i,j} x_i^2 x_j^2 M_{ij}$ is nonnegative on $\mathbb{R}^n$

M **completely positive** if $M = \sum_i u_i u_i^T$ with $u_i \geq 0$

Parrilo [2000] relaxes copositivity by:

$$\mathcal{K}^{(t)} := \{M \mid p_M(x)(\sum_{i=1}^n x_i^2)^{t-1} \;\; \text{SOS}\} \subseteq \mathcal{C}$$

- $\mathcal{K}^{(1)} = \{P + N \mid P \succeq 0, \; N \geq 0\}$
- $\bigcup_{t \geq 1} \mathcal{K}^{(t)} = \text{int}(\mathcal{C})$ [Pólya 1974]

Copositive programming formulations for $\alpha(G)$ [de Klerk-Pasechnik 02] and $\chi^*(G)$ [Dukanovic-Rendl 06]

$$\alpha(G) = \max \langle J, X\rangle \text{ s.t. } \mathrm{Tr}(X) = 1,\ \langle A_G, X\rangle = 0,\ X \in \mathcal{C}^* \stackrel{(\bullet)}{=} \min \lambda \text{ s.t. } \lambda(I + A_G) - J \in \mathcal{C}$$

$$\chi^*(G) = \min \lambda \text{ s.t. } X_{ii} = \lambda\ (i \in V),\ \langle A_G, X\rangle = 0, X \in \mathcal{C}^*,\ X - J \succeq 0$$

- $X = \chi^S(\chi^S)^T$ is **completely positive** and $(\bullet)$ follows using [Motzkin-Straus 1965]:

$$\frac{1}{\alpha(G)} = \min x^T(I + A_G)x \text{ s.t. } \sum_{i\in V} x_i = 1,\ x \geq 0$$

- If $\mathbf{1} = \sum_S \lambda_S \chi^S$ $(\lambda_S \geq 0)$ is a fractional coloring and $\lambda := \sum_S \lambda_S$, then $X = \lambda \sum_S \lambda_S(\chi^S)(\chi^S)^T$ is **completely positive** and $X - J \succeq 0$

SDP hierarchies for $\alpha(G)$ [dKP 02] and $\chi^*(G)$ [DR 06]

Replace the copositive cone $\mathcal{C}$ by the subcone $\mathcal{K}^{(t)}$:

$\leadsto$ **Reciprocal pair** $(\vartheta^{(t)}, \kappa^{(t)})$ of bounds for α, χ^*

Theorem: [dKP 02] $\lfloor \vartheta^{(t)}(G) \rfloor = \alpha(G)$ for $t \geq \alpha(G)^2 + 1$

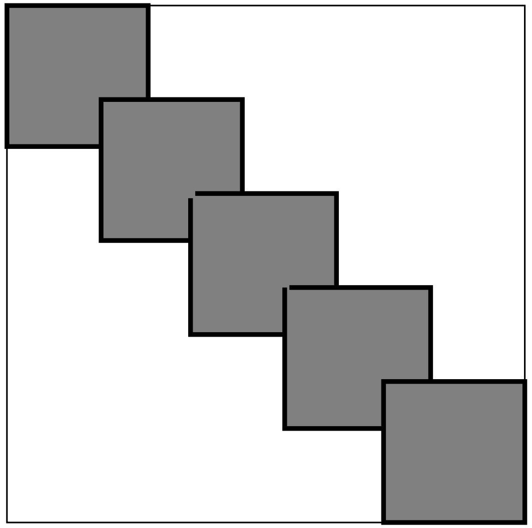
Conjecture: $\vartheta^{(t)}(G) = \alpha(G)$ for $t \geq \alpha(G)$

Equivalently: For $M = \alpha(I + A_G) - J$, $\alpha = \alpha(G)$,
$(\sum_{i,j} x_i^2 x_j^2 M_{i,j})(\sum_i x_i^2)^{\alpha-1}$ is a sum of squares of polynomials

Partial answer: [GL05] **Yes** if $\alpha(G) \leq 8$

Comparison: $\mathrm{las}_+^{(t)} \leq \vartheta^{(t)}$

How to obtain more economical bounds ?



Idea: Instead of **one** matrix $M_t(y)$ with **large** indexset $\mathcal{P}_t(V)$, consider **several** principal submatrices $M(T;y)$ **(**$T \in \mathcal{P}_{t-1}(V)$**)** with **small** indexsets:

$$\bigcup_{S \subseteq T} \{S, S \cup \{i\}\ (i \in V)\} =: \bigcup_{S \subseteq T} S \cdot \mathcal{P}_1(V)$$

$\rightsquigarrow O(n^{t+1})$ variables, instead of $O(n^{2t})$

The matrix $M(T;y)$ can be block-diagonalized

For $T = \{1,2\}$, $M(T;y)$ has the block-structure:

$$\begin{array}{cc} & \begin{array}{cccc} \mathbf{0} & 1 & 2 & 12 \end{array} \\ \begin{array}{c} \mathbf{0} \\ 1 \\ 2 \\ 12 \end{array} & \begin{pmatrix} A_{\mathbf{0}} & A_1 & A_2 & A_{12} \\ A_1 & A_1 & A_{12} & A_{12} \\ A_2 & A_{12} & A_2 & A_{12} \\ A_{12} & A_{12} & A_{12} & A_{12} \end{pmatrix} \end{array} \succeq 0 \Longleftrightarrow \begin{cases} A_{\mathbf{0}} - A_1 - A_2 + A_{12} \succeq 0 \\ A_1 - A_{12} \succeq 0 \\ A_2 - A_{12} \succeq 0 \\ A_{12} \succeq 0 \end{cases}$$

$$\boxed{M(T;y) \succeq 0 \Longleftrightarrow \sum_{T \supseteq S' \supseteq S} (-1)^{|S' \setminus S|} A_{S'} \succeq 0 \quad \forall S \subseteq T}$$

where $A_{S'}$ is indexed by $\mathcal{P}_1(V)$

$\rightsquigarrow$ Replace the matrix $M(T;y)$ of size $2^{|T|}(n+1)$ by $2^{|T|}$ matrices each of size $n+1$

Block-diagonal hierarchies for $\alpha(G)$, $\chi^*(G)$

$$\ell^{(t)}(G) := \max \sum_{i \in V} y_i \ \text{s.t. } M(T;y) \succeq 0 \ (|T| = t-1), \ y_{\mathbf{0}} = 1, \ y_{ij} = 0 \ (ij \in E)$$

$$\psi^{(t)}(G) := \min y_{\mathbf{0}} \ \text{s.t. } M(T;y) \succeq 0 \ (|T| = t-1), y_i = 1 \ \forall i, \ y_{ij} = 0 \ (ij \in E)$$

- **Reciprocal pair** $(\ell^{(t)}, \psi^{(t)})$
- Weaker bounds than $\text{las}^{(t)}$, $\psi^{(t)}_{\text{las}}$, but with the **same finite convergence** in $\alpha(G)$ steps
- $\ell^{(t)}(G)$ refines the bound obtained from $N_+^{t-1}(\text{TH}(G))$

General fact: [GLV 08] *The block-diagonal construction refines the SDP Lovász-Schrijver hierarchy, while being less costly to compute*

Complexity comparison

	$\text{las}^{(t)}(G)$ Lasserre relax.	$\ell^{(t)}(G)$ block-diagonal relax.	$N_+^{t-1}(\text{TH}(G))$ LS N_+-operator
# var.	$O(n^{2t})$	$\frac{1}{(t+1)!}n^{t+1} + O(n^t)$	$2^{t-2}n^{t+1} + O(n^t)$
size SDP	one matrix of size $O(n^{2t})$	$\frac{2^{t-1}}{(t-1)!}n^{t-1} + O(n^t)$ matrices of size $n+1$	$2^{t-1}n^{t-1} + O(n^t)$ matrices of size $n+1$
# linear eq.	m	m	$O(mn^{t-1})$

Note: $\ell^{(2)}$ needs n matrices: $M(\{i\}; y) \succeq 0$ **(**$i \in V$**)**

But **one** matrix suffices if G is **vertex-transitive**

Recap on symmetry reduction

$G = (V, E)$
$\mathcal{G} \subseteq \text{Aut}(G)$: group of permutations of V preserving edges

$g \in \mathcal{G}$ acts on V, $\mathcal{P}(V)$, $\mathbb{R}^V$, $\mathbb{R}^{\mathcal{P}(V)}$, etc.

$$y = (y_i, y_{\{i,j\}}, y_{\{i,j,k\}}, \cdots) \rightsquigarrow gy = (y_{g(i)}, y_{\{g(i),g(j)\}}, y_{\{g(i),g(j),g(k)\}}, \cdots)$$

Fact: If y is feasible for the SDP defining e.g. $\ell^{(2)}(G)$, then gy too, and thus $\frac{1}{|\mathcal{G}|}\sum_{g\in\mathcal{G}} gy$ too, *because* $M(\{i_0\}; y)$ *is permutation equivalent to* $M(\{g(i_0)\}; y)$

- We may assume that y is **invariant under action of** $\mathcal{G}$
$\rightsquigarrow$ **less variables**

- G is **vertex-transitive** if $\forall i, j \in V \; \exists g \in \mathcal{G} \;\; g(i) = j$
$\rightsquigarrow$ Enough to require $M(\{i_0\}; y) \succeq 0$ for **one** $i_0 \in V$

Numerical results for Paley graphs

P_q:= graph on $\mathbb{F}_q$, $q = 1$ (mod 4), with ij edge if $i - j$ is a square

- P_q is **self-complementary**

$\rightsquigarrow \vartheta(P_q) = \sqrt{q} \quad (= \vartheta'(G))$

- P_q is **vertex-transitive**

$\rightsquigarrow$ For $\ell^{(2)}(P_q)$, we need only **one** matrix $M(\{i_0\}, y) \succeq 0$

- $\text{Aut}(P_q)$ **acts transitively on edges and on non-edges**

$\rightsquigarrow$ For $\ell^{(3)}(P_q)$, we need only **one** matrix $M(\{i_1, i_2\}, y) \succeq 0$ with $i_1 i_2$ edge and **one** with $i_1 i_2$ non-edge

q	$\vartheta(P_q) = \sqrt{q}$	$N_+(\mathrm{TH}(P_q))$	$\ell^{(2)}(P_q)$	$\ell^{(3)}(P_q)$	$\alpha(P_q)$
101	10.050	7.290	6.611	5.496	5
149	12.207	9.188	8.231	7.136	7
241	15.524	11.595	9.891	8.275	7
257	16.031	11.558	10.247	8.131	7
269	16.401	12.307	10.624	8.778	8
277	16.643	12.469	10.340	8.670	8
281	16.763	11.902	10.605	8.397	7
313	17.692	13.128	11.630	9.458	8
337	18.358	13.724	11.658	9.464	9
401	20.025	14.927	12.753	10.023	9
509	22.561	16.580	14.307	11.196	9
601	24.515	17.999	16.077	12.484	11
701	26.476	19.332	16.857	12.822	10
809	28.443	20.636	17.371	13.499	11

Another simple strengthening of $\bar{\vartheta}$ toward χ^*

Pick a clique K of G

Consider the principal submatrix X of $M_2(y)$ indexed by

$$\mathcal{P}_1(V) \cup \bigcup_{h \in K} \{\{h\}, \{i, h\}\ (i \in V)\} = \mathcal{P}_1(V) \cup \bigcup_{h \in K} \{h\} \cdot \mathcal{P}_1(V)$$

$$X = \begin{pmatrix} A_{\mathbf{0}} & A_1 & A_2 & \dots & A_k \\ A_1 & A_1 & 0 & \dots & 0 \\ A_2 & 0 & A_2 & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ A_k & 0 & \dots & 0 & A_k \end{pmatrix} \Longleftrightarrow \begin{cases} A_0 - \sum_{h \in K} A_h \succeq 0 \\ A_1, \dots, A_k \succeq 0 \end{cases}$$

$\leadsto$ Bound $\psi_K(G) \le \chi^*(G)$

DIMACS instances DSJCa.b [Random graph on a nodes, edge probability $b/10$]

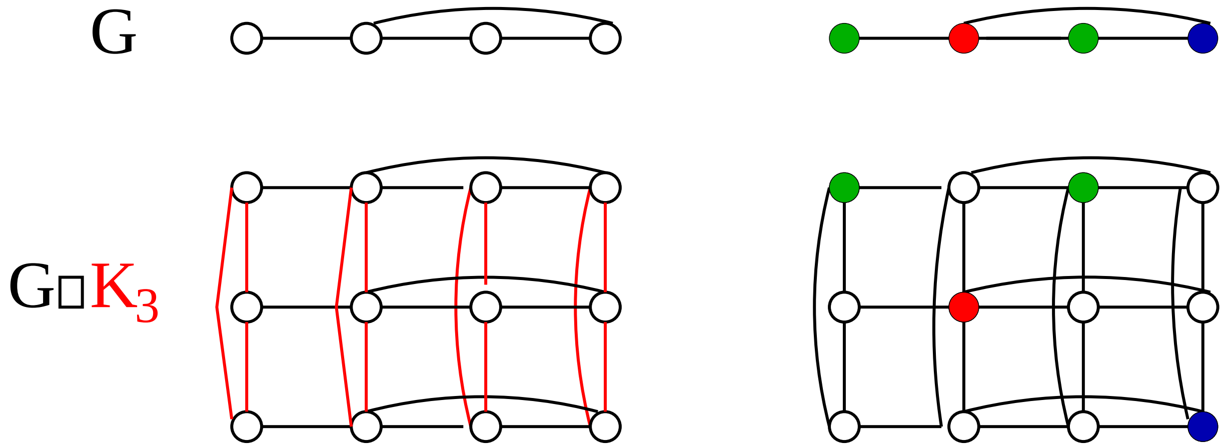
Graph	LB	$\overline{\vartheta}(G)$	$\lceil\overline{\vartheta}(G)\rceil$	$\lvert K\rvert$	$\psi_K(G)$	$\lceil\psi_K(G)\rceil$	UB
DSJC125.1	5	4.1062	5	4	4.337	**5**	5
DSJC125.5	14 (17)	11.7844	12	10	13.942	**14**	17
DSJC125.9	42	37.768	38	34	42.53	**43***	43
DSJC250.1	6 (8)	4.906	5	4	5.208	**6**	8
DSJC250.5	14	16.234	17	12	19.208	**20**	28
DSJC250.9	48	55.152	56	43	66.15	**67**	72
DSJC500.1	6	6.217	**7**	5	6.542	7	12
DSJC500.5	13 (16)	20.542	21	13	27.791	**28**	48
DSJC500.9	59	84.04	85	56	100.43	**101**	126
DSJC1000.1	6	8.307	**9**	5	-	-	20
DSJC1000.5	15 (17)	31.89	**32**	14	-	-	83
DSJC1000.9	66	122.67	**123**	65	-	-	224
DSJR500.1c	82 (83)	83.74	84	77	84.12	**85***	85

LB: [DesRosiers-Gallinier-Hertz 08, Mendez-Diaz-Zabala 06, Caramia-Dell’Olmo 04]

UB: [Caramia-Dell’Olmo 08, Gallinier-Hertz-Zufferey 08, Gallinier-Hao 07]

How to go beyond the fractional chromatic number ?

$G \square K_t$: the Cartesian product of G and K_t



Chvátal [1973]: $$\alpha(G \square K_t) = n \iff \chi(G) \leq t$$

Thus: $$\chi(G) = \min_{t \in \mathbb{N}} \; t \; \text{ s.t. } \alpha(G \square K_t) = n$$

The operator Ψ

Given a graph parameter $\beta(\cdot)$ s.t. $\frac{|V(\cdot)|}{\chi(\cdot)} \leq \beta(\cdot) \leq \bar{\chi}(\cdot)$

define the new graph parameter $\Psi_\beta(\cdot)$ by

$$\Psi_\beta(G) := \min_{t \in \mathbb{N}} \; t \;\; \text{s.t.} \;\; \beta(G \Box K_t) = n$$

Then: $\omega(\cdot) \leq \Psi_\beta(\cdot) \leq \chi(\cdot)$

- β poly-time computable $\Longrightarrow \Psi_\beta$ poly-time computable
- Ψ is monotone nonincreasing

Action of the operator Ψ

$$
\begin{array}{ccccccccccccc}
\frac{|V|}{\chi} & \leq & \frac{|V|}{\chi^*} & \leq & \alpha & \leq & \vartheta' & \leq & \vartheta & \leq & \bar{\chi}^* & \leq & \bar{\chi} \\
 & \searrow & \downarrow & \swarrow & & & \downarrow & & \downarrow & & \downarrow & \swarrow & \\
 & & \chi & & \geq & & \lceil \bar{\vartheta}^+ \rceil & \geq & \lceil \bar{\vartheta} \rceil & \geq & \omega & & \\
 & & & & & & & & & \nearrow & \uparrow & \nwarrow & \\
 & & & & & & & & \frac{|V|}{\omega} & \leq & \bar{\chi}^* & \leq & \bar{\chi}
\end{array}
$$

Hence: Ψ maps a hierarchy toward α to a hierarchy toward χ
For example, $\Psi_{\ell(t)} = \chi$ if $t \geq n$

Hard interval around the fractional chromatic number:
A graph parameter $\beta \in \left[\frac{|V|}{\omega}, \bar{\chi}\right]$ cannot be computed in polynomial time, unless P=NP

Examples of graph parameters in $[\chi^*, \chi]$

- [Vince 1988] The **circular chromatic number**:

$$\chi_c(G) := \min\ r \text{ s.t. } \quad \exists \text{ proper coloring } c \text{ s.t. } \quad 1 \le |c(i) - c(j)| \le r - 1 \ \forall ij \in E$$

- [Hahn-Hell-Poljak 1995] The **ultimate independence ratio:**

$$I(G) := \lim_{k\to\infty} \frac{\alpha(G^{\square k})}{|V|^k}$$

- [Körner-Pilotto-Simonyi 2005] **Local chromatic number:**

$$\psi(G) := \min_{c \text{ proper coloring}} \ \max_{v\in V} |\{c(u) \mid u \in N_G(v) \cup \{v\}\}|$$

$$\chi^*(G) \le \frac{1}{I(G)} \le \chi_c(G) \le \lceil \chi_c(G) \rceil = \chi(G)$$

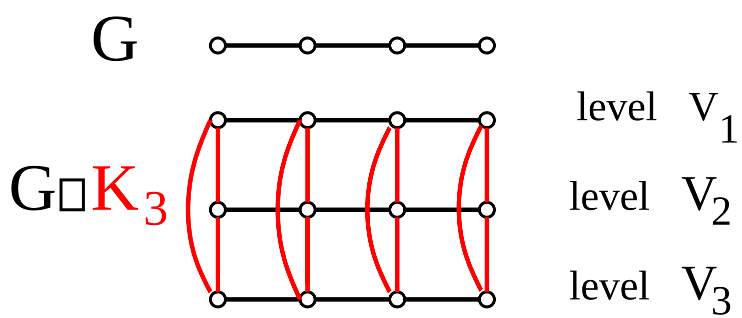
$$\chi^*(G) \le \psi(G) \le \chi(G)$$

Using symmetry reduction to compute Ψ_β for $\beta = \ell^{(2)}$

$$\Psi_{\ell^{(2)}}(G) = \min\ t \ \text{ s.t. } \ell^{(2)}(G_t) = n \quad \geq \psi^{(2)} \geq \frac{|V|}{\ell^{(2)}}$$

with $G_t := G \square K_t$

$$\ell^{(2)}(G_t) = \max \sum_{i \in V(G_t)} y_i \ \text{ s.t. } \quad y_{\mathbf{0}} = 1,\ y_{ij} = 0\ (ij \in E(G_t))$$
$$\text{(*)}\ M(\{u\}; y) \succeq 0\ (u \in V(G_t))$$



We may assume that y *is invariant under action of the symmetric group* S_t, thus it is enough to require (*) for $u \in V_1$ (**just one level**) and for **just one** $u \in V_1$ if G is vertex-transitive

Action of $K_t \rightsquigarrow$ Symmetry structure in $M(\{u\};y)$

$$M(\{u\};y) = \begin{array}{c} \\ \mathbf{0} \\ V(G_t) \\ V(G_t) \end{array} \begin{array}{c} \begin{array}{ccc} \mathbf{0} & V(G_t) & V(G_t) \end{array} \\ \begin{pmatrix} y_{\mathbf{0}} & a^T & b^T \\ a & A & B \\ b & B & B \end{pmatrix} \end{array}$$

$$A = \begin{array}{c} \\ V_1 \\ V_2 \\ \vdots \\ V_t \end{array} \begin{array}{c} \begin{array}{cccc} V_1 & V_2 & \cdots & V_t \end{array} \\ \begin{pmatrix} A_1 & A_2 & \cdots & A_2 \\ A_2 & A_1 & \cdots & A_2 \\ \vdots & \vdots & \ddots & \vdots \\ A_2 & A_2 & \cdots & A_1 \end{pmatrix} \end{array} \quad B = \begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ \vdots \\ V_t \end{array} \begin{array}{c} \begin{array}{ccccc} V_1 & V_2 & V_3 & \cdots & V_t \end{array} \\ \begin{pmatrix} B_1 & B_2 & B_2 & \cdots & B_2 \\ (B_2)^T & B_3 & B_4 & \cdots & B_4 \\ (B_2)^T & B_4 & B_3 & \cdots & B_4 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (B_2)^T & B_4 & B_4 & \cdots & B_3 \end{pmatrix} \end{array}$$

$\rightsquigarrow \ell^{(2)}(G_t)$ can be reformulated via a SDP with four matrices of sizes $2n+1$, $2n$, n, n, for G vertex-transitive

Numerical results for Hamming graphs $H(n, \mathcal{D})$

$V = \{0,1\}^n, \ \mathcal{D} \subseteq [1,n]$

edge ij if $d_H(i,j) = |i \oplus j| \in \mathcal{D}$

The coding problem: Find $\alpha(H(n,\mathcal{D}))$

- LP bound of [Delsarte 73]

$\rightsquigarrow \vartheta'(H(n,\mathcal{D}))$, computed via an LP of size n

- SDP bound of [Schrijver 05]

+ small improvment $\ell^{(2)}(H(n,\mathcal{D}))$ [L 07]

$\rightsquigarrow$ computed via an SDP of size $O(n^3)$

Exploit graph symmetry: May assume that y is invariant under action of $\mathcal{G} \subseteq \mathbf{Aut}(G)$

$$M(\{i_0\};y) = \begin{array}{c} \\ \mathbf{0} \\ V \\ i_0 \cdot V \end{array} \begin{array}{c} \begin{array}{ccc} \mathbf{0} & V & i_0 \cdot V \end{array} \\ \begin{pmatrix} y_\mathbf{0} & a^T & b^T \\ a & A & B \\ b & B & B \end{pmatrix} \end{array}$$

$$A_{i,j} = y_{\{i,j\}} \qquad B_{ij} = y_{\{i_0,i,j\}}$$

- $A_{i,j} = A_{i',j'}$ if $\exists\, g \in \mathcal{G} \;\; g(i) = i'$, $g(j) = j'$

$\rightsquigarrow A \in \mathcal{A}(\mathcal{G})$: algebra of matrices invariant under $\mathcal{G}$

- $B_{i,j} = B_{i',j'}$ if $\exists\, g \in \mathcal{G} \;\; g(i) = i'$, $g(j) = j'$ and $g(i_0) = i_0$

$\rightsquigarrow B \in \mathcal{A}(\mathcal{G}_{i_0})$: algebra of matrices invariant under $\mathcal{G}_{i_0} := \{g \in \mathcal{G} \mid g(i_0) = i_0\}$

Fact: *These are matrix $*$-algebras, which can thus be block-diagonalized* (by **Wedderburn theorem**)

Wedderburn theorem [1907]

Theorem: Let $\mathcal{A}$ be a matrix $*$-algebra over $\mathbb{C}$ with $I \in \mathcal{A}$.
There is a unitary matrix Q and $s, n_1, \ldots, n_s \in \mathbb{N}$ such that

$$Q^* \mathcal{A} Q = \begin{pmatrix} \mathcal{A}_1 & 0 & \cdots & 0 \\ 0 & \mathcal{A}_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \mathcal{A}_s \end{pmatrix}$$

where each $\mathcal{A}_i \sim \mathbb{C}^{n_i \times n_i}$ and takes the form

$$\mathcal{A}_i = \left\{ \begin{pmatrix} A & 0 & \cdots & 0 \\ 0 & A & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & A \end{pmatrix} \mid A \in \mathbb{C}^{n_i \times n_i} \right\}$$

Application to Hamming graphs $H(n,\mathcal{D})$

$\mathcal{G} = \mathrm{Aut}(H(n,\mathcal{D}))$: All permutations of $[1,n]$ combined with all 'switchings': $i \mapsto i \oplus i_0$

- $A_{i,j} = A_{i',j'} \iff |i \oplus j| = |i' \oplus j'|$

$\rightsquigarrow$ (commutative) Bose-Mesner algebra, with dimension $n+1$
$\rightsquigarrow$ LP of size n to compute $\vartheta'(H(n,\mathcal{D}))$

- $B_{i,j} = B_{i',j'} \iff |i|, |j|, |i \oplus j| = |i'|, |j'|, |i' \oplus j'|$

$\rightsquigarrow$ $\mathcal{A}_{\mathcal{G}_{i_0}}$: Terwilliger algebra, with dimension $O(n^3)$, whose block-diagonalization is given by [Schrijver 05]
$\rightsquigarrow$ SDP of size $O(n^3)$ for $\ell^{(2)}(H(n,\mathcal{D}))$

Bounds on $\alpha(H(n,\mathcal{D}))$ for $\mathcal{D} = \{1,\ldots,d-1\}$

n	d	LB	Delsarte ϑ'	UB	Schrijver	$\ell_+^{(2)}$
19	6	1024	1289	1288	**1280**	
23	6	8192	13,775	13,774	**13,766**	
25	6	16,384	48,148	48,148	47,998	**47,997**
19	8	128	145	144	**142**	
20	8	256	290	279	274	
25	8	4096	6474	5557	**5477**	
27	8	8192	18,189	17,804	**17,768**	
28	8	16,384	32,206	32,204	**32,151**	
22	10	64	95	88	**87**	
25	10	192	551	549	**503**	
26	10	384	1040	989	**886**	

Bounds on $\alpha(H(n, \mathcal{D}))$ **for** $\mathcal{D} = \{n/2\}$

Orthogonality graphs [de Klerk-Pasechnik 2005]

n	LB	ϑ'	Schrijver	$\ell_+^{(2)}$
16	2304	4096	2304	2304
20	20,144	52,428	20,166.98	20,166.62
24	178,208	699,050	184,194	**183,373**
28	406,336	9,586,980	1,883,009	**1,848,580**
32	14,288,896	134,217,728	21,723,404	**21,103,609**

Bounds on $\chi(H(n,\mathcal{D}))$ for $\mathcal{D} = \{d\}$

graph	$\bar{\vartheta}$	$\bar{\vartheta}^+$	$\kappa^{(2)}$	$\psi^{(2)}$	$\Psi_{\ell^{(2)}}$	$\psi^{(2)}_+$	$\Psi_{\ell^{(2)}_+}$
$H(10,6)$	6	8.72	**10.5**	10.43	11	10.89	11
$H(10,8)$	2.66	3.2	3.4	3.92	**5**	3.92	**5**
$H(11,4)$	16	21.56	24.7	**25.73**	26	25.73	26
$H(11,6)$	12	12	14.1	12	12	**15.28**	16
$H(11,8)$	3.2	4.93	5.4	**5.78**	6	5.78	6
$H(13,8)$	5.33	9.41	12.5	12.14	13	**13.65**	14
$H(15,6)$	27.76	30.73	43.0	46.43	47	**50.30**	51
$H(16,8)$	16	16	24.1	16	16	**28.44**	29
$H(17,6)$	35	48.22	62.5	86.30	87	**88.32**	89
$H(17,8)$	18	18	34.5	32	32	**46.51**	47
$H(17,10)$	6.66	12.63	20.5	15.87	16	**25.84**	26
$H(18,10)$	10	16	28.8	18.30	19	**38.88**	-
$H(20,6)$	59.37	59.37		**140.95**	141	140.95	-
$H(20,8)$	41.71	60.95		107.14	-	**136.41**	-
$H(10,[8,10])$	3.2	3.2		3.92	**5**	3.92	**5**

Numerical results for Kneser graphs $K(n, r)$

V : all r-subsets of $[1, n]$, with an edge between disjoint sets

$\alpha = \vartheta = \binom{n-1}{r-1}$ [Lovász 79]

$\chi^* = \frac{n}{r}$

$\omega = \lfloor \frac{n}{r} \rfloor$

$\chi = n - 2r + 2$ [Lovász 78]

As $\alpha = \vartheta$, the full hierarchy $\ell^{(t)}$ collapses to α, and the hierarchy $\psi^{(t)}$ collapses to χ^*, which is far from χ !

Thus the Ψ_β bounds may help ..

Numerical results for Kneser graphs

Graph	$\lceil \chi^* \rceil = \lceil n/r \rceil$	$\Psi_{\ell(2)}$	$\Psi_{\ell_+^{(2)}}$	$\chi = n - 2r + 2$
$K(6,2)$	3	**4**	**4**	**4**
$K(7,2)$	4	4	**5**	**5**
$K(8,3)$	3	**4**	**4**	**4**
$K(9,3)$	3	4	4	5
$K(10,4)$	3	3	**4**	**4**
$K(11,4)$	3	4	4	5
$K(12,3)$	4	5	6	8
$K(12,4)$	3	4	4	6
$K(12,5)$	3	3	**4**	**4**
$K(13,5)$	3	4	4	5
$K(15,3)$	5	6	6	11
$K(16,4)$	4	5	6	10
$K(25,5)$	5	6	7	17
$K(34,7)$	5	6	7	22
$K(36,6)$	6	7	9	26

Much work not covered in this talk

Exploiting symmetry is crucial to get compact SDP's

• *Bounds for the crossing number of* $K_{n,m}$ [de Klerk-Maharry-Pasechnik-Richter-Salazar 06] [de Klerk-Pasechnik-Schrijver 07] (using regular $*$-representation)

• *Bounds for the kissing number* (using harmonic analysis) [Bachoc-Vallentin 08]

• *QAP, truss topology optimization, polynomial optimization ...* [Gaterman-Parrilo 04], de Klerk & al., Murota, Kojima & al.

Some recent surveys:

• [de Klerk] *Exploiting special structure in semidefinite programming: A survey of theory and applications*

• [Vallentin] *Symmetry in semidefinite programming* & *Lecture Notes: Semidefinite programs and harmonic analysis*

Quadratic and Copositive Formulations for $\chi(G)$

Motzkin-Straus formulation for α + reduction of χ to α $\Longrightarrow$

$$\chi(G) = \min \sum_t t(e^T x_t)^2 \text{ s.t. } \begin{array}{l} \sum_t (e^T x_t)^2 = 1 \\ \sum_t x_t^T (I + A_{G\Box K_t}) x_t = \frac{1}{n} \\ x_t \in \mathbb{R}_+^{V(G\Box K_t)} \end{array}$$

$$\chi(G) = \min \sum_t t\langle J, X_t\rangle \text{ s.t. } \begin{array}{l} \sum_t \langle J, X_t\rangle = 1 \\ \sum_t \langle I + A_{G\Box K_t}, X_t\rangle = \frac{1}{n} \\ X_t \in \mathcal{C}^* \quad (t \in [1, \Delta(G)]) \\ X_t \text{ indexed by } V(G\Box K_t) \end{array}$$