Lower Bound for the Number of Iterations in Semidefinite Hierarchies for the Cut Polytope

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Abstract

Hierarchies of semidefinite relaxations for 0/1 polytopes have been constructed by Lasserre (2001a) and by Lovász and Schrijver (1991), permitting to find the cut polytope of a graph on n nodes in n steps. We show that $\lceil \frac{n}{2} \rceil$ iterations are needed for finding the cut polytope of the complete graph K_n.

Key words: cut polytope, semidefinite relaxation, moment matrix, rank of lift-and-project method.

1 Introduction

1.1 Preamble

Given a graph $G = (V, E)$ and a subset $A \subseteq V$, the *cut* determined by A is the vector $\delta(A) \in \{\pm 1\}^E$ with ij-th entry -1 if and only if $|A \cap \{i, j\}| = 1$. The *cut polytope* $\mathrm{CUT}(G)$ is the polytope in $\mathbb{R}^E$ defined as the convex hull of all cuts $\delta(A)$ ($A \subseteq V$). Given edge weights $w \in \mathbb{Q}^E$, the *max-cut problem* is the problem of finding a cut whose weight $\sum_{ij \in E | i \in A, j \notin A} w_{ij}$ is maximum. Hence it can be formulated as the linear programming problem:

$$\max \ \frac{1}{2} \sum_{ij \in E} w_{ij}(1 - x_{ij}) \ \text{ subject to } x \in \mathrm{CUT}(G)$$

over the cut polytope.

The max-cut problem being NP-hard, extensive research has been done for finding good and efficient relaxations of the cut polytope. Research has focused, in particular, on finding large classes of valid inequalities and facets for the cut polytope by exploiting the combinatorial structure of the specific max-cut problem (see Deza and Laurent (1997) for a detailed account). Next to that, research has also focused on developing general purpose methods applying to arbitrary 0/1 (or ± 1) problems. Let us mention, in particular, the lift-and-project method of Balas, Ceria and Cornuéjols (1993), the matrix-cut method of Lovász and Schrijver (1991), the linearization-reformulation technique of Sherali and Adams (1990,1999), and the more recent real-algebraic method of Lasserre (2001a,2001b) (and related work of Nesterov (2000), Parrilo (2000), Shor (1987)). A common feature of these methods is the construction of a hierarchy of linear or semidefinite relaxations for a given 0/1 (or ± 1) polytope P, converging to P in d steps (if P lies in the d-space). The various constructions have been compared in (Laurent 2001b) and it is shown there that the Lasserre hierarchy of semidefinite relaxations refines all the other hierarchies.

Let us define the *rank* of a given method as the smallest number of iterations needed for finding the integer polytope P. The dimension of the ambient space is a common upper bound for the rank of the various procedures and lower bounds have been established for several examples; for instance, for the Sherali-Adams procedure (Laurent 2001b), and for the Lovász-Schrijver procedure applied to the matching polytope (Stephen and Tunçel 1999), to the knapsack polytope and to the 'pigeon hole principle' polytope (Grigoriev et al. 2001), to the traveling salesman polytope and to some other examples (Cook and Dash 2001, Dash 2000, Goemans and Tunçel 2001). To the best of our knowledge, no lower bound has yet been established for the rank of the Lasserre procedure.

When $P = \mathrm{CUT}(G)$ is the cut polytope of a graph G, let $Q_t(G)$ (resp., $N^t(G)$, $N_+^t(G)$) denote the hierarchy of relaxations for $\mathrm{CUT}(G)$ obtained using the Lasserre construction (resp., using the N and N_+ operators in the Lovász-Schrijver construction). The sets $N^t(G)$ are *linear* relaxations while $N_+^t(G)$ and $Q_t(G)$ are *semidefinite* relaxations of $\mathrm{CUT}(G)$; these hierarchies of relaxations are studied in detail in (Laurent 2001a, 2002). The sets $Q_t(G)$ will be defined in Section 1.2, but we do not need here a precise definition of the sets $N^t(G)$ and $N_+^t(G)$; let us simply mention that they are obtained by applying the Lovász-Schrijver N and N_+ operators to the relaxation of the cut polytope defined by the triangle inequalities. It is shown in (Laurent 2002) that

$$Q_{t+2}(G) \subseteq N_+^t(G) \subseteq N^t(G) \tag{1}$$

(for $t \geq 0$) and in (Laurent 2001a) that $\mathrm{CUT}(G) = N^t(G)$ if G has t edges whose contraction produces a graph with no K_5-minor. Define the rank $\rho_L(G)$ of graph G with respect to the Lasserre procedure as the smallest integer t for which $\mathrm{CUT}(G) = Q_t(G)$; analogously, define $\rho_N(G)$ (resp., $\rho_{N_+}(G)$) as the smallest t for which $\mathrm{CUT}(G) = N^t(G)$ (resp., $N_+^t(G)$). Thus,

$$\rho_L(G) - 2 \leq \rho_{N_+}(G) \leq \rho_N(G) \leq \max(0, n - \alpha(G) - 3)$$

and, in particular, $\rho_N(K_n) \leq n-4$ for $n \geq 4$. It is conjectured in (Laurent 2001a) that $\rho_N(K_n) = n-4$; that is, $n-4$ iterations are in fact needed for finding $\mathrm{CUT}(K_n)$ using the N operator. Equality $\rho_N(K_n) = n-4$ is known to hold for $n \leq 7$.

The main contribution of this paper is to show the lower bound

$$\rho_L(K_n) \geq \left\lceil \frac{n}{2} \right\rceil$$

for the rank of the complete graph K_n with respect to the Lasserre procedure. As a consequence, this implies the lower bound

$$\rho_N(K_n) \geq \rho_{N_+}(K_n) \geq \left\lceil \frac{n}{2} \right\rceil - 2$$

for the rank of the Lovász-Schrijver procedure.

1.2 Semidefinite relaxations for the cut polytope

Let $G = (V, E)$ be a graph with $V = \{1, \ldots, n\}$ and let $t \geq 1$ be an integer; $\mathcal{U}_t(V)$ denotes the collection of subsets of V whose cardinality is $\leq t$ and has the same parity as t, and $\mathcal{E}(V)$ the collection of even subsets of V. Given a vector $y \in \mathbb{R}^{\mathcal{E}(V)}$, its *moment matrix of order t* is the matrix $M_t(y)$ indexed by $\mathcal{U}_t(V)$ with (I,J)-th entry $y(I\Delta J)$.

It is useful to observe that $M_t(y)$ can alternatively be indexed by the set $\{I\Delta O \mid I \in \mathcal{U}_t(V)\}$ (for any given set $O \subseteq V$). From this follows, in particular, that

$$M_n(y) = M_{n-1}(y).$$

Indeed, say, n is even. Then, $M_n(y)$ is indexed by all even subsets while $M_{n-1}(y)$ is indexed by all odd subsets. As the collection of odd subsets of V is equal to $\{I\Delta O \mid I \in \mathcal{E}(V)\}$ where O is any given odd set, $M_{n-1}(y)$ can be assumed to be indexed by all even sets and, therefore, $M_{n-1}(y) = M_n(y)$.

Let $\mathcal{F}_t(n)$ denote the set of moment matrices of order t that are positive semidefinite and have an all ones main diagonal. Finally, let $Q_t(G)$ denote the projection of $\mathcal{F}_t(n)$ on the subspace $\mathbb{R}^E$ indexed by the edge set of G. Then the set $Q_t(G)$ is a semidefinite relaxation of the cut polytope, i.e.,

$$\mathrm{CUT}(G) \subseteq Q_t(G) \tag{2}$$

and

$$\mathrm{CUT}(G) = Q_{n-1}(G). \tag{3}$$

The inclusion (2) follows from the fact that each cut $\delta(A)$ is equal to the projection on $\mathbb{R}^E$ of the moment matrix of the vector $y^A \in \mathbb{R}^{\mathcal{E}(V)}$ defined by

$$y^A(I) := (-1)^{|I\cap A|} \quad \text{for } I \in \mathcal{E}(V) \tag{4}$$

and from the fact that

$$M_{n-1}(y^A) = y^A(y^A)^T \tag{5}$$

is therefore positive semidefinite. Equality (3) is proved by Lasserre (2001b) using results about moment sequences. The following elementary proof was given by Laurent (2001b,2002).

LEMMA 1 *Given $y \in \mathbb{R}^{\mathcal{E}(V)}$, the eigenvectors of matrix $M_{n-1}(y)$ are the 2^{n-1} distinct vectors y^A ($A \subseteq V$) with respective eigenvalues $y^T y^A$.*

PROOF. Direct verification. ∎

Hence any matrix $Y = M_{n-1}(y) \in \mathcal{F}_{n-1}(n)$ can be written as

$$Y = \sum_{A\subseteq V\setminus\{n\}} \lambda_A y^A (y^A)^T$$

where $\lambda_A := \frac{1}{2^{n-1}} y^T y^A \geq 0$ and $\sum_A \lambda_A = 1$. Therefore, $\mathcal{F}_{n-1}(n)$ is a 2^{n-1}-dimensional simplex with the cut matrices $M_{n-1}(y^A) = y^A(y^A)^T$ as vertices and its projection on the edge subspace $\mathbb{R}^E$ is the cut polytope $\mathrm{CUT}(G)$. The following properties of the Lasserre rank $\rho_L(G)$ are shown in (Laurent 2002).

Proposition 2 **(i)** *If G has an edge e whose contraction produces a graph G/e with $\rho_L(G/e) \leq t$, then $\rho_L(G) \leq t+1$.*

(ii) *The class of graphs G with $\rho_L(G) \leq t$ is closed under taking minors.*

In view of Proposition 2 (ii) (by the results of Robertson and Seymour (1988)), the class of graphs with bounded rank t can be characterized by a *finite* list of minimal forbidden minors. Such a list is known only for $t \leq 2$; namely,

$$\rho_L(G) \leq 1 \Longleftrightarrow G \text{ has no } K_3 - \text{minor};$$

$$\rho_L(G) \leq 2 \Longleftrightarrow G \text{ has no } K_5 - \text{minor}.$$

The exact value of the rank of K_n is known for $n \leq 7$:

$$\rho_L(K_2) = 1,\ \rho_L(K_3) = \rho_L(K_4) = 2,\ \rho_L(K_5) = \rho_L(K_6) = 3,\ \rho_L(K_7) = 4.$$

Hence, $\rho_L(K_n) \leq n-3$ for $n \geq 6$. Generally, we conjecture:

Conjecture 3 *$\rho_L(K_n) = \lceil \frac{n}{2} \rceil$ for all $n \geq 2$.*

We show in this paper that $\lceil \frac{n}{2} \rceil$ is a lower bound for the Lasserre rank of K_n.

Theorem 4 *$\rho_L(K_n) \geq \lceil \frac{n}{2} \rceil$ for all $n \geq 2$.*

Observe that, in view of Proposition 2, it suffices to show Theorem 4 for all odd values of n and Conjecture 3 for all even values of n.

1.3 Method of proof for Theorem 4

We indicate here the proof method for Theorem 4. As observed earlier we can assume that n is odd, $n \geq 3$. Our objective is to show that the inclusion $\mathrm{CUT}(K_n) \subseteq Q_{\lceil \frac{n}{2} \rceil - 1}(K_n)$ is strict. For this we consider the inequality:

$$\sum_{1 \leq i < j \leq n} x_{ij} \geq \frac{1-n}{2} \tag{6}$$

which is valid for the cut polytope $\mathrm{CUT}(K_n)$, since $e^T x x^T e = (e^T x)^2 \geq 1$ for all $x \in \{\pm 1\}^n$ (with e denoting the all ones vector). Define

$$\begin{array}{rllcll} p_t^* := & \min & \displaystyle\sum_{1 \leq i < j \leq n} x_{ij} & = & \min & \displaystyle\sum_{1 \leq i < j \leq n} y_{ij} \\ & \text{s.t.} & x \in Q_t(K_n) & & \text{s.t.} & M_t(y) \succeq 0 \\ & & & & & y_\emptyset = 1 \end{array} \tag{7}$$

Then, for $1 \leq t \leq n$,

$$-\frac{n}{2} \leq p_t^* \leq \frac{1-n}{2}. \tag{8}$$

The lower bound $-\frac{n}{2}$ follows from the fact that $M_t(y) \succeq 0$ implies that $M_1(y) \succeq 0$ and thus $e^T M_1(y) e = n + 2\sum_{1\le i<j\le n} y_{ij} \ge 0$. The inequality (6) is valid for $N_+^{\lceil \frac{n}{2}\rceil -2}(K_n)$ ((Laurent 2001a), Proposition 5.1) and thus for $Q_{\lceil \frac{n}{2}\rceil}(K_n)$ by (1). That is,

$$p_t^* = \frac{1-n}{2} \quad \text{for } t \ge \left\lceil \frac{n}{2} \right\rceil .$$

We show here that

$$p_t^* = -\frac{n}{2} \quad \text{for } t \le \left\lceil \frac{n}{2} \right\rceil - 1. \tag{9}$$

For this we have to construct a positive semidefinite moment matrix $M_t(y)$ with $y_\emptyset = 1$ and $\sum_{ij} y_{ij} = -\frac{n}{2}$. By symmetry, we can assume that all y_{ij} are equal; that is, $y_{ij} = -\frac{1}{n-1}$ for all $1 \le i < j \le n$. Direct inspection tells us what the remaining coordinates of y should be. Namely, define the scalars $a_0, a_2, \ldots, a_{n-1}$ by

$$a_0 := 1, \; a_{2r+2} := -a_{2r}\frac{2r+1}{n-2r-1} \quad \text{for } r = 0, 1, \ldots, \frac{n-3}{2}; \tag{10}$$

in other words, $a_{2r} = (-1)^r \prod_{\substack{i \text{ odd} \\ 1\le i\le 2r-1}} \frac{i}{n-i} = \left(-\frac{1}{4}\right)^r \frac{\binom{2r}{r}}{\binom{\frac{n-1}{2}}{r}}$; and define the vector $y \in \mathbb{R}^{\mathcal{E}(V)}$ by

$$y_I := a_{|I|} \quad \text{for all even subsets } I \subseteq V. \tag{11}$$

Then Theorem 4 will follow from the following result, whose proof is given in the next section.

THEOREM 5 *The matrix $M_{\lceil \frac{n}{2}\rceil -1}(y)$ is positive semidefinite, where y is given by* (10), (11).

2 Proof of Theorem 5

Set $k := \lceil \frac{n}{2} \rceil - 1 = \frac{n-1}{2}$ and $Y := M_k(y)$ where y is given by (10), (11). Thus Y is indexed by the set $\mathcal{U}_k(V)$ and has entries $Y(I,J) = y(I\Delta J) = a_{|I\Delta J|}$. The order of Y is

$$N := \sum_{\substack{i\le k \\ i\equiv k \bmod 2}} \binom{n}{i}. \tag{12}$$

We proceed as follows for proving that Y is positive semidefinite. First we show that Y has a large kernel, namely, that

$$\dim \ker Y \ge D := \sum_{\substack{i\le k \\ i\not\equiv k \bmod 2}} \binom{n}{i} \tag{13}$$

(see Corollary 8). Second we identify a principal submatrix Z of Y of order $N - D$ which is positive definite (see Proposition 10). By the interlacing property of eigenvalues (see, e.g., Theorem 4.3.15 in (Horn and Johnson 1990)), this implies that Y has at least $N - D$ positive eigenvalues. Therefore, Y has exactly D zero eigenvalues and $N - D$ positive eigenvalues which shows that $Y \succeq 0$, thus proving Theorem 5.

2.1 Identifying a large kernel of Y

For $H \subseteq V$, define the vector $z^H \in \{0,1\}^{\mathcal{U}_k(V)}$ by

$$z^H(I) := 1 \text{ if and only if } |I\Delta H| = 1, \text{ i.e., if } I = H\Delta\{i\} \text{ for some } i \in V$$

and set $\mathcal{U}'_k(V) := \{I \subseteq V \mid |I| \leq k \text{ and } |I| \not\equiv k \bmod 2\}$.

Lemma 6 *We have: $Yz^H = 0$ for all $H \in \mathcal{U}'_k(V)$.*

Proof. For $H \in \mathcal{U}'_k(V)$ and $I \in \mathcal{U}_k(V)$, we have:

$$Yz^H(I) = \sum_{J \in \mathcal{U}_k(V)} a_{|I\Delta J|} z^H(J) = \sum_{i=1}^{n} a_{|I\Delta H\Delta\{i\}|}.$$

As the cardinalities of the sets I and H have distinct parities, $|I\Delta H| =: 2r+1$ for some $0 \leq r < k$. Then,

$$Yz^H(I) = (2r+1)a_{2r} + (n-2r-1)a_{2r+2}$$

which is equal to 0, by the definition (10) of a. ∎

Lemma 7 *The vectors z^H ($H \in \mathcal{U}'_k(V)$) are linearly independent.*

Proof. For $T \subseteq V$, define the vector $e_T \in \{0,1\}^{\mathcal{U}_k(V)}$ with entries $e_T(I) = 1$ if and only if $T \subseteq I$. Then we have:

$$(e_T)^T(z^H) = \begin{cases} 0 & \text{if } |H\cap T| \leq |T|-2 \\ 1 & \text{if } |H\cap T| = |T|-1 \\ n-|T| & \text{if } H \supseteq T. \end{cases}$$

Consider a linear dependency $\sum_{H \in \mathcal{U}'_k(V)} \alpha_H z^H = 0$. We show that all α_H are equal to 0. For this we first show by induction on $t = 0, 1, \ldots, k$ that

$$\sum_{H \in \mathcal{U}'_k(V) | H \supseteq T} \alpha_H = 0 \text{ for any set } T \subseteq V \text{ with } |T| = t. \tag{14}$$

Taking the scalar product of both sides of $\sum_H \alpha_H z^H = 0$ with the vector $e_\emptyset = (1,\ldots,1)^T$ we find that $\sum_H \alpha_H = 0$; that is, (14) holds for the base of induction $t = 0$. Suppose that (14) holds for $t-1$; we show that it also holds for t. Given a set $T \subseteq V$ of size t, take the inner product of $\sum_H \alpha_H z^H = 0$ with the vector e_T. This yields:

$$\sum_{H | |H\cap T| = |T|-1} \alpha_H + (n-t) \sum_{H | H \supseteq T} \alpha_H = 0. \tag{15}$$

We have:

$$\sum_{H||H\cap T|=t-1} \alpha_H = \sum_{i\in T} \sum_{H|H\cap T=T\setminus\{i\}} \alpha_H = \sum_{i\in T} \left(\sum_{H|H\supseteq T\setminus\{i\}} \alpha_H - \sum_{H|H\supseteq T} \alpha_H \right)$$

which is equal to $-t \sum_{H|H\supseteq T} \alpha_H$ since $\sum_{H|H\supseteq T\setminus\{i\}} \alpha_H = 0$ by the induction assumption. Combining with (15), this implies that $0 = (n-2t)\sum_{H|H\supseteq T} \alpha_H$ and thus (14) holds.

One can now easily derive that $\alpha_H = 0$ for all $H \in \mathcal{U}'_k(V)$ by induction on $|H| \le k-1$. ∎

COROLLARY 8 *The dimension of the kernel of Y is at least the value D given in* (13).

2.2 Identifying a large positive definite principal submatrix of Y

Let Z denote the principal submatrix of Y indexed by the collection of all subsets $I \subseteq \{1, \ldots, n-1\}$ of cardinality k.

LEMMA 9 *The order of Z is equal to $N - D$, where N and D are given by (12) and* (13), *respectively.*

PROOF. The order of Z being equal to $\binom{n-1}{k}$, we have to show the identity:

$$\sum_{\substack{i\le k\\ i\equiv k \bmod 2}} \binom{n}{i} = \binom{n-1}{k} + \sum_{\substack{i\le k\\ i\not\equiv k \bmod 2}} \binom{n}{i}. \tag{16}$$

The collection of sets $I \subseteq V$ with $|I| \le k$ and $|I| \equiv k \bmod 2$ can be partitioned into the following two classes:

- the sets $I \subseteq V \setminus \{n\}$ with $|I| \le k$ and $|I| \equiv k \bmod 2$,
- and the sets $I = J \cup \{n\}$ for $J \subseteq V \setminus \{n\}$ with $|J| \le k-1$ and $|J| \not\equiv k \bmod 2$.

Therefore, the left hand side of (16) is equal to

$$\sum_{\substack{i\le k\\ i\equiv k \bmod 2}} \binom{n-1}{i} + \sum_{\substack{i\le k-1\\ i\not\equiv k \bmod 2}} \binom{n-1}{i}$$

which is equal to $\binom{n-1}{k} + \sum_{i\le k-1} \binom{n-1}{i}$. Using the identity: $\binom{n-1}{i} + \binom{n-1}{i+1} = \binom{n}{i+1}$, we can now conclude that

$$\sum_{i\le k-1} \binom{n-1}{i} = \sum_{\substack{j\le k-1\\ j\not\equiv k \bmod 2}} \binom{n}{j}$$

and thus (16) holds. ∎

Remains now to prove that matrix Z is positive definite. For this we use the fact that Z belongs to the Bose-Mesner algebra of the Johnson scheme $J(n-1,k)$, which implies that its eigenvalues can be explicitely expressed in terms of the Eberlein polynomials.

We briefly recall some definitions and refer to Bannai and Ito (1984) or van Lint and Wilson (1992) for details about the Johnson scheme. Given integers k, v with $k \leq \frac{v}{2}$, the Johnson scheme $J(v,k)$ is defined on the set $\mathcal{X}$ of all k-subsets of $\{1,\ldots,v\}$. Its adjacency matrices are the 0/1 matrices $A_0, A_1, \ldots, A_k$ indexed by $\mathcal{X}$ with the (I,J)-th entry of A_i being equal to 1 if and only if $|I \Delta J| = 2i$; thus $A_0 = I$ and $A_0 + \ldots + A_k = J$. A fundamental property is that the linear span of $A_0, \ldots, A_k$ is closed under matrix multiplication and forms a commutative algebra, called the *Bose-Mesner algebra* of the scheme. Therefore, there exists a common orthonormal basis of eigenvectors for all matrices in the Bose-Mesner algebra. For $\ell = 0, 1, \ldots, k$, the distinct eigenvalues of A_ℓ are given by the Eberlein polynomial:

$$P_\ell(u) = \sum_{j=0}^{\ell} (-1)^j \binom{u}{j} \binom{k-u}{\ell-j} \binom{v-k-u}{\ell-j} \tag{17}$$

for $u = 0, 1, \ldots, k$. Therefore, the distinct eigenvalues of a matrix $X := \sum_\ell x_\ell A_\ell$ in the Bose-Mesner algebra of $J(v,k)$ are $\sum_\ell x_\ell P_\ell(u)$ for $u = 0, \ldots, k$.

In our case, we have $v = n-1 = 2k$ and matrix Z having the form $\sum_{\ell=0}^{k} a_{2\ell} A_\ell$ therefore belongs to the Bose-Mesner algebra of the Johnson scheme $J(n-1,k)$. Hence the distinct eigenvalues of Z are

$$\lambda_u := \sum_{\ell=0}^{k} a_{2\ell} P_\ell(u) = \sum_{\ell=0}^{k} a_{2\ell} \sum_{j=0}^{\ell} (-1)^j \binom{u}{j} \binom{k-u}{\ell-j}^2 \tag{18}$$

for $u = 0, \ldots, k$.

Proposition 10 *We have: $\lambda_u > 0$ for all $u = 0, \ldots, k$; that is, matrix Z is positive definite.*

The rest of the section gives the proof of Proposition 10. In order to evaluate λ_u, we use the tool of hypergeometric series. A detailed account on hypergeometric series and on how they can be used for explicitely computing rational series is given in the book by Petkovsek et al. (1996). We thank Dimitrii Pasechnik for bringing to our attention this book and a result in (Grigoriev et al. 2001, section 8) whose proof technique uses hypergeometric series and has inspired our proof below.

Let us briefly recall some facts we need. A *hypergeometric series* is a series $\sum_{i\geq 0} \alpha_i$ where $\alpha_0 = 1$ and the ratio $\frac{\alpha_{i+1}}{\alpha_i}$ is the quotient of two polynomials in i, say

$$\frac{\alpha_{i+1}}{\alpha_i} = \frac{(i+a_1)\cdots(i+a_p)}{(i+b_1)\cdots(i+b_q)} \frac{x}{i+1}$$

where $x, a_1, \ldots, a_p, b_1, \ldots, b_q$ are complex numbers such that $b_1, \ldots, b_q$ are not nonpositive integers. Equivalently, the series $\sum_{i\geq 0} \alpha_i$ has the form

$$\sum_{i\geq 0} \frac{(a_1)_i \cdots (a_p)_i}{(b_1)_i \cdots (b_q)_i} \frac{x^i}{i!} \tag{19}$$

which is commonly abbreviated as ${}_pF_q\left[\begin{matrix} a_1 \dots a_p \\ b_1 \dots b_q \end{matrix} ; x\right]$. Here $(a)_i$ denotes the *rising factorial function* defined by

$$(a)_i := \begin{cases} a(a+1)\cdots(a+i-1) & \text{if } i \geq 1, \\ 1 & \text{if } i = 0. \end{cases}$$

The monograph by Petkovsek et al. (1996) contains a number of identities permitting to compute explicitely certain hypergeometric series; in particular, the *Gauss identity*:

$$ {}_2F_1\left[\begin{matrix} a \; b \\ c \end{matrix} ; 1\right] = \frac{\Gamma(c-a-b)\Gamma(c)}{\Gamma(c-a)\Gamma(c-b)} \tag{20}$$

if b is a nonpositive integer or if $c-a-b$ has a positive real part. Here $\Gamma(.)$ is the Gamma function which is defined on the complex plane except on the nonpositive integers. Recall that

$$\Gamma(n+1) = n! = (1)_n, \; \frac{\Gamma(a+n)}{\Gamma(a)} = (a)_n \tag{21}$$

for an integer $n \geq 0$.

We now proceed to computing the value of λ_u in (18). Let us first rewrite λ_u by changing the summation in the following way. Set $i = \ell - j$. Then,

$$\lambda_u = \sum_{i=0}^{k-u} \binom{k-u}{i}^2 \left[\sum_{j=0}^{k-i} a_{2i+2j}(-1)^j \binom{u}{j}\right].$$

In a first step, we compute the inner sum

$$\Sigma_i := \sum_{j=0}^{k-i} a_{2i+2j}(-1)^j \binom{u}{j} =: \sum_{j\geq 0} \alpha_j.$$

We have that $\alpha_0 = a_{2i}$ and

$$\frac{\alpha_{j+1}}{\alpha_j} = \frac{(j+i+1/2)}{(j+i-k)} \frac{(j-u)}{(j+1)}.$$

Therefore,

$$\Sigma_i = a_{2i} \cdot {}_2F_1\left[\begin{matrix} -u & i+1/2 \\ i-k & \end{matrix} ; 1\right].$$

(This hypergeometric series is well-defined although $i-k$ is a negative integer; indeed, $(-u)_h = 0$ for all $h \geq u+1$ and thus the summation obtained by expanding the above hypergeometric series as in (19) can be restricted to $0 \leq h \leq u$ in which case $(i-k)_h \neq 0$.) Using Gauss formula (20) and relation (21), we find that

$$ {}_2F_1\left[\begin{matrix} -u & i+1/2 \\ i-k & \end{matrix} ; 1\right] = \frac{\Gamma(u-k-1/2)\Gamma(i-k)}{\Gamma(-k-1/2)\Gamma(i-k+u)} = \frac{(-k-1/2)_u}{(i-k)_u}.$$

$n=3$	$n=5$	$n=7$	$n=9$	$n=11$
0	0	0	0	0
$\lambda_1 := \frac{3}{2}$	$\lambda_2 := \frac{5}{4}\lambda_1$	$\lambda_4 := \frac{7}{6}\lambda_2$	$\lambda_6 := \frac{9}{8}\lambda_4$	$\lambda_9 := \frac{11}{10}\lambda_6$
	$\lambda_3 := \frac{13}{8}$ (*)	$\lambda_5 := \frac{7}{6}\lambda_3$	$\lambda_7 := \frac{9}{8}\lambda_5$	$\lambda_{10} := \frac{11}{10}\lambda_7$
			$\lambda_8 := \frac{263}{128}$ (*)	$\lambda_{11} := \frac{11}{10}\lambda_8$

Figure 1: Distinct eigenvalues of Y_n

Therefore,

$$\lambda_u = \sum_{i=0}^{k-u} a_{2i} \binom{k-u}{i}^2 \frac{(-k-1/2)_u}{(i-k)_u} =: \sum_{i\geq 0} \beta_i.$$

We compute this sum using again hypergeometric series. We have that $\beta_0 = \frac{(-k-1/2)_u}{(-k)_u}$ and

$$\frac{\beta_{i+1}}{\beta_i} = \frac{i+1/2}{i-k} \left(\frac{i+u-k}{i+1}\right)^2 \frac{i-k}{i-k+u} = \frac{(i+1/2)(i+u-k)}{(i+1)^2}.$$

Therefore,

$$\lambda_u = \beta_0 \cdot {}_2F_1 \left[\begin{matrix} 1/2 & u-k \\ 1 & \end{matrix} ; 1 \right].$$

Using again Gauss formula (20) and (21), we find that λ_u is equal to

$$\beta_0 \frac{\Gamma(k-u+1/2)\Gamma(1)}{\Gamma(1/2)\Gamma(k-u+1)} = \frac{(-k-1/2)_u}{(-k)_u} \frac{(1/2)_{k-u}}{(k-u)!} = \frac{1\cdot 3\cdots(2k+1)}{2^k \cdot k! \cdot (2k-2u+1)}.$$

Therefore, $\lambda_u > 0$ for all $u = 0, \ldots, k$. This concludes the proof of Proposition 10 and thus of Theorem 5.

2.3 More on the moment matrix Y

The matrix $Y_n := M_{\frac{n-1}{2}}(y)$ whose positive semidefiniteness was proven in Theorem 5, seems to have quite remarkable structural properties. Let us mention a few properties observed in some earlier attempts for proving Theorem 5.

About the spectrum of Y_n. We have computed the eigenvalues of Y_n for odd $n = 3, \ldots, 11$. The distinct eigenvalues are displayed in Figure 1. All eigenvalues have multiplicity greater than 1, except the eigenvalues λ_3, λ_8 marked with a star (*) which have multiplicity one. Interestingly, the distinct eigenvalues of Y_n for $\frac{n-1}{2}$ odd appear to be of the form $\frac{n}{n-1}\lambda$, where the λ's are the distinct eigenvalues of Y_{n-2}. The same holds when $\frac{n-1}{2}$ is even except that a new eigenvalue with multiplicity one appears (namely, λ_3 for $n=5$ and λ_8 for $n=9$). Although we could not prove it, it seems quite likely that this behaviour of the spectrum of Y_n holds for any odd n. If true, this could lead to an alternative proof of positive semidefiniteness of Y_n.

A tentative iterative proof. Matrix Y_n has a natural block decomposition into the blocks Y_{rs} ($r,s \leq k,\ r,s \equiv k \bmod 2$), where Y_{rs} is the submatrix of Y_n whose rows are indexed by all r-subsets of V and columns by all s-subsets.

Let $V_0^{(r)} \cup V_1^{(r)} \cup \ldots \cup V_r^{(r)}$ denote the orthogonal decomposition of the space $\mathbb{R}^{\mathcal{X}}$ ($\mathcal{X} = \binom{n}{r}$) corresponding to the distinct eigenvalues of matrices in the Bose-Mesner algebra of the Johnson scheme $J(n,r)$. Then the orthogonal projections $E_i^{(r)}$ of $\mathbb{R}^{\mathcal{X}}$ onto $V_i^{(r)}$ form another basis of the Bose-Mesner algebra of $J(n,r)$.

Suppose (to fix ideas) that k is odd. Then, Y_n has the form:

$$Y_n = \begin{pmatrix} Y_{11} & Y_{13} & Y_{15} & \ldots \\ Y_{31} & Y_{33} & Y_{35} & \ldots \\ Y_{51} & Y_{53} & Y_{55} & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

The upper left corner Y_{11} of Y_n belongs to the Bose-Mesner algebra of $J(n,1)$. Moreover,

$$Y_{11} = \frac{n}{n-1} E_1^{(1)},\ Y_{i1}E_0^{(1)} = 0 \ \text{ for odd } i \geq 3. \tag{22}$$

Therefore, $Y_n \succeq 0$ if and only if the matrix

$$Y_n^{(1)} := \begin{pmatrix} Y_{33} & Y_{35} & \ldots \\ Y_{53} & Y_{55} & \ldots \\ \vdots & \vdots & \ddots \end{pmatrix} - \frac{n-1}{n} \begin{pmatrix} Y_{31} \\ Y_{51} \\ \vdots \end{pmatrix} \begin{pmatrix} Y_{31} \\ Y_{51} \\ \vdots \end{pmatrix}^T$$

is positive semidefinite. We verified that the upper left corner $Y_{33}^{(1)}$ of $Y_n^{(1)}$ belongs to the Bose-Mesner algebra of $J(n,3)$ and

$$Y_{33}^{(1)} = \beta E_3^{(3)},\ Y_{i3}^{(1)} E_u^{(3)} = 0 \ \text{ for odd } i \geq 5,\ u = 0,1,2 \tag{23}$$

(with $\beta := \frac{n(n-2)(n-4)}{(n-1)(n-3)(n-5)}$). Analogously, $Y_n^{(1)} \succeq 0$ if and only if the matrix

$$Y_n^{(2)} := \begin{pmatrix} Y_{55}^{(1)} & Y_{57}^{(1)} & \ldots \\ Y_{75}^{(1)} & Y_{77}^{(1)} & \ldots \\ \vdots & \vdots & \ddots \end{pmatrix} - \frac{1}{\beta} \begin{pmatrix} Y_{53}^{(1)} \\ Y_{73}^{(1)} \\ \vdots \end{pmatrix} \begin{pmatrix} Y_{53}^{(1)} \\ Y_{73}^{(1)} \\ \vdots \end{pmatrix}^T$$

is positive semidefinite. Again the upper left corner of $Y_n^{(2)}$ is a positive multiple of the last idempotent $E_5^{(5)}$, etc. Quite probably, this reasoning can be carried out throughout, thus 'peeling off' one block layer of Y_n at each iteration until finding $Y_n^{(\frac{n-3}{4})}$ which should be proven to be a positive multiple of $E_k^{(k)}$. If true, this would show that Y_n is positive semidefinite. We did not succeed to carry out this type of proof. Checking (23) and its further analogues becomes indeed technically very complicated. A difficulty being that although the entries in all the blocks of the initial matrix Y_n are governed by a single sequence a, at a later iteration $t \geq 1$ the entries in each of the blocks of $Y_n^{(t)}$ are governed by distinct sequences.

3 Algebraic Interpretation

The Lasserre hierarchy of semidefinite relaxations was originally motivated by results about moment sequences and the dual theory of nonnegative polynomials and their representation as sums of squares of polynomials. It applies to general polynomial programming problems of the form:

$$p^* := \min\ p_0(x) \ \text{ subject to } x \in K := \{x \in \mathbb{R}^n \mid p_\ell(x) \geq 0\ \forall \ell = 1, \ldots, m\} \tag{24}$$

where $p_0, p_1, \ldots .p_m$ are polynomials in $x = (x_1, \ldots, x_n)$. Given $x \in \mathbb{R}^n$, define the vector $y := (x^\alpha)_{\alpha \in \mathbb{Z}_+^n}$ whose components are the monomials $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ and, given an integer $t \geq 1$, define $M_t(y)$ as the matrix indexed by the sequences $\alpha \in \mathbb{Z}_+^n$ with $\sum_i \alpha_i \leq t$ and whose (α, β)-entry is equal to $y_{\alpha+\beta}$. Observe that the matrix $M_t(y)$ is positive semidefinite as well as each matrix $M_t(p_\ell * y)$, setting $p_\ell * y := (\sum_\beta (p_\ell)_\beta y_{\alpha+\beta})_\alpha$. This observation leads to defining the following semidefinite relaxations of problem (24):

$$\begin{aligned} p_t^* := \quad & \min \quad \sum_\alpha (p_0)_\alpha y_\alpha \\ & \text{s.t.} \quad M_t(y) \succeq 0 \\ & \qquad\ \ M_{t-v_\ell}(p_\ell * y) \succeq 0 \ (\ell = 1, \ldots, m) \\ & \qquad\ \ y_0 = 1 \end{aligned} \tag{25}$$

for any $t \geq \max(v_\ell)$, where $v_\ell := \lceil \frac{\deg(p_\ell)}{2} \rceil$. The dual semidefinite program to (25) takes the form:

$$\begin{aligned} \rho_t^* := \quad & \max \rho \\ & \text{s.t. } p_0(x) - \rho = q_0(x) + \sum_\ell q_\ell(x) p_\ell(x) \\ & \text{where } q_0, q_\ell \text{ are sums of squares of polynomials and} \\ & \deg(q_0) \leq 2t,\ \deg(q_\ell) \leq 2(t - v_\ell). \end{aligned} \tag{26}$$

(See Lasserre (2001a,2002) for details.) We have: $\rho_t^* \leq p_t^* \leq p^*$ by weak semidefinite duality. Lasserre (2001a) shows asymptotic convergence of the bound ρ_t^* to p^* as t goes to infinity. His proof is based on a result of Putinar (1993) asserting that, if K is compact and satisfies some additional technical condition, then any polynomial *positive* on K has a decomposition of the form $q_0(x) + \sum_\ell q_\ell(x) p_\ell(x)$, where q_0, q_ℓ are sums of squares of polynomials. Putinar's result holds, in particular, in the ± 1 case when the polynomials $x_i^2 = 1$ are present in the description of K. Moreover, in this case, there is finite convergence in n steps. This was proved by Lasserre (2001b) using a result about rank extensions of moment matrices from Curto and Fialkow (2000), but the elementary proof mentioned earlier based on Lemma 1 also applies (see Laurent (2001b)). Indeed, the presence of the constraints $x_i^2 - 1 = 0$ in the description of K permits to replace the moment matrices indexed by integer sequences by their principal submatrices indexed by $0-1$ sequences (or equivalently by subsets of V) as considered earlier in this paper.

In the case of max-cut considered in this paper, we have the problem of finding the minimum value p^* of a quadratic polynomial $p_0(x) := \sum_{1\le i<j\le n} c_{ij}x_ix_j$ over the set $K := \{x \in \mathbb{R}^n \mid x_i^2 - 1 = 0 \ \forall i = 1,\ldots,n\}$. Thus the semidefinite program (25) gives the lower bound:

$$p_t^* := \min \sum_{1\le i<j\le n} c_{ij}y_{ij} \ \text{ subject to } M_t(y) \succeq 0, \ y_\emptyset = 1$$

(as in (7)) and its dual (26) gives the bound ρ_t^* defined as the maximum value of ρ for which the polynomial $\sum_{1\le i<j\le n} c_{ij}x_ix_j - \rho$ has a decomposition

$$q_0(x) + \sum_{i=1}^n q_i(x)(x_i^2 - 1) \tag{27}$$

where q_0 is a sum of squares of polynomials and q_0, q_i have degree $\le 2t$. In fact, there is no duality gap, i.e., $\rho_t^* = p_t^*$ (since the Slater condition holds). As

$$\sum_{1\le i<j\le n} x_ix_j + \frac{n}{2} = \frac{1}{2}\left(\sum_{i=1}^n x_i\right)^2 + \frac{1}{2}\sum_{i=1}^n (1 - x_i^2),$$

we find that the value $\rho = -\frac{n}{2}$ is dual feasible and thus $p_t^* \ge -\frac{n}{2}$ for any $t \ge 1$ (as in (8)). The result from relation (9) (or equivalently from Theorem 5) shown earlier can be reformulated in the following way: If the polynomial $\sum_{1\le i<j\le n} x_ix_j - \rho$ has a decomposition (27) where q_0 is a sum of squares and q_0, q_i have degree $\le 2\left(\lceil \frac{n}{2}\rceil - 1\right)$, then $\rho \le -\frac{n}{2}$.

Similarly, Conjecture 3 amounts to proving the following result: If $\sum_{1\le i<j\le n} c_{ij}x_{ij} \ge p^*$ is an inequality valid for $\mathrm{CUT}(K_n)$, then the polynomial: $\sum_{1\le i<j\le n} c_{ij}x_ix_j - p^*$ has a decomposition (27) where q_0 is a sum of squares and q_0, q_i have degree $\le 2\lceil \frac{n}{2}\rceil$. This is known to be true for $n \le 7$. For $n \ge 7$, such a decomposition is known to exist in which q_0, q_i have degree at most $2(n-3)$ (since $\rho_L(K_n) \le n-3$).

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