

THE OPERATOR Ψ FOR THE CHROMATIC NUMBER OF A GRAPH*

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Abstract. We investigate hierarchies of semidefinite approximations for the chromatic number $\chi(G)$ of a graph G . We introduce an operator Ψ mapping any graph parameter $\beta(G)$, nested between the stability number $\alpha(G)$ and $\chi(\overline{G})$, to a new graph parameter $\Psi_\beta(G)$, nested between $\alpha(\overline{G})$ and $\chi(G)$; $\Psi_\beta(G)$ is polynomial time computable if $\beta(G)$ is. As an application, there is no polynomial time computable graph parameter nested between the fractional chromatic number $\chi^*(\cdot)$ and $\chi(\cdot)$ unless $P = NP$. Moreover, based on the Motzkin–Straus formulation for $\alpha(G)$, we give (quadratically constrained) quadratic and copositive programming formulations for $\chi(G)$. Under some mild assumptions, $n/\beta(G) \leq \Psi_\beta(G)$, but, while $n/\beta(G)$ remains below $\chi^*(G)$, $\Psi_\beta(G)$ can reach $\chi(G)$ (e.g., for $\beta(\cdot) = \alpha(\cdot)$). We also define new polynomial time computable lower bounds for $\chi(G)$, improving the classic Lovász theta number (and its strengthenings obtained by adding nonnegativity and triangle inequalities); experimental results on Hamming graphs, Kneser graphs, and DIMACS benchmark graphs will be given in the follow-up paper [N. Gvozdencović and M. Laurent, *SIAM J. Optim.*, 19 (2008), pp. 592–615].

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1. Introduction. The chromatic number $\chi(G)$ of a graph $G = (V, E)$ is the minimum number of colors needed to color the nodes of G in such a way that adjacent nodes receive distinct colors. Computing $\chi(G)$ is an NP-hard problem [11], and it is also hard to approximate $\chi(G)$ within $|V(G)|^{1/14-\epsilon}$ for any $\epsilon > 0$ [1]. An obvious lower bound for $\chi(G)$ is the clique number $\omega(G)$, defined as the maximum size of a clique (i.e., a set of pairwise adjacent nodes) in G ; computing $\omega(G)$ is also hard [11] as well as approximating $\omega(G)$ within $|V(G)|^{1/6-\epsilon}$ for any $\epsilon > 0$ [1]. A well-known stronger lower bound for $\chi(G)$ is $\overline{\vartheta}(G) := \vartheta(\overline{G})$, the theta number of the complementary graph, introduced by Lovász [23] (see (2.3)). The theta number satisfies the “sandwich inequality”:

$$\omega(G) \leq \overline{\vartheta}(G) \leq \chi(G),$$

and it can be computed to any arbitrary precision in polynomial time since it can be formulated via a semidefinite program. It can also be used for approximately coloring the graph (see [5, 8, 17]). Intensive research has been done for strengthening the bound $\overline{\vartheta}(G)$ towards $\omega(G)$ or, equivalently, $\vartheta(G)$ towards the stability number $\alpha(G)$; see, e.g., [6, 19, 20, 21, 24, 26, 30, 32, 34]. Here $\alpha(G) = \omega(\overline{G})$, the maximum size of a stable set (i.e., a set of pairwise nonadjacent nodes) in G . In particular, hierarchies of semidefinite (or linear) bounds were constructed that find $\alpha(G)$ in $\alpha(G)$ steps [19, 20, 24, 34]. As $\chi(G)$ can be formulated via a 0/1 linear program (see, e.g.,

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[7]), the lift-and-project methods of [19, 24, 34] can in principle be applied to derive hierarchies of semidefinite approximations finding $\chi(G)$ in finitely many steps. To the best of our knowledge such hierarchies have not been investigated in detail so far.

In this paper we propose a systematic investigation of semidefinite approximations for $\chi(G)$. One of our main contributions is a simple construction permitting one to derive from any graph parameter $\beta(G)$ nested between $\alpha(G)$ and $\bar{\chi}(G)$ a new graph parameter $\Psi_\beta(G)$ nested between $\omega(G)$ and $\chi(G)$. For this, given an integer $t \geq 0$, let $K_t \square G$ denote the Cartesian product of the two graphs G and K_t , with node set

$$(1.1) \quad V(K_t \square G) := V(K_t) \times V(G) = \bigcup_{p=1}^t V_p, \quad \text{where } V_p := \{pi \mid i \in V(G)\},$$

and having an edge (pi, qj) if $(p \neq q \text{ and } i = j)$ or if $(p = q \text{ and } ij \in E(G))$. Chvátal [4] observed the following useful reduction of the chromatic number to the stability number:

$$(1.2) \quad \chi(G) \leq t \iff \alpha(K_t \square G) = |V(G)|.$$

(Reverse reductions, from the stability number to the chromatic number, can be found in Poljak [31] and in Schrijver [33].) Given a graph parameter $\beta(\cdot)$ nested between $\alpha(\cdot)$ and $\bar{\chi}(\cdot)$, relation (1.2) motivates the introduction of the new graph parameter $\Psi_\beta(\cdot)$, defining $\Psi_\beta(G)$ as the smallest integer $t \geq 0$ for which $\beta(K_t \square G) = |V(G)|$. Among other properties, $\Psi_\alpha(G) = \chi(G)$, $\Psi_{\bar{\chi}}(G) = \Psi_{\bar{\chi}^-}(G) = \omega(G)$, $\Psi_\vartheta(G) = \lceil \bar{\vartheta}(G) \rceil$, and $\Psi_{\vartheta'}(G) = \lceil \bar{\vartheta}^+(G) \rceil$. Here χ^* is the fractional chromatic number, and ϑ' and ϑ^+ are variations of ϑ obtained by adding certain nonnegativity conditions; see section 2.1. Moreover, the operator Ψ is monotone nonincreasing and, if $\beta(G)$ is polynomial time computable (resp., given by a semidefinite program), then the same holds for $\Psi_\beta(G)$. A somewhat surprising application is that there does *not* exist a polynomial time computable graph parameter nested between the fractional chromatic number and the chromatic number unless $P = NP$ (see Theorem 2.6). As another application we can give (quadratically constrained) quadratic and copositive programming formulations for $\chi(G)$ based on the Motzkin–Straus formulation for $\alpha(G)$ (see section 2.5).

The operator Ψ permits one to transform any hierarchy of upper bounds for $\alpha(G)$ into a hierarchy of lower bounds for $\chi(G)$. In this paper we study in particular hierarchies of lower bounds for $\chi(G)$ related to the Lasserre hierarchy $\text{las}^{(r)}(G)$ ($r \in \mathbb{N}$) for $\alpha(G)$ [19], which finds $\alpha(G)$ at order $r = \alpha(G)$ and refines several other known hierarchies for $\alpha(G)$. More precisely, we consider two hierarchies $\psi^{(r)}(G)$ and $\Psi_{\text{las}^{(r)}}(G)$ of lower bounds for the chromatic number $\chi(G)$, which satisfy $\psi^{(1)}(G) = \bar{\vartheta}(G)$ and $\psi^{(2)}(G) \geq \bar{\vartheta}^{\Delta}(G)$ (Meurdesoif strengthening—see section 2.1), and

$$\frac{|V(G)|}{\text{las}^{(r)}(G)} \leq \psi^{(r)}(G) \leq \Psi_{\text{las}^{(r)}}(G) \leq \chi(G).$$

The parameter $\psi^{(r)}(G)$ has the same computational cost as $\text{las}^{(r)}(G)$, but it cannot go beyond the fractional chromatic number; in fact, $\psi^{(r)}(G) = \chi^*(G)$ for $r \geq \alpha(G)$. The parameter $\Psi_{\text{las}^{(r)}}(G)$ has a higher computational cost than $\text{las}^{(r)}(G)$ (one has to evaluate $\text{las}^{(r)}(K_t \square G)$ for $O(\log n)$ queries on $t \leq n$), but it finds $\chi(G)$ at step $r = n$. Dukanovic and Rendl [9] introduced recently another hierarchy for $\chi(G)$, which is related to the hierarchy of de Klerk and Pasechnik [6] for $\alpha(G)$, both being based on

copositive programming. The hierarchy of Dukanovic and Rendl remains, however, bounded by the fractional chromatic number; see section 3.5 for details.

Although polynomial time computable for any fixed r , the parameters $\psi^{(r)}(G)$ and $\Psi_{\text{las}^{(r)}}(G)$ are yet too costly to compute for large values of n already for order $r = 2$. We propose some variations $\psi(G)$ and $\Psi_\ell(G)$ of the order 2 bounds, which are at least as good as $\bar{\vartheta}^+(G)$. As will be shown in the follow-up paper [14], for vertex-transitive graphs, the computation of $\psi(G)$ involves a semidefinite program with two matrices of sizes $n + 1$ and n , while the computation of $\Psi_\ell(G)$ can be reduced to $O(\log n)$ semidefinite programs with matrices of sizes $2n + 1, 2n, n$, and n ; these formulations are obtained by exploiting symmetries in the structure of the semidefinite programs and symmetries arising from the permutation group $\text{Sym}(t)$ acting on the complete graph K_t .

More details about the results of this paper can also be found in [12].

Contents of the paper. In section 2 we present the operator Ψ and its main properties, we discuss various ways for computing $\Psi_\beta(G)$, and we give (quadratically constrained) quadratic and copositive programming formulations for $\chi(G)$. In section 3 we investigate two hierarchies of lower bounds for $\chi(G)$ related to the hierarchy of Lasserre for $\alpha(G)$ and converging, respectively, to $\chi^*(G)$ and $\chi(G)$. This leads to two bounds $\psi(G)$ and $\Psi_\ell(G)$ formulated via semidefinite programs involving matrices of size $O(n)$. Finally we explore the link between our bounds and the copositive programming-based hierarchies of de Klerk and Pasechnik [6] for $\alpha(G)$ and of Dukanovic and Rendl [9] for $\chi(G)$.

Notation. Given a graph $G = (V, E)$, \bar{G} denotes its complementary graph whose edges are the pairs $uv \notin E(G)$ ($u, v \in V(G)$, $u \neq v$). Throughout we set $V := V(G)$, $n = |V|$, and to avoid trivial technicalities we assume that $G \neq K_n$ and $G \neq \bar{K}_n$, where K_n denotes the complete graph on n nodes. For two graphs G and G' , their Cartesian product $G \square G'$ has node set $V(G) \times V(G')$, with two nodes $uu', vv' \in V(G) \times V(G')$ being adjacent in $G \square G'$ if and only if ($u = v$ and $u'v' \in E(G')$) or ($uv \in E(G)$ and $u' = v'$). For an integer $t \geq 1$, we sometimes set $G_t = K_t \square G$ as a shorthand notation for the Cartesian product of G and K_t , whose node set is as in (1.1). Given a graph parameter $\beta(\cdot)$, $\bar{\beta}(\cdot)$ is the graph parameter defined by $\bar{\beta}(G) := \beta(\bar{G})$ for any graph G .

Throughout, the letters \mathbf{I}, \mathbf{J} , and e denote, respectively, the identity matrix, the all-ones matrix, and the all-ones vector (of the suitable size); \mathbb{N} is the set of nonnegative integers. For $n \times n$ matrices A, B , $\text{Tr}(A) = \sum_{i=1}^n A_{ii}$ and $\langle A, B \rangle = \text{Tr}(A^T B) = \sum_{i,j=1}^n A_{ij} B_{ij}$. Moreover, the notation $A \succeq 0$ means that A is a symmetric positive semidefinite matrix.

Given a finite set V , $\mathcal{P}(V)$ denotes the collection of all subsets of V . Given an integer r , set $\mathcal{P}_r(V) := \{I \in \mathcal{P}(V) \mid |I| \leq r\}$. $\mathcal{P}_r(V)$ contains the empty subset of V which we will denote as $\mathbf{0}$; thus, for instance, $\mathcal{P}_1(V) = \{\mathbf{0}, \{i\} \mid (i \in V)\}$. We sometimes identify $\mathcal{P}_1(V) \setminus \{\mathbf{0}\}$ with V ; i.e., we write $\{i\}$ as i and $\{i, j\}$ as ij , and, given a vector $x \in \mathbb{R}^{\mathcal{P}(V)}$ we also set $x_i := x_{\{i\}}$, $x_{ij} := x_{\{i,j\}}$, $x_{ijk} := x_{\{i,j,k\}}$, etc.

Let V be a finite set, and let \mathcal{G} be a subgroup of $\text{Sym}(V)$, the group of permutations of V , also denoted as $\text{Sym}(n)$ if $|V| = n$. \mathcal{G} acts on $\mathcal{P}(V)$ by letting $\sigma(I) := \{\sigma(i) \mid i \in I\}$ for $I \subseteq V$, $\sigma \in \mathcal{G}$. Moreover, \mathcal{G} acts on vectors and matrices indexed by V (and thus on vectors and matrices indexed by $\mathcal{P}(V)$). Namely, for $\sigma \in \mathcal{G}$, $x \in \mathbb{R}^V$, and $M \in \mathbb{R}^{V \times V}$, set $\sigma(x) := (x_{\sigma(i)})_{i \in V}$ and $\sigma(M) := (M_{\sigma(i), \sigma(j)})_{i, j \in V}$. One says that M is invariant under the action of \mathcal{G} if $\sigma(M) = M$ for all $\sigma \in \mathcal{G}$: The matrix $\frac{1}{|\mathcal{G}|} \sum_{\sigma \in \mathcal{G}} \sigma(M)$, the “symmetrization” of M obtained by applying the Reynolds

operator, is invariant under the action of \mathcal{G} . The same holds analogously for vectors. A semidefinite program is said to be invariant under action of \mathcal{G} if, for any feasible matrix X and any $\sigma \in \mathcal{G}$, the matrix $\sigma(X)$ is again feasible with the same objective value; then the optimum value of the program remains unchanged if we restrict to invariant feasible solutions and, in particular, there is an invariant optimal solution.

The automorphism group $\text{Aut}(G)$ of a graph $G = (V, E)$ consists of all $\sigma \in \text{Sym}(V)$ preserving the set of edges. G is said to be vertex-transitive when, given any two nodes $i, j \in V$, there exists $\sigma \in \text{Aut}(G)$, with $\sigma(i) = j$.

2. New parameters and formulations.

2.1. Some known graph parameters. We review here some classic bounds for the stability number $\alpha(G)$ and the chromatic number $\chi(G)$ of a graph $G = (V, E)$. We give some equivalent formulations for the bounds. Some work may be required to derive some of them; for details see, e.g., [22, 33].

- The *fractional clique cover number*, also known as the *fractional chromatic number* of \bar{G} :

$$(2.1) \quad \begin{aligned} \overline{\chi}^*(G) := \max \quad & e^T x & = \min \quad & e^T \lambda \\ \text{s.t.} \quad & \sum_{i \in C} x_i \leq 1 \text{ (} C \text{ clique)} & \text{s.t.} \quad & \sum_{C \text{ clique}} \lambda_C \chi^C = e, \\ & x \in \mathbb{R}_+^V, & & \lambda \geq 0. \end{aligned}$$

It is well known (and easy to verify) that $\alpha(G) \leq \overline{\chi}^*(G) \leq \overline{\chi}(G)$, and

$$(2.2) \quad \omega(G)\overline{\chi}^*(G) \geq |V(G)|, \text{ with equality when } G \text{ is vertex-transitive.}$$

It is hard to compute the fractional chromatic number, and, for some $\epsilon > 0$, there is no polynomial time algorithm to approximate $\overline{\chi}^*(G)$ within $|V(G)|^\epsilon$ unless $P = NP$ [25].

- *Lovász's theta number* (introduced in [23]):

$$(2.3) \quad \begin{aligned} \vartheta(G) := \max \quad & \langle \mathbf{J}, X \rangle & = \min \quad & t \\ \text{s.t.} \quad & \text{Tr}(X) = 1 & \text{s.t.} \quad & U_{ii} = 1 \text{ (} i \in V \text{),} \\ & X_{ij} = 0 \text{ (} ij \in E(G) \text{),} & & U_{ij} = -\frac{1}{t-1} \text{ (} ij \in E(\bar{G}) \text{),} \\ & X \succeq 0, & & U \succeq 0, t \geq 2, \end{aligned}$$

where X and U are symmetric matrices indexed by V . The minimization program in the above definition of $\vartheta(G)$ is used, e.g., in [17] for constructing a vector k -coloring. We will also use the following equivalent formulation:

$$(2.4) \quad \begin{aligned} \vartheta(G) = \max \quad & \sum_{i \in V} X_{ii} \\ \text{s.t.} \quad & X_{\mathbf{00}} = 1, X_{ij} = 0 \text{ (} ij \in E \text{),} \\ & X_{ii} = X_{\mathbf{0}i} \text{ (} i \in V \text{), } X \succeq 0, \end{aligned}$$

where the matrix variable X is indexed by the set $\mathcal{P}_1(V)$. Lovász [23] proved the following analogue of (2.2) for the pair $(\vartheta, \overline{\vartheta})$:

$$(2.5) \quad \vartheta(G)\overline{\vartheta}(G) \geq |V(G)|, \text{ with equality when } G \text{ is vertex-transitive.}$$

- The *strengthening of the theta number* of [26, 32]:

$$(2.6) \quad \begin{aligned} \vartheta'(G) &:= \max_{\text{s.t.}} \langle \mathbf{J}, X \rangle &= \min_{\text{s.t.}} & t \\ & \text{Tr}(X) = 1 & U_{ii} = 1 \ (i \in V), \\ & X_{ij} = 0 \ (ij \in E(G)), & U_{ij} \leq -\frac{1}{t-1} \ (ij \in E(\overline{G})), \\ & X \succeq 0, \ X \geq 0, & U \succeq 0, \ t \geq 2. \end{aligned}$$

- *Szegedy's number* [36]:

$$(2.7) \quad \begin{aligned} \vartheta^+(G) &:= \max_{\text{s.t.}} \langle \mathbf{J}, X \rangle &= \min_{\text{s.t.}} & t \\ & \text{Tr}(X) = 1 & U_{ii} = 1 \ (i \in V), \\ & X_{ij} \leq 0 \ (ij \in E(G)), & U_{ij} = -\frac{1}{t-1} \ (ij \in E(\overline{G})), \\ & X \succeq 0, & U_{ij} \geq -\frac{1}{t-1} \ (ij \in E(G)), \\ & & U \succeq 0, \ t \geq 2. \end{aligned}$$

Szegedy [36] showed that the analogue of (2.2) and (2.5) also holds for the pair $(\vartheta', \overline{\vartheta^+})$:

$$(2.8) \quad \vartheta'(G)\overline{\vartheta^+}(G) \geq |V(G)|, \text{ with equality when } G \text{ is vertex-transitive.}$$

Thus one may see the pairs (α, χ^*) , $(\vartheta, \overline{\vartheta})$, and $(\vartheta', \overline{\vartheta^+})$ as “reciprocal” pairs of graph parameters. We will see later in this paper (see Theorem 3.1(e)) that they are in fact part of a more general hierarchy of reciprocal pairs.

- Meurdesoif [27] defines *the bound* $\vartheta^{+\Delta}(G)$ obtained by adding the “triangle inequalities” $U_{ij} + U_{jk} - U_{ik} \leq 1$ (for $ij, jk \in E$) to the minimization program defining $\vartheta^+(G)$ in (2.7).

The above parameters satisfy

$$\alpha(G) \leq \vartheta'(G) \leq \vartheta(G) \leq \vartheta^+(G) \leq \vartheta^{+\Delta}(G) \leq \overline{\chi^*}(G) \leq \overline{\chi}(G).$$

The inequality $\vartheta^{+\Delta}(G) \leq \overline{\chi^*}(G)$ will follow from Theorem 3.1(c) and (d), and the other inequalities follow directly by using the definitions.

2.2. The operator Ψ . By using relation (1.2), we see that the chromatic number of a graph G can be defined as the optimum solution of the following program:

$$(2.9) \quad \chi(G) = \min_{t \in \mathbb{N}} t \text{ s.t. } \alpha(K_t \square G) = |V(G)|.$$

This fact motivates the following definition.

DEFINITION 2.1. *Given a graph parameter $\beta(\cdot)$ satisfying*

$$(2.10) \quad \min \left(\alpha(\cdot), \frac{|V(\cdot)|}{\omega(\cdot)} \right) \leq \beta(\cdot) \leq \overline{\chi}(\cdot),$$

define the graph parameter $\Psi_\beta(\cdot)$ by

$$(2.11) \quad \Psi_\beta(G) := \min_{t \in \mathbb{N}} t \text{ s.t. } \beta(K_t \square G) = |V(G)|.$$

Note added in proof. The operator Ψ applies in fact to the larger range of graph parameters $\beta(\cdot)$ satisfying $\frac{|V(\cdot)|}{\chi(\cdot)} \leq \beta(\cdot) \leq \bar{\chi}(\cdot)$, thus including graph parameters satisfying relation (2.10). See [12] for the details.

LEMMA 2.2.

- (a) The graph parameter $\Psi_\beta(G)$ is well defined if $\beta(\cdot)$ satisfies (2.10).
- (b) The operator Ψ is monotone nonincreasing; that is, $\Psi_{\beta_2}(\cdot) \leq \Psi_{\beta_1}(\cdot)$ if $\beta_1(\cdot)$ and $\beta_2(\cdot)$ satisfy (2.10) and $\beta_1(\cdot) \leq \beta_2(\cdot)$.
- (c) $\Psi_\alpha(G) = \chi(G)$.
- (d) $\Psi_\beta(G) = \omega(G)$ for $\beta(\cdot) := \frac{|V(\cdot)|}{\omega(\cdot)}$.
- (e) $\Psi_{\bar{\chi}}(G) = \omega(G)$.
- (f) $\Psi_\beta(G) = \chi(G)$ for $\beta(\cdot) := \min(\alpha(\cdot), \frac{|V(\cdot)|}{\omega(\cdot)})$.
- (g) If $\beta(\cdot)$ satisfies (2.10), then

$$(2.12) \quad \omega(\cdot) \leq \Psi_\beta(\cdot) \leq \chi(\cdot).$$

Proof. (a) Assume that $\beta(\cdot)$ satisfies (2.10), and let $1 \leq t \leq n := |V(G)|$. As $\omega(K_t \square G) = \max(t, \omega(G))$, we have $\frac{|V(K_t \square G)|}{\omega(K_t \square G)} \geq t$; together with $\alpha(K_t \square G) \geq t$, this implies that $\beta(K_t \square G) \geq t$. On the other hand, $\beta(K_t \square G) \leq \bar{\chi}(K_t \square G) \leq n$. Therefore, $\beta(K_n \square G) = n$, thus showing that $\Psi_\beta(G)$ is well-defined.

(b) If $\beta_1(\cdot) \leq \beta_2(\cdot)$ satisfies (2.10), then $\beta_1(K_t \square G) = n$ implies that $\beta_2(K_t \square G) = n$, which gives $\Psi_{\beta_2}(G) \leq \Psi_{\beta_1}(G)$.

(c) The identity $\Psi_\alpha(G) = \chi(G)$ follows directly from (2.9).

(d) For $\beta(\cdot) := \frac{|V(\cdot)|}{\omega(\cdot)}$, the identity $\Psi_\beta(G) = \omega(G)$ follows from the fact that $\omega(K_t \square G) = \max(t, \omega(G))$.

(e) We verify that $\Psi_{\bar{\chi}}(G) = \omega(G)$. As $\bar{\chi}(\cdot) \geq \frac{|V(\cdot)|}{\omega(\cdot)}$, we deduce by using (b) and (d) that $\Psi_{\bar{\chi}}(G) \leq \Psi_{|V|/\omega}(G) = \omega(G)$. To show the reverse inequality, consider a clique C in G of size $\omega(G)$, and let C_t be the subset of $V(K_t \square G)$ consisting of all of the copies of the nodes in C . Thus C_t is covered by t cliques of $K_t \square G$. As the remaining nodes of $K_t \square G$ can be covered by $n - |C|$ cliques, we have $\bar{\chi}(K_t \square G) \leq t + n - |C|$. Therefore $\bar{\chi}(K_t \square G) = n$ implies that $t \geq |C| = \omega(G)$, which shows that $\Psi_{\bar{\chi}}(G) \geq \omega(G)$.

(f) Consider now the parameter $\beta(\cdot) := \min(\alpha(\cdot), \frac{|V(\cdot)|}{\omega(\cdot)})$. As $\beta(\cdot) \leq \alpha(\cdot)$, we deduce by using (b) that $\Psi_\beta(G) \geq \Psi_\alpha(G) = \chi(G)$, and equality holds since one can easily verify that $\beta(K_t \square G) = n$ for $t := \chi(G)$.

(g) Relation (2.12) now follows directly by using again (b). \square

COROLLARY 2.3. If $\beta(\cdot)$ is a graph parameter satisfying $\frac{|V(\cdot)|}{\omega(\cdot)} \leq \beta(\cdot) \leq \bar{\chi}(\cdot)$, then $\Psi_\beta = \omega$. In particular, $\Psi_{\bar{\chi}^*} = \omega$.

Proof. The proof follows directly from Lemma 2.2(b), (d), and (e) and (2.2). \square

Therefore, the operator Ψ takes a graph parameter $\beta(G)$ (nested, e.g., between $\alpha(G)$ and $\bar{\chi}(G)$) and produces the integer lower bound $\Psi_\beta(G)$ (nested between $\omega(G)$ and $\chi(G)$) for the chromatic number $\chi(G)$; Figure 2.1 illustrates how the operator Ψ acts on various parameters. As $\alpha(G)\chi^*(G) \geq |V(G)|$,

$$\beta(G) \geq \alpha(G) \implies \chi(G) \geq \chi^*(G) \geq \frac{|V(G)|}{\beta(G)}.$$

The next lemma shows that, under the mild assumption (2.13), $\Psi_\beta(G)$ is at least as good as the obvious lower bound $|V(G)|/\beta(G)$ for $\chi(G)$. However, $\Psi_\beta(G)$ may be equal to $\chi(G)$, while $\frac{|V(G)|}{\beta(G)}$ always remains below the fractional chromatic number $\chi^*(G)$. One can easily verify that condition (2.13) holds for the graph parameters

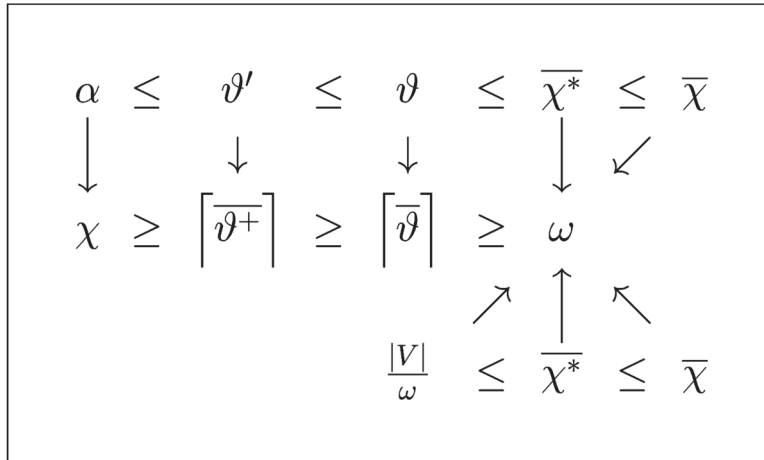


FIG. 2.1. Converting graph parameters by the operator Ψ .

considered in this paper, e.g., for $\beta(\cdot) = \alpha(\cdot), \chi(\cdot), \chi^*(\cdot), \vartheta(\cdot), \vartheta'(\cdot)$ and the parameter $\text{las}^{(r)}(\cdot)$ defined later in (3.1) (see [12] for details).

LEMMA 2.4. Assume that the graph parameter $\beta(\cdot)$ satisfies $\alpha(\cdot) \leq \beta(\cdot) \leq \bar{\chi}(\cdot)$ and

$$(2.13) \quad \beta(K_t \square G) \leq t\beta(G) \text{ for all } t \in \mathbb{N}.$$

Then $\Psi_\beta(G) \geq \frac{|V(G)|}{\beta(G)}$.

Proof. If $\beta(K_t \square G) = |V(G)|$, then $|V(G)| \leq t\beta(G)$, i.e., $t \geq \frac{|V(G)|}{\beta(G)}$. \square

REMARK 2.5. If $\beta(\cdot) \in [\alpha(\cdot), \bar{\chi}(\cdot)]$, then $\Psi_\beta(G) - \frac{|V(G)|}{\beta(G)} \leq \chi(G) - \frac{|V(G)|}{\bar{\chi}(G)}$, with equality, e.g., when G is a perfect graph (since then $\alpha(G) = \bar{\chi}(G) = \beta(G)$ and $\omega(G) = \chi(G) = \Psi_\beta(G)$). Hence the gap $\Psi_\beta(G) - \frac{|V(G)|}{\beta(G)}$ can be made arbitrarily large. For instance, this gap is equal to $n - \frac{2n}{n+1} = n \frac{n-1}{n+1}$ when G is the disjoint union of a clique of size n and n isolated points.

We will investigate in the next section how the operator Ψ applies to the theta number $\vartheta(\cdot)$ and its strengthening $\vartheta'(\cdot)$. We now present an easy but quite surprising consequence of Lemma 2.2 concerning the complexity of graph parameters nested between the fractional chromatic and chromatic numbers or, more generally, in the interval $[\frac{|V(\cdot)|}{\omega(\cdot)}, \bar{\chi}(\cdot)]$. The key observation is that the operator Ψ maps the whole interval $[\frac{|V(\cdot)|}{\omega(\cdot)}, \bar{\chi}(\cdot)]$ to a single graph parameter (namely, the clique number $\omega(\cdot)$), which is hard to compute.

THEOREM 2.6. If $\beta(\cdot)$ is a graph parameter satisfying $\frac{|V(\cdot)|}{\omega(\cdot)} \leq \beta(\cdot) \leq \bar{\chi}(\cdot)$, then there is no algorithm permitting one to compute $\beta(G)$ in time polynomial in $|V(G)|$ unless $P = NP$. As $\frac{|V(\cdot)|}{\omega(\cdot)} \leq \bar{\chi}^*(\cdot) \leq \bar{\chi}(\cdot)$, the same conclusion holds if $\bar{\chi}^*(\cdot) \leq \beta(\cdot) \leq \bar{\chi}(\cdot)$.

Proof. By applying Lemma 2.2, we find that $\Psi_\beta(\cdot) = \omega(\cdot)$. Suppose that one can compute $\beta(G)$ in time $f(n)$, where f is a polynomial in $n = |V(G)|$. Then one can compute $\Psi_\beta(G) = \omega(G)$ in time $\sum_{l=1}^n f(ln)$, thus polynomial in n . As computing the clique number is an NP-hard problem [11], this implies that $P = NP$. \square

Let us mention a few graph parameters that are known to lie within the “hard” interval $[\chi^*, \chi]$. Hence none of them can be computed in polynomial time unless

$P = NP$; such a result was known already, e.g., for the circular chromatic number $\chi_c(G)$ [3].

The *circular graph chromatic number* (or *star chromatic number*) $\chi_c(G)$, introduced by Vince [37] and further studied, e.g., in [3, 39], is defined as follows. Given $r \in \mathbb{R}$, $r \geq 2$, a function $f : V(G) \rightarrow [0, r]$ is said to be a r -coloring if $1 \leq |f(u) - f(v)| \leq r - 1$ for all edges $uv \in E(G)$. Then $\chi_c(G)$ is defined as the infimum of all r for which G has a r -coloring. The following hold: $\chi(G) - 1 < \chi_c(G) \leq \chi(G)$ and $\chi^*(G) \leq \chi_c(G) \leq \chi(G)$ (see, e.g., [39]).

Another graph parameter lying in the hard interval $[\chi^*, \chi]$ is the *local chromatic number* $\chi_{\text{loc}}(G)$, introduced in [10] as the minimum over all proper colorings of G of the largest number of colors used to color the neighborhood $N_G(v) = \{w \in V(G) \mid vw \in E(G)\}$ of any vertex $v \in V(G)$. Obviously, $\chi_{\text{loc}}(G) \leq \chi(G)$ (the gap between the two parameters can in fact be arbitrarily large [10]), and Körner, Pilotto, and Simonyi [18] show that $\chi^*(G) \leq \chi_{\text{loc}}(G)$.

The *independence ratio* of a graph G is $i(G) := \frac{\alpha(G)}{|V(G)|}$, and its *Hall ratio* is $\rho(G) := \max_{H \subseteq G} \frac{|V(H)|}{\alpha(H)}$, where the maximum is taken over all subgraphs of G . For an integer $k \geq 1$, let $G^{\square k}$ denote the graph obtained by taking the Cartesian product of k copies of G . Then the *ultimate independence ratio* $I(G)$ and the *ultimate Hall ratio* $h_{\square}(G)$ are defined, respectively, as $I(G) := \lim_{k \rightarrow \infty} i(G^{\square k})$ and $h_{\square}(G) := \lim_{k \rightarrow \infty} \rho(G^{\square k})$. These graph parameters are studied, e.g., in [15, 16, 35]. In particular, the following relations with fractional and circular chromatic numbers are shown there:

$$\chi^*(G) \leq \frac{1}{I(G)} = h_{\square}(G) \leq \chi_c(G) \leq \chi(G)$$

(see [39] for the inequality $1 \leq I(G)\chi_c(G)$).

2.3. Action of the operator Ψ on the theta number. The next theorem shows that the operator Ψ maps the theta number $\vartheta(\cdot)$ to $\lceil \overline{\vartheta}(\cdot) \rceil$ and its strengthening $\vartheta'(\cdot)$ to $\lceil \overline{\vartheta}^+(\cdot) \rceil$. De Klerk, Pasechnik, and Warners [5] consider a graph parameter closely related to Ψ_{ϑ} for which they can also show that it coincides with $\lceil \overline{\vartheta}(\cdot) \rceil$.

THEOREM 2.7. *For any graph G the following hold:*

- (i) $\Psi_{\vartheta}(G) = \lceil \overline{\vartheta}(G) \rceil$,
- (ii) $\Psi_{\vartheta'}(G) = \lceil \overline{\vartheta}^+(G) \rceil$.

We first state two lemmas that we need for the proof of Theorem 2.7.

LEMMA 2.8. *Let X be a $t \times t$ block matrix, having an $n \times n$ matrix A as its diagonal blocks and an $n \times n$ matrix B as nondiagonal blocks, i.e.,*

$$(2.14) \quad X = \underbrace{\begin{pmatrix} A & B & \dots & B \\ B & A & \dots & B \\ \vdots & \vdots & \ddots & \vdots \\ B & B & \dots & A \end{pmatrix}}_{t \text{ blocks}}.$$

Then $X \succeq 0 \iff A - B \succeq 0$ and $A + (t - 1)B \succeq 0$.

Proof. We define a $t \times t$ block matrix U_t having the same block structure as the matrix X . For $p, q = 1, \dots, t$, let U_t^{pq} denote the (p, q) th block of U_t , defined by

$$(2.15) \quad U_t^{pq} := \begin{cases} \frac{1}{\sqrt{t}} \mathbf{I} & \text{if } p = 1 \text{ or } q = 1, \\ \left(\frac{1}{\sqrt{t+t}} - 1 \right) \mathbf{I} & \text{if } p = q \geq 2, \\ \frac{1}{\sqrt{t+t}} \mathbf{I} & \text{otherwise.} \end{cases}$$

Here \mathbf{I} stands for the identity matrix of order n . Notice that U_t is symmetric and orthogonal, i.e., $U_t(U_t)^T = \mathbf{I}$. Let $Y := (U_t)^T X U_t$. Then $Y \succeq 0$ if and only if $X \succeq 0$, and a simple calculation gives

$$(2.16) \quad Y = \begin{bmatrix} A + (t-1)B & 0 & \dots & 0 \\ 0 & A - B & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A - B \end{bmatrix},$$

which shows the lemma. \square

LEMMA 2.9. *For a positive semidefinite $n \times n$ matrix X , $n\text{Tr}(X) \geq \langle \mathbf{J}, X \rangle$, with equality if and only if $X = c\mathbf{J}$ for some nonnegative scalar c .*

Proof. As $X \succeq 0$, its entries satisfy $X_{ii} + X_{jj} \geq 2X_{ij}$ for all $i, j \in \{1, \dots, n\}$. Thus, $n \sum_{i=1}^n X_{ii} \geq \sum_{i,j=1}^n X_{ij}$. Equality holds if and only if $X_{ii} + X_{jj} = 2X_{ij}$ for all i, j , which gives $X_{ii} = X_{jj} = X_{ij}$ for all i, j . \square

Proof of Theorem 2.7. (i) As G has at least one edge, $\vartheta(G) < n$ and thus $\Psi_\vartheta(G) \geq 2$. Let (t, X) be a feasible solution for the program defining $\Psi_\vartheta(G)$; that is,

$$(2.17) \quad X \succeq 0, X_{uv} = 0 \ (uv \in E(K_t \square G)), \text{Tr}(X) = 1, \langle \mathbf{J}, X \rangle = n.$$

Here the matrix X is indexed by $V(K_t \square G) = \cup_{p=1}^t V_p$ (recall (1.1)) and $t \in \mathbb{N}, t \geq 2$. As the program (2.17) is invariant under action of the group $\text{Sym}(t)$, one may assume that X is invariant under action of $\text{Sym}(t)$. Then X has the block form (2.14). By using Lemma 2.8, (2.17) can be rewritten as

$$(2.18) \quad \begin{aligned} A - B \succeq 0, A + (t-1)B \succeq 0, A_{ij} = 0 \ (ij \in E(G)), \text{diag}(B) = 0, \\ \text{Tr}(A) = \frac{1}{t}, \langle \mathbf{J}, A + (t-1)B \rangle = \frac{n}{t}. \end{aligned}$$

Lemma 2.9 implies that $A + (t-1)B = \frac{1}{nt}\mathbf{J}$. By setting $U := nt(A - B)$, we find that

$$(2.19) \quad U = \frac{1}{t-1}(nt^2 A - \mathbf{J}).$$

One can verify that (t, U) is feasible for the program

$$(2.20) \quad \min t \text{ s.t. } \text{diag}(U) = e, U_{ij} = -\frac{1}{t-1} \ (ij \in E(G)), U \succeq 0, t \geq 2$$

defining the parameter $\bar{\vartheta}(G)$ (see (2.3)). As $t \in \mathbb{N}$, this implies that $\Psi_\vartheta(G) \geq \lceil \bar{\vartheta}(G) \rceil$. Conversely, let (t, U) be feasible for (2.20), with t an integer. Define the matrices A and B via the equations

$$(2.21) \quad A - B = \frac{1}{nt}U \text{ and } A + (t-1)B = \frac{1}{nt}\mathbf{J},$$

and let X be the corresponding block matrix as in (2.14). One can verify that (2.18) holds and thus (2.17) holds, too. That is, (t, X) is feasible for (2.17). Thus we have shown that

$$(2.22) \quad \Psi_\vartheta(G) = \min_{t \in \mathbb{N}} t \text{ s.t. } \text{diag}(U) = e, U_{ij} = -\frac{1}{t-1} \ (ij \in E(G)), U \succeq 0, t \geq 2.$$

We now show that $\Psi_\vartheta(G) \leq \lceil \bar{\vartheta}(G) \rceil$. For this, set $t := \bar{\vartheta}(G)$, and take an optimal solution U to the program (2.20). Then, by setting $Y := \frac{t-1}{\lceil t \rceil - 1}U + \frac{\lceil t \rceil - t}{\lceil t \rceil - 1}\mathbf{I}$, the pair $(\lceil t \rceil, Y)$ is feasible for (2.22) with objective value $\lceil t \rceil$, which implies that $\lceil t \rceil \geq \Psi_\vartheta(G)$. Thus equality $\lceil \bar{\vartheta}(G) \rceil = \Psi_\vartheta(G)$ holds.

The proof of (ii) is analogous to that of (i). Simply note that adding the condition $X \geq 0$ to (2.17) amounts to adding the condition $A, B \geq 0$ to (2.18) and thus, in view of (2.19), to adding the condition $U_{ij} \geq -\frac{1}{t-1}$ ($i, j \in V$) to (2.22). \square

2.4. Semidefinite programming formulation for the new bounds. We consider here issues related to the computation of $\Psi_\beta(G)$. We assume throughout that $\beta(\cdot)$ satisfies (2.10). There is an obvious way to find $\Psi_\beta(G)$, namely, by computing $\beta(K_t \square G)$ for each $t = 1, \dots, n$. We now observe that, when $\beta(\cdot)$ is monotone nondecreasing (with respect to taking induced subgraphs), one can use binary search, and it suffices to compute $\beta(K_t \square G)$ for $O(\log n)$ instances of t .

LEMMA 2.10. *Assume that*

$$(2.23) \quad \beta(K_t \square G) \leq \beta(K_{t+1} \square G) \text{ for all } t \in \mathbb{N}.$$

Then $\beta(K_t \square G) = n \iff \Psi_\beta(G) \leq t$.

Proof. The “only if” part follows from the definition of $\Psi_\beta(G)$. For the “if” part assume that $t_0 := \Psi_\beta(G) \leq t$. Then $\beta(K_{t_0} \square G) = n \leq \beta(K_t \square G)$ implies that $\beta(K_t \square G) = n$, since $\beta(K_t \square G) \leq \bar{\chi}(G) \leq n$. \square

Under assumption (2.23) one can use binary search for computing $\Psi_\beta(G)$. Namely, given $t_0 \in [1, n]$, compute $\beta(K_{t_0} \square G)$. There are two cases:

- $\beta(K_{t_0} \square G) < n$. Then $\Psi_\beta(G) \geq t_0 + 1$ (by the above lemma), and we can now restrict the search to $t \in [t_0 + 1, n]$.
- Or $\beta(K_{t_0} \square G) = n$. Then $\Psi_\beta(G) \leq t_0$, and we can restrict the search to $t \in [1, t_0]$.

Therefore, one can find $\Psi_\beta(G)$ by computing $\beta(K_t \square G)$ for $O(\log n)$ queries of t .

Observe that one may restrict the range of search for t . Suppose that we know a lower bound t_1 and an upper bound t_2 on $\chi(G)$; that is, $t_1 \leq \chi(G) \leq t_2$. Then we may assume that $t \leq t_2$ in the definition of $\Psi_\beta(G)$, and if we add the condition $t \geq t_1$, then one still obtains a lower bound for $\chi(G)$. Therefore, we may restrict the binary search to $t \in [t_1, t_2]$. For instance, one can choose $t_1 = 3$ if G is not bipartite, or $t_1 = \omega(G)$, and $t_2 = \Delta(G) + 1$ (or even $\Delta(G)$ by Brook’s theorem (see [33]) if G is not a clique or an odd circuit), $\Delta(G)$ being the maximum degree of G .

Next we show that $\Psi_\beta(G)$ can be formulated via a single semidefinite program when $\beta(\cdot)$ is given by a semidefinite program satisfying certain assumptions. Namely, our construction applies to the case when the semidefinite program defining $\beta(\cdot)$ involves at least one equality constraint of the form $\langle A, X \rangle = 1$, with $A \succeq 0$. Then one may assume without loss of generality (w.l.o.g.) that all other (in)equality constraints in the program are homogeneous, i.e., of the form $\langle B, X \rangle \geq 0$. (Write any equation $\langle B, X \rangle = 0$ as two opposite inequalities $\langle -B, X \rangle \geq 0$ and $\langle B, X \rangle \geq 0$.) So let us assume that, for an arbitrary graph H , we can express $\beta(H)$ as

$$(2.24) \quad \beta(H) = \max \langle C(H), X(H) \rangle \quad \text{s.t.} \quad \begin{aligned} \langle A(H), X(H) \rangle &= 1, \\ \mathcal{B}(H)(X(H)) &\geq 0, \\ X(H) &\succeq 0, \end{aligned}$$

where $C(H)$ and $A(H)$ are constant symmetric $n \times n$ matrices, $\mathcal{B}(H) : S_n \rightarrow \mathbb{R}^{d(H)}$ is a linear operator, and $X(H)$ is the matrix variable. Note that $d(\cdot)$ depends on H ,

e.g., $d(H) = 2|E(H)|$ in the formulation of $\vartheta(H)$. Moreover we assume that

$$(2.25) \quad A(H) \succeq 0,$$

$$(2.26) \quad \langle A(H), X(H) \rangle = 0 \implies \langle C(H), X(H) \rangle = 0.$$

Note that assumptions (2.23), (2.24), (2.25), and (2.26) hold, e.g., for $\vartheta(\cdot)$ or for the Lasserre hierarchy considered in section 3.1. Recall that our operator Ψ maps $\beta(\cdot)$ in the following way:

$$(2.27) \quad \begin{aligned} \Psi_\beta(G) := \min t & & = \min t \\ \text{s.t. } \beta(G_t) = n & & \text{s.t. } \langle C(G_t), X(G_t) \rangle = n, \\ & & \langle A(G_t), X(G_t) \rangle = 1, \\ & & \mathcal{B}(G_t)(X(G_t)) \geq 0, \\ & & X(G_t) \succeq 0. \end{aligned}$$

Here we use the more concise notation $G_t := K_t \square G$. Let us define

$$(2.28) \quad \begin{aligned} \Phi_\beta(G) := \min \sum_{t=1}^n t \langle A(G_t), X(G_t) \rangle & \text{ s.t. } \sum_{t=1}^n \langle C(G_t), X(G_t) \rangle = n, \\ & \sum_{t=1}^n \langle A(G_t), X(G_t) \rangle = 1, \\ & \mathcal{B}(G_t)(X(G_t)) \geq 0 \quad (t = 1, \dots, n), \\ & X(G_t) \succeq 0 \quad (t = 1, \dots, n). \end{aligned}$$

THEOREM 2.11. *Under assumptions (2.24), (2.25), and (2.26), $\Phi_\beta(G) = \Psi_\beta(G)$.*

Proof. Take a feasible solution $(t, X(G_t))$ for the program (2.27), and for $k \neq t$ set $X(G_k) := 0$. In this way one obtains a feasible solution for (2.28) with the same objective value as (2.27), which shows that $\Phi_\beta(G) \leq \Psi_\beta(G)$. Conversely, let $X(G_t)$ ($t = 1, \dots, n$) be a feasible solution for (2.28), and set $a_t := \langle A(G_t), X(G_t) \rangle$. Thus $a_t \geq 0$ since $A(G_t) \succeq 0$ (by assumption (2.25)) and $\sum_t a_t = 1$. Consider t for which $a_t > 0$. As $\langle A(G_t), \frac{X(G_t)}{a_t} \rangle = 1$, $\frac{X(G_t)}{a_t}$ is feasible for (2.24) (with $H = G_t$), which implies that $\langle C(G_t), \frac{X(G_t)}{a_t} \rangle \leq \beta(G_t) \leq n$; moreover, equality $\langle C(G_t), \frac{X(G_t)}{a_t} \rangle = n$ implies that $\beta(G_t) = n$ and thus $\Psi_\beta(G) \leq t$. Now we have

$$n = \sum_t \langle C(G_t), X(G_t) \rangle = \sum_{t|a_t>0} a_t \left\langle C(G_t), \frac{X(G_t)}{a_t} \right\rangle \leq \left(\sum_{t|a_t>0} a_t \right) n = n.$$

(Here we used assumption (2.26) for the second equality.) Therefore, equality holds throughout, which implies that $\Psi_\beta(G) \leq t$ whenever $a_t > 0$. Hence, $\sum_t t a_t = \sum_{t|a_t>0} t a_t \geq \Psi_\beta(G) (\sum_{t|a_t>0} a_t) = \Psi_\beta(G)$, which gives $\Phi_\beta(G) \geq \Psi_\beta(G)$. \square

Hence, under the assumptions (2.24), (2.25), and (2.26), the parameter $\Psi_\beta(G)$ can be formulated via the semidefinite program (2.28), which involves a block-diagonal matrix with diagonal blocks $X(G_1), \dots, X(G_n)$, each $X(G_t)$ being the matrix variable involved in the program (2.24) for the graph $H = G_t$. For instance, if (2.24) involves a matrix variable of order $f(V(H))$, then (2.28) involves a block-diagonal matrix with block sizes $f(n), f(2n), \dots, f(n^2)$. As explained above one can reduce the size of the program (2.28) by restricting the range of t in program (2.28) to $t \in [t_1, t_2]$, where $t_1 \leq \chi(G) \leq t_2$.

2.5. Copositive programming formulation for the chromatic number.

The technique used in section 2.4 can also be applied to derive (quadratically constrained) quadratic and copositive programming formulations for the chromatic number. Recall that a matrix X is *copositive* if $x^T X x \geq 0$ for all $x \geq 0$. A matrix X is *completely positive* if it belongs to the dual of the cone of copositive matrices, i.e., if it can be written as $X = \sum_i x_i x_i^T$ for some $x_i \geq 0$.

Our starting point is the theorem of Motzkin and Straus [28], which, for a graph G with adjacency matrix A_G , gives the following formulation for its stability number:

$$(2.29) \quad \frac{1}{\alpha(G)} = \min x^T (\mathbf{I} + A_G) x \text{ s.t. } x \in \mathbb{R}_+^{V(G)}, e^T x = 1,$$

or, equivalently (see [6]),

$$(2.30) \quad \alpha(G) = \min t \text{ s.t. } t(\mathbf{I} + A_G) - \mathbf{J} \text{ is copositive.}$$

By using (2.29), we can rewrite the program (2.9) as

$$(2.31) \quad \chi(G) = \min t \text{ s.t. } x_t^T (\mathbf{I} + A_{G_t}) x_t = \frac{1}{n}, e_t^T x_t = 1, x_t \in \mathbb{R}_+^{V(G_t)}.$$

Here and below e_t denotes the all-ones vector in $\mathbb{R}^{V(G_t)}$. By using the idea from section 2.4 let us define

$$(2.32) \quad \begin{aligned} \Phi_1(G) := \min & \sum_{t=1}^n t(e_t^T x_t)^2 \\ \text{s.t.} & \sum_{t=1}^n (e_t^T x_t)^2 = 1, \\ & \sum_{t=1}^n x_t^T (\mathbf{I} + A_{G_t}) x_t = \frac{1}{n}, \\ & x_t \in \mathbb{R}_+^{V(G_t)} \quad (t = 1, \dots, n). \end{aligned}$$

PROPOSITION 2.12. $\Phi_1(G) = \chi(G)$.

Proof. By taking a feasible solution (t, x_t) for the program (2.31) and setting $x_k = 0$ for $k \neq t$, we obtain a feasible solution for (2.32) with objective value t . Thus, $\Phi_1(G) \leq \chi(G)$. Conversely, let x_t ($t = 1, \dots, n$) be feasible for (2.32). Then

$$\frac{1}{n} = \sum_t x_t^T (\mathbf{I} + A_{G_t}) x_t = \sum_{t|x_t \neq 0} \frac{x_t^T}{e_t^T x_t} (\mathbf{I} + A_{G_t}) \frac{x_t}{e_t^T x_t} (e_t^T x_t)^2 \geq \frac{1}{n} \sum_{t|x_t \neq 0} (e_t^T x_t)^2 = \frac{1}{n}.$$

We have used $\frac{x_t^T}{e_t^T x_t} (\mathbf{I} + A_{G_t}) \frac{x_t}{e_t^T x_t} \geq \frac{1}{\alpha(G_t)} \geq \frac{1}{n}$. Hence equality holds throughout, which implies that $\alpha(G_t) = n$ if $x_t \neq 0$ and thus $\chi(G) \leq t$ if $x_t \neq 0$. Therefore,

$$\sum_t t(e_t^T x_t)^2 = \sum_{t|x_t \neq 0} t(e_t^T x_t)^2 \geq \chi(G) \sum_{t|x_t \neq 0} (e_t^T x_t)^2 = \chi(G).$$

This shows that $\Phi_1(G) \geq \chi(G)$. \square

Up to rescaling, we obtain the following formulation for $\chi(G)$ involving only quadratic constraints:

$$\begin{aligned}
 (2.33) \quad \chi(G) = \min \quad & \frac{1}{n^2} \sum_{t=1}^n t(e_t^T x_t)^2 \\
 \text{s.t.} \quad & \sum_{t=1}^n (e_t^T x_t)^2 = n^2, \\
 & \sum_{t=1}^n x_t^T (\mathbf{I} + A_{G_t}) x_t = n, \\
 & x_t \in \mathbb{R}_+^{V(G_t)} \quad (t = 1, \dots, n).
 \end{aligned}$$

It is not difficult to verify that the above program remains a formulation of $\chi(G)$ if we replace the condition $x_t \geq 0$ (for all t) by the condition that x_t is 0/1 valued (for all t). Therefore this gives a 0/1 (quadratically constrained) quadratic programming formulation for the chromatic number involving $O(n^3)$ variables.

By starting from (2.33), we can now derive a copositive programming formulation for $\chi(G)$. Namely, consider the program

$$\begin{aligned}
 (2.34) \quad \Phi_2(G) := \min \quad & \frac{1}{n^2} \sum_{t=1}^n t \langle \mathbf{J}, X_t \rangle \\
 \text{s.t.} \quad & \sum_{t=1}^n \langle \mathbf{J}, X_t \rangle = n^2, \\
 & \sum_{t=1}^n \langle \mathbf{I} + A_{G_t}, X_t \rangle = n, \\
 & X_t \text{ completely positive} \quad (t = 1, \dots, n).
 \end{aligned}$$

PROPOSITION 2.13. $\Phi_2(G) = \chi(G)$.

Proof. The formulation (2.33) for $\chi(G)$ implies directly that $\Phi_2(G) \leq \chi(G)$. Conversely, let X_t ($1 \leq t \leq n$) be a feasible solution for (2.34). Consider t for which $X_t \neq 0$. Say, $X_t = \sum_{i_t} x_{i_t} x_{i_t}^T$ where $x_{i_t} \geq 0$, $x_{i_t} \neq 0$ for all i_t . Thus $\lambda_{i_t} := \sqrt{\langle \mathbf{J}, x_{i_t} x_{i_t}^T \rangle} = e_t^T x_{i_t} > 0$. Set $y_{i_t} := \frac{x_{i_t}}{\lambda_{i_t}}$. By assumption, we have $\sum_t \langle n(\mathbf{I} + A_{G_t}) - \mathbf{J}, X_t \rangle = 0$. By (2.30), each matrix $n(\mathbf{I} + A_{G_t}) - \mathbf{J}$ is copositive, since $n \geq \alpha(G_t)$. This implies that $\langle n(\mathbf{I} + A_{G_t}) - \mathbf{J}, X_t \rangle = 0$ and thus $\langle n(\mathbf{I} + A_{G_t}) - \mathbf{J}, x_{i_t} x_{i_t}^T \rangle = 0$ for all i_t . From this follows that $\langle \mathbf{I} + A_{G_t}, y_{i_t} y_{i_t}^T \rangle = \frac{1}{n}$ for all i_t . As $e_t^T y_{i_t} = 1$, y_{i_t} is feasible for the program (2.31), implying that $\chi(G) \leq t$ whenever $X_t \neq 0$. Now $(1/n^2) \sum_t t \langle \mathbf{J}, X_t \rangle \geq (1/n^2) \chi(G) \sum_t \langle \mathbf{J}, X_t \rangle = \chi(G)$, giving $\Phi_2(G) \geq \chi(G)$. \square

By rewriting the condition $\sum_t \langle \mathbf{I} + A_{G_t}, X_t \rangle = n$ as $\sum_t \langle n(\mathbf{I} + A_{G_t}) - \mathbf{J}, X_t \rangle = 0$, the dual conic program of (2.34) reads:

$$(2.35) \quad \max_{y,z} y \text{ s.t. } \frac{1}{n^2} (t - y) \mathbf{J} + z (n(\mathbf{I} + A_{G_t}) - \mathbf{J}) \text{ copositive for } 1 \leq t \leq n.$$

There is no duality gap since the program (2.35) is strictly feasible. Thus (2.35) is yet another formulation of $\chi(G)$. This opens the road to another type of hierarchy of relaxations for $\chi(G)$, obtained by approximating the copositive cone by tractable subcones as suggested by Parrilo [29]. This type of approach based on copositive programming has been studied, e.g., in [2] for standard quadratic optimization problems, in [6, 13, 30] for the stable set problem, and recently in [9] for the coloring problem. We will come back to it in section 3.5.

3. Semidefinite hierarchies for (fractional) chromatic numbers. We have seen in the previous section how to construct semidefinite programming lower bounds for the chromatic number of a graph from semidefinite programming upper bounds on the stability number. Several hierarchies of such upper bounds for the stability number have been proposed in the literature, in particular, in [6, 19, 24, 30, 34]. These hierarchies were further studied and compared, e.g., in [13, 20]. It turns out that Lasserre’s hierarchy, proposed in [19], gives the tightest bounds. For this reason we focus in this section on this hierarchy, and we show how it can be used and transformed to produce hierarchies of lower bounds for the (fractional) chromatic number. We will also discuss the link with another hierarchy recently proposed by Dukanovic and Rendl [9] based on copositive programming.

3.1. Lasserre’s hierarchy towards the stability number. For a subset $S \subseteq V$ and an integer $r \geq 1$, define the vectors $\chi^S \in \{0, 1\}^V$ with i th entry 1 if and only if $i \in S$ (for $i \in V$) and $\chi^{S,r} \in \{0, 1\}^{\mathcal{P}_r(V)}$ with I th entry 1 if and only if $I \subseteq S$ (for $I \in \mathcal{P}_r(V)$). Given a vector $x = (x_I)_{I \in \mathcal{P}_r(V)}$, consider the matrix:

$$M_r(x) := (x_{I \cup J})_{I, J \in \mathcal{P}_r(V)}$$

indexed by $\mathcal{P}_r(V)$, known as the (combinatorial) moment matrix of x of order r . Consider the program:¹

$$(3.1) \quad \text{las}^{(r)}(G) := \max \sum_{i \in V} x_i \quad \text{s.t.} \quad M_r(x) \succeq 0, \quad x_{\mathbf{0}} = 1, \quad x_{ij} = 0 \quad (ij \in E),$$

with variable $x \in \mathbb{R}^{\mathcal{P}_r(V)}$. As the feasible region is bounded, the maximum is indeed attained in program (3.1). Obviously, $\text{las}^{(r+1)}(G) \leq \text{las}^{(r)}(G)$ (since $M_r(x)$ is a principal submatrix of $M_{r+1}(x)$) and, in view of (2.4), $\text{las}^{(1)}(G) = \vartheta(G)$. In this way one obtains a hierarchy of semidefinite programming bounds for the stability number, known as Lasserre’s hierarchy [19, 20]. Indeed, if S is a stable set, the vector $x := \chi^{S,2r}$ is feasible for (3.1) with objective value $|S|$, showing that $\alpha(G) \leq \text{las}^{(r)}(G)$. For fixed r , the parameter $\text{las}^{(r)}(G)$ can be computed in polynomial time (to an arbitrary precision) since the semidefinite program (3.1) involves matrices of size $O(n^r)$ with $O(n^{2r})$ variables (see, e.g., [38] for details on semidefinite programming). It is shown in [20] that, for $r \geq \alpha(G)$,

$$(3.2) \quad x \text{ is feasible for (3.1)} \iff x = \sum_{S \text{ stable}} \lambda_S \chi^{S,2r}, \quad \text{for some } \lambda \geq 0, \quad \sum_{S \text{ stable}} \lambda_S = 1.$$

This implies that

$$(3.3) \quad \alpha(G) = \text{las}^{(r)}(G) \text{ for } r \geq \alpha(G).$$

3.2. An analogous semidefinite programming hierarchy towards the fractional chromatic number. For an integer $r \geq 1$, define the parameter

$$(3.4) \quad \psi^{(r)}(G) := \min t \quad \text{s.t.} \quad M_r(x) \succeq 0, \quad x_{\mathbf{0}} = t, \quad x_i = 1 \quad (i \in V), \quad x_{ij} = 0 \quad (ij \in E),$$

where the variable x is indexed by $\mathcal{P}_r(V)$. Note that one can avoid the variable t simply by replacing t by $x_{\mathbf{0}}$ in the objective function. We choose this formulation in

¹One can easily verify that, under the condition $M_r(x) \succeq 0$, the edge condition $x_{ij} = 0$ for $ij \in E$ implies that $x_I = 0$ for any $I \in \mathcal{P}_r(V)$ containing an edge.

order to have a unified presentation of the various bounds; compare, e.g., with (2.9), (2.11), (3.9), (3.12), and (3.14). Again the minimum is attained in program (3.4), and, for fixed r , one can compute $\psi^{(r)}(G)$ to any arbitrary precision in polynomial time.

THEOREM 3.1. *The parameters $\psi^{(r)}(G)$ satisfy:*

- (a) $\psi^{(r)}(G) \leq \psi^{(r+1)}(G)$,
- (b) $\psi^{(1)}(G) = \bar{\vartheta}(G)$,
- (c) $\bar{\vartheta}^{+\Delta}(G) \leq \psi^{(2)}(G)$,
- (d) $\psi^{(r)}(G) \leq \chi^*(G)$, with equality if $r \geq \alpha(G)$,
- (e) $\psi^{(r)}(G)\text{las}^{(r)}(G) \geq |V(G)|$, with equality if G is vertex-transitive.

Proof. (a) is obvious. For (b), let $M_1(x) = \begin{pmatrix} t & e^T \\ e & M \end{pmatrix}$ be a matrix optimal for (3.4) with $r = 1$. Then $\psi^{(1)}(G) = t \geq 2$ (as G has an edge) and $M_1(x) \succeq 0$ or, equivalently, $M - \frac{1}{t}ee^T \succeq 0$. After setting $U := \frac{t}{t-1}(M - \frac{1}{t}ee^T) = \frac{t}{t-1}M - \frac{1}{t-1}ee^T$, we can rewrite the program for $\psi^{(1)}(G)$ in the following way:

$$\begin{aligned} \psi^{(1)}(G) = \min t \quad \text{s.t.} \quad & U_{ii} = 1, \\ & U_{ij} = -\frac{1}{t-1} \quad (ij \in E), \\ & U \succeq 0, \quad t \geq 2. \end{aligned}$$

Thus, in view of (2.3), $\psi^{(1)}(G) = \bar{\vartheta}(G)$.

(c) Assume that (t, x) is feasible for the program defining $\psi^{(2)}(G)$. Consider the principal submatrix X of $M_2(x)$ indexed by $\{k, ij, ik, jk\}$, where i, j, k are distinct elements of V and the vector $w := (1, 1, -1, -1)^T$. Then $w^T X w \geq 0$ gives $x_{ik} + x_{jk} - x_{ij} \leq 1$. By setting $U := \frac{t}{t-1}((x_{ij})_{i,j=1}^n - \frac{1}{t}\mathbf{J})$, one can now verify that (t, U) is feasible for the program defining $\bar{\vartheta}^{+\Delta}(G)$, which shows the result.

(d) Let λ be an optimum solution for the minimization program defining $\chi^*(G)$ (recall (2.1)). That is, $e^T \lambda = \chi^*(G)$, $\sum_{S \text{ stable}} \lambda_S \chi^S = e$, and $\lambda \geq 0$. For $r \in \mathbb{N}$, the vector $x := \sum_{S \text{ stable}} \lambda_S \chi^{S,r}$ is feasible for (3.4) with objective value $\chi^*(G)$, which shows that $\psi^{(r)}(G) \leq \chi^*(G)$. Assume now that $r \geq \alpha(G)$, and consider an optimum solution $M_r(x)$ for (3.4). By setting $y := \frac{1}{\psi^{(r)}(G)}x$, we have $M_r(y) \succeq 0$, $y_{\mathbf{0}} = 1$, and $y_{ij} = 0$ ($ij \in E$). By using (3.2) we derive $y = \sum_{S \text{ stable}} \lambda_S \chi^{S,2r}$ for some $\lambda_S \geq 0$, with $\sum_S \lambda_S = 1$. By rescaling and taking the projection onto the subspace \mathbb{R}^V , we find a decomposition $e = \psi^{(r)}(G) \sum_{S \text{ stable}} \lambda_S \chi^S$, with $\sum_S \lambda_S \psi^{(r)}(G) = \psi^{(r)}(G)$, which shows that $\chi^*(G) \leq \psi^{(r)}(G)$.

(e) Take again an optimum solution $M_r(x)$ for (3.4), and let $n = |V(G)|$. Since $M_r(\frac{1}{\psi^{(r)}(G)}x)$ is feasible for (3.1) with objective value $\frac{n}{\psi^{(r)}(G)}$, we get $\text{las}^{(r)}(G) \geq \frac{n}{\psi^{(r)}(G)}$. Assume that G is vertex-transitive. Then there exists an optimum solution x for (3.1) which is invariant under the action of the automorphism group of G . In particular, $x_i = x_j$ for all $i, j \in V$ and thus $x_i = \frac{\text{las}^{(r)}(G)}{n}$ for all $i \in V$. Then the matrix $\frac{n}{\text{las}^{(r)}(G)}M_r(x)$ is feasible for (3.4), yielding $\psi^{(r)}(G) \leq \frac{n}{\text{las}^{(r)}(G)}$. \square

Theorem 3.1 shows that the reciprocity relations (2.5) and (2.2) for the pairs $(\bar{\vartheta}, \bar{\vartheta}) = (\text{las}^{(1)}, \psi^{(1)})$ and $(\alpha, \chi^*) = (\text{las}^{(r)}, \psi^{(r)})$ (for r large, $r \geq \alpha(G)$) extend to any order r pair $(\text{las}^{(r)}, \psi^{(r)})$ in the hierarchy.

3.3. The hierarchy $\Psi_{\text{las}^{(r)}}(G)$ ($r \geq 0$) towards the chromatic number.

By applying the operator Ψ to the hierarchy $\text{las}^{(r)}(\cdot)$ introduced in section 3.1, we

obtain the following hierarchy of lower bounds for $\chi(G)$:

$$(3.5) \quad \begin{aligned} \Psi_{\text{las}^{(r)}}(G) &= \min t \text{ s.t. } \text{las}^{(r)}(G_t) = n \\ &= \min t \text{ s.t. } y_{\mathbf{0}} = 1, \sum_{u \in V(G_t)} y_u = n, \\ &\quad y_{uv} = 0 \ (uv \in E(G_t)), \ M_r(y) \succeq 0, \end{aligned}$$

where the variable y is indexed by $\mathcal{P}_{2r}(V(G_t))$. As $\alpha(G_t) \leq n$, we deduce by using (3.3) that $\text{las}^{(n)}(G_t) = \alpha(G_t)$ for all $t \in \mathbb{N}$. Therefore, (1.2) implies the following.

PROPOSITION 3.2. $\Psi_{\text{las}^{(n)}}(G) = \chi(G)$.

In fact, this new hierarchy $\Psi_{\text{las}^{(r)}}$ refines the hierarchy $\psi^{(r)}$.

PROPOSITION 3.3. For any integer $r \geq 1$, $\psi^{(r)}(G) \leq \Psi_{\text{las}^{(r)}}(G)$.

Proof. Let (t, y) be feasible for the program defining the parameter $\Psi_{\text{las}^{(r)}}(G)$; that is, $y \in \mathbb{R}^{\mathcal{P}_{2r}(V(G_t))}$ satisfies $y_{\mathbf{0}} = 1$, $y_{uv} = 0$ ($uv \in E(G_t)$), $\sum_{u \in V(G_t)} y_u = n$, and $M_r(y) \succeq 0$. We may assume w.l.o.g. that y is invariant under the action of the symmetric group $\text{Sym}(t)$. The next claim determines y_u for $u \in V(G_t)$.

CLAIM 3.4. $y_u = \frac{1}{t}$ for all $u \in V(G_t)$.

Proof. Let X denote the principal submatrix of $M_r(y)$ indexed by $\mathcal{P}_1(V(G_t))$. With respect to the partition of $\mathcal{P}_1(V(G_t)) \sim \{\mathbf{0}\} \cup V(G_t)$ into $\{\mathbf{0}\} \cup V_1 \cup \dots \cup V_t$ (recall (1.1)), the matrix X has the block form

$$(3.6) \quad \begin{pmatrix} 1 & a^T & a^T & \dots & a^T \\ a & A & B & \dots & B \\ a & B & A & \dots & B \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a & B & B & \dots & A \end{pmatrix},$$

$\underbrace{\hspace{15em}}_{t \text{ blocks}}$

where $a = \text{diag}(A)$, $\text{diag}(B) = 0$, $A_{ij} = 0$ for $ij \in E(G)$, and $e^T a = \frac{n}{t}$. By taking the Schur complement with respect to the left upper corner and using Lemma 2.8, we have $A + (t-1)B - taa^T \succeq 0$. This implies that $\langle \mathbf{J}, A + (t-1)B \rangle \geq t(e^T a)^2 = \frac{n^2}{t}$. On the other hand, by Lemma 2.9, $\langle \mathbf{J}, A + (t-1)B \rangle \leq n \text{Tr}(A + (t-1)B) = n \text{Tr}(A) = \frac{n^2}{t}$. Hence equality holds, implying that $A + (t-1)B = \frac{1}{t} \mathbf{J}$ and thus $a = \frac{1}{t} e$. This shows that $y_u = \frac{1}{t}$ for all $u \in V(G_t)$. \square

Define the vector $x \in \mathbb{R}^{\mathcal{P}_{2r}(V)}$ with I th entry $x_I := ty_{\{pi|i \in I\}}$ for $I \in \mathcal{P}_{2r}(V) \setminus \{\mathbf{0}\}$ (where p is any fixed integer in $\{1, \dots, t\}$) and $x_{\mathbf{0}} = t$. Then $M_r(x) \succeq 0$, since it coincides with the principal submatrix of $M_r(ty)$ indexed by $\{\mathbf{0}\} \cup \{\{pi \mid i \in I\} \mid I \in \mathcal{P}_r(V) \setminus \{\mathbf{0}\}\}$. Moreover, $x_{\mathbf{0}} = t$ and $x_i = 1$ for $i \in V$. Thus, (t, x) is feasible for the program (3.4), which implies that $\psi^{(r)}(G) \leq \Psi_{\text{las}^{(r)}}(G)$. \square

In summary, we have shown the following relations among the graph parameters $\text{las}^{(r)}(G)$, $\psi^{(r)}(G)$, and $\Psi_{\text{las}^{(r)}}(G)$:

$$(3.7) \quad \frac{|V(G)|}{\text{las}^{(r)}(G)} \leq \psi^{(r)}(G) \leq \Psi_{\text{las}^{(r)}}(G) \leq \chi(G).$$

Let us point out again that, while $\psi^{(r)}(G)$ remains below the fractional chromatic number $\chi^*(G)$, $\Psi_{\text{las}^{(r)}}(G)$ may reach the chromatic number $\chi(G)$.

3.4. Variations of the second order bounds. As observed in Theorem 3.1 and Proposition 3.3, we have

$$\overline{\vartheta}^+(G) \leq \overline{\vartheta}^{+\Delta}(G) \leq \psi^{(2)}(G) \leq \Psi_{\text{las}^{(2)}}(G).$$

To compute $\psi^{(2)}(G)$ one needs to solve a semidefinite program with matrix size $O(n^2)$ and with $O(n^4)$ variables. We now introduce some variations of the parameters $\psi^{(2)}(G)$ and $\Psi_{\text{las}^{(2)}}(G)$ which are less costly to compute but still at least as good as $\overline{\vartheta}^+(G)$. The idea is to consider, instead of the full moment matrix of order 2, some principal submatrix of it. Namely, given $h \in V$, let $M_2(h; x)$ denote the principal submatrix of $M_2(x)$ indexed by the subset $\mathcal{P}_1(V) \cup \{\{h, i\} \mid i \in V\}$ of $\mathcal{P}_2(V)$. Thus in order to define the matrices $M_2(h; x)$ for all $h \in V$, one needs only the components of x indexed by $\mathcal{P}_3(V)$. By following [21], define the following upper bound for the stability number $\alpha(G)$:

$$(3.8) \quad \ell(G) := \max \sum_{i \in V} x_i \text{ s.t. } M_2(h; x) \succeq 0 \ (h \in V), \ x_{\mathbf{0}} = 1, \ x_{ij} = 0 \ (ij \in E(G)),$$

with variable $x \in \mathbb{R}^{\mathcal{P}_3(V)}$. Obviously,

$$\text{las}^{(2)}(G) \leq \ell(G) \leq \text{las}^{(1)}(G) = \vartheta(G).$$

Next, define the graph parameter

$$(3.9) \quad \psi(G) := \min t \text{ s.t. } M_2(h; x) \succeq 0 \ (h \in V), \ x_{ij} = 0 \ (ij \in E(G)), \\ x_{\mathbf{0}} = t, \ x_i = 1 \ (i \in V),$$

where the variable x is indexed by $\mathcal{P}_3(V)$. Again one can avoid variable t by replacing t by $x_{\mathbf{0}}$ in the objective function. We first observe that the pair (ℓ, ψ) satisfies the analogue of the reciprocity relation from Theorem 3.1(e) for the pairs $(\text{las}^{(r)}, \psi^{(r)})$.

PROPOSITION 3.5. *We have*

$$(3.10) \quad \ell(G)\psi(G) \geq |V(G)|, \text{ with equality if } G \text{ is vertex-transitive,}$$

$$(3.11) \quad \overline{\vartheta}^+(G) \leq \psi(G) \leq \psi^{(2)}(G).$$

Proof. The proof for (3.10) is analogous to that of Theorem 3.1(e), and the right inequality in (3.11) is obvious. For the left inequality, let (t, x) be feasible for (3.9). Observe first that $x_{hi} \geq 0$ for all $h, i \in V$, since x_{hi} is the diagonal entry of $M_2(h; x)$ at the $\{h, i\}$ th position and $M_2(h; x) \succeq 0$. Let A denote the principal submatrix of $M_2(h; x)$ indexed by V . Then $A = (x_{ij})_{i,j \in V} \geq 0$ and $A - \frac{1}{t}\mathbf{J} \succeq 0$, which implies that $U := \frac{t}{t-1}(A - \frac{1}{t}\mathbf{J})$ is feasible for the program defining $\overline{\vartheta}^+(G)$ (recall (2.7)). \square

By applying the operator Ψ to the parameter $\ell(\cdot)$ (introduced in (3.8)), one obtains the lower bound $\Psi_\ell(G)$ for $\chi(G)$, defined as

$$(3.12) \quad \Psi_\ell(G) = \min_{t \in \mathbb{N}} t \text{ s.t. } \ell(K_t \square G) = n \\ = \min_{t \in \mathbb{N}} t \text{ s.t. } \sum_{u \in V(G_t)} y_u = n, \ y_{uv} = 0 \ (uv \in E(G_t)), \\ y_{\mathbf{0}} = 1, \ M_2(u; y) \succeq 0 \ (u \in V(G_t)),$$

where the variable y is indexed by $\mathcal{P}_3(V(G_t))$. (Recall that $G_t = K_t \square G$.)

PROPOSITION 3.6. $\psi(G) \leq \Psi_\ell(G) \leq \Psi_{\text{las}^{(2)}}(G)$.

Proof. The right inequality follows from Lemma 2.2(b), and the proof for the left inequality is analogous to that of Proposition 3.3. \square

In summary, we have the following analogue of (3.7) about $\ell(G)$, $\psi(G)$, and $\Psi_\ell(G)$:

$$(3.13) \quad \frac{|V(G)|}{\ell(G)} \leq \psi(G) \leq \Psi_\ell(G) \leq \chi(G).$$

Again, $\psi(G) \leq \chi^*(G)$ since $\psi^{(2)}(G) \leq \chi^*(G)$, but $\Psi_\ell(G)$ may sometimes reach $\chi(G)$. The bound $\Psi_\ell(G)$ can be especially useful when the gap between $\chi^*(G)$ and $\chi(G)$ is large, e.g., when $\chi^*(G) \sim \omega(G) < \chi(G)$. We refer to the follow-up paper [14], where such graph instances will be considered (e.g., Kneser graphs) with experimental results. One can easily verify that the graph parameter $\ell(\cdot)$ is monotone nondecreasing with respect to induced subgraphs. Therefore, as explained in section 2.4, one can compute $\Psi_\ell(G)$ by evaluating $\ell(G_t)$ for $O(\log n)$ queries of t . We will show in the follow-up paper [14] how to give a more compact reformulation for the program (3.12) when G is a vertex-transitive graph. Namely, we will show there that each $\ell(G_t)$ can be computed via a semidefinite program involving four matrices of size $2n + 1$, $2n$, n , and n , respectively.

3.5. Link with copositive programming-based hierarchies. We have just seen one possible construction for hierarchies of bounds towards $\alpha(G)$ and $\chi^*(G)$, based on the method of Lasserre. As mentioned earlier in this section there are several other possible constructions for approximating the stable set problem. However, to the best of our knowledge, such constructions were much less investigated for the coloring problem. Recently Dukanovic and Rendl [9] investigated a hierarchy of lower bounds for $\chi^*(G)$, which is closely related to the hierarchy of de Klerk and Pasechnik [6] for $\alpha(G)$; both are based on copositive programming and some of its tractable relaxations in terms of sums of squares of polynomials, proposed by Parrilo [29]. Let \mathcal{C}_n denote the cone of $n \times n$ copositive matrices and \mathcal{C}_n^* its dual cone, consisting of the completely positive matrices. Thus $M \in \mathcal{C}_n$ if and only if $p_M(x) := \sum_{i,j=1}^n M_{ij}x_i^2x_j^2 \geq 0$ for all $x \in \mathbb{R}^n$. Obviously if, for some $r \in \mathbb{N}$, the polynomial $p_M(x)(\sum_{i=1}^n x_i^2)^r$ can be written as a sum of squares of polynomials (s.o.s. for short), then $M \in \mathcal{C}_n$. By following Parrilo [29], for an integer $r \geq 0$, define the cone

$$K_n^{(r)} := \left\{ M \in \mathbb{R}^{n \times n} \mid p_M(x) \left(\sum_{i=1}^n x_i^2 \right)^r \text{ is s.o.s.} \right\}.$$

Thus, $K_n^{(r)} \subseteq K_n^{(r+1)} \subseteq \mathcal{C}_n$. By following [6], define the graph parameter

$$\vartheta^{(r)}(G) := \min t \text{ s.t. } t(\mathbf{I} + A_G) - \mathbf{J} \in K_n^{(r)}.$$

In view of (2.30), $\alpha(G) \leq \vartheta^{(r)}(G)$. Moreover, it is proved in [6] that $\vartheta^{(0)}(G) = \vartheta'(G)$ (defined in (2.6)) and $\lfloor \vartheta^{(r)}(G) \rfloor = \alpha(G)$ for $r \geq (\alpha(G))^2$. Dukanovic and Rendl [9] propose an analogous hierarchy toward the fractional chromatic number. To start with, they show the following copositive programming formulation for $\chi^*(G)$:

$$(3.14) \quad \chi^*(G) = \min t \text{ s.t. } \begin{aligned} X_{ii} &= t \ (i \in V), \ X_{ij} = 0 \ (ij \in E(G)), \\ X &\in \mathcal{C}_n^*, \ X - \mathbf{J} \succeq 0. \end{aligned}$$

For an integer $r \geq 0$, let $\kappa^{(r)}(G)$ denote the graph parameter obtained by replacing the cone \mathcal{C}_n by its subcone $K_n^{(r)}$ in (3.14). Thus, $\kappa^{(r)}(G) \leq \kappa^{(r+1)}(G) \leq \chi^*(G)$. Moreover, it is proved in [9] that $\kappa^{(0)}(G) = \overline{\vartheta}^+(G)$ (defined in (2.7)) and that the pair $(\vartheta^{(r)}, \kappa^{(r)})$ satisfies the reciprocity relation:

$$(3.15) \quad \vartheta^{(r)}(G)\kappa^{(r)}(G) \geq |V(G)|, \text{ with equality if } G \text{ is vertex-transitive,}$$

thus extending (2.8) for the case $r = 0$.

Now one may wonder what the link is between the two hierarchies $\text{las}^{(r)}$ and $\vartheta^{(r)}$ for α and between the two hierarchies $\psi^{(r)}$ and $\kappa^{(r)}$ for χ^* . Here is what we can say about this. In order to be able to compare the various bounds we have to add nonnegativity to the definition of $\text{las}^{(r)}$ and $\psi^{(r)}$; namely, let $\text{las}_{\geq 0}^{(r)}(G)$ (resp., $\psi_{\geq 0}^{(r)}(G)$, $\ell_{\geq 0}(G)$, and $\psi_{\geq 0}(G)$) denote the parameter obtained by adding the condition $x \geq 0$ to program (3.1) (resp., to (3.4), (3.8), and (3.9)). The analogue of Theorem 3.1(e) holds for the pairs $(\text{las}_{\geq 0}^{(r)}, \psi_{\geq 0}^{(r)})$ and $(\ell_{\geq 0}, \psi_{\geq 0})$ as well, and we have $\text{las}_{\geq 0}^{(1)}(G) = \vartheta'(G) = \vartheta^{(0)}(G)$ and $\psi_{\geq 0}^{(1)}(G) = \bar{\vartheta}^+(G) = \kappa^{(0)}(G)$. It is shown in [13] that, for any graph G ,

$$\text{las}_{\geq 0}^{(r)}(G) \leq \vartheta^{(r-1)}(G) \quad \text{for all } r \geq 1,$$

and the same proof technique also shows that $\ell_{\geq 0}(G) \leq \vartheta^{(1)}(G)$ (see [12] for details). In view of the reciprocity relations for the pairs $(\ell_{\geq 0}, \psi_{\geq 0})$, $(\text{las}_{\geq 0}^{(r)}, \psi_{\geq 0}^{(r)})$, and $(\vartheta^{(r)}, \kappa^{(r)})$, this implies that

$$\kappa^{(1)}(G) \leq \psi_{\geq 0}(G), \quad \kappa^{(r-1)}(G) \leq \psi_{\geq 0}^{(r)}(G) \quad (r \geq 1), \quad \text{when } G \text{ is vertex-transitive.}$$

It is an open question to determine whether the above inequalities remain valid when G is not vertex-transitive. See [9, 14] for instances of Hamming graphs (which are indeed vertex-transitive) having a substantial gap between the two bounds $\kappa^{(1)}(G)$ and $\psi_{\geq 0}(G)$.

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