On weak regularity $\iff$ strong regularity

Notes for our seminar — Lex Schrijver

Let X be an inner product space and let R be a bounded subset of X spanning X. (So each element of X is a linear combination of finitely many elements of R.) Let G be the group of orthogonal transformations π of X with $\pi(R) = R$. Let $B(X)$ denote the unit ball in X. For any k, let $R_k := \{\pm r_1 \pm \cdots \pm r_k \mid r_1, \ldots, r_k \in R\}$.

Call R *weakly regular* if for each k there exists a finite set $Z \subseteq X$ such that for each $x \in R_k$ there exist $z \in Z$ and $\pi \in G$ satisfying $\langle r, x - z^\pi \rangle^2 \leq 1$ for each $r \in R$.

Call R *strongly regular* if for each $\varepsilon > 0$ and $f : X \to \{1, 2, \ldots\}$ there exists a finite set $Z \subseteq X$ such that for each $x \in B(X)$ there exist $z \in Z$ and $\pi \in G$ satisfying

$$(1) \qquad \sum_{j=1}^{f(z)} \langle r_j, x - z^\pi \rangle^2 < \varepsilon$$

for all orthogonal $r_1, \ldots, r_{f(z)} \in R$.

Theorem 1. *R is weakly regular $\iff$ R is strongly regular.*

Proof. $\Longleftarrow$ being easy, we prove $\Longrightarrow$. As R is bounded, by scaling we can assume that $\|r\| \leq 1$ for each $r \in R$. Let H be the completion of X. For $x, y \in H$ define:

$$(2) \qquad d_R(x, y) := \sup_{r \in R} |\langle r, x - y \rangle|.$$

Then weak regularity of R implies that for each k, the set $\{\lambda_1 r_1 + \cdots + \lambda_k r_k \mid r_1, \ldots, r_k \in R, \lambda_1, \ldots, \lambda_k \in [-1, +1]\}/G$ is totally bounded. Hence, by [2], the space $(B(H), d_R)/G$ is compact.

Choose $\varepsilon > 0$ and $f : X \to \{1, 2, \ldots\}$. For each $z \in X$ define

$$(3) \qquad U_z := \{x \in H \mid \sup_{\substack{\text{orthogonal} \\ r_1, \ldots, r_{f(z)} \in R}} \sum_{i=1}^{f(z)} \langle r_i, x - z \rangle^2 < \varepsilon\}.$$

Then U_z is open in $(B(H), d_R)$, for choose $x \in U_z$. Let s be the supremum in (3). Let $\delta := (\sqrt{\varepsilon} - \sqrt{s})/f(z)$. Then if $d_R(y, x) < \delta$, $y \in U_z$.

Moreover, the U_z for $z \in B(X)$ cover $B(H)$. Indeed, for any $x \in B(H)$ we have $\|x - z\| < \varepsilon$ for some $z \in B(X)$, implying $x \in U_z$.

So finitely many U_z cover $B(H)/G$, which gives the strong regularity of R. ∎

Applications. Since R spans X, X is fully determined by the positive semidefinite $R \times R$ matrix giving the inner products of pairs from R. Then G is given by the group of permutations of R that leave the matrix invariant. It is convenient to realize that R is weakly regular if (but not only if) the orbit space R^k/G is compact for each k.

1. Szemerédi's regularity lemma. Let R be the collection of sets $I \times J$, with I and J each being a union of finitely many subintervals of $[0, 1]$, with inner product equal to the measure of the intersection. Then Theorem 1 gives strong regularity for step functions $[0,1]^2 \to [0,1]$, with all steps being intervals, hence (with rounding) for graphs.

2. "Interval regularity". Let R be the collection of sets $I \times J$, with I and J subintervals of $[0,1]$, with inner product given by the measure of the intersection. Then Theorem 1 gives an "interval regularity theorem" for graphs (it can also be proved with Szemerédi's classical combinatorial method):

Corollary 1a. *For each $\varepsilon > 0$ and $p \in \mathbb{N}$ there exists $k_{p,\varepsilon} \in \mathbb{N}$ such that for each n, each graph $G = ([n], E)$ and each partition P of $[n]$ into intervals with $|P| \leq p$, P has a refinement to a partition Q into at most $k_{p,\varepsilon}$ intervals such that all intervals in Q have the same size except for some of them covering $\leq \varepsilon n$ vertices and such that*

$$(4) \qquad \sum_{A,B \in Q} \max_{\substack{I \subseteq A, J \subseteq B \\ I,J \text{ intervals}}} |I||J||d(I,J) - d(A,B)| < \varepsilon n^2.$$

Here $d(I,J)$ and $d(A,B)$ are the densities of the corresponding subgraphs of G.

3. Polynomial approximation. Let $k \leq n$. Each polynomial $p \in \mathbb{R}[x_1, \ldots, x_n]$ can be uniquely written as $p = \sum_\mu \mu p_\mu$, where μ ranges over the set M of all monomials in $\mathbb{R}[x_1, \ldots, x_k]$ and where $p_\mu \in \mathbb{R}[x_{k+1}, \ldots, x_n]$. If p is homogeneous of degree d, we say that p is *ε-concentrated on the first k variables* if

$$(5) \qquad \sum_{\substack{\mu \in M \\ \deg(\mu) < d}} \max_{\substack{x \in \mathbb{R}^{n-k} \\ \|x\|=1}} p_\mu(x)^2 \leq \varepsilon \|p\|^2,$$

where $\|p\|$ is the square root of the sum of the squares of the coefficients of p.

Corollary 1b. *For each $\varepsilon > 0$ and $d \in \mathbb{N}$ there exists $k_{d,\varepsilon}$ such that for each n, each homogeneous polynomial of degree d in n variables is ε-concentrated on the first k variables after some orthogonal transformation of $\mathbb{R}^n$, for some $k \leq k_{d,\varepsilon}$.*

This can be derived by setting R to be the set of all polynomials $(a^\mathsf{T} x)^d$, with $a \in \mathbb{R}^n$ and $\|a\| = 1$ for some n (setting $x = (x_1, x_2, \ldots)$), taking the inner product of $(a^\mathsf{T} x)^d$ and $(b^\mathsf{T} x)^d$ equal to $(a^\mathsf{T} b)^d$. (This corollary strengthens a 'weak regularity' result of Fernandez de la Vega, Kannan, Karpinski, and Vempala [1].)

References

[1] W. Fernandez de la Vega, R. Kannan, M. Karpinski, S. Vempala, Tensor decomposition and approximation schemes for constraint satisfaction problems, in: *Proceedings of the 37th Annual ACM Symposium on Theory of Computing* (STOC'05), pp. 747–754, ACM, New York, 2005.

[2] G. Regts, A. Schrijver, Compact orbit spaces in Hilbert spaces and limits of edge-colouring models, preprint, 2012. ArXiv `http://arxiv.org/abs/1210.2204`